Application of symmetry analysis to a PDE arising in the car windshield design *

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Abstract

A new approach to parameter identification problems from the point of view of symmetry analysis theory is given. A mathematical model that arises in the design of car windshield represented by a linear second order mixed type partial differential equation is considered. Following a particular case of the direct method (due to Clarkson and Kruskal), we introduce a method to study the group invariance between the parameter and the data. The equivalence transformations associated with this inverse problem are also found. As a consequence, the symmetry reductions relate the inverse and the direct problem and lead us to a reduced order model.

Keywords: symmetry reductions, parameter identification problems
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1 Introduction

Symmetry analysis theory links differential geometry to PDEs theory [18], symbolic computation [9] and more recently to numerical analysis theory [3], [6]. Sophus Lie [14] introduced the notion of continuous transformations groups and applied them to differential equations. Over the years, Lie's method has been proven to be a powerful tool for studying a remarkable number of PDEs arising in mathematical physics (more details can be found e.g. in [2], [10] and [21]). In the last years a variety of methods have been developed in order to find special classes of solutions to PDEs, which cannot be determined by applying the classical Lie method. Olver and Rosenau [20] showed that the common theme of all these methods has been the appearance of some form of group invariance. On the other hand, parameter identification problems arising in the inverse problems theory are concerned with the identification of physical parameters from observations of the evolution of a system. The iterative approach of studying parameter identification problems is a functional-analytic setup with a special emphasis on iterative regularization methods [8]. The aim of this paper is to show how parameter identification problems can be analyzed with the tools of group analysis theory. This is a new direction of research in the theory of inverse problems, although the symmetry analysis theory is a common approach for studying PDEs. We restrict ourselves to the case of a parameter identification problem modelled by a PDE of the form

$$F(x, w^{(m)}, E^{(n)}) = 0, \tag{1}$$

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where the unknown function $E = E(x)$ is called *parameter*, and respectively, the arbitrary function $w = w(x)$ is called *data*, with $x = (x_1, .., x_p) \in \Omega \subset R^p$ a given domain.Here $w^{(m)}$ denote the function w together with its partial derivatives up to order m. Assume that the parameters and the data are analytical functions. The PDE (1) sometimes augmented with certain boundary conditions is called *the inverse problem* associated to a *direct problem*. The direct problem is the same equation but the unknown function is the data, for which certain boundary conditions are required.

Sophus Lie's method, known today as the *classical Lie method*, is based on the notion of *symmetry group* related to a PDE. This is a (local) Lie group of transformations acting on the space of the independent variables and the space of the dependent variables of the equation with the property that it leaves the set of all analytical solutions invariant. Knowledge of these *classical symmetries* allows us to reduce the order of the studied PDE, and to determine *group-invariant solutions* (or *similarity solutions*) which are invariant under certain subgroups of the full symmetry group (for more details see [18]). Bluman and Cole [1] introduced the *nonclassical method* that allows one to find the *conditional symmetries* (also called *nonclassical symmetries*) associated with a PDE. These are transformations that leave only a subset of the set of all analytical solutions invariant. Note that any classical symmetry is a nonclassical symmetry but not conversely. Another procedure for finding symmetry reductions is the *direct method* (due to Clarkson and Kruskal [5]). The relation between these last two methods has been studied by Olver [19]. Moreover, for a PDE with coefficients depending on an arbitrary function, Ovsiannikov [21] introduced the notion of *equivalence transformations*, which are (local) Lie group of transformations acting on the space of the independent variables, the space of the dependent variables and the space of the arbitrary functions that leave the equation unchanged. Notice that these techniques based on group theory do not take into account the boundary conditions attached to a PDE.

