Formal Geometry: A Case Study in Theory Exploration Using Theorema

Günther Fuchs
Pedagogic Academy, A6422 Stams, Stiftshof, Austria
and *Theorema* Group, RISC, Johannes Kepler University, A4040 Linz, Austria

■ Abstract

In this paper, we give a sequence of definitions and theorems that builds up in a formal way plane geometry from geometric intuition. The cultivation of geometric intuition, in particular the clear understanding of the difference between observing and proving, should play a key role in any mathematical education.

Our account is the initial part of a major case study in formal theory exploration. It covers the basis "thread" in the four thread model of *Theorema* theory exploration proposed by B. Buchberger. Thus, our case study is both an exercise a preparatory exercise for the didactics of geometry and a case study for the use of algorithmic tools, like *Theorema*, in mathematical knowledge management.

Acknowledgement: Sponsored by SFB 1302 (*Theorema* Project) of the FWF (Österreichischer Fonds zur Förderung der Wissenschaftlichen Forschung; Austrian Science Foundation) under guidance of B. Buchberger.

■ Introduction

Formal Theory exploration is a main goal of the *Theorema* project. According to the basic philosophy of *Theorema* theory exploration outlined in [Buchberger 2000, 2003a, 20003b], elaboration of major case studies in the built-up of formal theories is very important in order to classify the basic formal tools future mathematical software systems must provide in order to computer-supportall phases of the theory exploration process.

In this paper, we give a first account of a major case study which we carried out in the past two years for building up geometry from a few concepts that are appropriate for teaching geometry in high scool with a particular emphasis of claryfying the subtle transition from an approach based on geometric intuition to a presentation in coordinate geometry. The cultivation of geometric intuition, in particular the clear understanding of the difference between observing and proving, should play a key role in any mathematical education as discussed in [Fuchs 1999].

This account covers the basis "thread" in the thread model of theory exploration in [Buchberger 2003b] (namely the theory formation thread). Starting from this, we will soon embark on the elaboration of the other three threads for this particular case study.

The body of the paper is a sequence of definitions and theorems that builds up a particular version of geometry in a formal way. Since the formulae should be self-explanatory, we do not add any explanatory natural language text. Also note that the axioms introduced do not strive for minimality. Rather, we want to start from a platform that should be intuitively appropriate for what high-scoolstudents consider as evident. All the rest should be formally provable from this platform.

Plane Geometry is a very old theory of mathematics to describe interesting "plane figures" and to solve problems related with them. We suppose each figure to be composed of "points". Among the figures the "(straight) lines" play an essential role. Points, lines and figures are then the interesting types of objects which we want to decribe within the Plane Geometry based on set theory.

We will follow the metric approach to geometry where the knowledge about the field of real numbers is assumed to be available. As usual, "ℝ" denotes the set of the real numbers. "ℙ", "𝕃", "𝔽" denote the set of points, lines and figures respectively. Some of the steps in building up the theory follow suggestions in [Hartshorne], [Martin], [Millmann−Parker],and [Moise].

In order to shorten the notation, we will introduce typed variables:

A, B, C ... points
a, b, c ... lines
$\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ ... figures
a, b, c ... real numbers

▪ Incidence

▫ The Axiom of Incidence

Axiom["I0",
$\mathbb{F} \subseteq \mathcal{P}[\mathbb{P}] \wedge \mathbb{L} \subseteq \mathbb{F}$]

Axiom["I1",
$\underset{A,B}{\forall} \underset{g}{\exists} (A \in g \wedge B \in g)$]

Axiom["I2",
$\underset{\substack{A,B \\ A \neq B}}{\forall} \underset{g,h}{\forall} (A \in g \wedge B \in g \wedge A \in h \wedge B \in h \Rightarrow (g = h))$]

Axiom["I3",
$\underset{g}{\forall} \underset{\substack{A,B \\ A \neq B}}{\exists} (A \in g \wedge B \in g)$]

Definition["G2.1.1",
$\underset{g,A,B,C}{\forall} (\text{is-col}[g, A, B, C] \Leftrightarrow (A \in g \wedge B \in g \wedge C \in g))$]

Definition["G2.1.2",
$\underset{A,B,C}{\forall} \left(\text{is-col}[A, B, C] \Leftrightarrow \underset{g}{\exists} \text{is-col}[g, A, B, C]\right)$]

Definition["G2.1.3",
$\underset{A,B,C}{\forall} (\text{is-tc}[A, B, C] \Leftrightarrow \neg \text{is-col}[A, B, C])$]

Axiom["I4",
$\underset{A,B,C}{\exists} \text{is-tc}[A, B, C]$]

Propositions on Incidence

Definition["G2.2.1", any[$\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$],
is-pd[$\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$] $\Leftrightarrow (\mathfrak{a} \neq \mathfrak{b} \wedge \mathfrak{a} \neq \mathfrak{c} \wedge \mathfrak{b} \neq \mathfrak{c})$]

Proposition["G2.2.2", any[A, B, C]
is-tc[A, B, C] $\Rightarrow$ is-pd[A, B, C]]

Proposition["G2.2.3", any[A, B, g, h], with[$g \neq h$],
$A \in g \wedge B \in g \wedge A \in h \wedge B \in h \Rightarrow (A = B)$]

Proposition["G2.2.4", any[A, B, g, h],
$A \notin h \wedge A \in g \wedge B \in h \Rightarrow A \neq B$]

Proposition["G2.2.5", any[P, g, h],
$P \notin g \wedge P \in h \Rightarrow g \neq h$]

Proposition["G2.2.6", any[P, g, h], with[$g \neq h$],
$P \in g \wedge P \in h \Rightarrow g \bigcap h = \{P\}$]

Proposition["G2.2.7", any[P, g, h],
$g \bigcap h = \{P\} \Rightarrow g \neq h$]

Proposition["G2.2.8", any[A, P, g, h], with[$g \bigcap h = \{P\}$],
$A \in h \wedge A \neq P \Rightarrow A \notin g$]

Proposition["G2.2.9", any[A, B], with[A ≠ B],
$\underset{g}{\exists!} (A \in g \wedge B \in g)$]

Definition["G2.2.10",
$\underset{\substack{A,B \\ A \neq B}}{\forall} \left(l[A, B] = \underset{g}{\iota} (A \in g \wedge B \in g) \right)$]

Proposition["G2.2.11", any[A, B], with[A ≠ B],
$l[A, B] \in \mathbb{L} \wedge A \in l[A, B] \wedge B \in l[A, B]$]

Proposition["G2.2.12", any[g, A, B], with[A ≠ B],
$(g = l[A, B]) \Leftrightarrow (A \in g \wedge B \in g)$]

Proposition["G2.2.13", any[A, B], with[A ≠ B],
$l[A, B] = l[B, A]$]

Proposition["G2.2.14", any[A, B, C], with[A ≠ B],
$C \notin l[A, B] \Rightarrow C \neq A \wedge C \neq B$]

Proposition["G2.2.15", any[A, B, C], with[$A \neq B \wedge B \neq C$],
$A \in l[B, C] \Rightarrow (l[B, C] = l[A, B])$]

Proposition["G2.2.16", any[A, B, C],
$(A \neq B \wedge C \notin l[A, B]) \Leftrightarrow$ is-tc[A, B, C]]

Proposition["G2.2.17", any[A, B, C], with[is-pd[A, B, C]],
is-col[A, B, C] $\Rightarrow l[A, B] = l[A, C]$]

Proposition["G2.2.18", any[A, B, C],
is-tc[A, B, C] $\Rightarrow l[A, B] \neq l[A, C]$]

Proposition["G2.2.19", any[A, B, P, Q], with[$A \neq B \wedge P \neq Q$],
$P \in l[A, B] \wedge Q \in l[A, B] \Rightarrow (l[A, B] = l[P, Q])$]

Proposition["G2.2.20",
$\underset{g}{\forall} \underset{P}{\exists} P \notin g$]

Proposition["G2.2.21",
$\forall_{P} \exists_{g,h} (g \neq h \wedge (g \bigcap h = \{P\}))$]

Proposition["G2.2.22",
$\forall_{P} \exists_{g} P \notin g$]

Proposition["G2.2.23", any[g, h],
$g \subseteq h \Rightarrow g = h$]

Parallel Lines

Definition["G2.3.1",
$\forall_{g,h} (g \shortparallel h \Leftrightarrow ((g = h) \bigvee (g \bigcap h = \emptyset)))$]

Proposition["G2.3.2", any[g, h],
$\neg\, g \shortparallel h \Rightarrow \exists!_{S} (S \in g \wedge S \in h)$]

Definition["G2.3.3",
$\forall_{\substack{g,h \\ \neg g \shortparallel h}} \left(i[g, h] = \iota_{S} (S \in g \wedge S \in h)\right)$]

Proposition["G2.3.4", any[g, h], with[$\neg\, g \shortparallel h$],
$i[g, h] \in g \wedge i[g, h] \in h \wedge ((S = i[g, h]) \Leftrightarrow (S \in g \wedge S \in h))$]

Proposition["G2.3.5", any[A, B, C],
$\text{is-tc}[A, B, C] \Rightarrow \neg\, l[A, B] \shortparallel l[A, C] \wedge \neg\, l[A, B] \shortparallel l[B, C] \wedge \neg\, l[A, C] \shortparallel l[B, C]$]

Distance and Betweenness

Ruler Axiom and Distance Function

Definition["G3.1.1",
$\forall_{\Gamma,g} \left(\text{is-cos}[\Gamma, g] \Leftrightarrow \left(\left(\Gamma : g \xrightarrow{\text{bij}} \mathbb{R}\right) \bigwedge \forall_{\substack{A,B \\ A\in g \wedge B\in g}} (d[A, B] = |\, \Gamma[\![B]\!] - \Gamma[\![A]\!]\, |)\right)\right)$]

Definition["G3.1.2",
$\forall_{\substack{x,X,g,\Gamma \\ X\in g}} (\text{is-co}[x, X, g, \Gamma] \Leftrightarrow (\text{is-cos}[\Gamma, g] \wedge (x = \Gamma[\![X]\!])))$]

Axiom["RA",
$\forall_{g} \exists_{\Gamma} \text{is-cos}[\Gamma, g]$]

Proposition["G3.1.3", any[A, B],
$d[A, B] \geq 0$
$(d[A, B] = 0) \Leftrightarrow (A = B)$]
$d[A, B] = d[B, A]$

Proposition$\Big[$"G3.1.4", any[g, Γ_1, Γ_2], with[is-cos[Γ_1, g] $\wedge$ $\Gamma_2 : g \longrightarrow \mathbb{R}$],

$\underset{a}{\exists} \underset{\substack{P \\ P \in g}}{\forall} (\Gamma_2[\![P]\!] = \Gamma_1[\![P]\!] + a) \Rightarrow$ is-cos[Γ_2, g]

$\underset{\substack{P \\ P \in g}}{\forall} (\Gamma_2[\![P]\!] = -\Gamma_1[\![P]\!]) \Rightarrow$ is-cos[Γ_2, g] $\Big]$

Proposition$\Big[$"G3.1.5", any[A, B], with[A ≠ B],

$\underset{\Gamma}{\exists}$ (is-cos[Γ, l[A, B]] ∧ (Γ⟦A⟧ = 0) ∧ Γ⟦B⟧ > 0)$\Big]$

Proposition["G3.1.6", any[A, B, P, Q], with[A ≠ B],
(d[A, P] = d[A, Q]) ∧ (d[B, P] = d[B, Q]) ⇒ (P = Q)]

Ordering the Points on a Line

Definition$\Big[$"G3.2.1",

$\underset{A,B,C}{\forall}$ (is-b[A, B, C] ⇔ (is-pd[A, B, C] ∧ is-col[A, B, C] ∧ d[A, B] + d[B, C] = d[A, C]))$\Big]$

Proposition["G3.2.2", any[A, B, C],
is-b[A, B, C] ⇒ is-b[C, B, A]]

Proposition["G3.2.3", any[A, B, C],
is-b[A, B, C] ⇒ (¬ is-b[A, C, B] ∧ ¬ is-b[C, A, B])]

Definition$\Big[$"G3.2.4",

$\underset{a,b,c}{\forall}$ (is-b[a, b, c] ⇔ (a < b < c ∨ c < b < a))$\Big]$

Proposition["G3.2.5", any[A, B, C, g, Γ], with[is-col[g, A, B, C] ∧ is-cos[Γ, g]],
is-b[Γ⟦A⟧, Γ⟦B⟧, Γ⟦C⟧] ⇔ is-b[A, B, C]]

Proposition$\Big[$"G3.2.6", any[A, B, C], with[is-pd[A, B, C] ∧ is-col[A, B, C]],

$\overset{!}{\vee}$ [is-b[A, B, C], is-b[A, C, B], is-b[C, A, B]]$\Big]$

Proposition$\Big[$"G3.2.7", any[A, B], with[A ≠ B],

$\underset{X}{\forall}\Big($X ∈ l[A, B] ⇔ is-b[X, A, B] $\overset{!}{\vee}$ X = A $\overset{!}{\vee}$ is-b[A, X, B] $\overset{!}{\vee}$ X = B $\overset{!}{\vee}$ is-b[A, B, X]$\Big)\Big]$

Proposition$\Big[$"G3.2.8", any[A, C], with[A ≠ C],

$\underset{B,D}{\exists}$ (is-b[A, B, C] ∧ is-b[A, C, D])$\Big]$

Proposition["G3.2.9", any[A, B, C, D],
is-b[A, B, C] ∧ is-b[A, B, D] ⇒ ((C = D) ∨ is-b[B, C, D] ∨ is-b[B, D, C])]

Definition["G3.2.10", any[A, B, C, D],
is-b[A, B, C, D] ⇔ (is-b[A, B, C] ∧ is-b[A, B, D] ∧ is-b[A, C, D] ∧ is-b[B, C, D])]

Proposition["G3.2.11", any[A, B, C, D],
is-b[A, B, C] ∧ is-b[B, C, D] ⇒ is-b[A, B, C, D]]

Definition["G3.2.12", any[$\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$],
is-pd[$\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$] ⇔ ($\mathfrak{a} \neq \mathfrak{b} \wedge \mathfrak{a} \neq \mathfrak{c} \wedge \mathfrak{a} \neq \mathfrak{d} \wedge \mathfrak{b} \neq \mathfrak{c} \wedge \mathfrak{b} \neq \mathfrak{d} \wedge \mathfrak{c} \neq \mathfrak{d}$)]

Definition$\Big[$"G3.2.13",

$\underset{g,A,B,C,D}{\forall}$ (is-col[g, A, B, C, D] ⇔ (A ∈ g ∧ B ∈ g ∧ C ∈ g ∧ D ∈ g))$\Big]$

Definition["G3.2.14",

$$\underset{A,B,C,D}{\forall}\left(\text{is-col}[A, B, C, D] \Leftrightarrow \underset{g}{\exists}\ \text{is-col}[g, A, B, C, D]\right)]$$

Congruence of Line Segments and Midpoint

Definition["G3.3.1",

$$\underset{\substack{A,B\\A\neq B}}{\forall}\ (\text{s}[A, B] = \{A, B\} \bigcup \{X \mid \text{is-b}[A, X, B]\})]$$

Definition["G3.3.2",
$\mathbb{S} = \{\text{s}[A, B] \mid A \neq B\}$]

Proposition["G3.3.3", any[A, B], with[A ≠ B],
$\text{s}[A, B] = \text{s}[B, A]$]

Proposition["G3.3.4", any[A, B, C, D], with[$A \neq B \wedge C \neq D$],
$\text{s}[A, B] = \text{s}[C, D] \Leftrightarrow \{A, B\} = \{C, D\}$]

Definition["G3.3.5",

$$\underset{\substack{\mathcal{S}\\\mathcal{S}\in\mathbb{S}}}{\forall}\left(\text{le}[\mathcal{S}] = \underset{y}{\iota}\ \underset{X,Y}{\exists}\ (X \neq Y \wedge \mathcal{S} = \text{s}[X, Y] \wedge y = \text{d}[X, Y])\right)]$$

Proposition["G3.3.6", any[A, B], with[A ≠ B],
$\text{le}[\text{s}[A, B]] = \text{d}[A, B]$]

Definition["G3.3.7",

$$\underset{\mathcal{S}_1,\mathcal{S}_2}{\forall}\ (\mathcal{S}_1 \simeq \mathcal{S}_2 \Leftrightarrow (\mathcal{S}_1 \in \mathbb{S} \wedge \mathcal{S}_2 \in \mathbb{S} \wedge (\text{le}[\mathcal{S}_1] = \text{le}[\mathcal{S}_2])))]$$

Proposition["G3.3.8", any[A, B, C, D], with[$A \neq B \wedge C \neq D$],
$\text{s}[A, B] \simeq \text{s}[C, D] \Leftrightarrow (\text{d}[A, B] = \text{d}[C, D])$]

Proposition["G3.3.9", any[A, B, C, D, E, F], with[$A \neq B \wedge C \neq D \wedge E \neq F$],
$\text{s}[A, B] \simeq \text{s}[A, B] \wedge (\text{s}[A, B] \simeq \text{s}[C, D] \Rightarrow \text{s}[C, D] \simeq \text{s}[A, B]) \wedge (\text{s}[A, B] \simeq \text{s}[C, D] \wedge \text{s}[C, D] \simeq \text{s}[E, F] \Rightarrow \text{s}[A, B] \simeq \text{s}[E, F])$

Proposition["G3.3.10", any[A, B, C, D, E, F],
$\text{is-b}[A, B, C] \wedge \text{is-b}[D, E, F] \wedge \text{s}[A, B] \simeq \text{s}[D, E] \wedge \text{s}[B, C] \simeq \text{s}[E, F] \Rightarrow \text{s}[A, C] \simeq \text{s}[D, F]$]

Proposition["G3.3.11", any[A, B, C, D, E, F],
$\text{is-b}[A, B, C] \wedge \text{is-b}[D, E, F] \wedge \text{s}[A, B] \simeq \text{s}[D, E] \wedge \text{s}[A, C] \simeq \text{s}[D, F] \Rightarrow \text{s}[B, C] \simeq \text{s}[E, F]$]

Definition["G3.3.12",

$$\underset{A,M,B}{\forall}\ (\text{is-mip}[A, M, B] \Leftrightarrow (\text{is-b}[A, M, B] \wedge \text{s}[A, M] \simeq \text{s}[M, B]))]$$

