A Derivative Free Landweber Method for Parameter Identification in Elliptic PDEs

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Abstract

We consider the identification of a parameter in an elliptic equation which - in its weak formulation - can be described by a strictly monotone and Lipschitz continuous operator from knowledge of the physical state. Taking advantage of the special structure, we develop a derivative free Landweber iteration for solving this nonlinear inverse problem in a stable way. Thereby, the Fréchet differentiability of the parameter-to-output map as well as conditions restricting its nonlinearity are no longer required. Instead, the convergence analysis is performed under natural assumptions already associated to the solvability of the direct problem allowed to be also nonlinear. Numerical experiments are presented.

1 Introduction

This paper focuses on an iterative method for solving the inverse problem

$$F(q) = z, \tag{1.1}$$

where the nonlinear *parameter-to-output map* F maps the parameter $q \in X$ onto the (unique) solution $u_q \in Y$ of a possibly nonlinear elliptic state equation that can be weakly formulated by means of a strictly monotone and Lipschitz continuous operator. Thereby, X and Y are Hilbert spaces. Given an observed or desired physical state $z \in Y$, the parameter identification problem is to determine a parameter q_* for which $u_{q_*} = z$ holds. Problem (1.1) can also be understood as the problem of controlling the physical state in an optimal way.

As example, the parameter identification problem of estimating the positive parameter $q = q(x, y)$ in the elliptic equation

$$(q(z_{xx} + \nu z_{yy}))_{xx} + (q(z_{yy} + \nu z_{xx}))_{yy} + (qz_{xy})_{xy} \;=\; f \;\text{ on } \Omega \subset \mathbb{R}^2 \;\text{ in }\; \Omega \tag{1.2}$$

(where ν and f are known constants) from knowledge of its solution z is considered. In a first model for the sag bending process related to the manufacture of car windshields, z represents the displacement of a glass sheet corresponding to a desired windshield and q

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denotes the sought for glass material parameter that contains the information needed in order to achieve the target z by that process, see [19] for more details. Considering (1.2) as an equation for the unknown parameter q, this inverse problem leads - depending on the fastening of the glass sheet, i.e., depending on the boundary conditions on z - to a second order pde that changes its type between elliptic and hyperbolic, we refer to [24], [11] and Section 4 for examples. Facing a mixed type situation, neither existence, uniqueness of a solution q nor its continuous dependence on the data z could so far be proven. Furthermore, a numerical strategy for *directly* solving equation (1.2) for q is missing. A possible resort is based on the abstract formulation (1.1) of the inverse problem (we shall return to the windshield problem in Section 4).

Though in the introductory example only a linear state equation is considered (i.e., (1.2) considered as an equation for z), one already can draw a general conclusion for the parameter identification problems to be treated in this paper: The inverse problem (1.1) is nonlinear due to the nonlinearity of the forward operator F. Furthermore, the existence of a solution cannot be guaranteed in general, and if a solution exists, it may not depend continuously on the data. Hence, parameter identification problems typically belong to the class of nonlinear inverse and ill-posed problems.

In the following, it is always assumed that the (exact) data z are attainable, i.e., that a solution q_* to (1.1) exists, and focus on the aspect of stability. In practice, the data z may not be known exactly due to measurement or model errors, but only a rough approximation z^δ with

$$\|z - z^\delta\| \leq \delta \tag{1.3}$$

may be available. Then, a numerically stable and reliable approximation of q_* can only be sought by the use of regularization techniques, see [6] for a general introduction. Facing a nonlinear ill-posed problem as (1.1), the probably most widely used method is Tikhonov-regularization, where the approximate solution is sought as the minimizer of

$$\|z^\delta - F(q)\|^2 + \beta\|q\|^2, \tag{1.4}$$

see [7]. Since - opposed to the linear case - this functional is no longer convex and the determination of an appropriate regularization parameter can be rather numerically expensive, iterative methods are an attractive alternative. Based on a successive minimization of

$$q \to \frac{\lambda}{2}\|z^\delta - F(q)\|^2,$$

see [8] for a comprehensive survey (λ is a scaling parameter), the regularization effect now is simply obtained by stopping the iteration at the right time. Denoting the iterates by q_k^δ, the discrepancy principle for instance determines the stopping index $k_*(\delta)$ by

$$\|z^\delta - F(q_{k_*}^\delta)\| \leq \tau\delta < \|z^\delta - F(q_k^\delta)\|, \quad 0 \leq k < k_*, \tag{1.5}$$

for some sufficiently large $\tau > 0$. The (final) residual $z^\delta - F(q^\delta_{k_*})$ for the regularized solution $q^\delta_{k_*}$ then is of the order of the noise level δ, which is the best one should ask for. Note that without an estimate for the data error δ a regularization method in the strict sense of the definition cannot be constructed, see [6].

However, given an initial guess in the neighbourhood of the true solution q_*, i.e.,

$$q_* \in \mathcal{B}_{\rho/2}(q_0) \subset \mathcal{D}(F) \tag{1.6}$$

for some $\rho > 0$, convergence of the iterates, both for exact data, i.e., $q_k \to q_*$ as $k \to \infty$, as well as for perturbed data, i.e., $q^\delta_{k_*} \to q_*$ as $\delta \to 0$, could so far only be guaranteed under severe restrictions on the parameter-to-output map F and its Fréchet derivative. The most straightforward method is the Landweber iteration

$$q^\delta_{k+1} = q^\delta_k + \lambda F'(q^\delta_k)^*(z^\delta - F(q^\delta_k)), \tag{1.7}$$

where $F'(q^\delta_k)^*$ denotes the Hilbert space adjoint operator to $F'(q^\delta_k)$. Still, convergence of (1.7) requires the local boundedness

$$\|F'(q)\| \le L, \quad q \in \mathcal{B}_\rho(q_0) \tag{1.8}$$

of the iteration operator and the nonlinearity condition

$$\|F(\tilde{q}) - F(q) - F'(q)(\tilde{q} - q)\| \le \eta \|F(\tilde{q}) - F(q)\|, \quad q, \tilde{q} \in \mathcal{B}_\rho(q_0) \tag{1.9}$$

with $\eta < 1/2$ (indepently of the scaling parameter λ) or the Newton-Mysovskii condition

$$\|(F'(q) - F'(q_*))F'(q_*)^\sharp\| \le C_{NM}\|q - q_*\|, \quad q \in \mathcal{D}(F). \tag{1.10}$$

to be satisfied, where $F'(q_*)^\sharp$ denotes a left inverse of $F'(q_*)$, see [10] and [5]. In [10] and [6], (1.9) has been discussed for underlying linear elliptic state equations, then meaning a strong smallness assumption on ρ in (1.6) due to $\eta < 1/2$. Cases with nonlinear direct problems have not been addressed. In [11], we failed to verify (1.9) for this kind of inverse problems, no matter, if the nonlinearity of the direct problem was due to known terms of the differential operator or the unknown parameter itself, see the next section for examples. The alternative condition (1.10) has - to our knowledge - not been discussed at all in the context of parameter identification.