To find symmetry reductions associated with the parameter identification problem (1) one can seek classical and nonclassical symmetries related to this equation. Two cases can occur when applying the classical Lie method or the nonclassical method, depending if the data w is known or not. From the symbolic computation point of view, the task of finding symmetry reductions for a PDE depending on an arbitrary function might be a difficult one, due to the lack of the symbolic manipulation programs that can handle this kind of equations. Another method to determine symmetry reductions for (1) might be a particular case of the direct method, that has been applied by Zhdanov [24] to certain multidimensional PDEs arising in mathematical physics. Based on this method and taking into account that (1) depends on an arbitrary function, we introduce a procedure to find the relation between the data and the parameter in terms of a similarity variables (see Section 2). As a consequence, it follows that the equivalence transformations related to (1) must be considered as well. These last symmetry reductions are found by using any symbolic manipulation program designed to determine classical symmetries for a PDE system — now both the data and the parameter are unknown functions in the equation (1). The equivalence transformations relate the direct problem and the inverse problem. Moreover, one can find special classes of data and parameters respectively, written in terms of the invariants of the group action; the order of the studied PDE can be reduced at least by one, and analytical solutions to (1) can be found.

At the first step, the group approach of the free boundary problem related to (1) can be considered, and afterwards, the invariance of the boundary conditions under particular group actions has to be analyzed (see [2]). In the case of parameter identification problems we sometimes have to deal with two pair of boundary conditions, for data and the parameter too, or we might only know the boundary conditions for the data, so that the problem of finding symmetry reductions for a given data can be more complicated. At least, by finding the equivalence transformations related to the problem, the invariants of the group actions can be used to establish suitable domains Ω on which the order of the model can be reduced.

In this paper we consider a mathematical model arising in the car windshield design. Let us briefly explain the *gravity sag bending process*, one of the main industrial processes used in the manufacture of car windshields. A piece of glass is placed over a rigid frame with the desired edge curvature and heated from below. The glass becomes viscous due to the temperature rise and sags under its own weight. The final shape depends on the viscosity distribution of the glass obtained from varying the temperature. It has been shown that the sag bending process can also be controlled (in a first approximation) in the terms of Young modulus E, a spatially varying glass material parameter, and the displacement of the glass w can be described by the theory of a thin linear elastic plate theory (see [11], [16] and [17] and references from there). The model is based on the linear plate equation

$$(E(w_{xx}+\nu w_{yy}))_{xx}+2(1-\nu)(Ew_{xy})_{xy}+(E(w_{yy}+\nu w_{xx}))_{yy}=\frac{12(1-\nu^2)f}{h^3} \quad \text{on} \quad \Omega, \qquad (2)$$

where $w=w(x,y)$ represents the displacement of the glass sheet (the target shape) occupying a domain $\Omega \subset R^2$, $E=E(x,y)$ is the Young modulus, a positive function that can be influenced by adjusting the temperature in the process of heating the glass, f is the gravitational force, $\nu \in \left(0, \frac{1}{2}\right)$ is the glass Poisson ratio, and h is thickness of the plate. The *direct problem* (or the *forward problem*) is the following: for a given Young modulus E, find the displacement w of a glass sheet occupying a domain Ω before the heating process. Note that the PDE (2) is an elliptic fourth order linear PDE for the function w. Since now, two problems related to (2) have been studied: the *clamped plate case* and the *simply supported plate case* (more details can be found e.g. in [15]). In this paper we consider the clamped case, in which the following boundary conditions are required: the plate is placed over a rigid frame, i.e.,

$$w(x,y)|_{\partial\Omega}=0, \qquad (3)$$

and respectively,

$$\frac{\partial w}{\partial n}|_{\partial\Omega}=0, \qquad (4)$$

that means the (outward) normal derivative of w must be zero, i.e., the sheet of glass is not allowed to freely rotate around the tangent to $\partial\Omega$. The associated *inverse problem* consists of finding the Young modulus E for a given data w in (2). This is a linear second order PDE for the Young modulus that can be written in the equivalent form

$$(w_{xx}+\nu w_{yy})E_{xx}+2(1-\nu)w_{xy}E_{xy}+(w_{yy}+\nu w_{xx})E_{yy} \\ +2(\Delta w)_x E_x+2(\Delta w)_y E_y+(\Delta^2 w)E=1, \qquad (5)$$