Proposition["G3.3.13", any[A, B], with[A ≠ B],

$$\underset{M,N}{\exists!}\ (\text{is-mip}[A, M, B] \wedge \text{is-mip}[A, B, N])]$$

Definition["G3.3.14",

$$\underset{\substack{A,B\\A\neq B}}{\forall}\left(\text{mp}[A, B] = \underset{M}{\iota}\ \text{is-mip}[A, M, B]\right)]$$

Proposition["G3.3.15", any[A, B, C],
$\text{is-pd}[A, B, C] \wedge \text{is-col}[A, B, C] \wedge \text{s}[A, B] \simeq \text{s}[B, C] \Rightarrow \text{is-b}[A, B, C]$]

Proposition["G3.3.16", any[g, A, B, C], with[A ≠ B], $g \bigcap \text{l}[A, B] = \{C\} \wedge C \notin \text{s}[A, B] \Rightarrow g \bigcap \text{s}[A, B] = \emptyset$]

■ Rays, Angles and Triangles

▫ Rays and Half Lines

Definition["G4.1.1",
$\underset{\substack{V,A\\V\neq A}}{\forall}$ (r[V, A] = {X | X ∈ l[V, A] ∧ ¬ is-b[X, V, A]})]

Proposition["G4.1.2", any[A, B], with[A ≠ B],
r[A, B] ≠ r[B, A]]

Proposition["G4.1.3", any[A, B], with[A ≠ B],
r[A, B] = s[A, B] ⋃ {X | [A, B, X]}   "a"
s[A, B] ⊂ r[A, B] ⊂ l[A, B]   "b"]

Proposition["G4.1.4", any[A, B], with[A ≠ B],
s[A, B] = r[A, B] ⋂ r[B, A]   "a"
l[A, B] = r[A, B] ⋃ r[B, A]   "b"]

Proposition["G4.1.5", any[V, A], with[V ≠ A],
$\underset{\Gamma}{\exists!}$ (is-cos[Γ, l[V, A]] ∧ (Γ⟦V⟧ = 0) ∧ (r[V, A] = {X | Γ⟦X⟧ ≥ 0}))]

Proposition["G4.1.6", any[A, B, V, C], with[A ≠ B ∧ V ≠ C],
$\underset{D}{\exists!}$ (D ∈ r[V, C] ∧ s[A, B] ≃ s[V, D])]

Definition["G4.1.7",
$\underset{\substack{V,C,A,B\\V\neq C\wedge A\neq B}}{\forall}$ (lo[V, C, A, B] = $\underset{X}{\iota}$ (X ∈ r[V, C] ∧ s[V, X] ≃ s[A, B]))]

Definition["G4.1.8",
$\underset{\substack{V,A\\V\neq A}}{\forall}$ (hl[V, A] = r[V, A] ⊖ {V})]

Proposition["G4.1.9", any[A, V], with[V ≠ A],
(r[V, A] = {V} ⋃ hl[V, A])]

Proposition["G4.1.10", any[A, B, V], with[V ≠ A],
B ∈ hl[V, A] ⇒ (r[V, B] = r[V, A])]

Proposition["G4.1.11", any[A, B, V, W], with[V ≠ A ∧ W ≠ B],
(r[V, A] = r[W, B]) ⇒ (V = W)]

Definition["G4.1.12",
$\underset{\substack{A,B\\A\neq B}}{\forall}$ (int[A, B] = s[A, B] ⊖ {A, B})]

Proposition["G4.1.13", any[A, B], with[A ≠ B],
int[A, B] = hl[A, B] ⋂ hl[B, A]]

Definition["G4.1.14",
$\underset{\substack{V,A\\V\neq A}}{\forall}$ (or[V, A] = l[V, A] ⊖ hl[V, A])]

Definition$\Big[$"G4.1.15",

$$\underset{\substack{V,A \\ V\neq A}}{\forall} (\text{ohl}[V, A] = \text{or}[V, A] \ominus \{V\})\Big]$$

Proposition$\Big[$"G4.1.16", any[A, V], with[V ≠ A],

$$\underset{P}{\forall} (P \in \text{ohl}[V, A] \Leftrightarrow \text{is-b}[P, V, A])\Big]$$

Proposition$\Big[$"G4.1.17", any[V, A], with[V ≠ A],

$$\begin{array}{ll} \text{l}[V, A] = \text{ohl}[V, A] \bigcup \{V\} \bigcup \text{hl}[V, A] & \text{"a"} \\ \text{ohl}[V, A] \bigcap \text{hl}[V, A] = \emptyset & \text{"b"} \end{array}\Big]$$

Proposition["G4.1.18", any[V, A, B], with[A ≠ V ∧ V ≠ B],
(r[V, B] = or[V, A]) ⇔ is-b[B, V, A]]

Proposition["G4.1.19", any[A, V, B], with[A ≠ V ∧ V ≠ B],
r[V, B] = or[V, A] ⇔ r[V, A] = or[V, B]]

▫ Angles

Definition$\Big[$"G4.2.1",

$$\underset{\substack{A,V,B \\ \text{is-tc}[A,V,B]}}{\forall} (\angle[A, V, B] = \text{r}[V, A] \bigcup \text{r}[V, B])\Big]$$

Proposition["G4.2.2", any[A, V, B], with[is-tc[A, V, B]],
(∠[A, V, B] = ∠[B, V, A]) ∧ ∠[A, V, B] ≠ ∠[A, B, V]]

Proposition["G4.2.3", any[A, V, B, C, D], with[is-tc[A, V, B]],
C ∈ hl[V, A] ∧ D ∈ hl[V, B] ⇒ (∠[A, V, B] = ∠[C, V, D])]

Proposition["G4.2.4", any[A, V, B, C, D], with[is-tc[A, V, B] ∧ is-tc[C, V, D]],
(∠[A, V, B] = ∠[C, V, D]) ⇒ (r[V, A] = r[V, C]) ∨ (r[V, A] = r[V, D])]

Proposition["G4.2.5", any[A, V, B, W], with[is-tc[A, V, B] ∧ is-tc[A, W, B]],
(∠[A, V, B] = ∠[A, W, B]) ⇒ (V = W)]

Proposition["G4.2.6", any[A, V, B, C, W, D, P, Q], with[is-tc[A, V, B] ∧ is-tc[C, W, D] ∧ (∠[A, V, B] = ∠[C, W, D])],
P ∈ hl[W, C] ∧ Q ∈ hl[W, D] ⇒ ((P ∈ hl[V, A] ∧ Q ∈ hl[V, B]) ∨ (P ∈ hl[V, B] ∧ Q ∈ hl[V, A]))]

Proposition["G4.2.7", any[A, V, B, C, W, D], with[is-tc[A, V, B] ∧ is-tc[C, W, D]],
(∠[A, V, B] = ∠[C, W, D]) ⇒ ((V = W) ∧ ((r[V, A] = r[V, C]) ∨ (r[V, A] = r[V, D])))]

Definition$\Big[$"G4.2.8",

$$\underset{\mathcal{A}_1,\mathcal{A}_2}{\forall}\left(\text{is-pv}[\mathcal{A}_1, \mathcal{A}_2] \Leftrightarrow \underset{\substack{A_1,V,B_1,A_2,B_2 \\ \text{is-tc}[A_1,V,B_1]}}{\exists} (\text{is-b}[A_1, V, A_2] \wedge \text{is-b}[B_1, V, B_2] \wedge (\mathcal{A}_1 = \angle[A_1, V, B_1]) \wedge (\mathcal{A}_2 = \angle[A_2, V, B_2]))\right)\Big]$$

Proposition["G4.2.9", any[A_1, V, B_1, A_2, B_2], with[is-tc[A_1, V, B_1] ∧ is-b[A_1, V, A_2] ∧ is-b[B_1, V, B_2]],
is-pv[∠[A_1, V, B_2], ∠[A_2, V, B_1]]]

Definition$\Big[$"G4.2.10",

$$\underset{\mathcal{A}_1,\mathcal{A}_2}{\forall}\left(\text{is-lp}[\mathcal{A}_1, \mathcal{A}_2] \Leftrightarrow \underset{\substack{A_1,V,B,A_2 \\ \text{is-tc}[A_1,V,B]}}{\exists} (\text{is-b}[A_1, V, A_2] \wedge (\mathcal{A}_1 = \angle[A_1, V, B]) \wedge (\mathcal{A}_2 = \angle[A_2, V, B]))\right)\Big]$$

Proposition["G4.2.11", any[C_1, W, D_1, C_2, D_2], with[is-tc[C_1, W, D_1] ∧ is-b[C_1, W, C_2] ∧ is-b[D_1, W, D_2]],
is-lp[∠[C_1, W, D_1], ∠[C_1, W, D_2]] "a"
is-lp[∠[C_2, W, D_1], ∠[C_2, W, D_2]] "b"]
is-lp[∠[C_1, W, D_2], ∠[C_2, W, D_2]] "c"

Triangles

Definition["G4.3.1",
$$\underset{\substack{A,B,C \\ \text{is-tc}[A,B,C]}}{\forall} (\triangle[A, B, C] = s[A, B] \bigcup s[B, C] \bigcup s[C, A])]$$

Proposition["G4.3.2", any[A, B, C], with[is-tc[A, B, C]],
(△[A, B, C] = △[C, B, A]) ∧ (△[A, B, C] = △[A, C, B]) "a"
s[A, B] = △[A, B, C] ⋂ l[A, B] "b"]

Proposition["G4.3.3", any[A, B, C], with[is-tc[A, B, C] ∧ is-tc[D, E, F]],
(△[A, B, C] = △[D, E, F]) ⇒ ({A, B, C} = {D, E, F})]

Convexity and Plane Separation

Convex Figures

Definition["G5.1.1",
$$\underset{\mathcal{F}}{\forall}\left(\text{is-cv}[\mathcal{F}] \Leftrightarrow \underset{\substack{A,B \\ A\neq B}}{\forall} (A \in \mathcal{F} \wedge B \in \mathcal{F} \Rightarrow s[A, B] \subseteq \mathcal{F})\right)]$$

Proposition["G5.1.2", any[$\mathcal{F}_1$, $\mathcal{F}_2$],
is-cv[$\mathcal{F}_1$, $\mathcal{F}_2$] ⇒ is-cv[$\mathcal{F}_1 \bigcap \mathcal{F}_2$]]

Proposition["G5.1.3",
$$\text{is-cv}[\emptyset] \bigwedge \text{is-cv}[\mathbb{P}] \bigwedge \underset{A}{\forall} \text{is-cv}[\{A\}]]$$

Proposition["G5.1.4", any[A, B], with[A ≠ B],
is-cv[l[A, B]] "a"
is-cv[r[A, B]] "b"]
is-cv[s[A, B]] "c"

Proposition["G5.1.5", any[A, B], with[A ≠ B],
is-cv[hl[A, B]] "a"
is-cv[int[A, B]] "b"]

Proposition["G5.1.6", any[g, V], with[V ∈ g],
$$\underset{\substack{\mathcal{H}_1,\mathcal{H}_2 \\ \text{is-cv}[\mathcal{H}_1,\mathcal{H}_2]}}{\exists}\left((g \ominus \{V\} = \mathcal{H}_1 \bigcup \mathcal{H}_2) \bigwedge \underset{P,Q}{\forall} (P \in \mathcal{H}_1 \wedge Q \in \mathcal{H}_2 \Rightarrow s[P, Q] \bigcap \{V\} \neq \emptyset)\right)]$$

▫ The Plane Separation Axiom

Axiom["PSA",

$$\underset{g}{\forall}\ \underset{\substack{\mathcal{H}_1,\mathcal{H}_2\\ \text{is-cv}[\mathcal{H}_1,\mathcal{H}_2]}}{\exists}\left((\mathbb{P}\ominus g=\mathcal{H}_1\cup\mathcal{H}_2)\bigwedge\underset{\substack{P,Q\\ P\neq Q}}{\forall}(P\in\mathcal{H}_1\wedge Q\in\mathcal{H}_2\Rightarrow s[P,Q]\cap g\neq\emptyset)\right)]$$

Proposition["G5.2.1", any[g, $\mathcal{H}_1$, $\mathcal{H}_2$, A, B], with[is-cv[$\mathcal{H}_1$, $\mathcal{H}_2$] $\wedge$ $(\mathbb{P}\ominus g=\mathcal{H}_1\cup\mathcal{H}_2)\wedge A\neq B\wedge A\notin g\wedge B\notin g$],
$s[A,B]\cap g\neq\emptyset\Rightarrow(A\notin\mathcal{H}_1\vee B\notin\mathcal{H}_1)\wedge(A\notin\mathcal{H}_2\vee B\notin\mathcal{H}_2)$]

Proposition["G5.2.2", any[g, $\mathcal{H}_1$, $\mathcal{H}_2$], with[is-cv[$\mathcal{H}_1$, $\mathcal{H}_2$] $\wedge$ $(\mathbb{P}\ominus g=\mathcal{H}_1\cup\mathcal{H}_2)$],
$\mathcal{H}_1\neq\emptyset\bigwedge\mathcal{H}_2\neq\emptyset\bigwedge(\mathcal{H}_1\cap\mathcal{H}_2=\emptyset)$]

Proposition["G5.2.3", any[g, $\mathcal{H}_1$, $\mathcal{H}_2$], with[is-cv[$\mathcal{H}_1$, $\mathcal{H}_2$] $\wedge$ $(\mathbb{P}\ominus g=\mathcal{H}_1\cup\mathcal{H}_2)$],
$(\mathbb{P}=\mathcal{H}_1\cup g\cup\mathcal{H}_2)\wedge(\mathcal{H}_2=\mathbb{P}\ominus(\mathcal{H}_1\cup g))\wedge(\mathcal{H}_2=(\mathbb{P}\ominus g)\ominus\mathcal{H}_1)$]

Proposition["G5.2.4", any[g, $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$, $\mathcal{H}_4$],

$$\text{is-cv}[\mathcal{H}_1,\mathcal{H}_2,\mathcal{H}_3,\mathcal{H}_4]\bigwedge(\mathbb{P}\ominus g=\mathcal{H}_1\cup\mathcal{H}_2)\bigwedge\underset{\substack{P,Q\\ P\neq Q}}{\forall}(P\in\mathcal{H}_1\wedge Q\in\mathcal{H}_2\Rightarrow s[P,Q]\cap g\neq\emptyset)\bigwedge$$

$$(\mathbb{P}\ominus g=\mathcal{H}_3\cup\mathcal{H}_4)\bigwedge\underset{\substack{P,Q\\ P\neq Q}}{\forall}(P\in\mathcal{H}_3\wedge Q\in\mathcal{H}_4\Rightarrow s[P,Q]\cap g\neq\emptyset)\Rightarrow(\{\mathcal{H}_1,\mathcal{H}_2\}=\{\mathcal{H}_3,\mathcal{H}_4\})]$$

Definition["G5.2.5",

$$\underset{\substack{g,A\\ A\notin g}}{\forall}\left(\text{hp}[g,A]=\left\{\underset{X\notin g}{X\mid}(X=A)\vee s[A,X]\cap g=\emptyset\right\}\right)]$$

Definition["G5.2.6",

$$\underset{\substack{g,A\\ A\notin g}}{\forall}\left(\text{ohp}[g,A]=\left\{\underset{X\notin g}{X\mid}X\neq A\bigwedge s[A,X]\cap g\neq\emptyset\right\}\right)]$$

Proposition["G5.2.7", any[g, A], with[$A\notin g$],

$$\text{is-cv}[\text{hp}[g,A],\text{ohp}[g,A]]\bigwedge(\mathbb{P}\ominus g=\text{hp}[g,A]\cup\text{ohp}[g,A])\bigwedge\underset{\substack{P,Q\\ P\neq Q}}{\forall}(P\in\text{hp}[g,A]\wedge Q\in\text{ohp}[g,A]\Rightarrow s[P,Q]\cap g\neq\emptyset)]$$

Definition["G5.2.8",

$$\underset{g,A,B}{\forall}(\text{is-ss}[g,A,B]\Leftrightarrow(A\notin g\wedge B\in\text{hp}[g,A]))]$$

Definition["G5.2.9",

$$\underset{g,A,B}{\forall}(\text{is-os}[g,A,B]\Leftrightarrow(A\notin g\wedge B\in\text{ohp}[g,A]))]$$

Definition["G5.2.10",

$$\underset{\substack{A,B,P\\ \text{is-tc}[A,B,P]}}{\forall}(\text{hp}[A,B,P]=\text{hp}[l[A,B],P])]$$

Definition["G5.2.11",

$$\underset{\substack{A,B,P\\ \text{is-tc}[A,B,P]}}{\forall}(\text{ohp}[A,B,P]=\text{ohp}[l[A,B],P])]$$

Proposition["G5.2.12", any[g, h, A], with[$A\notin g$],
$(\text{hp}[g,A]=\text{hp}[h,A])\Rightarrow(g=h)$]

Incidence Theorems based on PSA

Proposition["G5.3.1", any[g, A, B], with[A ∉ g],
B ∈ hp[g, A] ⇒ (hp[g, A] = hp[g, B]) "a"
B ∈ ohp[g, A] ⇒ (hp[g, A] = ohp[g, B]) "b"]

Proposition["G5.3.2", any[g, A, B], with[A ∉ g ∧ B ∉ g],
(hp[g, B] = ohp[g, A]) ⇒ (hp[g, A] = ohp[g, B])]

Proposition["G5.3.3", any[g, A, B],
is-os[g, A, B] ⇒ ¬ is-ss[g, A, B]]

Proposition["G5.3.4", any[g, A, B, C], with[is-os[g, A, B]],
is-os[g, B, C] ⇒ is-ss[g, A, C] "a"
is-ss[g, B, C] ⇒ is-os[g, A, C] "b"]

Proposition["G5.3.5", any[g, A, B], with[A ∈ g ∧ B ∉ g],
$\forall_{P}$ (P ∈ hl[A, B] ⇒ is-ss[g, B, P])]

Proposition["G5.3.6", any[g, A, B, C], with[is-tc[A, B, C] ∧ A ∉ g ∧ B ∉ g ∧ C ∉ g],
(s[A, B] ⋂ g = ∅) ⋁ (s[B, C] ⋂ g = ∅) ⋁ (s[C, A] ⋂ g = ∅)]

Proposition["G5.3.7", any[g, A, B, C], with[is-tc[A, B, C] ∧ A ∉ g ∧ B ∉ g ∧ C ∉ g],
g ⋂ s[A, B] ≠ ∅ ⇒ (g ⋂ s[B, C] ≠ ∅ ⋁ g ⋂ s[A, C] ≠ ∅)]