In order to relax condition (1.9), the modified Landweber iteration

$$q^\delta_{k+1} = q^\delta_k + \lambda G(q^\delta_k)^*(z^\delta - F(q^\delta_k)) \tag{1.11}$$

was suggested in [21], for which convergence could be established without the Fréchet differentiability of F but still requiring

$$\|F(\tilde{q}) - F(q) - G(q)(\tilde{q} - q)\| \le \eta \|F(\tilde{q}) - F(q)\|, \quad q, \tilde{q} \in \mathcal{B}_\rho(q_0) \tag{1.12}$$

with $\eta < 1/2$ and

$$\|G(q)\| \leq L, \quad q \in \mathcal{B}_\rho(q_0). \tag{1.13}$$

In [21], (1.11) was again applied to (1.1) only in connection with a linear direct problem. The discussions in [3] and [11] show that, when given a nonlinear direct problem, the chances for constructing an operator $G(\cdot)$ suitable for (1.12) are very limited. Finally, we mention that the convergence analysis of more advanced iterative regularization methods, see [8], is based on even stronger conditions on F and F', i.e., they then would imply (1.9).

In this paper, we present a derivative free Landweber method for solving (1.1) in the presence of (possibly) perturbed data. Though it formally looks like (1.11), the modified nonlinearity condition (1.12) is not needed for proving convergence. Instead, our choice of the iteration operator $G(\cdot)$ (for which (1.12) holds under no circumstances) allows to derive in Section 3 the desired convergence results under natural conditions already associated to the solvability of the underlying direct problem to be discussed in Section 2. Since we no longer have to mind the nonlinearity of the latter, our theory then uniformly applies to a wide range of identification problems covered by (1.1). In addition, the numerical effort compared to (1.7) will be reduced since solutions to auxiliary pde-problems and/or derivatives of the iterates with respect to the physical state become redundant, see also the numerical experiments in Section 4.

Since we focused on limits of the classical theory for Landweber iteration and its variants suggested in the literature, we want to emphasize that this theory has been developed for a widely larger range of inverse problems than considered in this paper, see [10]. Concentrating only on the subclass of parameter identification in elliptic pdes as introduced above, it is not surprising that the classical theory leaves space for improvements.

2 The Direct Problem

Let Y_0 be a closed (not necessarily strict) subspace of Y. Both for the Hilbert spaces X and Y, the inner products and norms are denoted by $(\cdot, \cdot)$ and $\|\cdot\|$, their meaning can always be identified from the context in which they appear. Furthermore, let Y_0^* be the dual space to Y_0, equipped with the duality product $\langle \cdot, \cdot \rangle$ and the duality map $J : Y_0^* \to Y_0$.

Given a parameter q out of an admissible set $Q \subset X$, the direct problem consists in solving the abstract elliptic state equation

$$C(q)u = f \quad \text{in } Y_0^*, \tag{2.14}$$

for which we shall assume

Assumption 1. *Let $Q \subset X$ be a set of admissible parameters. For $q \in Q$ the operator $C(q)$ maps from Y_0 into the dual space Y_0^*, i.e.*

$$C(q) : Y_0 \to Y_0^*.$$

Furthermore, there exist positive constants α_1 and α_2 such that

$$\alpha_1 \|v - w\|^2 \leq \langle C(q)v - C(q)w, v - w\rangle \quad v, w \in Y_0 \tag{2.15}$$

and

$$\langle C(q)v - C(q)w, y\rangle \leq \alpha_2 \|v - w\| \|y\| \quad v, w, y \in Y \tag{2.16}$$

hold for all $q \in Q$.

Under Assumption 1, which states the strict monotonicity and the Lipschitz continuity of $C(q)$, the direct problem (2.14) is uniquely solvable for $f \in Y_0^*$ and any $q \in Q$, see, e.g., [25]. In order to emphasis the dependence on the parameter, the solution is denoted by $u_q \in Y_0$.

Already with respect to the parameter identification problem, we also assume

Assumption 2. *For all $p \in X$ and $u \in Y_0$ the operator $C(p)$ satisfies*

$$C(p) = B + A(p)$$

with

$$A(\cdot)u \in \mathcal{L}(X, Y_0^*) \tag{2.17}$$

and a parameter independent, possibly nonlinear operator B acting from Y_0 to Y_0^*.

Hence, on the one hand the parameter q shall appear linearly in the direct problem (2.14). On the other hand, we also require $A(p)u \in Y_0^*$ not only for $p \in Q$ - which would already be given by Assumption 1 - but also for $p \in X$. Note that $C(p) : Y_0 \to Y_0^*$ still only has to be invertible if $p \in Q$. Furthermore, we emphasize that despite of (2.17), the nonlinearity of (2.14) may be due to the parameter q itself.

Simple examples of partial differential equations that can be treated in this abstract framework with $Y = H^1(\Omega)$ and $Y_0 = H_0^1(\Omega)$ are

Example 1.

$$\begin{aligned} -\Delta u + q(u) &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

with

$$\langle A(q)u, v\rangle = \int_\Omega q(u)v \, dx, \tag{2.18}$$

$$\langle Bu, v\rangle = \int_\Omega \nabla u \nabla v \, dx, \tag{2.19}$$

$X = H^1(I)$ for an appropriate real interval I, and

$$Q = \left\{ q \in X \mid \underline{\gamma} \leq q' \leq \bar{\gamma} \right\},$$

or

Example 2.

$$\begin{aligned} -\nabla(q(x)\nabla u) + b(u) &= f \quad in\ \Omega, \\ u &= 0 \quad on\ \partial\Omega, \end{aligned}$$

with

$$\begin{aligned} \langle A(q)u, v\rangle &= \int_\Omega q(x)\nabla u \nabla v\, dx, \\ \langle Bu, v\rangle &= \int_\Omega b(u) v\, dx, \end{aligned}$$

$X \subset L^\infty(\Omega)$ *and*

$$Q = \left\{q \in X \mid \underline{\gamma} \le q \le \bar{\gamma}\right\}. \tag{2.20}$$

A priori, there are no restrictions on the type of nonlinearity, i.e., functions in (2.14) may depend on x and u, ∇u, $|\nabla u|$, ... Furthermore, our assumptions are not limited to second order pdes, see for instance (1.2) and Section 4, and the unknown parameter may also appear in higher order terms of the pde-operator. Besides, neither the Dirichlet-type nor the homogeneity of the boundary condition are essential for the forthcoming theory.

Turning to Example 1, the choice of Q implies the Lipschitz continuity and strict monotonicity of the parameter q. These properties also hold for the Nemyckii operator

$$N_q : \tilde{Y} \to \tilde{Y}, v \to q(v) \tag{2.21}$$

with $\tilde{Y} = L_2(\Omega)$. Hence, we have

$$\begin{aligned} \alpha_1\|v-w\|^2 &\le \alpha_1\|v-w\|^2 + \underline{\gamma}\int_\Omega (v-w)^2\, dx \\ &\le \langle B(v-w), v-w\rangle + \int_\Omega (q(v)-q(w))(v-w)\, dx \\ &= \langle C(q)v - C(q)w, v-w\rangle \end{aligned}$$

with $\alpha_1 = 1/(1+C_F)$ and C_F denoting the constant in the Friedrichs-inequality

$$\int_\Omega u^2\, dx \le C_F^2 \int_\Omega |\nabla u|^2\, dx$$

which holds for all $u \in H_0^1(\Omega)$. Furthermore,

$$\begin{aligned} & \langle B(v-w), y\rangle + \langle A(q)v - A(q)w, y\rangle \\ \le\ & \|v-w\|\|y\| + \|q(v)-q(w)\|_{\tilde{Y}}\|y\|_{\tilde{Y}} \\ \le\ & \alpha_2\|v-w\|\|y\| \end{aligned}$$

holds with $\alpha_2 = 1 + \bar{\gamma}$. Hence, Assumption 1 is fulfilled. Since $X = H^1(I)$ can be embedded into $C_b(I)$ (the set of continuous and bounded functions) and the linearity of $A(\cdot)u$ is obvious, also Assumption 2 is satisfied.