after a scaling transformations of the form $w \to \frac{1}{k}w$ or $E \to \frac{1}{k}E$, with $k=\frac{12(1-\nu^2)f}{h^3}$. In (5), Δ denotes the Laplace operator. The main problem in the car windshield design is that the prescribed target shape w is frequently such that the discriminant

$$D=(1-\nu)^2 w_{xy}^2-(w_{xx}+\nu w_{yy})(w_{yy}+\nu w_{xx})$$

of the equation (5) changes sign in the domain Ω, so that we get a mixed type PDE. This is one of the reason for which optical defects might occur during the process. Note that the equation (5) would naturally call for boundaries conditions for E on $\partial\Omega$ in the purely elliptic case (when $D<0$), and Cauchy data on a suitable (non-characteristic part) $\Gamma \subset \partial\Omega$ in the purely hyperbolic part (for

$D > 0$). There is a recent interest in studying this inverse problem (see e.g., [13]). It is known [15] that a constant Young modulus corresponds to a data which satisfies the nonhomogeneous biharmonic equation (33). A survey on this subject can be found in [23]. Salazar and Westbrook [22] studied the case when the data and the parameter are given by radial functions, Kügler [12] used a derivative free iterative regularization method for analyzing the problem on rectangular frames, and a simplified model for the inverse problem on circular domains was considered by Engl and Kügler in their paper [7].

So far it is not obvious which shapes can be made by using this technique. Hence, we try to answer this question by finding out the symmetry reductions related to the PDE (5) hidden by the nonlinearity that occurs between the data and the parameter. In this sense, we determine (see Section 3) the group of transformations that leave the equation unchanged, and so, its mixed type form. Knowledge of the invariants of these group actions allows us to write the target shape and the parameter in terms of them, and therefore, to reduce the order of the studied equation. We find again the obvious result that a Young modulus constant corresponds to a data which is a solution of an homogeneous biharmonic equation. The circular case problem considered by Salazar and Westbrook is in fact, a particular case of our study, so we show that other target shapes which are not radial functions can be considered. We prove that the equation (5) is invariant under scaling transformations. It follows that target shapes modelled by homogeneous functions can be analyzed as well. In particular, we are interested in target shapes modelled by homogeneous polynomials defined on elliptical domains or square domains with rounded corners.

The paper is structured as follows. In Section 2 we propose a method to study the relation between the data and the parameter that might occur in terms of a similarity variable that allows one to reduce the order of the PDE (5). The equivalence transformations related to this equation are given in Section 3. The symbolic manipulation program DESOLV, authors Carminati and Vu [4] has been used for this purpose. Table 1 contains a complete classification of these symmetry reductions. In the last section, we discuss the PDE (5) augmented with the boundary conditions (3) and (4). Certain examples of data are given in order to show to relate the invariants of the group action and suitable domains Ω on which the order of model can be reduced.

2 Conditional symmetries

The direct method approach to a second order partial differential equation

$$\mathcal{F}(x, y, E^{(2)}) = 0,$$

consists of seeking solutions written in the form

$$E(x, y) = \Phi(x, y, F(z)), \quad \text{where} \quad z = z(x, y), \quad (x, y) \in \Omega. \tag{6}$$

In this case the function z is called *similarity variable* and its level sets $\{z = k\}$ are named *similarity curves*. After substituting (6) into the studied second order PDE, one requires that the result to be an ODE for the arbitrary function $F = F(z)$, and hence, certain conditions are imposed upon the functions Φ, z and their partial derivatives. The particular case

$$E(x, y) = F(z(x, y)) \tag{7}$$

consists of looking for solutions depending only on the similarity variable, so that, if z is an invariant of the group action then the solutions of the form (7) are as well. Assume that the similarity variable is such that $\|\nabla z\| \neq 0$ on $\bar{\Omega}$.