Proposition["G5.3.8", any[A, B, C, D, E], with[is-tc[A, B, C]],
is-b[A, B, D] ∧ is-b[B, E, C] ⇒ $\exists_{F}$ (F ∈ l[D, E] ∧ is-b[A, F, C])]

Proposition["G5.3.9", any[A, B, C, D, F], with[is-tc[A, B, C]],
is-b[A, B, D] ∧ is-b[A, F, C] ⇒ $\exists_{E}$ (E ∈ l[D, F] ∧ is-b[B, E, C] ∧ is-b[D, E, F])]

Proposition["G5.3.10", any[A, B, C, P], with[is-tc[A, B, C]],
$\exists_{\substack{Q,R \\ Q\neq R}}$ (Q ∈ △[A, B, C] ∧ R ∈ △[A, B, C] ∧ P ∈ l[Q, R])]

Proposition["G5.3.11", any[$\mathcal{F}$, g, A], with[is-cv[$\mathcal{F}$] ⋀ $\mathcal{F}$ ≠ ∅ ⋀ A ∈ $\mathcal{F}$],
$\mathcal{F}$ ⋂ g = ∅ ⇒ $\mathcal{F}$ ⊆ hp[g, A]]

More Incidence Theorems and Quadrilaterals

Interior of Angles and Triangles

Definition["G6.1.1",
$\forall_{\substack{A,V,B \\ \text{is-tc}[A,V,B]}}$ (int[A, V, B] = hp[V, A, B] ⋂ hp[V, B, A])]

Proposition["G6.1.2", any[A, V, B, P], with[is-tc[A, V, B]],
P ∈ int[A, V, B] ⇔ (is-ss[l[V, B], A, P] ∧ is-ss[l[V, A], B, P]) "a"
is-b[A, P, B] ⇒ P ∈ int[A, V, B] "b"]

Proposition["G6.1.3", any[A, B, C], with[is-tc[A, B, C]],
int[A, B] ⊆ int[A, C, B]]

Proposition["G6.1.4", any[A, V, B, P, D], with[is-tc[A, V, B] $\wedge$ P $\in$ int[A, V, B]],
r[A, P] $\bigcap$ r[V, B] = {D} $\Rightarrow$ is-b[A, P, D]]

Proposition["G6.1.5", any[A, V, B, P], with[is-tc[A, V, B]],
P $\in$ int[A, V, B] $\Rightarrow$ r[V, P] $\bigcap$ int[A, B] $\neq \emptyset$]

Proposition["G6.1.6", any[A, V, B, P], with[is-tc[A, V, B]],
r[V, P] $\bigcap$ int[A, B] $\neq \emptyset \Rightarrow$ P $\in$ int[A, V, B]]

Proposition["G6.1.7", any[A, V, B, P], with[is-tc[A, V, B] $\wedge$ P $\in$ hp[V, A, B]],
P $\in$ int[A, V, B] $\Leftrightarrow$ is-os[l[V, P], A, B]]

Proposition["G6.1.8", any[A, V, B, C, P], with[is-tc[A, V, B] $\wedge$ P $\in$ hp[V, A, B] $\wedge$ is-b[A, V, C]],
P $\in$ int[A, V, B] $\Leftrightarrow$ B $\in$ int[C, V, P]]

Proposition["G6.1.9", any[A, V, B, P], with[is-tc[A, V, B] $\wedge$ P $\in$ hp[V, A, B]],
$\overset{!}{\vee}$ [(r[V, P] = r[V, B]), P $\in$ int[A, V, B], B $\in$ int[A, V, P]]]

Proposition["G6.1.10", any[A, V, B, C, D, E], with[is-tc[A, V, B] $\wedge$ is-tc[C, V, D]],
($\angle$[A, V, B] = $\angle$[C, V, D]) $\bigwedge$ r[V, E] $\bigcap$ int[C, D] $\neq \emptyset \Rightarrow$ r[V, E] $\bigcap$ int[A, B] $\neq \emptyset$]

Proposition["G6.1.11", any[A, B, C, D, E, F], with[is-tc[A, B, C]],
is-b[B, C, D] $\wedge$ is-b[A, E, C] $\wedge$ is-b[B, E, F] $\Rightarrow$ F $\in$ int[A, C, D]]

Definition["G6.1.12",
$\underset{\substack{A,V,B \\ \text{is-tc}[A,V,B]}}{\forall}$ (ar[A, V, B] = $\angle$[A, V, B] $\bigcup$ int[A, V, B])]

Definition["G6.1.13",
$\underset{\substack{A,V,B \\ \text{is-tc}[A,V,B]}}{\forall}$ (ext[A, V, B] = $\mathbb{P} \ominus$ ar[A, V, B])]

Definition["G6.1.14",
$\underset{\substack{A,B,C \\ \text{is-tc}[A,B,C]}}{\forall}$ (it[A, B, C] = hp[A, B, C] $\bigcap$ hp[B, C, A] $\bigcap$ hp[C, A, B])]

Proposition["G6.1.15", any[A, B, C, V], with[is-tc[A, V, B] $\wedge$ is-tc[A, B, C]],
is-cv[int[A, V, B]] $\wedge$ is-cv[it[A, B, C]]]

Proposition["G6.1.16", any[A, B, C], with[is-tc[A, B, C]],
it[A, B, C] = int[C, A, B] $\bigcap$ int[A, B, C] $\bigcap$ int[B, C, A]]

Proposition["G6.1.17", any[g, A, B, C], with[is-tc[A, B, C]],
$g \bigcap$ it[A, B, C] $\neq \emptyset \Rightarrow \underset{\substack{P,Q \\ P \neq Q}}{\exists}$ ($g \bigcap \triangle$[A, B, C] = {P, Q})]

▫ Quadrilaterals

Definition["G6.2.1",
$\underset{A,B,C,D}{\forall}$ (is-qc[A, B, C, D] $\Leftrightarrow$ (is-tc[A, B, C] $\wedge$ is-tc[B, C, D] $\wedge$
is-tc[C, D, A] $\wedge$ is-tc[D, A, B] $\wedge$ int[A, B] $\bigcap$ int[C, D] = $\emptyset \bigwedge$ int[B, C] $\bigcap$ int[A, D] = $\emptyset$))]

Definition["G6.2.2",
$\underset{\substack{A,B,C,D \\ \text{is-qc}[A,B,C,D]}}{\forall}$ ($\square$[A, B, C, D] = s[A, B] $\bigcup$ s[B, C] $\bigcup$ s[C, D] $\bigcup$ s[D, A])]

Proposition["G6.2.3", any[A, B, C, D], with[is-qc[A, B, C, D]],
$(\square[A, B, C, D] = \square[D, C, B, A]) \wedge (\square[A, B, C, D] = \square[B, C, D, A]) \wedge (\square[A, B, C, D] = \square[C, D, A, B]) \wedge (\square[A, B, C, D] = \square[D, A, B, C])$]

Proposition["G6.2.4", any[A, B, C, D],
$\text{is-qc}[A, B, C, D] \wedge \text{is-qc}[A, B, D, C] \Rightarrow \square[A, B, C, D] \neq \square[A, B, D, C]$]

Proposition["G6.2.5", any[A, B, C, D, E, F, G, H], with[is-qc[A, B, C, D] ∧ is-qc[E, F, G, H]],
$(\square[A, B, C, D] = \square[E, F, G, H]) \Rightarrow (\{A, B, C, D\} = \{E, F, G, H\})$]

Definition["G6.2.6",
$\forall_{A,B,C,D} (\text{is-cqc}[A, B, C, D] \Leftrightarrow (\text{is-qc}[A, B, C, D] \wedge A \in \text{hp}[B, C, D] \wedge B \in \text{hp}[C, D, A] \wedge C \in \text{hp}[D, A, B] \wedge D \in \text{hp}[A, B, C]))$]

Proposition["G6.2.7", any[A, B, C, D],
$\text{is-cqc}[A, B, C, D] \Leftrightarrow (\text{is-qc}[A, B, C, D] \wedge A \in \text{int}[B, C, D] \wedge B \in \text{int}[C, D, A] \wedge C \in \text{int}[D, A, B] \wedge D \in \text{int}[A, B, C])$]

Proposition["G6.2.8", any[A, B, C, D],
$\text{is-cqc}[A, B, C, D] \Leftrightarrow (\text{is-qc}[A, B, C, D] \bigwedge s[A, C] \bigcap s[B, D] \neq \emptyset)$]

Definition["G6.2.9",
$\forall_{A,B,C,D} (\text{is-trc}[A, B, C, D] \Leftrightarrow (\text{is-qc}[A, B, C, D] \wedge l[A, B] \parallel l[C, D]))$]

Definition["G6.2.10",
$\forall_{A,B,C,D} (\text{is-itc}[A, B, C, D] \Leftrightarrow (\text{is-trc}[A, B, C, D] \wedge s[A, D] \simeq s[B, C]))$]

Definition["G6.2.11",
$\forall_{A,B,C,D} (\text{is-pc}[A, B, C, D] \Leftrightarrow (\text{is-trc}[A, B, C, D] \wedge l[B, C] \parallel l[A, D]))$]

Proposition["G6.2.12", any[A, B, C, D],
$\text{is-pc}[A, B, C, D] \Rightarrow \text{is-cqc}[A, B, C, D]$]

Angular Measure

Degree Measure of Angles and the Protractor Axiom

Definition["G7.1.1",
$\mathbb{A} = \{\angle[A, V, B] \mid \text{is-tc}[A, V, B]\}$]

Axiom["PA1",
$\alpha : \mathbb{A} \longrightarrow \{x \mid 0 < x < 180\}$]

Definition["G7.1.2",
$\forall_{\substack{A,V,B \\ \text{is-tc}[A,V,B]}} (m[A, V, B] = \alpha[\![\angle[A, V, B]]\!])$]

Definition["G7.1.3",
$\forall_{\mathcal{H},V} \left(\text{is-hl}[\mathcal{H}, V] \Leftrightarrow \exists_{\substack{A \\ A \neq V}} (\mathcal{H} = \text{hl}[V, A])\right)$]

Axiom["PA2",
$\forall_{\substack{V,A,P \\ \text{is-tc}[V,A,P]}} \forall_{\substack{r \\ 0<r<180}} \exists!_{\substack{\mathcal{H} \\ \text{is-hl}[\mathcal{H},V]}} \left(\mathcal{H} \subseteq \text{hp}[V, A, P] \bigwedge \forall_{X \in \mathcal{H}} (m[A, V, X] = r)\right)$]

Axiom$\Big[$"PA3",

$$\underset{\substack{V,A,B,C\\ \text{is-tc}[A,V,C]}}{\forall} (B \in \text{int}[A, V, C] \Rightarrow (m[A, V, B] + m[B, V, C] = m[A, V, C]))\Big]$$

Proposition["G7.1.4", any[A, V, B, C], with[is-tc[A, V, C] $\wedge$ B $\in$ hp[V, A, C]],
m[A, V, B] < m[A, V, C] $\Rightarrow$ B $\in$ int[A, V, C]]

Definition$\Big[$"G7.1.5",

$$\underset{\mathcal{A}_1,\mathcal{A}_2}{\forall} (\text{is-sup}[\mathcal{A}_1, \mathcal{A}_2] \Leftrightarrow (\mathcal{A}_1 \in \mathbb{A} \wedge \mathcal{A}_2 \in \mathbb{A} \wedge (\alpha[\![\mathcal{A}_1]\!] + \alpha[\![\mathcal{A}_2]\!] = 180)))\Big]$$

Definition$\Big[$"G7.1.6",

$$\underset{\mathcal{A}_1,\mathcal{A}_2}{\forall} (\text{is-com}[\mathcal{A}_1, \mathcal{A}_2] \Leftrightarrow (\mathcal{A}_1 \in \mathbb{A} \wedge \mathcal{A}_2 \in \mathbb{A} \wedge (\alpha[\![\mathcal{A}_1]\!] + \alpha[\![\mathcal{A}_2]\!] = 90)))\Big]$$

Proposition["G7.1.7", any[$\mathcal{A}_1$, $\mathcal{A}_2$],
is-lp[$\mathcal{A}_1$, $\mathcal{A}_2$] $\Rightarrow$ is-sup[$\mathcal{A}_1$, $\mathcal{A}_2$]]

Proposition["G7.1.8", any[A, V, B, C], with[is-tc[A, V, B] $\wedge$ is-os[l[V, B], A, C]],
(m[A, V, B] + m[B, V, C] = 180) $\Rightarrow$ is-b[A, V, C]]

Proposition["G7.1.9", any[A, V, B, C], with[is-tc[A, V, C] $\wedge$ is-tc[B, V, C]],
(m[A, V, B] + m[B, V, C] = m[A, V, C]) $\Rightarrow$ B $\in$ int[A, V, C]]

Definition$\Big[$"G7.1.10",

$$\underset{\mathcal{A}}{\forall} (\text{is-aa}[\mathcal{A}] \Leftrightarrow (\mathcal{A} \in \mathbb{A} \wedge \alpha[\![\mathcal{A}]\!] < 90))\Big]$$

Definition$\Big[$"G7.1.11",

$$\underset{\mathcal{A}}{\forall} (\text{is-oa}[\mathcal{A}] \Leftrightarrow (\mathcal{A} \in \mathbb{A} \wedge \alpha[\![\mathcal{A}]\!] > 90))\Big]$$

▫ Congruence of Angles and Angle Bisector

Definition$\Big[$"G7.2.1",

$$\underset{\mathcal{A}_1,\mathcal{A}_2}{\forall} (\mathcal{A}_1 \cong \mathcal{A}_2 \Leftrightarrow (\mathcal{A}_1 \in \mathbb{A} \wedge \mathcal{A}_2 \in \mathbb{A} \wedge (\alpha[\![\mathcal{A}_1]\!] = \alpha[\![\mathcal{A}_2]\!])))\Big]$$

Proposition["G7.2.2", any[A, V, B, C, W, D], with[is-tc[A, V, B] $\wedge$ is-tc[C, W, D]],
$\angle$[A, V, B] $\cong$ $\angle$[C, W, D] $\Leftrightarrow$ (m[A, V, B] = m[C, W, D])]

Proposition["G7.2.3", any[A, V, B, C, W, D, E, X, F], with[is-tc[A, V, B] $\wedge$ is-tc[C, W, D] $\wedge$ is-tc[E, X, F]],
$\angle$[A, V, B] $\cong$ $\angle$[A, V, B] $\wedge$ ($\angle$[A, V, B] $\cong$ $\angle$[C, W, D] $\Rightarrow$ $\angle$[C, W, D] $\cong$ $\angle$[A, V, B]) $\wedge$
($\angle$[A, V, B] $\cong$ $\angle$[C, W, D] $\wedge$ $\angle$[C, W, D] $\cong$ $\angle$[E, X, F] $\Rightarrow$ $\angle$[A, V, B] $\cong$ $\angle$[E, X, F])]

Proposition$\Big[$"G7.2.4", any[A, V, B, W, C, P], with[is-tc[A, V, B] $\wedge$ is-tc[W, C, P]],

$$\underset{\mathcal{H}}{\exists!}\left(\text{is-hl}[\mathcal{H}, W] \bigwedge \mathcal{H} \subseteq \text{hp}[W, C, P] \bigwedge \underset{\substack{D\\ D\in\mathcal{H}}}{\forall} \angle[A, V, B] \cong \angle[C, W, D]\right)\Big]$$

Proposition$\Big[$"G7.2.5", any[A, B, P, a, r], with[is-tc[A, B, P] $\wedge$ 0 < a < 180 $\wedge$ r > 0],

$$\underset{C}{\exists!} (C \in \text{hp}[A, B, P] \wedge (m[A, B, C] = a) \wedge (d[B, C] = r))\Big]$$

Proposition["G7.2.6", any[A_1, V_1, B_1, C_1, A_2, V_2, B_2, C_2],
with[is-tc[A_1, V_1, B_1] $\wedge$ is-tc[A_2, V_2, B_2] $\wedge$ $C_1 \in$ int[A_1, V_1, B_1] $\wedge$
$C_2 \in$ int[A_2, V_2, B_2] $\wedge$ $\angle$[A_1, V_1, C_1] $\cong$ $\angle$[A_2, V_2, C_2]],
$\angle$[A_1, V_1, B_1] $\cong$ $\angle$[A_2, V_2, B_2] $\Leftrightarrow$ $\angle$[C_1, V_1, B_1] $\cong$ $\angle$[C_2, V_2, B_2]]

Proposition["G7.2.7", any[$\mathcal{A}_1$, $\mathcal{A}_2$],
is-pv[$\mathcal{A}_1$, $\mathcal{A}_2$] $\Rightarrow$ $\mathcal{A}_1 \cong \mathcal{A}_2$]

Proposition["G7.2.8", any[A, B, C, D, V], with[is-b[A, V, C] ∧ is-os[l[A, C], B, D]],
$\angle$[A, V, B] ≅ $\angle$[C, V, D] ⇒ is-b[B, V, D]]

Definition["G7.2.9",

$$\underset{\substack{\mathcal{B},A,V,B \\ \text{is-tc}[A,V,B]}}{\forall}\left(\text{is-ab}[\mathcal{B}, A, V, B] \Leftrightarrow \underset{\substack{C \\ C\in\text{int}[A,V,B]}}{\exists}((\mathcal{B} = \text{r}[V, C]) \wedge \angle[A, V, C] \cong \angle[C, V, B])\right)]$$

Proposition["G7.2.10", any[A, V, B], with[is-tc[A, V, B]],
$\underset{\mathcal{B}}{\exists!}$ is-ab[$\mathcal{B}$, A, V, B]]

Definition["G7.2.11",

$$\underset{\substack{A,V,B \\ \text{is-tc}[A,V,B]}}{\forall}\left(\text{ab}[A, V, B] = \underset{\mathcal{B}}{\iota}\ \text{is-ab}[\mathcal{B}, A, V, B]\right)]$$

Proposition["G7.2.12", any[A, V, B, P],
P ≠ V ∧ P ∈ ab[A, V, B] ⇒ P ∈ int[A, V, B]]

Right Angles and Perpendicularity

Definition["G7.3.1",
$\underset{\mathcal{A}}{\forall}$ (is-ra[$\mathcal{A}$] ⇔ ($\mathcal{A} \in \mathbb{A} \wedge (\alpha[\![\mathcal{A}]\!] = 90)$))]