Especially for problems involving space dependent parameters, assumption (2.17) may be understood as a condition on the Hilbert space X. Considering, e.g., Example 2 in higher dimensions, (2.17) requires a more regular Hilbert space X than needed for Assumption 1 due to $X \subset L^\infty(\Omega)$. However, this discrepancy can also be found in the classical theory of Landweber iteration already due to (1.6), and hence is not specific to our strategy. Assumption 1 is then easily satisfied for Example 2 with any monotone and Lipschitz continuous b.

In the next section, we introduce a derivative free Landweber algorithm for the identification of the parameter q in (2.14) from knowledge of its solution. Convergence can easily be obtained under Assumptions 1 and 2, then - as opposed to (1.9) or (1.12) - uniformly applicable to a wide range of identification problems of that type.

3 The Derivative Free Landweber Iteration

We set $\mathcal{D}(F) = Q$ and define the parameter-to-output map

$$F : \mathcal{D}(F) \subset X \to Y, q \to u_q, \tag{3.22}$$

where u_q denotes the solution of problem (2.14). Then, our parameter identification problem is given by

$$F(q) = z, \tag{3.23}$$

where $z \in Y_0$ denotes a solution of (2.14) for some $q_* \in Q$. We recall that F is a nonlinear operator, even if the direct problem is linear. Furthermore, it is important to distinguish between the pde-operator $C(q)$ and the forward operator F. Though both are associated to the direct problem (2.14), their meaning is not the same. While $C(q)$ simply describes the direct problem, $F(q)$ actually represents its solution. As a consequence, neither the invertibility nor the monotonicity and Lipschitz properties of $C(q)$ have to hold for F.

Considering also possible data perturbations, we assume that the noisy data z^δ belong to the solution space Y_0 and satisfy the error bound (1.3). Of course, the perturbed data have not to be attainable.

Now we are able to introduce to the derivative free Landweber iteration

$$q_{k+1}^\delta = q_k^\delta + \lambda L(q_k^\delta)^*(z^\delta - u_{q_k^\delta}) \tag{3.24}$$

for solving problem (1.1) with perturbed data (1.3) in combination with the discrepancy principle (1.5). Thereby, the linear operator $L(q) : X \to Y_0$ is defined by

$$L(q)p = -JA(p)u_q, \tag{3.25}$$

which exists according to Assumption 2 and hence admits a Hilbert space adjoint. The construction of the iteration operator (3.25) is in fact motivated by the ideas presented in [9] and [17]. For illustration, we consider the iteration for our two reference Examples 1 and 2. In the first case, (3.24) translates into

$$(q_{k+1}^\delta, p) = (q_k^\delta, p) - \lambda \int_\Omega p(u_{q_k^\delta}) \cdot (z^\delta - u_{q_k^\delta})\, dx,$$

in the second case, we obtain

$$(q_{k+1}^\delta, p) = (q_k^\delta, p) - \lambda \int_\Omega p(x) \nabla u_{q_k^\delta} \cdot \nabla (z^\delta - u_{q_k^\delta})\, dx, \tag{3.26}$$

where $p \in X$ is a testfunction. In general, derivatives of the parameter or of known functions in B with respect to the solution u are no longer required by our method since $F(\cdot)'$ is not involved. Furthermore, only (2.14) has to be solved once per iteration step in order to obtain $u_{q_k^\delta}$, while (1.7) would also require to solve the linearized direct problem with $z^\delta - u_{q_k^\delta}$ as right-hand side, see [11].

Thinking about convergence of the iteration algorithm (3.24), the first idea of course is to choose $G(q) = L(q)$ in (1.11) and to consider the modified nonlinearity condition (1.12). Though we can show that

$$(u_{\tilde{q}} - u_q - L(q)(\tilde{q} - q), u_{\tilde{q}} - u_q) \le (1 - \alpha_1)\|u_{\tilde{q}} - u_q\|^2,$$

we failed in proving

$$(u_{\tilde{q}} - u_q - L(q)(\tilde{q} - q), u_{\tilde{q}} - u_q) \ge c\|u_{\tilde{q}} - u_q - L(q)(\tilde{q} - q)\|\|u_{\tilde{q}} - u_q\|$$

for some (positive) constant c. Based on the discussion led in [11], we think that (1.12) cannot be satisfied by $L(q)$ even if the direct problem is linear. Another idea is to consider (3.24) (at least for $\delta = 0$) as a fixed point iteration for

$$\tilde{L}(q) = q + L(q)^*(z - u_q),$$

but also this approach seems to fail. Finally, we emphasize that the monotonicity of $C(q)$ does not yield monotonicity of F.

Hence, none of the available theories can be applied in order to establish convergence of method (3.24). Nevertheless we next show that Assumptions 1 and 2 suffice to derive the desired results.

3.1 Convergence Analysis

As in the case of the classical Landweber iteration (and all the variants discussed in the literature), the method (3.24) can only converge if the iteration operator $L(\cdot)^*$ is (locally) uniformly bounded and the scaling parameter λ is properly chosen. Hence, we assume for our analysis (and shall verify for our examples) that

$$\|L(q)\| \leq \hat{L}, \quad q \in \mathcal{B}_\rho(q_0) \tag{3.27}$$

with a ball $\mathcal{B}_\rho(q_0)$ of radius ρ around q_0 satisfying

$$\mathcal{B}_\rho(q_0) \subset \mathcal{D}(F). \tag{3.28}$$

Now, Assumptions 1 and 2 yield that

$$\begin{aligned} \alpha_1 \|u_q - u_{\tilde{q}}\|^2 &\leq \langle C(\tilde{q})u_q - C(\tilde{q})u_{\tilde{q}}, u_q - u_{\tilde{q}}\rangle \\ &= \langle A(\tilde{q})u_q - A(q)u_q, u_q - u_{\tilde{q}}\rangle \\ &= (L(q)(q-\tilde{q}), u_q - u_{\tilde{q}}) \end{aligned} \tag{3.29}$$

for q, $\tilde{q}$ in $\mathcal{B}_\rho(q_0)$. Hence, (3.27) can be understood as sufficient condition for the Lipschitz continuity of the parameter-to-output map F with Lipschitz constant $\hat{L}/\alpha_1$. Condition (3.27) does not imply the Fréchet differentiability of F since the operator B is not involved. However, we emphasize that (3.27) is also needed if one wants to guarantee the boundedness of F' in the classical iterations, see [11].

Regarding Example 1, the embedding $X \subset C_b(I)$ yields the existence of a constant $\tilde{c}$ such that $\|p\|_\infty \leq \tilde{c}\|p\|$ for $p \in X$. Hence, we get

$$|(L(q)p, v)| \leq \|p\|_\infty \|v\| \leq \tilde{c}\|p\|\|v\|,$$

i.e., condition (3.27) holds with $\hat{L} = \tilde{c}$. Turning to Example 2, we have

$$\langle A(p)v, w\rangle \leq c\|p\|\|v\|\|w\|$$

because of $X \subset L^\infty(\Omega)$. Then, by means of Assumption 1, we obtain (3.27) with $\hat{L} = c\|f\|/\alpha_1$.