In this section we apply this particular approach to the equation (5) in order to study if the parameter and the data are functionally independent, that means if they can depend or not on the same similarity variable. Assume that the Young modulus takes the form (7). In this case we get the relation

$$
\begin{aligned}
F''(z)\left[z_x^2(w_{xx}+\nu w_{yy})+2z_xz_y(1-\nu)w_{xy}+z_y^2(w_{yy}+\nu w_{xx})\right] \\
+F'(z)\left[z_{xx}(w_{xx}+\nu w_{yy})+2(1-\nu)z_{xy}w_{xy}++z_{yy}(w_{yy}+\nu w_{xx})\right. \\
\left.+2z_x(\Delta w)_x+2z_y(\Delta w)_y\right]+F(z)(\Delta^2 w)=1, \quad (8)
\end{aligned}
$$

which must be an ODE for the unknown function $F=F(z)$. This condition is satisfied if the coefficients of the partial derivatives of F are function of z only (note that these coefficients are also invariant under the same group action). Denote them by

$$
\begin{aligned}
\Gamma_1(z) &= z_x^2(w_{xx}+\nu w_{yy})+2z_xz_y(1-\nu)w_{xy}+z_y^2(w_{yy}+\nu w_{xx}), \\
\Gamma_2(z) &= z_{xx}(w_{xx}+\nu w_{yy})+2(1-\nu)z_{xy}w_{xy}+z_{yy}(w_{yy}+\nu w_{xx}) \\
&\quad +2z_x(\Delta w)_x+2z_y(\Delta w)_y, \\
\Gamma_3(z) &= \Delta^2 w.
\end{aligned} \tag{9}
$$

If these relations hold the PDE (5) is reduced to the second order linear ODE

$$
\Gamma_1(z)F''(z)+\Gamma_2(z)F'(z)+\Gamma_3(z)F(z)=1. \tag{10}
$$

2.1 Data and parameter invariant under the same group

If the target shape is invariant under the same group action as the Young modulus then

$$
w(x,y)=G(z(x,y)), \tag{11}
$$

where $G=G(z)$ is an arbitrary function. Substituting (11) into the relations (9), these coefficients take the form

$$
\begin{aligned}
\Gamma_1 &= G''(z_x^2+z_y^2)^2+G'\left[(z_x^2+\nu z_y^2)z_{xx}+2(1-\nu)z_xz_yz_{xy}+(z_y^2+\nu z_x^2)z_{yy}\right], \\
\Gamma_2 &= 2G'''(z_x^2+z_y^2)^2+G''\left\{[7z_x^2+(\nu+2)z_y^2]z_{xx}+2(5-\nu)z_xz_yz_{xy}\right. \\
&\quad \left.+[7z_y^2+(\nu+2)z_x^2]z_{yy}\right\}+G'\left\{(\Delta z)^2+2(1-\nu)(z_{xy}^2-z_{xx}z_{yy})\right. \\
&\quad \left.+2\left[z_x(\Delta z)_x+z_y(\Delta z)_y\right]\right\}, \\
\Gamma_3 &= G''''(z_x^2+z_y^2)^2+2G'''\left[(3z_x^2+z_y^2)z_{xx}+4z_xz_yz_{xy}+(z_x^2+3z_y^2)z_{yy}\right. \\
&\quad +G''\left\{3(\Delta z)^2+4(z_{xy}^2-z_{xx}z_{yy})+4\left[z_x(\Delta z)_x+z_y(\Delta z)_y\right]\right\}+G'\Delta^2 z.
\end{aligned} \tag{12}
$$