Proposition["G7.3.2", any[$\mathcal{A}_1$, $\mathcal{A}_2$],
$\mathcal{A}_1 \cong \mathcal{A}_2$ ∧ is-lp[$\mathcal{A}_1$, $\mathcal{A}_2$] ⇒ is-ra[$\mathcal{A}_1$, $\mathcal{A}_2$]]

Proposition["G7.3.3", any[A, V, B, C, D], with[is-tc[A, V, B] ∧ is-b[A, V, C] ∧ is-b[B, V, D]],
is-ra[$\angle$[A, V, B]] ⇒ is-ra[$\angle$[C, V, B], $\angle$[A, V, D], $\angle$[C, V, D]]]

Definition["G7.3.4",
$\underset{g,h}{\forall}\left(g \perp h \Leftrightarrow \underset{\mathcal{A}}{\exists}(\text{is-ra}[\mathcal{A}] \wedge \mathcal{A} \subseteq g \cup h)\right)$]

Proposition["G7.3.5", any[g, h],
$g \perp h \Rightarrow h \perp g$]

Proposition["G7.3.6", any[g, P], with[P ∈ g],
$\underset{h}{\exists!}$ (P ∈ h ∧ $h \perp g$)]

Congruence for Triangles

The Penultimate Axiom SAS

Definition["G8.1.1",
$\underset{A,B,C,D,E,F}{\forall}$ (is-ctc[A, B, C, D, E, F] ⇔ (is-tc[A, B, C] ∧ is-tc[D, E, F] ∧ s[A, B] ≃ s[D, E] ∧ s[A, C] ≃ s[D, F] ∧
s[B, C] ≃ s[E, F] ∧ $\angle$[C, A, B] ≅ $\angle$[F, D, E] ∧ $\angle$[A, B, C] ≅ $\angle$[D, E, F] ∧ $\angle$[B, C, A] ≅ $\angle$[E, F, D]))]

Proposition["G8.1.2", any[A, B, C, D, E, F],
(is-ctc[A, B, C, D, E, F] ⇔ is-ctc[D, E, F, A, B, C]) ∧
(is-ctc[A, B, C, D, E, F] ⇔ is-ctc[B, C, A, E, F, D]) ∧ (is-ctc[A, B, C, D, E, F] ⇔ is-ctc[C, B, A, F, E, D])]

Proposition["G8.1.3", any[A, B, C, D, E, F, G, H, I],
is-ctc[A, B, C, A, B, C] ∧ (is-ctc[A, B, C, D, E, F] ⇒ is-ctc[D, E, F, A, B, C]) ∧
(is-ctc[A, B, C, D, E, F] ∧ is-ctc[D, E, F, G, H, I] ⇒ is-ctc[A, B, C, G, H, I])]

Definition["G8.1.4",

$\forall_{\mathcal{T}_1,\mathcal{T}_2} \left(\text{is-cg}[\mathcal{T}_1, \mathcal{T}_2] \Leftrightarrow \right.$

$\exists_{A,B,C,D,E,F} ((\mathcal{T}_1 = \triangle[A, B, C]) \wedge (\mathcal{T}_2 = \triangle[D, E, F]) \wedge (\text{is-ctc}[A, B, C, D, E, F] \vee \text{is-ctc}[A, C, B, D, E, F] \vee$

$\left. \text{is-ctc}[B, A, C, D, E, F] \vee \text{is-ctc}[B, C, A, D, E, F] \vee \text{is-ctc}[C, A, B, D, E, F] \vee \text{is-ctc}[C, B, A, D, E, F]))\right)$]

Proposition["G8.1.5", any[A, B, C, D, E, F], with[is-tc[A, B, C] ∧ is-tc[D, E, F]],
is-cg[△[A, B, C], △[D, E, F]] ⇔ (is-ctc[A, B, C, D, E, F] ∨ is-ctc[A, C, B, D, E, F] ∨
is-ctc[B, A, C, D, E, F] ∨ is-ctc[B, C, A, D, E, F] ∨ is-ctc[C, A, B, D, E, F] ∨ is-ctc[C, B, A, D, E, F])]

Axiom["SAS",

$\forall_{\substack{A,B,C,D,E,F \\ \text{is-tc}[A,B,C] \wedge \text{is-tc}[D,E,F]}} (s[A, B] \simeq s[D, E] \wedge \angle[C, A, B] \cong \angle[F, D, E] \wedge s[A, C] \simeq s[D, F] \Rightarrow \text{is-ctc}[A, B, C, D, E, F])$]

▫ The Basic Congruence Theorems

Definition["G8.2.1",

$\forall_{A,B,C} (\text{is-iso}[A, B, C] \Leftrightarrow (\text{is-tc}[A, B, C] \wedge s[A, C] \simeq s[B, C]))$]

Definition["G8.2.2",

$\forall_{A,B,C} (\text{is-sca}[A, B, C] \Leftrightarrow (\text{is-tc}[A, B, C] \wedge \neg \text{is-iso}[A, B, C] \wedge \neg \text{is-iso}[B, C, A] \wedge \neg \text{is-iso}[C, A, B]))$]

Proposition["G8.2.3", any[A, B, C],
is-iso[A, B, C] ⇒ ∠[C, A, B] ≅ ∠[C, B, A]]

Definition["G8.2.4",

$\forall_{A,B,C} (\text{is-eql}[A, B, C] \Leftrightarrow (\text{is-iso}[A, B, C] \wedge s[A, B] \simeq s[A, C]))$]

Definition["G8.2.5",

$\forall_{A,B,C} (\text{is-eqa}[A, B, C] \Leftrightarrow \text{is-tc}[A, B, C] \wedge \angle[C, A, B] \cong \angle[A, B, C] \wedge \angle[A, B, C] \cong \angle[B, C, A])$]

Proposition["G8.2.6", any[A, B, C],
is-eql[A, B, C] ⇒ is-eqa[A, B, C]]

Proposition["ASA", any[A, B, C, D, E, F], with[is-tc[A, B, C] ∧ is-tc[D, E, F]],
∠[C, A, B] ≅ ∠[F, D, E] ∧ s[A, B] ≃ s[D, E] ∧ ∠[C, B, A] ≅ ∠[F, E, D] ⇒ is-ctc[A, B, C, D, E, F]]

Proposition["G8.2.7", any[A, B, C], with[is-tc[A, B, C]],
∠[C, A, B] ≅ ∠[C, B, A] ⇒ is-iso[A, B, C]]

Proposition["G8.2.8", any[A, B, C],
is-eqa[A, B, C] ⇒ is-eql[A, B, C]]

Proposition["SSS", any[A, B, C, D, E, F], with[is-tc[A, B, C] ∧ is-tc[D, E, F]],
s[A, B] ≃ s[D, E] ∧ s[B, C] ≃ s[E, F] ∧ s[C, A] ≃ s[F, D] ⇒ is-ctc[A, B, C, D, E, F]]

▪ Geometric Inequalities

▫ Exterior Angle Theorem and its Consequences, Perpendicular Bisector

Proposition["G9.1.1", any[A, B, C, D], with[is-tc[A, B, C]],
is-b[D, A, B] ⇒ m[D, A, C] > m[B, C, A] "a"
is-b[D, A, B] ⇒ m[D, A, C] > m[A, B, C] "b"]

Proposition["G9.1.2", any[A, B, C],
is-iso[A, B, C] ⇒ is-aa[∠[C, A, B]]]

Proposition["SAA", any[A, B, C, D, E, F], with[is-tc[A, B, C] ∧ is-tc[D, E, F]],
s[A, B] ≃ s[D, E] ∧ ∠[A, B, C] ≅ ∠[D, E, F] ∧ ∠[B, C, A] ≅ ∠[E, F, D] ⇒ is-ctc[A, B, C, D, E, F]]

Proposition["G9.1.3", any[g, P],
$\exists!_{h}$ (P ∈ h ∧ $h \perp g$)]

Definition["G9.1.4",
$\forall_{P,g}$ (pp[g, P] = ι_{h} (P ∈ h ∧ $h \perp g$))]

Definition["G9.1.5",
$\forall_{P,g}$ (ft[g, P] = i[g, pp[g, P]])]

Definition["G9.1.6",
$\forall_{g,A,B}$ (is-pbs[g, A, B] ⇔ (A ≠ B ∧ mp[A, B] ∈ g ∧ $g \perp$ l[A, B]))]

Proposition["G9.1.7", any[A, B], with[A ≠ B],
$\exists!_{g}$ is-pbs[g, A, B]]

Definition["G9.1.8",
$\forall_{\substack{A,B \\ A \neq B}}$ (pb[A, B] = ι_{g} is-pbs[g, A, B])]

Proposition["G9.1.9", any[A, B, X], with[A ≠ B],
d[X, A] = d[X, B] ⇔ X ∈ pb[A, B]]

Proposition["G9.1.10", any[P, g],
P ∉ g ⇒ $\exists!_{Q}$ is-pbs[g, P, Q]]

Proposition["G9.1.11", any[g, h, k],
$h \perp g \wedge k \perp g \Rightarrow h \parallel k$]

Proposition["G9.1.12", any[P, Q, R], with[is-pd[P, Q, R]],
is-col[P, Q, R] ⇒ pb[P, Q] ∥ pb[Q, R]]

Proposition["G9.1.13", any[A, B, C, F], with[is-tc[A, B, C] ∧ (F = ft[l[A, B], C])],
is-aa[∠[B, A, C]] ⇒ F ∈ hl[A, B]]

▫ Inequalities and Right Triangles

Proposition["9.2.1", any[A, B, C], with[is-tc[A, B, C]],
d[A, B] > d[A, C] ⇒ m[A, C, B] > m[A, B, C]]

Proposition["9.2.2", any[A, B, C], with[is-tc[A, B, C]],
m[B, C, A] > m[A, B, C] ⇒ d[A, B] > d[A, C]]

Proposition["G9.2.3", any[A, B, C],
is-tc[A, B, C] ⇒ d[A, B] + d[B, C] > d[A, C]]

Proposition["G9.2.4", any[A, B, C],
d[A, B] + d[B, C] ≥ d[A, C]]

Proposition["G9.2.5", any[A, B, C],
(d[A, B] + d[B, C] = d[A, C]) ⇔ ((A ≠ C ∧ B ∈ s[A, C]) ∨ (A = B = C))]

Proposition["G9.2.6", any[A, B, M], with[A ≠ B],
$\left(d[A, M] = \frac{d[A, B]}{2}\right) \bigwedge \left(d[B, M] = \frac{d[A, B]}{2}\right) \Rightarrow (M = mp[A, B])$]

Proposition["G9.2.7", any[A, B], with[A ≠ B],
s[A, B] = {X | (d[A, X] + d[X, B] = d[A, B])} "a"
r[A, B] = {X | (d[B, X] = | d[A, X] − d[A, B] |)} "b"]

Proposition["G9.2.8", any[A, B, C],
is-tc[A, B, C] ⇔ (d[A, B] + d[B, C] > d[A, C] ∧ d[A, C] + d[C, B] > d[A, B] ∧ d[B, A] + d[A, C] > d[B, C])]

Proposition["G9.2.9", any[A, B, C, D], with[is-tc[A, B, C]],
D ∈ tri[A, B, C] ⇒ d[A, D] + d[D, B] < d[A, C] + d[C, B] ∧ m[A, D, B] > m[A, C, B]]

Proposition["G9.2.10", any[A, B, C, D], with[is-tc[A, B, C]],
is-b[A, D, B] ∧ d[B, C] ≥ d[A, C] ⇒ d[D, C] < d[B, C]]

Proposition["G9.2.11", any[A, B, C, D, E, F], with[is-tc[A, B, C] ∧ is-tc[D, E, F]],
s[A, B] ≃ s[D, E] ∧ s[A, C] ≃ s[D, F] ∧ m[C, A, B] > m[F, D, E] ⇒ s[B, C] > d[E, F]]

Proposition["G9.2.12", any[A, B, C, D, E, F], with[is-tc[A, B, C] ∧ is-tc[D, E, F]],
s[A, B] ≃ s[D, E] ∧ s[A, C] ≃ s[D, F] ∧ d[B, C] > d[E, F] ⇒ m[C, A, B] > m[F, D, E]]

Proposition["G9.2.13", any[A, B, C], with[is-tc[A, B, C]],
m[B, C, A] ≥ 90 ⇒ is-aa[∠[C, A, B], ∠[A, B, C]]]

Definition["G9.2.14",
$\forall_{A,B,C}$ (is-rtc[A, B, C] ⇔ (is-tc[A, B, C] ∧ l[A, C] ⊥ l[B, C]))]

Proposition["G9.2.15", any[g, A, B, C],
(A ∉ g ∧ (C = ft[g, A]) ∧ B ≠ C ∧ B ∈ g) ⇒ (d[A, C] < d[A, B] ∧ d[B, C] < d[A, B])]

Definition["G9.2.16",
$\forall_{P,g}$ (d[g, P] = d[P, ft[g, P]])]

Proposition["G9.2.17", any[A, B, C, F], with[is-tc[A, B, C] ∧ (F = ft[l[A, B], C])],
d[A, B] ≥ d[A, C] ∧ d[A, B] ≥ d[B, C] ⇒ is-b[A, F, B]]

Proposition["G9.2.18", any[A, B, C, D, E, F], with[is-rtc[A, B, C] ∧ is-rtc[D, E, F]],
s[B, C] ≃ s[E, F] ∧ s[A, B] ≃ s[D, E] ⇒ is-ctc[A, B, C, D, E, F]]

Proposition["G9.2.19", any[P, Q, R],
is-pd[P, Q, R] ⇒ pb[P, Q] ≠ pb[Q, R]]

Proposition["G9.2.20", any[A, V, B, P], with[P ≠ V],
P ∈ ab[A, V, B] ⇒ (d[l[V, A], P] = d[l[V, B], P])]

Proposition["G9.2.21", any[A, V, B, P], with[P ∈ int[A, V, B]],
(d[l[V, A], P] = d[l[V, B], P]) ⇒ (ft[l[V, A], P] ∈ hl[V, A] ∧ ft[l[V, B], P] ∈ hl[V, B])]

Proposition["G9.2.22", any[A, V, B, P], with[P ∈ int[A, V, B]],
(d[l[V, A], P] = d[l[V, B], P]) ⇒ P ∈ ab[A, V, B]]

Proposition["G9.2.23", any[A, B, C], with[is-tc[A, B, C]],
$\exists_P$ (ab[C, A, B] ⋂ ab[A, B, C] ⋂ ab[B, C, A] = {P})]

Definition["G9.2.24",
$\forall_{\substack{A,B,C \\ \text{is-tc}[A,B,C]}}$ (inc[A, B, C] = ι_P (ab[C, A, B] ⋂ ab[A, B, C] ⋂ ab[B, C, A] = {P}))]

Proposition["G9.2.25", any[A, B, C, P], with[is-tc[A, B, C]],
$(\mathrm{P} = \mathrm{inc}[\mathrm{A}, \mathrm{B}, \mathrm{C}]) \Rightarrow (\mathrm{P} \in \mathrm{it}[\mathrm{A}, \mathrm{B}, \mathrm{C}] \wedge (\mathrm{d}[\mathrm{l}[\mathrm{A}, \mathrm{B}], \mathrm{P}] = \mathrm{d}[\mathrm{l}[\mathrm{B}, \mathrm{C}], \mathrm{P}]) \wedge (\mathrm{d}[\mathrm{l}[\mathrm{B}, \mathrm{C}], \mathrm{P}] = \mathrm{d}[\mathrm{l}[\mathrm{A}, \mathrm{C}], \mathrm{P}]))$]

▪ Reflections

▫ Introducing Isometries

Definition["G10.1.1",
$\underset{\varphi}{\forall}\left(\text{is-iso}[\varphi] \Leftrightarrow \left((\varphi : \mathbb{P} \longrightarrow \mathbb{P}) \bigwedge \underset{\mathrm{P},\mathrm{Q}}{\forall} (\mathrm{d}[\varphi[\![\mathrm{P}]\!], \varphi[\![\mathrm{Q}]\!]] = \mathrm{d}[\mathrm{P}, \mathrm{Q}])\right)\right)$]

Definition["G10.1.2",
$\mathbb{I} = \{\varphi \mid \text{is-iso}[\varphi]\}$]

Definition["G10.1.3",
$\mathrm{id} = \{\langle \mathrm{X}, \mathrm{X}\rangle \mid \mathrm{X} \in \mathbb{P}\}$]

Proposition["G10.1.4", any[φ],
$\text{is-iso}[\varphi] \Rightarrow \left(\varphi : \mathbb{P} \overset{\mathrm{inj}}{\longrightarrow} \mathbb{P}\right)$]

Proposition["G10.1.5", any[φ, A, B, C],
$\text{is-iso}[\varphi] \Rightarrow (\text{is-b}[\mathrm{A}, \mathrm{B}, \mathrm{C}] \Leftrightarrow \text{is-b}[\varphi[\![\mathrm{A}]\!], \varphi[\![\mathrm{B}]\!], \varphi[\![\mathrm{C}]\!]])$]

Proposition["G10.1.6", any[φ, A, B, C],
$\text{is-iso}[\varphi] \Rightarrow (\text{is-tc}[\mathrm{A}, \mathrm{B}, \mathrm{C}] \Leftrightarrow \text{is-tc}[\varphi[\![\mathrm{A}]\!], \varphi[\![\mathrm{B}]\!], \varphi[\![\mathrm{C}]\!]])$]

Proposition["G10.1.7", any[φ, A, B, C], with[is-tc[A, B, C]],
$\text{is-iso}[\varphi] \Rightarrow (\mathrm{m}[\mathrm{A}, \mathrm{B}, \mathrm{C}] = \mathrm{m}[\varphi[\![\mathrm{A}]\!], \varphi[\![\mathrm{B}]\!], \varphi[\![\mathrm{C}]\!]])$]

Proposition["G10.1.8", any[φ],
$\text{is-iso}[\varphi] \Rightarrow \left(\varphi : \mathbb{P} \overset{\mathrm{surj}}{\longrightarrow} \mathbb{P}\right)$]

Definition["G10.1.9",
$\underset{\substack{\varphi,\mathcal{F} \\ \varphi:\mathbb{P}\longrightarrow\mathbb{P}}}{\forall} (\mathrm{a}[\varphi, \mathcal{F}] = \{\varphi[\![\mathrm{P}]\!] \mid \mathrm{P} \in \mathcal{F}\})$]