In general, the analysis of an iterative regularization method follows a basic scheme. First, one proves that the error $e_k = \|q_k^\delta - q_*\|$ is monotonically decreasing, i.e., $e_{k+1} \leq e_k$, as long as the stopping rule is obeyed. Afterwards, one shows convergence of the iterates q_k in the noise free situation based on an estimate of boundedness for the output residues. In the presence of noisy data, the stopping rule is used in order to obtain the desired regularization property of the iterative method.

In our convergence analysis of (3.24) we adopt the approach of [10]. However, all estimates concerning the iteration operator now have to be done differently, since we neither require the Fréchet differentiability of F nor the nonlinearity conditions (1.9) or (1.12). For the sake of brevity we denote the solutions of the direct problem (2.14) corresponding to the k-th iterate q_k^δ in the following simply by u_k.

Proposition 3.1. *Let Assumptions 1 and 2 hold, let $L(\cdot)$ satisfy* (3.27)*. Furthermore, assume that q_* is a solution of* (3.23) *in $\mathcal{B}_{\rho/2}(q_0)$ and let λ and τ be chosen such that*

$$2\left(\alpha_1 - \frac{\alpha_2}{\tau}\right) - \lambda \hat{L}^2 \geq D \tag{3.30}$$

holds, where D is a fixed positive constant. In case of noisy data z^δ satisfying (1.3)*, we denote by k_* the stopping index of the iteration according to the discrepancy principle* (1.5) *with τ satisfying* (3.30)*. Then, we have*

$$\|q_* - q_{k+1}^\delta\| \leq \|q_* - q_k^\delta\|, \quad 0 \leq k < k_*, \tag{3.31}$$

and

$$\sum_{k=0}^{k_*-1} \|z^\delta - u_k\|^2 \leq \frac{\rho^2}{4\lambda D}. \tag{3.32}$$

For $\delta = 0$ (with $\tau = \infty$ in (3.30)*), we have*

$$\sum_{k=0}^{\infty} \|z - u_k\|^2 \leq \frac{\rho^2}{4\lambda D}. \tag{3.33}$$

Proof. Given $\|q_0 - q_*\| \leq \rho/2$, we assume

$$\|q_k^\delta - q_*\| \leq \rho/2$$

for $k < k_*(\delta)$ and argue by induction. Then, the iteration step (3.24) is well-defined, and it follows

$$\begin{aligned} &\|q_* - q_{k+1}^\delta\|^2 - \|q_* - q_k^\delta\|^2 \\ = \;& -2\lambda(L(q_k^\delta)(q_* - q_k^\delta), z^\delta - u_k) + \lambda^2 \|L(q_k^\delta)^*(z^\delta - u_k)\|^2. \end{aligned} \tag{3.34}$$

The following considerations play the decisive role in our analysis and are only possible for the special iteration operator (3.25). Because of its definition, (2.17) and

$$A(q_*)z + Bz = A(q_k^\delta)u_k + Bu_k \quad \text{in } Y_0^*,$$

we get

$$\begin{aligned} & -(z^\delta - u_k, L(q_k^\delta)(q_* - q_k^\delta)) \\ = \;& \langle z^\delta - u_k, A(q_* - q_k^\delta)u_k \rangle \\ = \;& \langle z^\delta - u_k, A(q_*)u_k - A(q_*)z \rangle + \langle z^\delta - u_k, Bu_k - Bz \rangle \\ = \;& \langle z^\delta - u_k, C(q_*)u_k - C(q_*)z \rangle \\ = \;& -\langle z^\delta - u_k, C(q_*)z^\delta - C(q_*)u_k \rangle + \langle z^\delta - u_k, C(q_*)z^\delta - C(q_*)z \rangle \\ \leq \;& -\alpha_1 \|z^\delta - u_k\|^2 + \alpha_2 \|z^\delta - u_k\| \|z^\delta - z\|, \end{aligned} \tag{3.35}$$

where the inequality holds because of (2.15) and (2.16). Estimation (3.35) makes the nonlinearity condition (1.12) used in the classical proof unnecessary, still one now can continue similarly to the latter. Using (3.35) in (3.34), one obtains

$$\begin{aligned} &\|q_* - q_{k+1}^\delta\|^2 - \|q_* - q_k^\delta\|^2 \\ \leq\ &\|z^\delta - u_k\|\lambda\left(2\alpha_2\delta - 2\alpha_1\|z^\delta - u_k\| + \lambda\hat{L}^2\|z^\delta - u_k\|\right). \end{aligned}$$

Following the discrepancy principle (1.5), one gets from (3.30) that

$$\|q_* - q_{k+1}^\delta\|^2 + \lambda D\|z^\delta - u_k\|^2 \leq \|q_* - q_k^\delta\|^2$$

for $k < k_* = k_*(\delta)$. This implies assertion (3.31) and $q_{k+1}^\delta \in \mathcal{B}_{\rho/2}(q_*) \subset \mathcal{B}_\rho(q_0)$. Furthermore, one can conclude that

$$\lambda D \sum_{k=0}^{k_*-1} \|z^\delta - u_k\|^2 \leq \sum_{k=0}^{k_*-1} \left(\|q_k^\delta - q_*\|^2 - \|q_{k+1}^\delta - q_*\|^2\right)$$

holds, which leads to the inequality

$$k_*\tau^2\delta^2 \leq \sum_{k=0}^{k_*-1} \|z^\delta - u_k\|^2 \leq \frac{\rho^2}{4\lambda D}$$

and hence to assertion (3.33). □

Regarding condition (3.30) we see that it can always be satisfied by choosing the λ sufficiently small and τ sufficiently large. Note that the use of a "large" τ in the discrepancy principle (1.5) might cause a too early termination of the iteration. However, this problem is not specific to the iteration (3.24) but also appears in the theories for (1.7) and (1.11) (where $\tau \to \infty$ as $\eta \to 1/2$ in (1.9) or (1.12)). Furthermore, our stopping rule no longer requires the (in practical situations) unknown constant η but only depends on quantities associated to the direct problem.

The estimation (3.33) shows that in the absence of data noise the residual norms of the iterates tend to zero for $k \to \infty$, hence - if the iteration converges - the limit certainly is a solution of problem (3.23). In the case of noisy data, (3.32) yields the existence of a unique stopping index k_* such that $\|z^\delta - u_k\| > \tau\delta$ holds for all $k < k_*$, but is violated at $k = k_*$. Together with (1.5), one obtains the relation

$$k_*(\delta) = \mathcal{O}(\delta^{-2})$$

between the stopping index k_* and the noise level δ.

Turning to convergence of (3.24), we first consider precise data and show that the iterates q_k tend to a solution of (1.1). The basic idea is to verify that q_k is a Cauchy sequence. Again we follow [10], but once more we only require the properties of the pde-operator $C(q)$.