Next we require that the coefficients of the partial derivatives of the function G in the above relations to depend only on z. Denote

$$
\begin{aligned}
\Gamma_1 &= \alpha^4G''+a_1G', \\
\Gamma_2 &= 2\alpha^4G'''+a_2G''+a_3G', \\
\Gamma_3 &= \alpha^4G''''+2a_4G'''+a_5G''+a_6G',
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha^2(z) &= z_x^2+z_y^2, \\
a_1(z) &= (z_x^2+\nu z_y^2)z_{xx}+2(1-\nu)z_xz_yz_{xy}+(z_y^2+\nu z_x^2)z_{yy}, \\
a_2(z) &= \left[7z_x^2+(\nu+2)z_y^2\right]z_{xx}+2(5-\nu)z_xu_yz_{xy}+\left[7z_y^2+(\nu+2)z_x^2\right]z_{yy}, \\
a_3(z) &= (\Delta z)^2+2(1-\nu)(z_{xy}^2-z_{xx}z_{yy})+2\left[z_x(\Delta z)_x+z_y(\Delta z)_y\right], \\
a_4(z) &= (3z_x^2+z_y^2)z_{xx}+4z_xz_yz_{xy}+(z_x^2+3z_y^2)z_{yy}, \\
a_5(z) &= 3(\Delta z)^2+4(z_{xy}^2-z_{xx}z_{yy})+4\left[z_x(\Delta z)_x+z_y(\Delta z)_y\right], \\
a_6(z) &= \Delta^2 z.
\end{aligned} \tag{13}
$$

Notice that the first relation in (13) which is a 2D eikonal equation implies

$$\begin{aligned}
&z_x^2 z_{xx} + 2z_x z_y z_{xy} + z_y^2 z_{yy} = \alpha^3(z)\alpha'(z),\\
&z_{xx} = \alpha(z)\alpha'(z) - \tfrac{z_y}{z_x} z_{xy},\\
&z_{yy} = \alpha(z)\alpha'(z) - \tfrac{z_x}{z_y} z_{xy}.
\end{aligned}$$

¿From the last two equations we get

$$z_y^2 z_{xx} - 2z_x z_y z_{xy} + z_x^2 z_{yy} = \alpha^3(z)\alpha'(z) - \alpha^4(z)\frac{z_{xy}}{z_x z_y}. \tag{14}$$

Assume that there is a function $\beta = \beta(z)$ such that

$$z_{xy} = \beta(z) z_x z_y. \tag{15}$$

Indeed, since the left hand side in (14) depends only on z, one can easily check that if z satisfies both the 2D eikonal equation in (13) and (15), then all the functions $a_i = a_i(z)$ defined by (13) are written in terms of α and β. Therefore, the problem of finding the similarity variable z is reduced to that of integrating the 2D eikonal equation and the PDE system

$$\begin{cases} z_{xx} = \alpha\alpha' - \beta z_y^2 \\ z_{xy} = \beta z_x z_y \\ z_{yy} = \alpha\alpha' - \beta z_x^2. \end{cases} \tag{16}$$

The compatibility condition of the above system is given by the ODE

$$\alpha\alpha'' + \alpha'^2 - 3\beta\alpha\alpha' + \alpha^2\left(\beta^2 - \beta'\right) = 0,$$

which can be written in the following equivalent form

$$\mu'' - 3\beta\mu' + 2\mu\left(\beta^2 - \beta'\right) = 0, \tag{17}$$

where we set $\mu = \frac{1}{2}\alpha^2$. On the other hand, writing the function β as

$$\beta(z) = -\frac{\lambda''(z)}{\lambda'(z)}, \tag{18}$$

where λ is a non-constant function, the equation (15) becomes

$$(\lambda(z))_{xy} = 0,$$

and its general solution is given by

$$\lambda(z(x,y)) = a(x) + b(y), \tag{19}$$

with a and b being arbitrary functions. Substituting β from (18) into the compatibility condition (17), and after integrating once, we get

$$\mu'\lambda' + 2\mu\lambda'' = k, \tag{20}$$

where k is an arbitrary constant.
Case 1. If $k \neq 0$ then after integrating (20) and substituting back $\mu = \frac{1}{2}\alpha^2$, we get

$$\alpha^2(z) = \frac{2k\lambda(z) + C_1}{\lambda'^2(z)}. \tag{21}$$

The relation (19) implies $\lambda'(z)z_x = a'(x)$, and $\lambda'(z)z_y = b'(y)$. We substitute these relations, (19), and (21) into the 2D eikonal equation (see (13)). It follows that the functions $a = a(x)$, and $b = b(y)$ are solutions to the ODEs