Proposition["G10.1.10", any[φ, g],
$\text{is-iso}[\varphi] \Rightarrow \mathrm{a}[\varphi, g] \in \mathbb{L}$]

Proposition["G10.1.11", any[φ, ψ], with[is-iso[φ]],
$(\psi = \{\langle g, \mathrm{a}[\varphi, g]\rangle \mid g \in \mathbb{L}\}) \Rightarrow \left(\psi : \mathbb{L} \overset{\mathrm{bij}}{\longrightarrow} \mathbb{L}\right)$]

Proposition["G10.1.12", any[φ, A, B], with[A ≠ B],
$\text{is-iso}[\varphi] \Rightarrow (\mathrm{a}[\varphi, \mathrm{s}[\mathrm{A}, \mathrm{B}]] = \mathrm{s}[\varphi[\![\mathrm{A}]\!], \varphi[\![\mathrm{B}]\!]])$]

Proposition["G10.1.13", any[φ, A, B], with[A ≠ B],
$\text{is-iso}[\varphi] \Rightarrow (\mathrm{a}[\varphi, \mathrm{r}[\mathrm{A}, \mathrm{B}]] = \mathrm{r}[\varphi[\![\mathrm{A}]\!], \varphi[\![\mathrm{B}]\!]])$]

Proposition["G10.1.14", any[φ, ψ],
$\text{is-iso}[\varphi, \psi] \Rightarrow \text{is-iso}[\psi \circ \varphi]$]

Proposition["G10.1.15", any[φ],
$\text{is-iso}[\varphi] \Rightarrow \text{is-iso}[\varphi^{-1}]$]

▫ Reflection in a Line

Definition["G10.2.1",

$$\underset{\rho,g}{\forall}\left(\text{is-rf}[\rho,\, g] \Leftrightarrow \left((\rho : \mathbb{P} \longrightarrow \mathbb{P}) \bigwedge \underset{\substack{P \\ P\in g}}{\forall} (\rho[\![P]\!] = P) \bigwedge \underset{\substack{P \\ P\notin g}}{\forall} \text{is-pbs}[g,\, P,\, \rho[\![P]\!]]\right)\right)]$$

Proposition["G10.2.2", any[g],

$$\underset{\rho}{\exists!}\ \text{is-rf}[\rho,\, g]]$$

Definition["G10.3",

$$\underset{g}{\forall}\left(\text{r}[g] = \underset{\rho}{\iota}\ \text{is-rf}[\rho,\, g]\right)]$$

Definition["G10.2.4",

$$\underset{\varphi}{\forall}\ (\text{is-inv}[\varphi] \Leftrightarrow ((\varphi : \mathbb{P} \longrightarrow \mathbb{P}) \wedge \varphi \neq \text{id}) \wedge (\varphi \circ \varphi = \text{id}))]$$

Proposition["G10.2.5", any[ρ, g],
is-rf[ρ, g] $\Rightarrow$ is-inv[ρ]]

Proposition["G10.2.6", any[ρ, g],

$$\text{is-rf}[\rho,\, g] \Rightarrow \underset{\substack{P \\ P\notin g}}{\forall}\ (\text{a}[\rho,\, \text{hp}[g,\, P]] = \text{ohp}[g,\, P]) \quad \text{"a"}$$

$$\text{is-rf}[\rho,\, g] \Rightarrow \underset{\substack{P \\ P\notin g}}{\forall}\ (\text{a}[\rho,\, \text{ohp}[g,\, P]] = \text{hp}[g,\, P]) \quad \text{"b"}]$$

Proposition["G10.2.7", any[ρ, g, h], with[is-rf[ρ, g]],

$$\underset{\substack{X \\ X\in h}}{\forall}\ (\rho[\![X]\!] = X) \Leftrightarrow (g = h)]$$

Proposition["G10.2.8", any[ρ, g, h], with[is-rf[ρ, g]],
$(\text{a}[\rho,\, h] = h) \Leftrightarrow ((g = h) \vee h \perp g)$]

Proposition["G10.2.9", any[ρ, g],
is-rf[ρ, g] $\Rightarrow$ is-iso[ρ]]

Proposition["G10.2.10", any[φ, g], with[is-iso[φ]],

$$\underset{\substack{A,B \\ A\neq B \wedge A\in g \wedge B\in g}}{\exists}\ ((\varphi[\![A]\!] = A) \wedge (\varphi[\![B]\!] = B)) \Rightarrow \underset{\substack{X \\ X\in g}}{\forall}\ (\varphi[\![X]\!] = X)]$$

Proposition["G10.2.11", any[φ], with[is-iso[φ]],

$$\underset{\substack{A,B,C \\ \text{is-tc}[A,B,C]}}{\exists}\ ((\varphi[\![A]\!] = A) \wedge (\varphi[\![B]\!] = B) \wedge (\varphi[\![C]\!] = C)) \Rightarrow (\varphi = \text{id})]$$

▫ The Most General Concept of Congruence, Symmetry

Proposition["G10.3.1", any[A, B, C, D, E, F],

$$\text{is-ctc}[A,\, B,\, C,\, D,\, E,\, F] \Rightarrow \underset{\substack{\varphi \\ \text{is-iso}[\varphi]}}{\exists!}\ ((\varphi[\![A]\!] = D) \wedge (\varphi[\![B]\!] = E) \wedge (\varphi[\![C]\!] = F))]$$

Proposition$\Big[$"G10.3.2", any[φ],

$$\text{is-iso}[\varphi] \Rightarrow \left((\varphi = \text{id}) \bigvee \underset{g}{\exists} (\varphi = \text{r}[g]) \bigvee \underset{\substack{g,h \\ g \neq h}}{\exists} (\varphi = \text{r}[g] \circ \text{r}[h]) \bigvee \underset{\substack{g,h,k \\ \text{is-pd}[g,h,k]}}{\exists} (\varphi = \text{r}[g] \circ \text{r}[h] \circ \text{r}[k])\right)\Big]$$

Proposition$\Big[$"G10.3.3", any[φ], with[is-iso[φ]],

$$\underset{\text{P}}{\exists} (\varphi[\![\text{P}]\!] = \text{P}) \Rightarrow (\varphi = \text{id}) \bigvee \underset{g}{\exists} (\varphi = \text{r}[g]) \bigvee \underset{\substack{g,h \\ g \neq h}}{\exists} (\varphi = \text{r}[g] \circ \text{r}[h])\Big]$$

Proposition$\Big[$"G10.3.4", any[φ, g], with[is-iso[φ]],

$$\left(\underset{\substack{\text{A,B} \\ \text{A} \neq \text{B} \wedge \text{A} \in g \wedge \text{B} \in g}}{\exists} ((\varphi[\![\text{A}]\!] = \text{A}) \wedge (\varphi[\![\text{B}]\!] = \text{B}))\right) \Leftrightarrow ((\varphi = \text{id}) \vee (\varphi = \text{r}[g]))\Big]$$

Proposition["G10.3.5", any[φ, A, B], with[is-iso[φ] $\wedge$ A $\neq$ B],
$(\varphi[\![\text{A}]\!] = \text{B}) \wedge (\varphi[\![\text{B}]\!] = \text{A}) \Rightarrow (\varphi[\![\text{mp}[\text{A, B}]]\!] = \text{mp}[\text{A, B}])$]

Proposition$\Big[$"G10.3.6", any[A, B, D, E], with[A $\neq$ B $\wedge$ D $\neq$ E],

$$\text{s}[\text{A, B}] \simeq \text{s}[\text{D, E}] \Leftrightarrow \underset{\substack{\varphi \\ \text{is-iso}[\varphi]}}{\exists} (\text{a}[\varphi, \text{s}[\text{A, B}]] = \text{s}[\text{D, E}])\Big]$$

Proposition$\Big[$"G10.3.7", any[A, B, C, D, E, F], with[is-tc[A, B, C] $\wedge$ is-tc[D, E, F]],

$$\angle[\text{A, B, C}] \cong \angle[\text{D, E, F}] \Leftrightarrow \underset{\substack{\varphi \\ \text{is-iso}[\varphi]}}{\exists} (\text{a}[\varphi, \angle[\text{A, B, C}]] = \angle[\text{D, E, F}])\Big]$$

Proposition$\Big[$"G10.3.8", any[A, B, C, D, E, F], with[is-tc[A, B, C] $\wedge$ is-tc[D, E, F]],

$$\text{is-cg}[\triangle[\text{A, B, C}], \triangle[\text{D, E, F}]] \Leftrightarrow \underset{\substack{\varphi \\ \text{is-iso}[\varphi]}}{\exists} (\text{a}[\varphi, \triangle[\text{A, B, C}]] = \triangle[\text{D, E, F}])\Big]$$

Definition$\Big[$"G10.3.9",

$$\underset{\mathcal{F}_1, \mathcal{F}_2}{\forall} \left(\mathcal{F}_1 \equiv \mathcal{F}_2 \Leftrightarrow \underset{\substack{\varphi \\ \text{is-iso}[\varphi]}}{\exists} (\mathcal{F}_2 = \text{a}[\varphi, \mathcal{F}_1])\right)\Big]$$

Proposition["G10.3.10", any[$\mathcal{F}_1$, $\mathcal{F}_2$, $\mathcal{F}_3$],
$\mathcal{F}_1 \equiv \mathcal{F}_1 \wedge (\mathcal{F}_1 \equiv \mathcal{F}_2 \Rightarrow \mathcal{F}_2 \equiv \mathcal{F}_1) \wedge (\mathcal{F}_1 \equiv \mathcal{F}_2 \wedge \mathcal{F}_2 \equiv \mathcal{F}_3 \Rightarrow \mathcal{F}_1 \equiv \mathcal{F}_3)$]

Definition$\Big[$"G10.3.11",

$$\underset{g, \mathcal{F}_1, \mathcal{F}_2}{\forall} (\text{is-los}[g, \mathcal{F}_1, \mathcal{F}_2] \Leftrightarrow (\mathcal{F}_2 = \text{a}[\text{r}[g], \mathcal{F}_1]))\Big]$$

Definition$\Big[$"GG10.3.12",

$$\underset{g, \mathcal{F}}{\forall} (\text{is-sym}[g, \mathcal{F}] \Leftrightarrow \text{is-los}[g, \mathcal{F}, \mathcal{F}])\Big]$$

■ Circles

▫ Introducing Circles

Definition$\Big[$"G11.1.1",

$$\underset{\substack{\text{C,r} \\ \text{r}>0}}{\forall} (\text{c}[\text{C, r}] = \{\text{X} \mid \text{d}[\text{C, X}] = \text{r}\})\Big]$$

Definition["G11.1.2",
$\forall_{\mathcal{R},C,A,r}$ (is-rad[$\mathcal{R}$, A, C, r] $\Leftrightarrow$ (r > 0 $\wedge$ A $\in$ c[C, r] $\wedge$ ($\mathcal{R}$ = s[C, A])))]

Definition["G11.1.3",
$\forall_{\mathcal{C},A,B,C,r}$ (is-cho[$\mathcal{C}$, A, B, C, r] $\Leftrightarrow$ (r > 0 $\wedge$ A $\neq$ B $\wedge$ A $\in$ c[C, r] $\wedge$ B $\in$ c[C, r] $\wedge$ ($\mathcal{C}$ = s[A, B])))]

Definition["G11.1.4",
$\forall_{\mathcal{D},A,B,C,r}$ (is-dia[$\mathcal{D}$, A, B, C, r] $\Leftrightarrow$ (is-cho[$\mathcal{D}$, A, B, C, r] $\wedge$ is-b[A, C, B]))]

Definition["G11.1.5",
$\forall_{d,\mathcal{C},C,r}$ (is-di[d, $\mathcal{C}$, C, r] $\Leftrightarrow$ (r > 0 $\wedge$ ($\mathcal{C}$ = c[C, r]) $\wedge$ (d = 2 r)))]

Definition["G11.1.6",
$\forall_{g,A,B,C,r}$ (is-sec[g, A, B, C, r] $\Leftrightarrow$ (r > 0 $\wedge$ A $\neq$ B $\wedge$ A $\in$ c[C, r] $\wedge$ B $\in$ c[C, r] $\wedge$ (g = l[A, B])))]

Definition["G11.1.7",
$\forall_{g,P,C,r}$ (is-tg[g, P, C, r] $\Leftrightarrow$ (r > 0 $\wedge$ $g \cap$ c[C, r] = {P}))]

Definition["G11.1.8",
$\forall_{\substack{C,r \\ r>0}}$ (int[C, r] = {X | d[C, X] < r})]

Definition["G11.1.9",
$\forall_{\substack{C,r \\ r>0}}$ (ext[C, r] = {X | d[C, X] > r})]

Definition["G11.1.10",
$\forall_{\mathcal{C}_1,\mathcal{C}_2,C,r_1,r_2}$ (is-con[$\mathcal{C}_1$, $\mathcal{C}_2$, C, r_1, r_2] $\Leftrightarrow$ (r_1 > 0 $\wedge$ r_2 > 0 $\wedge$ ($\mathcal{C}_1$ = c[C, r_1]) $\wedge$ ($\mathcal{C}_2$ = c[C, r_2])))]

Definition["G11.1.11",
$\forall_{P,Q,R,C,r}$ (is-coc[P, Q, R, C, r] $\Leftrightarrow$ (r > 0 $\wedge$ is-pd[P, Q, R] $\wedge$ P $\in$ c[C, r] $\wedge$ Q $\in$ c[C, r] $\wedge$ R $\in$ c[C, r]))]

Definition["G11.1.12",
$\forall_{\substack{A,B,C \\ \text{is-tc}[A,B,C]}}$ (inr[A, B, C] = d[l[A, B], inc[A, B, C]])]

Definition["G11.1.13",
$\forall_{\substack{A,B,C \\ \text{is-tc}[A,B,C]}}$ (icc[A, B, C] = c[inc[A, B, C], inr[A, B, C]])]

▫ First Propositions on Circles

Proposition["G11.2.1", any[$\mathcal{C}$, A, B, C, r],
is-cho[$\mathcal{C}$, A, B, C, r] $\Rightarrow$ C $\in$ pb[A, B]]

Proposition["G11.2.2", any[$\mathcal{C}$, A, B, M, C, r], with[is-cho[$\mathcal{C}$, A, B, C, r] $\wedge$ C $\notin \mathcal{C}$],
(M = mp[A, B]) $\Rightarrow$ l[C, M] $\perp$ l[A, B]]

Proposition["G11.2.3", any[$\mathcal{C}$, A, B, C, r], with[is-cho[$\mathcal{C}$, A, B, C, r]],
pp[l[A, B], C] $\cap \mathcal{C}$ = {mp[A, B]}]

Proposition["G11.2.4", any[$\mathcal{C}_1$, $\mathcal{C}_2$, P, Q, R, C, r], with[P ≠ R],
is-cho[$\mathcal{C}_1$, P, Q, C, r] ∧ is-cho[$\mathcal{C}_2$, Q, R, C, r] ⇒ pb[P, Q] ⋂ pb[Q, R] = {C}]

Proposition["G11.2.5", any[P, Q, R, C, r],
is-coc[P, Q, R, C, r] ⇒ is-tc[P, Q, R]]

Proposition["G11.2.6", any[P, Q, R, C_1, r_1, C_2, r_2],
is-coc[P, Q, R, C_1, r_1] ∧ is-coc[P, Q, R, C_2, r_2] ⇒ (($C_1 = C_2$) ∧ ($r_1 = r_2$))]

Proposition["G11.2.7", any[P, Q, R, C, r],
is-coc[P, Q, R, C, r] ∧ is-col[P, Q, R] ⇒ ((R = P) ∨ (R = Q))]

Proposition["G11.2.8", any[C, r], with[r > 0],
$\underset{P,Q,R}{\exists}$ (is-coc[P, Q, R, C, r])]

Proposition["G11.2.9", any[C_1, r_1, C_2, r_2], with[$r_1 > 0 \wedge r_2 > 0$],
(c[C_1, r_1] = c[C_2, r_2]) ⇒ (($C_1 = C_2$) ∧ ($r_1 = r_2$))]

Proposition["G11.2.10", any[P, Q, R, C_1, r_1, C_2, r_2], with[$r_1 > 0 \wedge r_2 > 0$ ∧ c[C_1, r_1] ≠ c[C_2, r_2]],
is-coc[P, Q, R, C_1, r_1] ∧ is-coc[P, Q, R, C_2, r_2] ⇒ ((R = P) ∨ (R = Q))]

Proposition["G11.2.11", any[P, C, r], with[r > 0],
P ∈ c[C, r] ⇒ $\underset{X \in c[C,r]}{\underset{X}{\forall}}$ $\underset{g}{\exists}$ (C ∈ g ∧ (X = r[g]⟦P⟧))]

Proposition["G11.2.12", any[X, P, C, r], with[r > 0 ∧ P ∈ c[C, r]],
$\underset{g}{\exists}$ (C ∈ g ∧ (X = r[g]⟦P⟧)) ⇒ X ∈ c[C, r]]

Proposition["G11.2.13", any[g, C, r], with[r > 0],
C ∈ g ⇔ is-sym[g, c[C, r]]]

Proposition["G11.2.14", any[g, P, C, r], with[r > 0 ∧ P ∈ c[C, r]],
g = pp[l[C, P], P] ⇔ is-tg[g, P, C, r]]

Proposition["G11.2.15", any[P, C, r], with[r > 0 ∧ P ∈ c[C, r]],
$\underset{g}{\exists!}$ is-tg[g, P, C, r]]

Proposition["G11.2.16", any[g, P, C, r], with[r > 0],
is-tg[g, P, C, r] ⇒ $\underset{X \in g}{\underset{X}{\forall}}$ X ∉ int[C, r]]

Proposition["G11.2.17", any[$\mathcal{C}$, A, B, C, r],
is-cho[$\mathcal{C}$, A, B, C, r] ⇒ d[A, B] ≤ 2 r]

Proposition["G11.2.18", any[$\mathcal{C}$, A, B, C, r], with[is-cho[$\mathcal{C}$, A, B, C, r]],
is-dia[$\mathcal{C}$, A, B, C, r] ⇔ (d[A, B] = 2 r)]

Proposition["G11.2.19", any[φ, C, r], with[r > 0],
is-iso[φ] ⇒ (a[φ, c[C, r]] = c[φ⟦C⟧, r])]

Proposition["G11.2.20", any[φ, g, P, C, r], with[is-iso[φ]],
is-tg[g, P, C, r] ⇒ is-tg[a[φ, g], φ⟦P⟧, φ⟦C⟧, r]]