Theorem 3.1 (Convergence). *Let $\delta = 0$ in (1.3). Furthermore, let (3.27), and Assumptions 1 and 2 hold. If (3.23) is solvable in $\mathcal{B}_{\rho/2}(q_0)$, then q_k converges to a solution $q_* \in \mathcal{B}_{\rho/2}(q_0)$ of (3.23).*

Proof. Let $\tilde{q}$ be any solution of (1.1) in $\mathcal{B}_{\rho/2}(q_0)$, i.e., $u_{\tilde{q}} = z$, and set

$$e_k := q_k - \tilde{q}.$$

Proposition 3.1 yields

$$\|e_k\| \to \epsilon \quad \text{for} \quad k \to \infty, \tag{3.36}$$

where ϵ is a nonnegative constant. The next step is to show that e_k is a Cauchy sequence. For $j \geq k$, we choose l with $j \geq l \geq k$ such that

$$\|z - u_l\| \leq \|z - u_i\|, \quad k \leq i \leq j. \tag{3.37}$$

We will use

$$\|e_j - e_k\| \leq \|e_j - e_l\| + \|e_l - e_k\| \tag{3.38}$$

and

$$\begin{aligned} \|e_j - e_l\|^2 &= 2(e_l - e_j, e_l) + \|e_j\|^2 - \|e_l\|^2, \\ \|e_l - e_k\|^2 &= 2(e_l - e_k, e_l) + \|e_k\|^2 - \|e_l\|^2. \end{aligned} \tag{3.39}$$

The last terms on each of the right-hand sides of (3.39) converge to 0 for $k \to \infty$ because of (3.36). From (3.24) it follows

$$|(e_l - e_j, e_l)| = \lambda \left| \sum_{r=l}^{j-1} (z - u_r, L(q_r)(\tilde{q} - q_l)) \right|.$$

Because of (3.25), (2.17) and

$$\begin{aligned} A(q_r)u_r + Bu_r &= A(\tilde{q})z + Bz \ \text{ in } \ Y_0^*, \\ A(q_r)u_r + Bu_r &= A(q_l)u_l + Bu_l \ \text{ in } \ Y_0^* \end{aligned}$$

one has

$$\begin{aligned} (z - u_r, L(q_r)(\tilde{q} - q_l)) &= -\langle A(\tilde{q} - q_l)u_r, z - u_r \rangle \\ &= -\langle A(\tilde{q} - q_r)u_r, z - u_r \rangle \\ &\quad -\langle A(q_r - q_l)u_r, z - u_r \rangle \\ &= \langle A(\tilde{q})z - A(\tilde{q})u_r, z - u_r \rangle \\ &\quad +\langle Bz - Bu_r, z - u_r \rangle \\ &\quad -\langle A(q_l)u_l - A(q_l)u_r, z - u_r \rangle \\ &\quad -\langle Bu_l - Bu_r, z - u_r \rangle. \end{aligned} \tag{3.40}$$

From this relation, (3.37) and (2.16) it follows that

$$\begin{aligned}
&\lambda\left|\sum_{r=l}^{j-1}(z-u_r, L(q_r)(\tilde{q}-q_l))\right| \\
\leq\ & \lambda\alpha_2\sum_{r=l}^{j-1}\|z-u_r\|(2\|z-u_r\|+\|z-u_l\|) \\
\leq\ & 3\lambda\alpha_2\sum_{r=l}^{j-1}\|z-u_r\|^2.
\end{aligned}$$

Hence, Proposition 3.1 yields

$$|(e_l-e_j, e_l)| \to 0$$

and analogously

$$|(e_l-e_k, e_l)| \to 0$$

as k tends to ∞ (j and l depend on k). From (3.36) it now follows that

$$0 \leq \lim_{k\to\infty}\|e_j-e_l\|^2 \leq \lim_{k\to\infty}(2(e_l-e_j,e_l)+\|e_j\|^2-\|e_l\|^2) = 0$$

and

$$0 \leq \lim_{k\to\infty}\|e_l-e_k\|^2 \leq \lim_{k\to\infty}(2(e_l-e_k,e_l)+\|e_k\|^2-\|e_l\|^2) = 0$$

Therefore the right-hand sides in (3.39) tend to zero for $k\to\infty$, from which one concludes with (3.38) that e_k and hence q_k are Cauchy sequences. Denoting the limit of q_k by q_* one observes that q_* is a solution of (1.1) since the residues $z-u_k$ converge to zero for $k\to\infty$, see Proposition 3.1. □

Regarding noisy data, the next theorem shows that the discrepancy principle (1.5) makes the derivative free Landweber iteration (3.24) a regularization method. In fact, this proof now is independent of the iteration operator, and hence identical to that given in [10].

Theorem 3.2 (Regularization). *Let* (3.27) *and Assumptions 1 and 2 hold. Furthermore, assume that* (3.23) *is solvable in* $\mathcal{B}_{\rho/2}(q_0)$ *and* z^δ *satisfies* (1.3)*. Then*

$$q^\delta_{k_*(\delta)} \to q_*, \quad \delta\to 0,$$

if the iteration (3.24) *is stopped at* $k_*(\delta)$ *according to the discrepancy principle* (1.5)*,* (3.30)*.*

Hence, focusing on the parameter identification problem (1.1), the use of an iteration operator closely coupled to the direct problem allows to establish convergence of the iterates towards a solution under so far minimal assumptions. Thereby, it is decisive to apply the discrepancy principle and to measure the data error in the full Y-norm. In contrast, the classical theory for Landweber iteration in [10] also applies to data errors measured in weaker norms - provided that the nonlinearity condition (1.9) on F is also satisfied in the weaker setting. However, the ideas presented above may serve as a basis for the development of derivative free approaches capable of data only available in a weaker setup, see [11] and [15] for preliminary results. Also the following lines can be understood as a first extension of our theory in that direction.

The derivative free approach is not restricted to cases where the direct problem (2.14) is described by a strictly monotone operator. For instance, consider

Example 3.

$$\begin{aligned} -\nabla(q(u)\nabla u) &= f \quad in \ \Omega, \\ u &= 0 \quad on \ \partial\Omega, \end{aligned}$$

with $B = 0$,

$$\langle A(q)u, v\rangle = \int_\Omega q(u)\nabla u \nabla v \, dx, \tag{3.41}$$

and

$$Q = \left\{ q \in H^1(I) \mid \underline{\gamma} \leq q \leq \bar{\gamma} \right\}. \tag{3.42}$$

Here, the pde-operator $C(q)$ does not satisfy the monotonicity condition (2.15) but can only be shown to be quasi-monotone. Still, the direct problem (2.14) admits a unique solution $u_q \in Y_0 = H_0^1(\Omega) \subset Y = H^1(\Omega)$, see [23].

However, if we replace Assumption 1 by

$$\alpha_1 \|v - w\|_{\tilde{Y}}^2 \leq \langle C(q)v - C(q)w, S(v - w)\rangle \quad v, w \in Y_0, \tag{3.43}$$

$$\langle C(q)v - C(q)w, Sy\rangle \leq \alpha_2 \|v - w\|_{\tilde{Y}} \|y\|_{\tilde{Y}} \quad v, w, y \in Y \tag{3.44}$$

for all $q \in Q$, where Y now is assumed to be a (not necessarily strict) subspace of the Hilbert space $\tilde{Y}$, the basic monotonicity property for the iterates of

$$q_{k+1}^\delta = q_k^\delta + \lambda L(q_k^\delta)^* S(z^\delta - u_{q_k^\delta}) \tag{3.45}$$

can be restored in combination with Assumption 2. Thereby, S denotes a linear operator acting from $\tilde{Y}$ to Y_0. The convergence proof is once more not based on the modified nonlinearity condition (1.12), instead all the desired results follow with