$$a'^2(x) - 2ka(x) = C_2, \quad \text{and} \quad b'^2(y) - 2kb(y) = C_3,$$

with $C_2 + C_3 = C_1$ (here C_i are real constants). The above ODEs admit the non-constants solutions

$$a(x) = \frac{1}{2k}\left[k^2(x - C_4)^2 - C_2\right] \quad \text{and} \quad b(y) = \frac{1}{2k}\left[k^2(y - C_5)^2 - C_3\right],$$

and so, (19) takes the form

$$\lambda(z(x,y)) = \frac{k}{2}\left[(x - C_4)^2 + (y - C_5)^2\right] - \frac{C_1}{2k}. \tag{22}$$

Notice that $\frac{1}{k_1}\lambda$ or $\lambda + k_2$ defines the same function β as the function λ does. Moreover, since the PDE (5) is invariant under translation in the (x, y)-space, we can consider

$$\lambda(z(x,y)) = x^2 + y^2. \tag{23}$$

If $\sqrt{\lambda}$ is a bijective function on a suitable interval, and if we denote by $\Phi = \left(\sqrt{\lambda}\right)^{-1}$ its inverse function, then the similarity variable written in the polar coordinates (r, θ) (where $x = r\cos(\theta)$, $y = r\sin(\theta)$) is given by

$$z(x,y) = \Phi(r). \tag{24}$$

For simplicity, we consider $\Phi = Id$, and then

$$E = F(r) \quad \text{and} \quad w = G(r), \quad \text{where} \quad z(x,y) = r, \tag{25}$$

and hence, the ODE (10) turns into

$$\left(G'' + \frac{\nu}{r}G'\right)F'' + \left(2G''' + \frac{\nu+2}{r}G'' - \frac{1}{r^2}G'\right)F' + \left(G'''' + \frac{2}{r}G''' - \frac{1}{r^2}G'' + \frac{1}{r^3}G'\right)F = 1, \tag{26}$$

which can be reduced to the first order ODE [15]

$$\left(G'' + \frac{\nu}{r}G'\right)F' + \left(G''' + \frac{1}{r}G'' - \frac{1}{r^2}G'\right)F = \frac{r}{2}. \tag{27}$$

Case 2. If $k = 0$, similarly we get

$$z(x,y) = \Phi(k_1x + k_2y), \tag{28}$$

where k_1 and k_2 are real constants such that $k_1^2 + k_2^2 > 0$. In this case, for $\Phi = Id$, the parameter and the data are written as

$$E = F(z) \quad \text{and} \quad w = G(z), \quad \text{where} \quad z(x,y) = k_1x + k_2y, \tag{29}$$

and the ODE (10) turns into

$$G''(z)F''(z) + 2G'''(z)F'(z) + G''''(z)F(z) = \frac{1}{(k_1^2 + k_2^2)^2}, \tag{30}$$

with $\{z | G''(z) = 0\}$ the associated set of singularities. Integrating the above ODE on the set $\{z | G''(z) \neq 0\}$ we obtain that the Young modulus is given by

$$E(x,y) = \frac{(k_1x + k_2y)^2 + C_1(k_1x + k_2y) + C_2}{2(k_1^2 + k_2^2)^2 G''(k_1x + k_2y)},$$

where C_i are arbitrary constants.