Proposition["G11.2.21", any[C_1, r_1, C_2, r_2], with[$r_1 > 0 \wedge r_2 > 0$],
c[C_1, r_1] ≡ c[C_2, r_2] ⇔ ($r_1 = r_2$)]

Proposition["G11.2.22", any[$\mathcal{C}_1$, $\mathcal{C}_2$, A, B, D, E, C_1, C_2, r], with[is-cho[$\mathcal{C}_1$, A, B, C_1, r] ∧ is-cho[$\mathcal{C}_2$, D, E, C_2, r]],
$\mathcal{C}_1 \simeq \mathcal{C}_2$ ⇔ (di[l[A, B], C_1] = di[l[D, E], C_2])]

Proposition["G11.2.23", any[C, r], with[r > 0],
is-cv[c[C, r]]]

Proposition["G11.2.24", any[U, V, W, X, Y, Z, g, h, k, C, r],
with[is-tc[U, V, W] ∧ (g = l[U, V]) ∧ (h = l[V, W]) ∧ (k = l[W, U]) ∧
(X = ft[g, C]) ∧ (Y = ft[h, C]) ∧ (Z = ft[k, C]) ∧ (C = inc[U, V, W]) ∧ (r = inr[U, V, W])],
(is-tg[g, X, C, r] ∧ is-tg[h, Y, C, r] ∧ is-tg[k, Z, C, r])]

▫ The Two-Circle-Theorem

Proposition["G11.3.1", any[C, P, Q, r], with[r > 0 ∧ is-tc[C, P, Q]],
l[P, Q] ⊥ l[C, P] ∧ d[C, P] < r ⇒ $\nexists_{T}$ (T ∈ r[P, Q] ∧ (d[C, T] = r))]

Proposition["G11.3.2", any[S, Q, C, r], with[r > 0],
S ∈ int[C, r] ∧ Q ∈ ext[C, r] ⇒ c[C, r] ⋂ s[S, Q] ≠ ∅]

Proposition["G11.3.3", any[g, C, r], with[r > 0],
g ⋂ int[C, r] ≠ ∅ ⇒ $\exists_{\substack{A,B \\ A \neq B}}$ (A ∈ g ⋂ c[C, r] ∧ B ∈ g ⋂ c[C, r])]

Proposition["G11.3.4", any[g, C, r], with[r > 0],
g ⋂ int[C, r] ≠ ∅ ⇒ $\exists_{\substack{A,B \\ A \neq B}}$ (g ⋂ c[C, r] = {A, B})]

Proposition["G11.3.5", any[V, P, C, r], with[r > 0 ∧ V ≠ P],
V ∈ int[C, r] ⇒ $\exists_{A}$ (r[V, P] ⋂ c[C, r] = {A})]

Definition["G11.3.6", any[V, P, C, r], with[V ≠ P ∧ r > 0 ∧ V ∈ int[C, r]],
irc[V, P, C, r] = ι_{X} (X ∈ r[V, P] ⋂ c[C, r])]

Proposition["G11.3.7", any[P, C, r], with[r > 0 ∧ P ∈ ext[C, r]],
$\exists_{\substack{P,Q,R \\ \text{is-tc}[P,Q,R]}}$ (is-tg[l[P, Q], Q, C, r] ∧ is-tg[l[P, R], R, C, r] ∧ s[P, Q] ≃ s[P, R])]

Proposition["G11.3.8", any[P, Q, R, S, C, r], with[r > 0 ∧ P ∈ ext[C, r] ∧ is-tc[P, Q, R] ∧ P ≠ S],
is-tg[l[P, Q], Q, C, r] ∧ is-tg[l[P, R], R, C, r] ∧ is-tg[l[P, S], S, C, r] ⇒ ((S = Q) ∨ (S = R))]

Proposition["G11.3.9", any[A, B, C, D, E, F], with[is-rtc[A, B, C] ∧ is-rtc[D, E, F]],
(d[A, B] = d[D, E]) ∧ d[A, C] > d[D, F] ⇒ d[B, C] < d[E, F]]

Proposition["G11.3.10", any[$\mathcal{D}$, $\mathcal{C}_1$, $\mathcal{C}_2$, A, B, P, R, Q, S, X, Y, C, r],
with[is-dia[$\mathcal{D}$, A, B, C, r] ∧ is-cho[$\mathcal{C}_1$, P, R, C, r] ∧ is-cho[$\mathcal{C}_2$, Q, S, C, r] ∧
l[P, R] ⊥ l[A, B] ∧ l[Q, S] ⊥ l[A, B] ∧ ($\mathcal{D}$ ⋂ $\mathcal{C}_1$ = {X}) ∧ ($\mathcal{D}$ ⋂ $\mathcal{C}_2$ = {Y})],
is-b[C, X, Y] ⇒ r > d[P, X] > d[Q, Y]]

Proposition["G11.3.11", any[a, b, c], with[a > 0 ∧ b > 0 ∧ c > 0],
$\left(\exists_{\substack{A,B,C \\ \text{is-tc}[A,B,C]}}\right.$ ((d[A, B] = c) ∧ (d[B, C] = a) ∧ (d[C, A] = b))$\left.\right)$ ⇔ (a < b + c ∧ b < a + c ∧ c < a + b)]

Proposition["G11.3.12", any[A, B, a, b, c], with[a > 0 ∧ b > 0 ∧ c > 0 ∧ (d[A, B] = c)],
a < b + c ∧ b < a + c ∧ c < a + b ⇒ $\exists_{\substack{P,Q \\ P \neq Q}}$ ((c[A, b] ⋂ c[B, a] = {P, Q}) ∧ is-os[l[A, B], P, Q])]

Definition["G11.3.13", any[P, C_1, C_2, r_1, r_2],
with[is-tc[C_1, C_2, P] ∧ $r_1 > 0$ ∧ $r_2 > 0$ ∧ d[C_1, C_2] < $r_1 + r_2$ ∧ $r_1 < r_2$ + d[C_1, C_2] ∧ $r_2 < r_1$ + d[C_1, C_2]],
icc[M_1, r_1, M_2, r_2, P] = ι_{X} (X ∈ c[C_1, r_1] ∧ X ∈ c[C_2, r_2] ∧ X ∈ hp[C_1, C_2, P])]

■ More of Absolute Geometry

▫ Sufficient Conditions for Parallelism

Proposition["G12.1.1", any[g, P], with[P ∉ g],
$\underset{h}{\exists}$ (P ∈ h ∧ g ∥ h)]

Definition["G12.1.2",
$\underset{g,h,\text{V},\text{W}}{\forall}$ (is-trc[g, h, V, W] ⇔ (V ≠ W ∧ V ∈ g ∧ W ∈ h ∧ is-pd[g, h, l[V, W]]))]

Definition["G12.1.3", any[$\mathcal{A}_1$, $\mathcal{A}_2$, g, h, V, W, X, Y],
is-pai[$\mathcal{A}_1$, $\mathcal{A}_2$, g, h, V, W, X, Y] ⇔
(is-trc[g, h, V, W] ∧ X ∈ g ∧ Y ∈ h ∧ is-os[l[V, W], X, Y] ∧ ($\mathcal{A}_1$ = ∠[X, V, W]) ∧ ($\mathcal{A}_2$ = ∠[V, W, Y]))]

Proposition["G12.1.4", any[$\mathcal{A}_1$, $\mathcal{A}_2$, g, h, V, W, X, Y], with[is-pai[$\mathcal{A}_1$, $\mathcal{A}_2$, g, h, V, W, X, Y]],
$\mathcal{A}_1 \cong \mathcal{A}_2$ ⇒ $\underset{k}{\exists}$ (k ⊥ g ∧ k ⊥ h)]

Proposition["G12.1.5", any[$\mathcal{A}_1$, $\mathcal{A}_2$, g, h, V, W, X, Y], with[is-pai[$\mathcal{A}_1$, $\mathcal{A}_2$, g, h, V, W, X, Y]],
$\mathcal{A}_1 \cong \mathcal{A}_2$ ⇒ g ∥ h]

Definition["G12.1.6", any[$\mathcal{C}_1$, $\mathcal{C}_2$, g, h, V, W, X, Y, $\mathcal{A}_1$, $\mathcal{A}_2$],
is-pca[$\mathcal{C}_1$, $\mathcal{C}_2$, g, h, V, W, X, Y, $\mathcal{A}_1$, $\mathcal{A}_2$] ⇔ (is-pai[$\mathcal{A}_1$, $\mathcal{A}_2$, g, h, V, W, X, Y] ∧ ($\mathcal{C}_1$ = $\mathcal{A}_1$) ∧ is-pva[$\mathcal{C}_2$, $\mathcal{A}_2$])]

Proposition["G12.1.7", any[$\mathcal{C}_1$, $\mathcal{C}_2$, g, h, V, W, X, Y, $\mathcal{A}_1$, $\mathcal{A}_2$], with[is-pca[$\mathcal{C}_1$, $\mathcal{C}_2$, g, h, V, W, X, Y, $\mathcal{A}_1$, $\mathcal{A}_2$]],
$\mathcal{C}_1 \cong \mathcal{C}_2$ ⇔ $\mathcal{A}_1 \cong \mathcal{A}_2$]

Proposition["G12.1.8", any[$\mathcal{C}_1$, $\mathcal{C}_2$, g, h, V, W, X, Y, $\mathcal{A}_1$, $\mathcal{A}_2$], with[is-pca[$\mathcal{C}_1$, $\mathcal{C}_2$, g, h, V, W, X, Y, $\mathcal{A}_1$, $\mathcal{A}_2$]],
$\mathcal{C}_1 \cong \mathcal{C}_2$ ⇒ g ∥ h]

Proposition["G12.1.9", any[A, B, C, D], with[B ≠ C ∧ is-ss[l[B, C], A, D]],
(m[A, B, C] + m[B, C, D] = 180) ⇒ l[A, B] ∥ l[C, D]]

Proposition["G12.1.10", any[A, B, C, D], with[is-qc[A, B, C, D]],
(d[A, B] = d[C, D]) ∧ (d[B, C] = d[D, A]) ⇒ is-pc[A, B, C, D]]

Proposition["G12.1.11", any[A, B, C, D], with[is-qc[A, B, C, D]],
∠[D, A, B] ≅ ∠[B, C, D] ∧ ∠[A, B, C] ≅ ∠[C, D, A] ⇒ is-pc[A, B, C, D]]

Proposition["G12.1.12", any[A, B, C, D], with[is-cqc[A, B, C, D]],
(mp[A, C] = mp[B, D]) ⇒ is-pc[A, B, C, D]]

Proposition["G12.1.13", any[A, B, C, D], with[is-trc[A, B, C, D]],
d[A, B] = d[B, C] ⇒ is-pc[A, B, C, D]]

▫ Saccheri Quadrilaterals

Proposition["G12.2.1", any[A, B, C, D], with[is-qc[A, B, C, D]],
is-ra[∠[B, A, D], ∠[A, D, C]] ⇒ is-cqc[A, B, C, D]]

Proposition["G12.2.2", any[A, B, C, D], with[is-qc[A, B, C, D] ∧ is-ra[∠[B, A, D], ∠[A, D, C]]],
(d[A, B] = d[C, D]) ⇒ (m[A, B, C] = m[D, C, B])]

Proposition["G12.2.3", any[A, B, C, D], with[is-qc[A, B, C, D] ∧ is-ra[∠[B, A, D], ∠[A, D, C]]],
d[A, B] < d[C, D] ⇒ m[A, B, C] > m[D, C, B]]

Proposition["G12.2.4", any[A, B, C, D], with[is-qc[A, B, C, D] ∧ is-ra[∠[B, A, D], ∠[A, D, C]]],
d[A, B] > d[C, D] ⇒ m[A, B, C] < m[D, C, B]]

Proposition["G12.2.5", any[A, B, C, D], with[is-qc[A, B, C, D] ∧ is-ra[∠[B, A, D], ∠[A, D, C]]],
(d[A, B] = d[C, D]) ⇔ (m[A, B, C] = m[D, C, B])]

Proposition["G12.2.6", any[A, B, C, D], with[is-qc[A, B, C, D] ∧ is-ra[∠[B, A, D], ∠[A, D, C]]],
d[A, B] < d[C, D] ⇔ m[A, B, C] > m[D, C, B]]

Proposition["G12.2.7", any[A, B, C, D], with[is-qc[A, B, C, D] ∧ is-ra[∠[B, A, D], ∠[A, D, C]]],
d[A, B] > d[C, D] ⇔ m[A, B, C] < m[D, C, B]]

Definition["G12.2.8",
$\forall_{A,B,C,D}$ (is-rec[A, B, C, D] ⇔ (is-qc[A, B, C, D] ∧ is-ra[∠[B, A, D], ∠[A, D, C], ∠[D, C, B], ∠[C, B, A]]))]

Definition["G12.2.9",
$\forall_{A,B,C,D}$ (is-sqc[A, B, C, D] ⇔ (is-qc[A, B, C, D] ∧ is-ra[∠[B, A, D], ∠[A, D, C]] ∧ (d[A, B] = d[C, D])))]

Proposition["G12.2.10", any[A, B, C, D],
is-sqc[A, B, C, D] ⇒ is-cqc[A, B, C, D]]

Proposition["G12.2.11", any[A, B, C, D],
is-sqc[A, B, C, D] ⇒ ((m[A, B, C] = m[D, C, B]) ∧ (d[A, C] = d[B, D]))]

Proposition["G12.2.12", any[A, B, C, D],
is-rec[A, B, C, D] ⇒ ((d[A, B] = d[C, D]) ∧ (d[A, D] = d[B, C]))]

Proposition["G12.2.13", any[A, B, C, D, M, N], with[is-sqc[A, B, C, D]],
(M = mp[B, C]) ∧ (N = mp[A, D]) ⇒ (l[M, N] ⊥ l[A, D] ∧ l[M, N] ⊥ l[B, C])]

Proposition["G12.2.14", any[A, B, C, D],
is-sqc[A, B, C, D] ⇒ (pb[A, D] = pb[B, C])]

Proposition["G12.2.15", any[A, B, C, D],
is-sqc[A, B, C, D] ⇒ is-trc[A, D, C, B]]

Proposition["G12.2.16", any[A, B, C, D],
is-sqc[A, B, C, D] ⇒ is-pc[A, B, C, D]]

Proposition["G12.2.17", any[A, B, C, D, M, N, F, G], with[is-sqc[A, B, C, D]],
(M = mp[B, C]) ∧ (N = mp[A, D]) ∧ (F = mp[A, B]) ∧ (G = mp[C, D]) ⇒ l[M, N] ⊥ l[F, G]]

Proposition["G12.2.18", any[g, P, Q], with[P ≠ Q ∧ is-ss[g, P, Q]],
(d[g, P] = d[g, Q]) ⇒ g ∥ l[P, Q]]

Definition["G12.2.19",
$\forall_{g,h}$ (is-eqd[g, h] ⇔ $\forall_{P,Q \atop P\in g \land Q\in g}$ (d[h, P] = d[h, Q]))]

Proposition["G12.2.20", any[g, h],
is-eqd[g, h] ⇒ is-eqd[h, g]]

Proposition["G12.2.21", any[g, h],
is-eqd[g, h] ⇒ g ∥ h]

Proposition["G12.2.22", any[A, B, C, X, Y, Z, h],
with[¬ is-col[h, A, B, C] ∧ is-b[A, B, C] ∧ X = ft[h, A] ∧ Y = ft[h, B] ∧ Z = ft[h, C]],
(d[A, X] = d[B, Y]) ∧ (d[B, Y] = d[C, Z]) ⇒ is-rec[X, A, C, Z]]

Proposition["G12.2.23", any[A, B, C, P, h], with[is-b[A, B, C] ∧ (d[h, A] = d[h, B]) ∧ (d[h, B] = d[h, C])],
is-b[A, P, C] ⇒ (d[h, P] = d[h, A])]

Proposition["G12.2.24", any[g, h],
($\exists_{A,B,C}$ (is-pd[A, B, C] ∧ is-col[g, A, B, C] ∧ (d[h, A] = d[h, B]) ∧ (d[h, B] = d[h, C]))) ⇒ is-eqd[g, h]]

Definition["G12.2.25",

$$\underset{\substack{g,h \\ \text{is-eqd}[g,h]}}{\forall}\left(d[g,\, h] = \underset{x}{\iota}\ \underset{\substack{P \\ P\in g}}{\forall}\ (x = d[h,\, P])\right)]$$

▫ The Theory of Parallels

Proposition["G12.3.1",

$$\underset{\substack{A,B,C,D \\ B\neq C\wedge \text{is-ss}[l[B,C],A,D]}}{\forall}(l[A,\, B] \parallel l[C,\, D] \Rightarrow (m[A,\, B,\, C] + m[B,\, C,\, D] = 180))]$$

Proposition["EPP",

$$\underset{\substack{A,B,C,D \\ B\neq C\wedge \text{is-ss}[l[B,C],A,D]}}{\forall}(m[A,\, B,\, C] + m[B,\, C,\, D] < 180 \Rightarrow r[B,\, A] \cap r[C,\, D] \neq \emptyset)]$$

Proposition["PPP",

$$\underset{\substack{g,P \\ P\notin g}}{\forall}\ \underset{h}{\exists!}\,(P \in h \wedge g \parallel h)]$$

Proposition["G12.3.2",

$$\underset{\substack{A,B,C,D \\ B\neq C\wedge \text{is-ss}[l[B,C],A,D]}}{\forall}(m[A,\, B,\, C] + m[B,\, C,\, D] < 180 \Rightarrow r[B,\, A] \cap r[C,\, D] \neq \emptyset) \Rightarrow$$

$$\underset{\substack{A,B,C,D \\ B\neq C\wedge \text{is-ss}[l[B,C],A,D]}}{\forall}(l[A,\, B] \parallel l[C,\, D] \Rightarrow (m[A,\, B,\, C] + m[B,\, C,\, D] = 180))]$$

Proposition["G12.3.3",

$$\underset{\substack{A,B,C,D \\ B\neq C\wedge \text{is-ss}[l[B,C],A,D]}}{\forall}(l[A,\, B] \parallel l[C,\, D] \Rightarrow (m[A,\, B,\, C] + m[B,\, C,\, D] = 180)) \Rightarrow \underset{\substack{g,P \\ P\notin g}}{\forall}\ \underset{h}{\exists!}\,(P \in h \wedge g \parallel h)]$$