$$\begin{aligned} & -(S(z^\delta - u_k), L(q_k^\delta)(q_* - q_k^\delta)) \\ = \ & \langle S(z^\delta - u_k), A(q_* - q_k^\delta)u_k\rangle \\ = \ & \langle S(z^\delta - u_k), A(q_*)u_k - A(q_*)z\rangle + \langle S(z^\delta - u_k), Bu_k - z\rangle \\ = \ & -\langle S(z^\delta - u_k), C(q_*)z^\delta - C(q_*)u_k\rangle + \langle S(z^\delta - u_k), C(q_*)z^\delta - C(q_*)z\rangle \\ \leq \ & -\alpha_1 \|z^\delta - u_k\|_{\tilde{Y}}^2 + \alpha_2 \|z^\delta - u_k\|_{\tilde{Y}} \delta, \end{aligned}$$

compare to (3.35) (the discrepancy principle and the data error estimate now have only to be considered with respect to the weaker $\tilde{Y}$-norm). Hence, the remaining question is if an appropriate operator S in (3.43) and (3.44) can be found appropriate for $C(q)$. In case of Example 3, $S = -\Delta^{-1}$ with homogeneous Dirichlet boundary conditions and $\tilde{Y} = L^2(\Omega)$ is the right choice, see [11]. Then, derivatives of the iterates with respect to the current state are again not required for (3.45).

Before numerically testing the iterative algorithm, we briefly comment on the (so-called) *direct approach* for solving the inverse problem (3.23): Given the solution z of the state equation (2.14), one might consider the latter (in its classical formulation) as an equation for the unknown parameter q. Considering Example 3, this would lead to

$$-q(z)\Delta z - q'(z)|\nabla z|^2 = f,$$

where it is for higher dimensions not clear how to proceed. But even if we restrict ourselves to the identification of space dependent parameters, the direct approach is not necessarily applicable, as it has been demonstrated by our introductory discussion of equation (1.2).

4 Numerical Experiments

For a first numerical test of (3.24) we return to our introductory parameter identification problem and complement the fourth order state equation (1.2) for z with boundary conditions on z, for instance,

$$z|_{\partial\Omega} = 0, \quad M_n = 0 \text{ on } \partial\Omega. \tag{4.46}$$

With M_n denoting the bending moment, (4.46) corresponds to the fastening of a simply supported plate - in case of a rectangular frame, the second condition in (4.46) reads as

$$\begin{aligned} z_{xx} + \nu z_{yy} &= 0 \quad \text{along the edges with y = constant} \\ z_{yy} + \nu z_{xx} &= 0 \quad \text{along the edges with x = constant.} \end{aligned}$$

Now, given a desired target shape $\hat{z}$ satisfying (4.46), the inverse problem is to determine a corresponding positive q_*. Since the operator formulation of the direct problem (1.2), (4.46) is based on

$$\langle A(q)u, v\rangle = \int_\Omega q\left[(u_{xx}+u_{yy})(v_{xx}+v_{yy}) - \frac{1}{2}(u_{xx}v_{yy}+u_{yy}v_{xx}-2u_{xy}v_{xy})\right]\,dxdy$$

with $Y = H^2(\Omega)$ and

$$Y_0 = \{w \in Y \mid w|_{\partial\Omega} = 0\}, \tag{4.47}$$

for which Assumptions 1 and 2 with Q as in (2.20) can be easily verified, see [18], the iterative method (3.24) is suited for solving the inverse windshield problem.

For a comparison of (3.24) to (1.7) we choose a simply supported target shape

$$\hat{z}_S = -(x - 2x^3 + x^4)(y - 2y^3 + y^4), \tag{4.48}$$

defined on $\Omega = [0,1] \times [0,1]$, and a true parameter

$$q_* = 1 + x + 2y. \tag{4.49}$$

Then, the right-hand side f in the direct problem is chosen such that

$$u_{q_*} = \hat{z}_S$$

holds. The use of a non-physical right-hand side f, i.e., not representing the gravity force, facilitates the construction of test examples for which the solution of the inverse problem is analytically known. Though the convergence analysis of (3.24) - and also of (1.7) - applied to the windshield problem would require a parameter space satisfying $X \subset L^\infty(\Omega)$, compare to (2.20), we choose $X = H^1(\Omega)$ for the numerics. On the one hand, this allows to keep the numerical efforts low (since the use of higher order elements for the parameter is avoided), on the other hand it responds to the natural wish for keeping the regularity that is sufficient for the solvability of the direct problem. All our tests have shown that the iterates remain in the domain of the parameter-to-output map F without the use of a projection operator.

All computations were done in MATLAB, based on the PDE Toolbox which uses the finite element method. For the representation of the parameter, we chose linear ansatz functions, the solutions of the direct problem were represented by the discrete Kirchhoff triangle, see [2]. For our test, we used a regular and uniform triangular mesh with 655 nodal points.

As initial guess and starting value for the iterations we use

$$q_0 = 4, \tag{4.50}$$

which means a relative deviation from q_* of approximately 80% measured with respect to the norm in X. The respective scaling parameters λ are co-ordinated such that the first updates are of the same magnitude. The performance of both iterations is documented in Figure 1, where the relative error

$$\frac{\|q_* - q_k\|}{\|q_*\|} \tag{4.51}$$

is plotted vs. the iteration index k. Method (3.24) (blue line) shows a similar convergence behaviour (also confirmed by other numerical tests) as Landweber iteration (red line), which is known to be a slow but reliable algorithm. The difference is that (3.24)

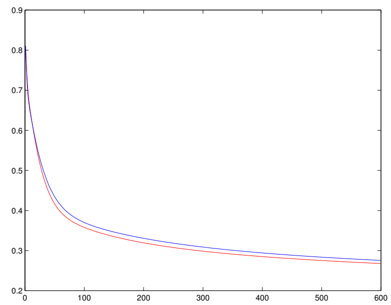

Figure 1: Landweber iteration vs. method (3.24)

only required to solve a sequence of direct problems, while for Landweber iteration, in addition solutions to the same number of auxiliary problems had to be computed.

The results obtained by the derivative free method (3.24) are shown in Figures 2 and 3. The simply supported target shape is approximated with a relative error smaller than

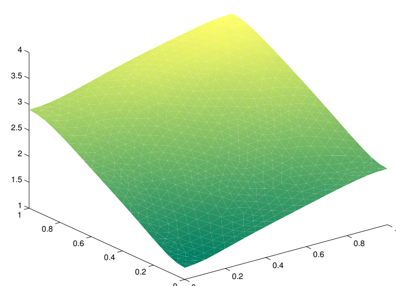

Figure 2: the identified parameter q_{450}

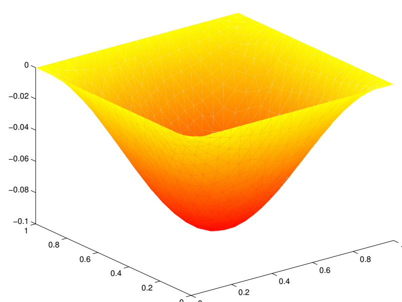

Figure 3: the corresponding output $u_{E_{450}}$

2%. This is remarkable since the deviation from (4.48) is measured with respect to $H^2(\Omega)$.

Hence, method (3.24) successfully passed our first test concerning the identification of a space dependent parameter. We mention that the inverse windshield problem requires special care since the unknown parameter q appears with its second derivatives in (1.2), then - in the direct approach - leading to a second order pde of mixed type in dependence on the given target shape. For a thorough discussion of the influence of the mixed type on the performance of the iterative algorithm (3.24) we refer to [12].

