COMPUTER PROOFS OF MATRIX PRODUCT IDENTITIES

PETER J. LARCOMBE, AXEL RIESE, AND BURKHARD ZIMMERMANN

Abstract. We introduce a straightforward but useful method for computing indefinite rational matrix products. The method is used to prove a certain identity involving definite sums and a definite integral.

1. Introduction

Matrix products such as (2) arise in certain problems of applied mathematics. Our object is to show that indefinite rational matrix products, i.e. indefinite products of square matrices with entries being rational functions, have *P-recursive* [5] entries. A function (or sequence) f from $\mathbb{N}$ to a field $\mathbb{F}$ is said to be *P-recursive* over $\mathbb{F}$ if there exist polynomials $p_0(n), \ldots, p_d(n) \in \mathbb{F}[n]$, not all zero, such that

$$p_d(n)f(n+d) + \cdots + p_0(n)f(n) = 0.$$

Such sequences are closed under various operations like addition and multiplication. In the following we show how to compute recurrences for (the entries of) indefinite rational matrix products algorithmically.

Theorem 1. *Let $\mathbb{F}$ be a field and let $M(x)$ be a $d \times d$ matrix over $\mathbb{F}(x)$. Let*

$$A(n) := \prod_{k=0}^{n} M(k) := M(0)M(1)\cdots M(n)$$

be its indefinite product. Fix indices $1 \le i, j \le d$. Then $a_{ij}(n)$ is P-recursive over $\mathbb{F}$. For computable $\mathbb{F}$ it is possible to compute such a recurrence algorithmically.

Proof. Since

$$A(n+1) = A(n)M(n+1), \tag{1}$$

the d^2 entries of $A(n)$ satisfy a coupled system of linear recurrence equations with polynomial coefficients. Such a system can always be decoupled [1, 2, 8]. Therefore each entry satisfies a recurrence in n with coefficients in $\mathbb{F}[n]$. □

It is easy to see that system (1) splits into d independent subsystems consisting of d equations in d variables. Each subsystem may be decoupled separately.

Date: July 9, 2003.

2000 *Mathematics Subject Classification.* 15A24, 33F10, 68W30.

Key words and phrases. Matrix product, computer algebra, definite integration, recurrence relations.

The second and third author were supported by SFB grant F1305 of the Austrian FWF.

2. A Simple Example

Consider the integral

$$\int_0^\infty e^{(ir-m)x}(1-e^{-x})^n \, dx =: u(n) + iv(n),$$

say, where $i^2 = -1$, n is a natural number, and m and r are real numbers with $m > 0$. By expanding the term $(1-e^{-x})^n$ binomially, it is easy to show that the integral has real and imaginary parts

$$u(n) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{m+j}{(m+j)^2 + r^2}$$

and

$$v(n) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \frac{r}{(m+j)^2 + r^2}.$$

On the other hand, integrating the given integral by parts and uncoupling the resulting recurrences for $u(n)$ and $v(n)$ leads eventually to the following theorem. Note that these lengthy analytic procedures yield only the first column of $A(n)$ in Eq. (3). We conjecture the full form of $A(n)$, and use our computer-based method to prove the entries stated.

Theorem 2. *Let n, m, r, $u(n)$, and $v(n)$ be as above. Define*

$$P(n) := \prod_{j=0}^{n} \begin{pmatrix} m+n-j & r \\ -r & m+n-j \end{pmatrix}. \tag{2}$$

Then $P(n)$ is invertible and its inverse $A(n) := P(n)^{-1}$ satisfies

$$A(n) = \frac{1}{n!} \begin{pmatrix} u(n) & -v(n) \\ v(n) & u(n) \end{pmatrix}. \tag{3}$$

Proof. We prove $a_{11}(n) = u(n)/n!$ only; the proofs of the remaining three parts of Eq. (3) are analogous. We proceed by induction on n. Clearly, the identity is true for $n = 0$ and $n = 1$. It remains to show that both sides satisfy the same second-order recurrence.

First, we try to compute a recurrence for $a_{11}(n)$ by Theorem 1. Unfortunately, the product in (2) is not indefinite: the multiplicand involves the upper bound n. To make the product indefinite we reparametrize (2) by $j = n - k$. Then we pull matrix inversion into the product. Since the matrices in (2) commute, we eventually obtain the indefinite product

$$A(n) = \prod_{k=0}^{n} \begin{pmatrix} m+k & r \\ -r & m+k \end{pmatrix}^{-1}.$$

By inverting the multiplicand we arrive at

$$A(n) = \prod_{k=0}^{n} \frac{1}{(m+k)^2 + r^2} \begin{pmatrix} m+k & -r \\ r & m+k \end{pmatrix},$$

which fits Theorem 1 with $d = 2$ and $\mathbb{F} = \mathbb{Q}(m, r)$. In this case recurrence (1) reads

$$A(n+1) = A(n) \, \frac{1}{(m+n+1)^2 + r^2} \begin{pmatrix} m+n+1 & -r \\ r & m+n+1 \end{pmatrix}. \tag{4}$$