Proposition["G12.3.4",

$$\underset{\substack{g,P \\ P\notin g}}{\forall}\ \underset{h}{\exists!}\,(P \in h \wedge g \parallel h) \Rightarrow \underset{\substack{A,B,C,D \\ B\neq C\wedge \text{is-ss}[l[B,C],A,D]}}{\forall}(m[A,\, B,\, C] + m[B,\, C,\, D] < 180 \Rightarrow r[B,\, A] \cap r[C,\, D] \neq \emptyset)]$$

▪ Classical Results of Euclidean Geometry

▫ The Euclidean Parallel Axiom and Immediate Consequences

Axiom["EPA",

$$\underset{\substack{P,g,h,k \\ P\notin g}}{\forall}(P \in h \wedge g \parallel h \wedge P \in k \wedge g \parallel k \wedge \Rightarrow (h = k))]$$

Proposition["PPP",

$$\underset{\substack{g,P \\ P\notin g}}{\forall}\ \underset{h}{\exists!}\,(P \in h \wedge g \parallel h)]$$

Definition["G13.1.1",

$$\underset{P,g}{\forall}\left(p[g,\, P] = \underset{h}{\iota}\,(P \in h \wedge g \parallel h)\right)]$$

Proposition["G13.1.2", any[g, h, k],
$g \parallel h \wedge h \parallel k \Rightarrow g \parallel k$]

Proposition["G13.1.3", any[$\mathcal{A}_1$, $\mathcal{A}_2$, g, h, V, W, X, Y], with[is-pai[$\mathcal{A}_1$, $\mathcal{A}_2$, g, h, V, W, X, Y]],
$g \parallel h \Rightarrow \mathcal{A}_1 \cong \mathcal{A}_2$]

Proposition["G13.1.4", any[$\mathcal{C}_1$, $\mathcal{C}_2$, g, h, V, W, X, Y, $\mathcal{A}_1$, $\mathcal{A}_2$], with[is-pca[$\mathcal{C}_1$, $\mathcal{C}_2$, g, h, V, W, X, Y, $\mathcal{A}_1$, $\mathcal{A}_2$]],
$g \parallel h \Rightarrow \mathcal{C}_1 \cong \mathcal{C}_2$]

Proposition["G13.1.5", any[g, h, k],
$g \neq k \bigwedge g \parallel h \bigwedge g \bigcap k \neq \emptyset \Rightarrow h \bigcap k \neq \emptyset$]

Proposition["G13.1.6", any[g, h, k],
$g \parallel h \wedge k \perp g \Rightarrow k \perp h$]

Proposition["G13.1.7", any[A, B, C, D],
is-sqc[A, B, C, D] ⇒ is-rec[A, B, C, D]]

Proposition["G13.1.8", any[g, h],
$g \parallel h \Rightarrow$ is-eqd[g, h]]

Proposition["G13.1.9", any[a, b, g, h],
$a \parallel b \wedge g \perp a \wedge h \perp b \Rightarrow g \parallel h$]

Proposition["G13.1.10", any[A, B, C],
is-tc[A, B, C] ⇒ $\underset{\mathrm{P}}{\exists}$ (pb[A, C] ⋂ pb[B, C] = {P})]

Proposition["G13.1.11", any[A, B, C],
is-tc[A, B, C] ⇒ (m[B, A, C] + m[A, B, C] + m[B, C, A] = 180)]

Theorems about Triangles and Quadrilaterals

Proposition["G13.2.1", any[A, B, C], with[is-tc[A, B, C]],
$\underset{\mathrm{P}}{\exists}$ (pb[A, B] ⋂ pb[B, C] ⋂ pb[C, A] = {P})]

Definition["G13.2.2",
$\underset{\substack{\mathrm{A,B,C} \\ \text{is-tc[A,B,C]}}}{\forall}$ (cc[A, B, C] = $\underset{\mathrm{P}}{\iota}$ (pb[A, B] ⋂ pb[B, C] ⋂ pb[C, A] = {P}))]

Proposition["G13.2.3", any[A, B, C, P], with[is-tc[A, B, C]],
(P = cc[A, B, C]) ⇒ ((d[A, P] = d[B, P]) ∧ (d[B, P] = d[C, P]))]

Definition["G13.2.4",
$\underset{\substack{\mathrm{A,B,C} \\ \text{is-tc[A,B,C]}}}{\forall}$ (ccr[A, B, C] = d[A, cc[A, B, C]])]

Definition["G13.2.5",
$\underset{\substack{\mathrm{A,B,C} \\ \text{is-tc[A,B,C]}}}{\forall}$ (ccc[A, B, C] = c[cc[A, B, C], ccr[A, B, C]])]

Proposition["G13.2.6", any[P, Q, R], with[is-tc[P, Q, R]],
$\underset{\mathrm{C,r}}{\exists!}$ is-coc[P, Q, R, C, r]]

Definition["G13.2.7",
$\underset{\substack{\mathrm{A,B,C} \\ \text{is-tc[A,B,C]}}}{\forall}$ (alt[A, B, C] = pp[l[B, C], A])]

Proposition["G13.2.8", any[A, B, C], with[is-tc[A, B, C]],
$\underset{P}{\exists}$ (alt[A, B, C] ⋂ alt[B, C, A] ⋂ alt[C, A, B] = {P})]

Definition["G13.2.9",
$\underset{\substack{A,B,C \\ \text{is-tc}[A,B,C]}}{\forall}$ (oc[A, B, C] = $\underset{P}{\iota}$ (alt[A, B, C] ⋂ alt[B, C, A] ⋂ alt[C, A, B] = {P}))]

Proposition["G13.2.10", any[A, B, C, D],
is-trc[A, B, C, D] ⇒ is-cqc[A, B, C, D]]

Proposition["G13.2.11", any[A, B, C, D],
is-pc[A, B, C, D] ⇒ is-ctc[A, B, C, C, D, A]]

Proposition["G13.2.12", any[A, B, C, D],
is-pc[A, B, C, D] ⇒ ((d[A, B] = d[C, D]) ∧ (d[B, C] = d[D, A]))]

Proposition["G13.2.13", any[A, B, C, D],
is-pc[A, B, C, D] ⇒ (∠[D, A, B] ≅ ∠[B, C, D] ∧ ∠[A, B, C] ≅ ∠[C, D, A])]

Proposition["G13.2.14", any[A, B, C, D], with[is-pc[A, B, C, D]],
is-pc[A, B, C, D] ⇒ (mp[A, C] = mp[B, D])]

▫ Angles in a Circle

Proposition["Thales", any[P, Q, R, C, r],
is-coc[P, Q, R, C, r] ∧ is-b[P, C, Q] ⇒ (m[P, R, Q] = 90)]

Proposition["G13.3.1", any[P, Q, R],
is-rtc[P, Q, R] ⇒ is-coc[P, Q, R, m[P, Q], $\frac{\text{d}[P, Q]}{2}$]]

Proposition["G13.3.2", any[P, Q, R, C, r], with[is-tc[P, Q, C]],
R ∈ hp[P, Q, C] ⇒ (m[P, R, Q] = $\frac{1}{2}$ m[P, C, Q]) "a"
R ∈ ohp[P, Q, C] ⇒ (m[P, R, Q] = $\frac{1}{2}$ (360 − m[P, C, Q])) "b"]

Proposition["G13.3.3", any[P, Q, R, S, C, r], with[is-coc[P, Q, R, C, r] ∧ S ∈ hp[P, Q, R]],
S ∈ c[C, r] ⇒ (m[P, S, Q] = m[P, R, Q])]

Proposition["G13.3.4", any[P, Q, R, S, C, r], with[is-coc[P, Q, R, C, r] ∧ S ∈ hp[P, Q, R]],
(m[P, S, Q] = m[P, R, Q]) ⇒ S ∈ c[C, r]]

Proposition["G13.3.5", any[g, P, Q, R, S, T, C, r],
with[is-coc[P, Q, R, C, r] ∧ S ∈ hp[P, Q, R] ∧ is-tg[g, P, C, r] ∧ T ∈ g ∧ T ∈ ohp[P, Q, R]],
S ∈ c[C, r] ⇒ (m[P, S, Q] = m[T, P, Q])]

Definition["G13.3.6",
$\underset{A,B,C,D}{\forall}$ (is-iqc[A, B, C, D] ⇔ (is-qc[A, B, C, D] ⋀ $\underset{\substack{M,r \\ r>0}}{\exists}$ (A ∈ c[M, r] ∧ B ∈ c[M, r] ∧ C ∈ c[M, r] ∧ D ∈ c[M, r])))]

Proposition["G13.3.7", any[A, B, C, D],
is-iqc[A, B, C, D] ⇔ is-sup[∠[D, A, B], ∠[B, C, D]]]

■ Similarity

▫ The Basic Similarity Theorem

Definition["G14.1.1", any[a, b, c, g, h, A, B, C, D, E, F],
is-sic[a, b, c, g, h, A, B, C, D, E, F] ⟺ (is-pd[a, b, c] ∧ $a \parallel b$ ∧ $b \parallel c$ ∧ $g \neq h$ ∧
$(g \cap a = \{A\}) \wedge (g \cap b = \{B\}) \wedge (g \cap c = \{C\}) \wedge (h \cap a = \{D\}) \wedge (h \cap b = \{E\}) \wedge (h \cap c = \{F\})$)]

Proposition["G14.1.2", any[a, b, c, g, h, A, B, C, D, E, F], with[is-sic[a, b, c, g, h, A, B, C, D, E, F]],
is-b[A, B, C] ⇒ is-b[D, E, F]]

Proposition["G14.1.3", any[a, b, c, g, h, A, B, C, D, E, F], with[is-sic[a, b, c, g, h, A, B, C, D, E, F]],
s[A, B] ≃ s[B, C] ⇒ s[D, E] ≃ s[E, F]]

Proposition["G14.1.4", any[a, b, c, g, h, A, B, C, D, E, F, p, q], with[is-sic[a, b, c, g, h, A, B, C, D, E, F] ∧ q ∈ ℕ],

$$\left(\text{is-b}[A, B, C] \wedge p = \max\left\{n \underset{n \in \mathbb{N}_0}{\Big|} n \leq q \cdot \frac{d[B, C]}{d[A, B]}\right\}\right) \Rightarrow \left(\frac{p}{q} \leq \frac{d[B, C]}{d[A, B]} < \frac{p+1}{q} \bigwedge \frac{p}{q} \leq \frac{d[E, F]}{d[D, E]} < \frac{p+1}{q}\right)]$$

Proposition["G14.1.5", any[a, b, c, g, h, A, B, C, D, E, F], with[is-sic[a, b, c, g, h, A, B, C, D, E, F]],

$$\text{is-b}[A, B, C] \Rightarrow \left(\frac{d[B, C]}{d[A, B]} = \frac{d[E, F]}{d[D, E]}\right)]$$

Proposition["G14.1.6", any[A, B, C, D, E], with[is-tc[A, B, C] ∧ is-b[A, D, C] ∧ is-b[B, E, C]],

$$l[D, E] \parallel l[A, B] \Rightarrow \left(\left(\frac{d[C, D]}{d[D, A]} = \frac{d[C, E]}{d[E, B]}\right) \bigwedge \left(\frac{d[C, D]}{d[C, A]} = \frac{d[C, E]}{d[C, B]}\right)\right)]$$

Proposition["G14.1.7", any[A, B, C, D, E], with[is-tc[A, B, C] ∧ is-b[A, D, C] ∧ is-b[B, E, C]],

$$\left(\frac{d[C, D]}{d[D, A]} = \frac{d[C, E]}{d[E, B]}\right) \Rightarrow l[D, E] \parallel l[A, B]]$$

▫ Similarities Between Triangles

Definition["G14.2.1",

$$\underset{A,B,C,D,E,F}{\forall} (\text{is-stc}[A, B, C, D, E, F] \Leftrightarrow$$
$$(\text{is-tc}[A, B, C] \wedge \text{is-tc}[D, E, F] \wedge \angle[C, A, B] \cong \angle[F, D, E] \wedge \angle[A, B, C] \cong \angle[D, E, F] \wedge \angle[B, C, A] \cong \angle[E, F, D]))]$$

Proposition["G14.2.2", any[A, B, C, D, E, F], with[is-tc[A, B, C] ∧ is-tc[D, E, F]],
∠[C, A, B] ≅ ∠[F, D, E] ∧ ∠[A, B, C] ≅ ∠[D, E, F] ⇒ is-stc[A, B, C, D, E, F]]

Proposition["G14.2.3", any[A, B, C, D, E, F, G, H, I],
is-stc[A, B, C, A, B, C] ∧ (is-stc[A, B, C, D, E, F] ⇒ is-stc[D, E, F, A, B, C]) ∧
(is-stc[A, B, C, D, E, F] ∧ is-stc[D, E, F, G, H, I] ⇒ is-stc[A, B, C, G, H, I])]

Proposition["G14.2.4", any[A, B, C, D, E], with[is-tc[A, B, C] ∧ is-b[A, D, C] ∧ is-b[B, E, C]],
l[D, E] ∥ l[A, B] ⇒ is-stc[D, E, C, A, B, C]]

Proposition["G14.2.5", any[A, B, C, D, E, F],

$$\text{is-stc}[A, B, C, D, E, F] \Rightarrow \left(\left(\frac{d[A, B]}{d[D, E]} = \frac{d[B, C]}{d[E, F]}\right) \bigwedge \left(\frac{d[B, C]}{d[E, F]} = \frac{d[A, C]}{d[D, F]}\right)\right)]$$

Proposition["G14.2.6", any[A, B, C, D, E], with[is-tc[A, B, C] ∧ is-b[A, D, C] ∧ is-b[B, E, C]],

$$l[D, E] \parallel l[A, B] \Rightarrow \left(\frac{d[D, E]}{d[A, B]} = \frac{d[C, D]}{d[C, A]}\right)]$$

Proposition$\Big[$"G14.2.7", any[A, B, C, D, E, F], with[is-tc[A, B, C] ∧ is-tc[D, E, F]],
$\left(\frac{d[A, B]}{d[D, E]} = \frac{d[B, C]}{d[E, F]}\right) \bigwedge \left(\frac{d[B, C]}{d[E, F]} = \frac{d[A, C]}{d[D, F]}\right) \Rightarrow$ is-stc[A, B, C, D, E, F]$\Big]$

Proposition$\Big[$"G14.2.8", any[A, B, C, D, E, F], with[is-tc[A, B, C] ∧ is-tc[D, E, F]],
∠[B, C, A] ≅ ∠[E, F, D] $\bigwedge \left(\frac{d[A, C]}{d[D, F]} = \frac{d[B, C]}{d[E, F]}\right) \Rightarrow$ is-stc[A, B, C, D, E, F]$\Big]$

Proposition["G14.2.9", any[A, B, C, D, E], with[is-tc[A, B, C] ∧ (D = ft[l[A, B], C]) ∧ (E = ft[l[A, C], B])],
d[C, D] d[A, B] = d[B, E] d[A, C]]

Proposition$\Big[$"G14.2.10", any[A, B, C, D, E, F, G, H],
is-stc[A, B, C, D, E, F] ∧ (G = ft[l[A, B], C]) ∧ (E = ft[l[D, E], F]) $\Rightarrow \left(\frac{d[C, G]}{d[F, H]} = \frac{d[A, C]}{d[D, F]}\right)\Big]$

Proposition["G14.2.11", any[A, B, C, D, E], with[is-tc[A, B, C] ∧ is-b[B, C, D] ∧ is-b[A, C, E]],
l[D, E] ∥ l[A, B] ⇒ is-stc[A, B, C, E, D, C]]

Proposition$\Big[$"G14.2.12", any[A, B, C, D, E], with[is-tc[A, B, C]],
(D = mp[A, C]) ∧ (E = mp[B, C]) $\Rightarrow \left(l[D, E] \parallel l[A, B] \bigwedge \left(d[D, E] = \frac{1}{2} d[A, B]\right)\right)\Big]$

Proposition$\Big[$"G14.2.13", any[A, B, C, D, E], with[is-tc[A, B, C] ∧ (D = mp[A, C]) ∧ (E = mp[B, C])],
$\underset{P}{\exists}\left((s[A, E] \cap s[B, D] = \{P\}) \wedge d[P, E] = \frac{1}{2} d[A, P] \bigwedge d[P, D] = \frac{1}{2} d[B, P]\right)\Big]$

Definition$\Big[$"G14.2.14",
$\underset{\substack{A,B,C \\ \text{is-tc}[A,B,C]}}{\forall}$ (med[A, B, C] = s[A, mp[B, C]])$\Big]$

Proposition$\Big[$"G14.2.15", any[A, B, C], with[is-tc[A, B, C]],
$\underset{P}{\exists}$ (med[A, B, C] ⋂ med[B, C, A] ⋂ med[C, A, B] = {P})$\Big]$

Definition$\Big[$"G14.2.16",
$\underset{\substack{A,B,C \\ \text{is-tc}[A,B,C]}}{\forall}$ $\left(\text{cto}[A, B, C] = \underset{P}{\iota}\,(\text{med}[A, B, C] \cap \text{med}[B, C, A] \cap \text{med}[C, A, B] = \{P\})\right)\Big]$

Proposition$\Big[$"G14.2.17", any[A, B, C, E, P], with[is-tc[A, B, C]],
(P = cto[A, B, C]) ∧ (E = mp[B, C]) $\Rightarrow \left(d[A, P] = \frac{2}{3} d[A, E]\right)\Big]$

Proposition["G14.2.18", any[A, B, C], with[is-tc[A, B, C]],
is-col[oc[A, B, C], cto[A, B, C], cc[A, B, C]]]

▫ The Pythagorean Theorem

Proposition["G14.3.1", any[A, B, C, D], with[is-rtc[A, B, C]],
(D = ft[l[A, B], C]) ⇒ is-b[A, D, B]]

Proposition["G14.3.2", any[A, B, C, D], with[is-rtc[A, B, C] ∧ (D = ft[l[A, B], C])],
is-stc[A, B, C, A, C, D] ∧ is-stc[A, B, C, C, B, D] ∧ is-stc[C, B, D, A, C, D]]

Proposition$\Big[$"G14.3.3", any[A, B, C, D, a, b, c, p, q, h],
with[is-tc[A, B, C] ∧ (D = ft[l[A, B], C]) ∧ (d[B, C] = a) ∧ (d[A, C] = b) ∧
(d[A, B] = c) ∧ (d[B, D] = p) ∧ (d[A, D] = q) ∧ (d[C, D] = h)],
is-rtc[A, B, C] ⇒ ((a^2 = cp) ⋀ (b^2 = cq)) "a"
is-rtc[A, B, C] ⇒ (h^2 = pq) "b"$\Big]$