4.1 Identification of a Nonlinearity

In this section we test our iterative parameter identification algorithm on a more traditional example. Given a sequence of perturbed solutions of the nonlinear direct problem in Example 3, we observe the respective courses of the inverse iteration (3.45) as the noise level tends to zero.

The Gauss-Newton-method is used in order to solve the nonlinear direct problem (2.14) in an iterative way, where an improved approximation for the solution is sought by solving a linearized problem. The exact data z for the inverse problem were generated by solving the direct problem with a true parameter

$$q_*(\tau) = 2 + \frac{1}{4}\cos(\pi\tau) \tag{4.52}$$

and a right-hand side

$$f = 7.8 + 2x - y \tag{4.53}$$

on the unit circle. Thereby, we again used the Matlab environment with linear ansatz functions for the solution of the direct problem. In order to avoid inverse crimes, see [4], the solution u_{q_*} - which is illustrated in Figure 4 - was computed on a certain mesh and

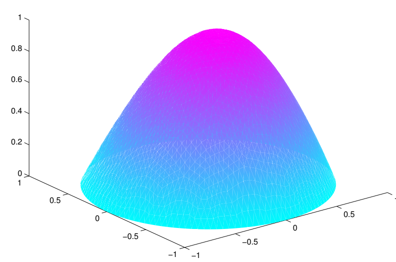

Figure 4: the exact data z

then interpolated to a regular mesh of 2129 nodal points for the unit circle used for the inverse iteration.

For the iterative method (3.45), we choose the real interval

$$I = [0, 1] \tag{4.54}$$

as domain of the parameters q and linear ansatz fnctions for its finite element representation. This interval covers the range of the exact data z but also that of the solution u_{q_0} of the direct problem (2.14) for our initial guess

$$q_0 = 2.25.$$

Then, I is "hopefully" large enough in order to cover all temperatures u_k appearing during the course of the iteration. As opposed to the identification of a space dependent parameter, the domain of a nonlinearity to be estimated is not a-priori known. For more details on this crucial topic and numerical strategies we refer to [14], [11]. Note that the parameter space $X = H^1(I)$ here is in perfect correspondence to Assumption 2 of the convergence theory.

In order to accelerate (3.45), we now use a line search algorithm resulting in an iteration index dependent "scaling" parameter λ_k. Figures 5 and 6 show the course of the

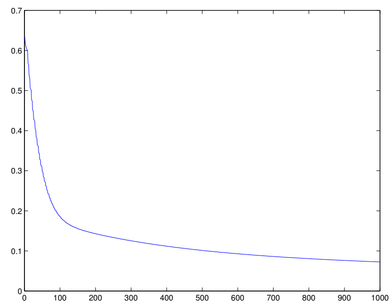

Figure 5: (4.55) with $\delta = 0$ vs. k

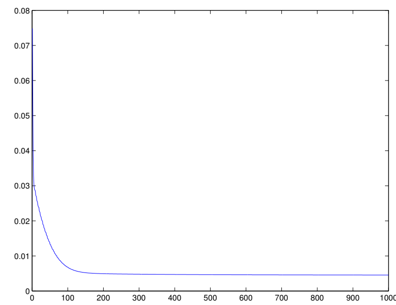

Figure 6: (4.56) with $\delta = 0$ vs. k

error

$$\|q_* - q_k^\delta\|_{H^1(I)} \tag{4.55}$$

in the parameter and the error

$$\|z^\delta - u_{q_k^\delta}\|_{L^2(\Omega)} \tag{4.56}$$

in the output during the iteration (3.45) with exact data, i.e., $\delta = 0$. We recall that - as opposed to an implementation of (1.7) - derivatives of the iterates q_k with respect to the current state now have not to be computed. After 100 steps of rather fast convergence the iteration slows down. Nevertheless, the error in the parameter continues to decrease significantly. The comparison of the computed parameter q_{1000} (blue line) with (4.52) (red line) as illustrated in Figure 7 shows a result that is typical for the identification of a nonlinearity. The parameter can be perfectly identified for temperature values - in our example ranging from 0 to approximately 0.8 - on a "sufficiently" large set covered by the data z. For temperature values that are less or - due to an interval I chosen too large - not at all covered by the data, the quality of the solution decreases and is more and more influenced by the initial guess q_0 due to the lack of data.

Next, we consider a sequence of noisy data z^δ obtained by a random perturbation of the exact data z and focus on their impact on the respective courses of the iteration. In Figures 9 and 10, the errors (4.55) and (4.56) are plotted vs. the iteration index k for noise levels $\delta = 0.0362$ (red line) and $\delta = 0.072$ (blue line), where the noise level is

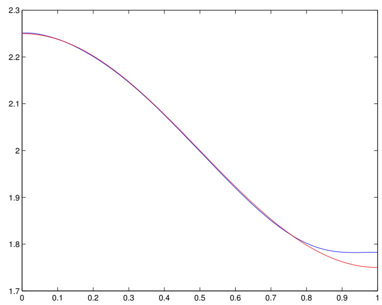

Figure 7: q_* (red) and q_{1000} (blue)

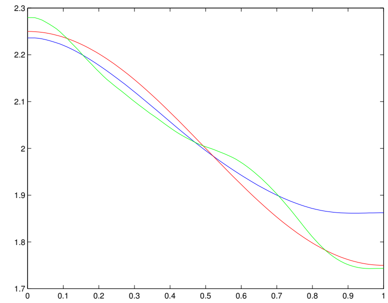

Figure 8: q_* (red), $q^\delta_{k_*}$ (blue) and q^δ_{1000} (green) with $\delta = 0.072$

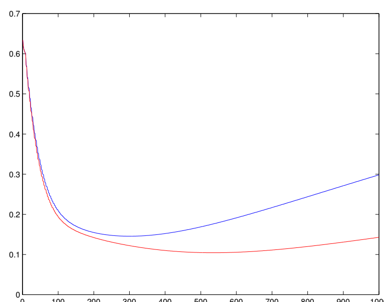

Figure 9: (4.55) vs. k for $\delta = 0.0362$ (red line) and $\delta = 0.072$ (blue line)

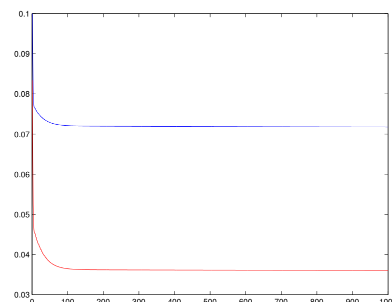

Figure 10: (4.56) vs. k for $\delta = 0.0362$ (red line) and $\delta = 0.072$ (blue line)

measured with respect to $L^2(\Omega)$. Though the error in the output is monotonically decreasing during the iteration, the error in the parameter increases after an initial decay. Thereby, the data noise propagation starts earlier and is stronger for the higher noise level. Hence, a stable approximation to the solution q_* of the inverse problem can only be obtained if the iteration is stopped at the right time. Applying the discrepancy principle (1.5) (in the $L^2(\Omega)$-norm) with $\tau = 1$, the error in the output drops below the noise level $\delta = 0.072$ after $k_*(\delta) = 119$ steps. The corresponding approximate solution $q^\delta_{k_*}$ as well as the parameter q^δ_{1000}, that is obtained if the stopping rule is ignored, are compared to q_* in Figure 8. Though the discrepancy principle yields a satisfactory result in the presence of data noise for lower temperature values, the blue line in Figure 9 shows that the iteration is still terminated at a too early step. Thereby, we have already chosen a parameter τ that is certainly smaller than required by the theory, see (3.30). On the other hand, without the discrepancy principle (or any other stopping rule) a reliable approximation of the parameter q_* cannot be obtained. The iterate q^δ_{1000} shows oscillations that are purely caused by the data noise and would even get stronger as the iteration continues.