Equation (4) is made up of two independent systems of two recurrences at each case. Here we have to consider only the one containing the sequences $a_{11}(n)$ and $a_{12}(n)$:

In[1]:= $\text{sys} = \Big\{a_{11}[n+1] == \frac{a_{11}[n]\,(m+n+1)}{(m+n+1)^2+r^2} + \frac{a_{12}[n]\,r}{(m+n+1)^2+r^2},$
$a_{12}[n+1] == \frac{-a_{11}[n]\,r}{(m+n+1)^2+r^2} + \frac{a_{12}[n]\,(m+n+1)}{(m+n+1)^2+r^2}\Big\};$

We uncouple this system by Stefan Gerhold's [3] Mathematica implementation OreSys[1] of Zürcher's [8] algorithm:

In[2]:= <<OreSys.m

OreSys Package by Stefan Gerhold — © RISC Linz — V 1.1 (12/02/02)

In[3]:= UncoupleDifferenceSystem[sys, $\{a_{11}[n], a_{12}[n]\}$, $\{a_{11}[n], a_{12}[n]\}$, n, Method → Zuercher][[1, 1]]

Out[3]= $-\frac{a_{11}[n]}{4+4m+m^2+4n+2mn+n^2+r^2} + \frac{(3+2m+2n)\,a_{11}[1+n]}{4+4m+m^2+4n+2mn+n^2+r^2} - a_{11}[2+n] == 0$

It suffices to show that $u(n)/n!$ satisfies the same recurrence. Indeed, Peter Paule's and Markus Schorn's [4] Mathematica implementation FastZeil[2] of Zeilberger's algorithm [6, 7] finds:

In[4]:= <<zb.m

Fast Zeilberger Package by Peter Paule, Markus Schorn, and Axel Riese — © RISC Linz — V 3.39 (03/14/03)

In[5]:= $\text{Zb}\Big[\frac{(-1)^j(m+j)}{j!(n-j)!((m+j)^2+r^2)}, \{j,0,n\}, n, 2\Big]$

If 'n' is a natural number, then:

Out[5]= $\{-\text{SUM}[n] + (3+2m+2n)\,\text{SUM}[1+n] - (4+4m+m^2+4n+2mn+n^2+r^2)\,\text{SUM}[2+n] == 0\}$

By clearing denominators both recurrences agree. Since our assumption on n, m, and r guarantees that the leading coefficient

$$(4+4m+m^2+4n+2mn+n^2+r^2) = (m+n+2+ir)(m+n+2-ir)$$

of the recurrence does not vanish for any critical n, the recurrence uniquely determines a sequence for given initial values for $n=0$ and $n=1$. This proves $a_{11}(n) = u(n)/n!$ by induction on n. □

Finally, we want to remark that uncoupling the two recurrence equations in our proof could have been done also by hand without much effort. However, for $d > 2$, uncoupling is usually no longer a simple task without computer algebra.

Acknowledgments. We would like to thank Stefan Gerhold for helpful comments.

[1] available at *http://www.risc.uni-linz.ac.at/research/combinat/risc/software/OreSys*
[2] available at *http://www.risc.uni-linz.ac.at/research/combinat/risc/software/PauleSchorn*

References

[1] S.A. Abramov and E.V. Zima, A universal program to uncouple linear systems, in *Proc. of the International Conference on Computational Modelling and Computing in Physics* (1996) pp. 16–26.

[2] M. Bronstein and M. Petkovšek, An introduction to pseudo-linear algebra, *Theoret. Comput. Sci.* **157** (1995) 3–33.

[3] S. Gerhold, Uncoupling systems of linear Ore operator equations, Diploma Thesis, RISC, J. Kepler University Linz, 2002.

[4] P. Paule and M. Schorn, A Mathematica version of Zeilberger's algorithm for proving binomial coefficient identities, *J. Symbolic Comput.* **20** (1995) 673–698.

[5] R. P. Stanley, Differentiably finite power series, *European J. Combin.* **1** (1980) 175–188.

[6] D. Zeilberger, A fast algorithm for proving terminating hypergeometric identities, *Discrete Math.* **80** (1990) 207–211.

[7] D. Zeilberger, The method of creative telescoping, *J. Symbolic Comput.* **11** (1991) 195–204.

[8] B. Zürcher, Rationale Normalformen von pseudo-linearen Abbildungen (in German), Diploma Thesis, Mathematik, ETH Zürich, 1994.

[P.J.L.] Derbyshire Business School, University of Derby, Kedleston Road, Derby DE22 1GB, U.K.

E-mail address: `P.J.Larcombe@derby.ac.uk`

[A.R.] Research Institute for Symbolic Computation, Johannes Kepler University, A–4040 Linz, Austria

E-mail address: `Axel.Riese@risc.uni-linz.ac.at`

[B.Z.] Research Institute for Symbolic Computation, Johannes Kepler University, A–4040 Linz, Austria

E-mail address: `Burkhard.Zimmermann@risc.uni-linz.ac.at`