Proposition["Pythagoras", any[A, B, C, a, b, c], with[is-tc[A, B, C] $\wedge$ (d[B, C] = a) $\wedge$ (d[A, C] = b) $\wedge$ (d[A, B] = c)], is-rtc[A, B, C] $\Rightarrow$ ($a^2 + b^2 = c^2$)]

Proposition["G14.3.4", any[A, B, C, a, b, c], with[is-tc[A, B, C] $\wedge$ (d[B, C] = a) $\wedge$ (d[A, C] = b) $\wedge$ (d[A, B] = c)], ($a^2 + b^2 = c^2$) $\Rightarrow$ is-rtc[A, B, C]]

■ Polygonal Regions and Their Areas

▫ The Area Function

Definition["G15.1.1",

$$\underset{\substack{A,B,C \\ \text{is-tc}[A,B,C]}}{\forall} (\blacktriangle[A, B, C] = \text{it}[A, B, C] \cup \triangle[A, B, C])]$$

Definition["G15.1.2",
$\mathbb{T} = \{\blacktriangle[A, B, C] \mid \text{is-tc}[A, B, C]\}$]

Definition["G15.1.3",

$$\underset{\substack{\mathcal{T} \\ \mathcal{T}\in\mathbb{T}}}{\forall} \left(\text{int}[\mathcal{T}] = \underset{\mathcal{F}}{\iota} \left(\underset{\substack{X,Y,Z \\ \text{is-tc}[X,Y,Z]}}{\exists} ((\mathcal{T} = \blacktriangle[X, Y, Z]) \wedge (\mathcal{F} = \text{it}[X, Y, Z]))\right)\right)]$$

Proposition["G15.1.4", any[A, B, C], with[is-tc[A, B, C]],
int[▲[A, B, C]] = it[A, B, C]]

Definition["G15.1.5",

$$\underset{n,\tau}{\forall} \left(\text{is-por}[n, \tau] \Leftrightarrow \left(n \in \mathbb{N} \bigwedge \tau : \mathbb{N}_n \longrightarrow \mathbb{T} \bigwedge \underset{\substack{i,j \\ i\in\mathbb{N}\wedge j\in\mathbb{N}}}{\forall} (1 \le i < j \le n \Rightarrow (\text{int}[\tau[\![i]\!]] \cap \text{int}[\tau[\![j]\!]] = \emptyset))\right)\right)]$$

Definition["G15.1.6",

$$\underset{\substack{n,\tau \\ \text{is-por}[n,\tau]}}{\forall} \left(\text{pr}[n, \tau] = \bigcup_{i=1,\ldots,n} \tau[\![i]\!]\right)]$$

Definition["G15.1.7",
$\mathbb{O} = \{\text{pr}[n, \tau] \mid \text{is-por}[n, \tau]\}$]

Definition["G15.1.8",

$$\underset{\substack{P,\mathcal{R} \\ \mathcal{R}\in\mathbb{O}}}{\forall} \left(\text{is-ins}[P, \mathcal{R}] \Leftrightarrow \underset{\substack{A,B,C \\ \text{is-tc}[A,B,C]}}{\exists} (\blacktriangle[A, B, C] \subseteq \mathcal{R} \wedge P \in \text{it}[A, B, C])\right)]$$

Definition["G15.1.9",

$$\underset{\substack{P,\mathcal{R} \\ \mathcal{R}\in\mathbb{O}}}{\forall} (\text{is-bp}[P, \mathcal{R}] \Leftrightarrow (P \in \mathcal{R} \wedge \neg\, \text{is-ins}[P, \mathcal{R}]))]$$

Definition["G15.1.10",

$$\underset{\substack{\mathcal{R} \\ \mathcal{R}\in\mathbb{O}}}{\forall} (\text{ins}[\mathcal{R}] = \{P \mid \text{is-ins}[P, \mathcal{R}]\})]$$

Definition["G15.1.11",

$$\underset{\substack{\mathcal{R} \\ \mathcal{R}\in\mathbb{O}}}{\forall} (\text{bd}[\mathcal{R}] = \{P \mid \text{is-bp}[P, \mathcal{R}]\})]$$

Definition["G15.1.12",

$$\underset{\substack{A,B,C,D \\ \text{is-cqc}[A,B,C,D]}}{\forall} (\text{int}[A, B, C, D] = \text{hp}[A, B, C] \bigcap \text{hp}[B, C, D] \bigcap \text{hp}[C, D, A] \bigcap \text{hp}[D, A, B])]$$

Definition["G15.1.13",

$$\underset{\substack{A,B,C,D \\ \text{is-cqc}[A,B,C,D]}}{\forall} (\blacksquare[A, B, C, D] = \text{int}[A, B, C, D] \bigcup \square[A, B, C, D])]$$

Proposition["G15.1.14", any[A, B, C], with[is-tc[A, B, C]],
$\blacktriangle[A, B, C] \in \mathbb{O} \wedge (\text{ins}[\blacktriangle[A, B, C]] = \text{it}[A, B, C]) \wedge (\text{bd}[\blacktriangle[A, B, C]] = \triangle[A, B, C])$]

Proposition["G15.1.15", any[A, B, C, D], with[is-cqc[A, B, C, D]],
$\blacksquare[A, B, C, D] \in \mathbb{O} \wedge (\text{ins}[\blacksquare[A, B, C, D]] = \text{int}[A, B, C, D]) \wedge (\text{bd}[\blacksquare[A, B, C, D]] = \square[A, B, C, D])$]

Axiom["A1",
$\mu : \mathbb{O} \longrightarrow \mathbb{R}^+$]

Axiom["A2",

$$\underset{A,B,C,D,E,F}{\forall} (\text{is-ctc}[A, B, C, D, E, F] \Rightarrow (\mu[\![\blacktriangle[A, B, C]]\!] = \mu[\![\blacktriangle[D, E, F]]\!]))]$$

Axiom["A3",

$$\underset{\substack{\mathcal{R}_1,\mathcal{R}_2 \\ \mathcal{R}_1 \in \mathbb{O} \wedge \mathcal{R}_2 \in \mathbb{O}}}{\forall} ((\mathcal{R}_1 \bigcap \mathcal{R}_2 = \text{bd}[\mathcal{R}_1] \bigcap \text{bd}[\mathcal{R}_1]) \Rightarrow (\mu[\![\mathcal{R}_1 \bigcup \mathcal{R}_2]\!] = \mu[\![\mathcal{R}_1]\!] + \mu[\![\mathcal{R}_2]\!]))]$$

Axiom["A4",

$$\underset{A,B,C,D}{\forall} (\text{is-sqc}[A, B, C, D] \wedge (d[A, B] = 1) \Rightarrow (\mu[\![\blacksquare[A, B, C, D]]\!] = 1))]$$

Definition["G15.1.16",

$$\underset{\substack{A,B,C \\ \text{is-tc}[A,B,C]}}{\forall} (A[A, B, C] = \mu[\![\blacktriangle[A, B, C]]\!])]$$

Definition["G15.1.17",

$$\underset{\substack{A,B,C,D \\ \text{is-cqc}[A,B,C,D]}}{\forall} (A[A, B, C, D] = \mu[\![\blacksquare[A, B, C, D]]\!])]$$

▫ Area Theorems for Triangles and Quadrilaterals

Proposition["G15.2.1", any[A, B, C, D, q], with[q ∈ ℕ],

$$\text{is-sqc}[A, B, C, D] \bigwedge \left(d[A, B] = \frac{1}{q}\right) \Rightarrow \left(A[A, B, C, D] = \frac{1}{q^2}\right)]$$

Proposition["G15.2.2", any[A, B, C, D, p, q], with[p ∈ ℕ ∧ q ∈ ℕ],

$$\text{is-sqc}[A, B, C, D] \bigwedge \left(d[A, B] = \frac{p}{q}\right) \Rightarrow \left(A[A, B, C, D] = \frac{p^2}{q^2}\right)]$$

Proposition["G15.2.3", any[A, B, C, D, a],
$\text{is-sqc}[A, B, C, D] \wedge (d[A, B] = a) \Rightarrow (A[A, B, C, D] = a^2)$]

Proposition["G15.2.4", any[A, B, C, D, a, b],
$\text{is-rec}[A, B, C, D] \wedge (d[A, B] = a) \wedge (d[A, D] = b) \Rightarrow (A[A, B, C, D] = ab)$]

Proposition["G15.2.5", any[A, B, C, a, b],

$$\text{is-rtc}[A, B, C, D] \wedge (d[B, C] = a) \wedge (d[A, C] = b) \Rightarrow \left(A[A, B, C] = \frac{ab}{2}\right)]$$

Proposition["G15.2.6", any[A, B, C, c, h], with[is-tc[A, B, C]],
$(\mathrm{d}[A, B] = c) \wedge (\mathrm{d}[\mathrm{l}[A, B], C] = h) \Rightarrow \left(A[A, B, C] = \frac{ch}{2}\right)$]

Proposition["G15.2.7", any[A, B, C, D, a, h],
$\text{is-pc}[A, B, C, D] \wedge (\mathrm{d}[A, B] = a) \wedge (\mathrm{d}[\mathrm{l}[A, B], C] = h) \Rightarrow (A[A, B, C, D] = ah)$]

Proposition["G15.2.8", any[A, B, C, D, a, c, h],
$\text{is-trc}[A, B, C, D] \wedge (\mathrm{d}[A, B] = a) \wedge (\mathrm{d}[C, D] = c) \wedge (\mathrm{d}[\mathrm{l}[A, B], C] = h) \Rightarrow \left(A[A, B, C, D] = \frac{(a + c)\, h}{2}\right)$]

Proposition["G15.2.9", any[A, B, C, D, E, F],
$\text{is-stc}[A, B, C, D, E, F] \Rightarrow \left(\frac{A[A, B, C]}{A[D, E, F]} = \left(\frac{\mathrm{d}[A, B]}{\mathrm{d}[D, E]}\right)^2\right)$]

■ Cartesian Coordinate Systems

▫ Introduction of Coordinates

Definition["G16.1.1",
$$\underset{\kappa, x_1, x_2, \Gamma_1, \Gamma_2}{\forall} \left(\text{is-ccs}[\kappa, x_1, x_2, \Gamma_1, \Gamma_2] \Leftrightarrow \left(x_1 \perp x_2 \bigwedge \text{is-cos}[\Gamma_1, x_1] \bigwedge \text{is-cos}[\Gamma_2, x_2] \bigwedge (\Gamma_1[\![\mathrm{i}[x_1, x_2]]\!] = 0) \bigwedge (\Gamma_2[\![\mathrm{i}[x_1, x_2]]\!] = 0) \bigwedge (\kappa : \mathbb{P} \longrightarrow \mathbb{R}^2) \bigwedge \underset{P}{\forall} ((\mathrm{p}_1^2[\kappa[\![P]\!]] = \Gamma_1[\![\mathrm{ft}[x_1, P]]\!]) \wedge (\mathrm{p}_2^2[\kappa[\![P]\!]] = \Gamma_2[\![\mathrm{ft}[x_2, P]]\!]))\right)\right)$$]

Proposition["G16.1.2", any[κ, x_1, x_2, Γ_1, Γ_2],
$\text{is-ccs}[\kappa, x_1, x_2, \Gamma_1, \Gamma_2] \Rightarrow \left(\kappa : \mathbb{P} \xrightarrow{\text{bij}} \mathbb{R}^2\right)$]

Proposition["G16.1.3", any[P, Q, p_1, p_2, q_1, q_2, κ, x_1, x_2, Γ_1, Γ_2], with[is-ccs[κ, x_1, x_2, Γ_1, Γ_2]],
$(\kappa[\![P]\!] = \langle \mathrm{p}_1, \mathrm{p}_2 \rangle) \wedge (\kappa[\![Q]\!] = \langle \mathrm{q}_1, \mathrm{q}_2 \rangle) \Rightarrow \left(\mathrm{d}[P, Q] = \sqrt{(\mathrm{q}_1 - \mathrm{p}_1)^2 + (\mathrm{q}_2 - \mathrm{p}_2)^2}\right)$]

▫ Graphs

Definition["G16.2.1",
$$\underset{\substack{\mathfrak{m}, \kappa, x_1, x_2, \Gamma_1, \Gamma_2 \\ \mathfrak{m} \subseteq \mathbb{R}^2 \wedge \text{is-ccs}[\kappa, x_1, x_2, \Gamma_1, \Gamma_2]}}{\forall} (\mathrm{gr}[\mathfrak{m}, \kappa, x_1, x_2, \Gamma_1, \Gamma_2] = \mathrm{a}[\kappa^{-1}, \mathfrak{m}])$$]

Proposition["G16.2.2", any[g, κ, x_1, x_2, Γ_1, Γ_2], with[is-ccs[κ, x_1, x_2, Γ_1, Γ_2]],
$$\underset{\substack{a, b, c \\ a \neq 0 \vee b \neq 0}}{\exists} (g = \mathrm{gr}[\{\langle x, y \rangle \mid (ax + by + c = 0)\}, \kappa, x_1, x_2, \Gamma_1, \Gamma_2])$$]

Definition["G16.2.3",
$$\underset{\substack{g, \kappa, x_1, x_2, \Gamma_1, \Gamma_2 \\ \text{is-ccs}[\kappa, x_1, x_2, \Gamma_1, \Gamma_2]}}{\forall} \left(\text{is-vrt}[g, \kappa, x_1, x_2, \Gamma_1, \Gamma_2] \Leftrightarrow \underset{a}{\exists} (g = \mathrm{gr}[\{\langle x, y \rangle \mid (x = a)\}, \kappa, x_1, x_2, \Gamma_1, \Gamma_2])\right)$$]

Proposition["G16.2.4", any[g, κ, x_1, x_2, Γ_1, Γ_2], with[is-ccs[κ, x_1, x_2, Γ_1, Γ_2] ∧ ¬ is-vrt[g, κ, x_1, x_2, Γ_1, Γ_2]],
$$\underset{k, d}{\exists} (g = \mathrm{gr}[\{\langle x, y \rangle \mid (y = kx + d)\}, \kappa, x_1, x_2, \Gamma_1, \Gamma_2])$$]

Proposition$\left[\text{"G16.2.5", any}[g, \kappa, x_1, x_2, \Gamma_1, \Gamma_2, \text{k, d, P, Q}, p_1, p_2, q_1, q_2], \text{with}[\text{is-ccs}[\kappa, x_1, x_2, \Gamma_1, \Gamma_2] \wedge (g = \text{gr}[\{\langle x, y\rangle \mid (y = kx + d)\}, \kappa, x_1, x_2, \Gamma_1, \Gamma_2]) \wedge P \neq Q \wedge P \in g \wedge Q \in g], (\kappa[\![P]\!] = \langle p_1, p_2\rangle) \wedge (\kappa[\![Q]\!] = \langle q_1, q_2\rangle) \Rightarrow \left(k = \frac{q_2 - p_2}{q_1 - p_1}\right)\right]$

Proposition$\left[\text{"G16.2.6", any}[g, h, \kappa, x_1, x_2, \Gamma_1, \Gamma_2, \text{k, d, h, e}], \text{with}[\text{is-ccs}[\kappa, x_1, x_2, \Gamma_1, \Gamma_2] \wedge k \neq 0 \wedge (g = \text{gr}[\{\langle x, y\rangle \mid (y = kx + d)\}, \kappa, x_1, x_2, \Gamma_1, \Gamma_2]) \wedge (h = \text{gr}[\{\langle x, y\rangle \mid (y = hx + e)\}, \kappa, x_1, x_2, \Gamma_1, \Gamma_2])], g \perp h \Rightarrow \left(h = -\frac{1}{k}\right)\right]$

Proposition$\left[\text{"G16.2.7", any}[\mathcal{C}, \text{C, r}, \kappa, x_1, x_2, \Gamma_1, \Gamma_2], \text{with}[\text{is-ccs}[\kappa, x_1, x_2, \Gamma_1, \Gamma_2] \wedge r > 0], (\mathcal{C} = \text{c}[\text{C, r}]) \Rightarrow \underset{a,b,c}{\exists} ((\mathcal{C} = \text{gr}[\{\langle x, y\rangle \mid (x^2 + y^2 + ax + by + c = 0)\}, \kappa, x_1, x_2, \Gamma_1, \Gamma_2]))\right]$

■ References

[Buchberger 2000] B. Buchberger.
Theory Exploration with *Theorema*.
Analele Universitatii Din Timisoara, Ser. Matematica−Informatica,Vol. XXXVIII, Fasc.2, 2000, (Proceedings of SYNASC 2000, 2nd International Workshop on Symbolic and Numeric Algorithms in Scientific Computing, Oct. 4−6,2000, University of the West at Timisoara, Rumania, T. Jebelean, V. Negru, A. Popovici eds.), ISSN 1124−970X,pp. 9−32.

[Buchberger 2003a] B. Buchberger.
Algorithm Invention and Verification by Lazy Thinking.
In: D. Petcu, V. Negru, D. Zaharie, T. Jebelean eds., Proceedings of the International SYNASC 03 Conference (Symbolic and Numeric Algorithms in Scientific Computing), University of the West at Timisoara, Romania, Oct. 1−3,2003, Mirton Publishing Company, ISBN 973−661−104−3pp. 2−26.

[Buchberger 2003b] B. Buchberger.
The Four Parallel Threads of Formal Theory Exploration.
Technical Report 2003−25,SFB 013 "Scientific Computing", Johannes Kepler University, October 2003.

[Fuchs 1999] G. Fuchs, Die Methode der Mathematik aus schulmathematischer und fachdidaktischer Sicht, Technical Report 99−1,Otto−von−Guericke−UniversitätMagdeburg, März 1999.

[Fuchs 2001] G. Fuchs, Algorithmische Aspekte der Elementargeometrie, Technical Report 2001−1,Otto−von-Guericke−UniversitätMagdeburg, Januar 2001.

[Hartshorne] R. Hartshorne, Geometry: Euclid and Beyond, Springer, New York, 2000.

[Martin] G.E. Martin, The Foundations of Geometry and the Non−EuclideanPlane, Springer, New York, 1975.

[Millmann−Parker]R.S. Millmann and G.D. Parker, Geometry, Springer, New York, 1991.

[Moise] E.E. Moise, Elementary Geometry from an Advanced Standpoint, Addison−Wesley,1990.