The results obtained for a sequence of noisy data with a noise level δ ranging from 0.0362 to 0.072 are summarized in Table 1. The last column indicates a convergence rate of order

δ	k_*	$\|z^\delta - u_{q^\delta_{k_*}}\|$	$\|q^\delta_{k_*} - q_*\|$	$\|q^\delta_{k_*} - q_*\|/\sqrt{\delta}$
0.0362	268	0.0362	0.1274	0.6698
0.0402	224	0.0402	0.1374	0.6855
0.0452	188	0.0452	0.1477	0.6946
0.0517	159	0.0517	0.1594	0.7014
0.0602	137	0.0602	0.1729	0.7047
0.0720	119	0.0720	0.1888	0.7035

Table 1: a sequence of noisy data

$\sqrt{\delta}$, i.e.,

$$\|q^\delta_{k_*} - q_*\| = \mathcal{O}(\sqrt{\delta}). \tag{4.57}$$

5 Conclusion and Outlook

Our numerical tests confirm that the derivative free Landweber iteration can be successfully applied to parameter identification in elliptic pdes from knowledge of the solution as predicted by our theory. Dealing with direct problems described by a monotone operator, the number of forward problems to be solved is cut in half compared to the classical Landweber method, in case of a state dependent parameter, derivatives of the current iterate with respect to the state have no longer to be computed.

Thereby, the theoretical fundament guaranteeing the convergence of the derivative free iteration can be laid under natural and easily revisable assumptions for a wide class of nonlinear elliptic state equations. Our ideas might also be extended to parameter identification in nonlinear pde systems, where the proof of the unique solvability again is based on fixpoint arguments. The fact that the differentiability of the direct problem is no longer required makes the method also a candidate for the identification of parameters that appear in (non-differentiable) variational inequalities, see [11] for preliminary results. Furthermore, the derivative free iteration allows an extension to a multi-level iteration, where the forward operator has to be only roughly evaluated in the early steps of the inverse iteration in order to reduce the global computational effort, again we refer to [11]. Note that the convergence of any iterative regularization method for inverse and ill-posed problems may be arbitrarily slow, see [22]. Hence, convergence rate results as, e.g., (4.57), can only be proven under additional strong assumptions, in general formulated in terms of the derivative of F, see [8]. Facilitations in this context that can be achieved by the derivative free Landweber method are subject of the forthcoming paper [13]. Finally, we mention that the ideas presented in this paper have meanwhile successfully been extended to the identification of $q = q(|\nabla u|)$ in (2.14) from only single boundary measurements, see [16].

References

[1] H.T. Banks, K. Kunisch, Estimation Techniques for Distributed Parameter Systems, Birkhäuser, Boston, Basel, Berlin, 1989.

[2] D. Braess, Finite Elements, Cambridge University Press, 2001.

[3] M. Burger, Direct and Inverse Problems in Polymer Crystallization Processes, PhD thesis, Johannes Kepler University, Linz, Austria, 2000.

[4] D. Colton, R. Kress, Inverse Acoustic and Electromagnetic Scattering Theory, Springer, Berlin-Heidelberg-New York, 1998.

[5] P. Deuflhard, H.W. Engl, O. Scherzer, A convergence analysis of iterative methods for the solution of nonlinear ill-posed problems under affinely invariant conditions, Inverse Problems 14 (1998), 1081–1106.

[6] H.W. Engl, M. Hanke, A. Neubauer, Regularization of Inverse Problems, Kluwer Academic Publishers, 1996.

[7] H.W. Engl, K. Kunisch, A. Neubauer, Convergence rates for Tikhonov regularisation of non-linear ill-posed problems, Inverse Problems 5 (1989), 523–540.

[8] H.W. Engl, O. Scherzer, Convergence rate results for iterative methods for solving nonlinear ill-posed problems, in: D. Colton, H.W. Engl, A.K. Louis, J. McLaughlin, W.F. Rundell (eds.), Survey on Solution Methods for Inverse Problems, Springer, Vienna/New York, 2000, 7-34.

[9] H.W. Engl, J. Zou, A new approach to convergence rate analysis of Tiknonov regularization for parameter identification in heat conduction, Inverse Problems 16 (2000), 1907–1923.

[10] M. Hanke, A. Neubauer, O. Scherzer, a convergence analysis of the Landweber iteration for nonlinear ill-posed problems, Numerische Mathematik 72 (1995), 21 – 37.

[11] P. Kügler, A Derivative Free Landweber Method for Parameter Identification in Elliptic Partial Differential Equations with Application to the Manufacture of Car Windshields, PhD Thesis, Johannes Kepler University, Linz, Austria, 2003.

[12] P. Kügler, A parameter identification problem of mixed type related to the manufacture of car windshields, SIAM Journal on Applied Mathematics, to appear

[13] P. Kügler, Convergence Rates for A Derivative Free Landweber Iteration for Parameter Identification in Elliptic PDEs, in preparation

[14] P. Kügler, Identification of a Temperature Dependent Heat Conductivity by Tikhonov Regularization, Diploma thesis, Johannes Kepler University, Linz, Austria, 2000.

[15] P. Kügler, Identification of a temperature dependent heat conductivity from single boundary measurements, SIAM Journal on Numerical Analysis, to appear

[16] P. Kügler, A derivative free iteration method for solving a parameter identification problem in nonlinear electrostatics from single boundary measurements, in preparation

[17] P. Kügler, H.W. Engl, Identification of a temperature dependent heat conductivity by Tikhonov regularization, Journal of Inverse and Ill-posed Problems 10 (2002), 67–90.

[18] W. Litvinov, Optimization in Elliptic Problems with Applications to Mechanics of Deformable Bodies and Fluid Mechanics, Birkhäuser Verlag, Basel, Boston, Berlin, 2001.

[19] S. Manservisi, Control and optimization of the sag bending process in glass windscreen design, in: L. Arkeryd, J. Bergh, P. Brenner, R. Pettersson, Progress in Industrial Mathematics at ECMI98, Teubner, Stuttgart, 1999, 97–105.

[20] G. Richter, An inverse problem for the steady state diffusion equation, Siam Journal on Applied Mathematics 41 (1981), 210 – 221.

[21] O. Scherzer, Convergence criteria of iterative methods based on Landweber iteration for solving nonlinear problems, Journal of Mathematical Analysis and Applications 194, (1995), 911–933.

[22] E. Schock, Approximate solution of ill-posed equations: arbitrarily slow convergence vs. superconvergence, in: G. Hämmerlin and K.H. Hoffmann, eds., Construcitve Methods for the Practical Treatment of Integral Equations, Birkhäuser, Basel, 1985, 234–243.

[23] R.E. Showalter, Monotone Operators in Banach Space and Nonlinear Partial Differential Equations, American Mathematical Society, 1997.

[24] D. Temple, An Inverse System - An Analysis Arising from Windscreen Manufacture, MSc Thesis, University of Oxford, England, 2002.

[25] E. Zeidler, Nonlinear Functional Analysis and its Applications II/b, Springer-Verlag, New York, Berlin, Heidelberg, 1980.