2.2 Data and parameter invariant under different groups

Consider two functionally independent functions on Ω, say $z = z(x, y)$ and $v = v(x, y)$, and let

$$w = H(v(x, y)), \tag{31}$$

be the target shape, where $H = H(z)$ is an arbitrary function. In this case, the data and the parameter do not share the same invariance. Similarly to above, substituting (31) into the relations (9) we get

$$\begin{aligned}
\Gamma_1 &= H'' \left[(z_x v_x + z_y v_y)^2 + \nu(z_y v_x - z_x v_y)^2\right] \\
&+ H' \left[z_x^2 v_{xx} + 2z_x z_y v_{xy} + z_y^2 v_{yy} + \nu \left(z_x^2 v_{yy} - 2z_x z_y v_{xy} + z_y^2 v_{xx}\right)\right] \\
\Gamma_2 &= H'''(v_x^2 + v_y^2)(z_x v_x + z_y v_y) + H'' \left[v_x^2 z_{xx} + 2v_x v_y u_{xy} + v_y^2 u_{yy}\right. \\
&+ \nu \left(v_y^2 u_{xx} - 2v_x v_y z_{xy} + v_x^2 z_{yy}\right) + 2z_x v_x v_{xx} + 2(z_x v_y + z_y v_x)v_{xy} + 2z_y v_y v_{yy} \\
&\left. + (z_x v_x + z_y v_y)(\Delta v)\right] + H' \left[z_{xx} v_{xx} + 2z_{xy} v_{xy} + z_{yy} v_{yy} + \nu \left(z_{xx} v_{yy} - 2z_{xy} v_{xy}\right.\right. \\
&\left.\left. + z_{yy} v_{xx}\right) + z_x (\Delta v)_x + z_y (\Delta v)_y\right] \\
\Gamma_3 &= H''''(v_x^2 + v_y^2)^2 + 2H''' \left[(3v_x^2 + v_y^2)v_{xx} + 4v_x v_y v_{xy} + (v_x^2 + 3v_y^2)v_{yy}\right] \\
&+ H'' \left[3v_{xx}^2 + 4v_{xy}^2 + 3v_{yy}^2 + 2v_{xx} v_{yy} + 4v_x (\Delta v)_x + 4v_y (\Delta v)_y\right] + H' \Delta^2 v.
\end{aligned} \tag{32}$$

Recall that Γ_i's are functions of $z = z(x, y)$ only. Since each right hand side in the above relations contains the function $H = H(v)$ and its derivatives, we require that the coefficients of the derivatives of H to be functions of v. It follows that Γ_i must be constant, and denote them by γ_i. Hence, the last condition in (32) becomes

$$\Delta^2(w) = \gamma_3, \tag{33}$$

which is the biharmonic equation. According to the above assumption, we seek solutions of (33) that are functions of v only. Similarly to Subsection 2.1, we get

$$v(x, y) = \Psi(r), \quad \text{or} \quad v(x, y) = \Psi(k_1 x + k_2 y), \tag{34}$$

and thus, for $\Psi = Id$, the target shape is written as

$$w(x, y) = H(r), \quad \text{or} \quad w(x, y) = H(k_1 x + k_2 y). \tag{35}$$

Since $z = z(x, y)$ and $v = v(x, y)$ are functionally independent, we get

$$z(x, y) = k_1 x + k_2 y, \quad v(x, y) = \sqrt{x^2 + y^2} \tag{36}$$

or

$$z(x, y) = \sqrt{x^2 + y^2}, \quad v(x, y) = k_1 x + k_2 y. \tag{37}$$

One can prove that if the coefficients γ_i are constant, and if z and v are given by (36) or (37) respectively, then $\gamma_1 = \gamma_2 = 0$, and $\gamma_3 \neq 0$. One the other hand, the solutions of the biharmonic equation (33) of the form (35), are the following

$$w(x, y) = \frac{\gamma_3}{64} z^4 + C_1 z^2 + C_2 \ln(z) + C_3 z^2 \ln(z) + C_4, \quad \text{for} \quad z = \sqrt{x^2 + y^2},$$

and respectively,

$$w(x, y) = \frac{\gamma_3}{24(k_1^2 + k_2^2)^2} v^4 + C_1 v^3 + C_2 v^2 + C_3 v + C_4, \quad \text{for} \quad v = k_1 x + k_2 y,$$

and these correspond to the constant Young modulus

$$E(x,y) = \frac{1}{\gamma_3}. \tag{38}$$

Notice that only particular solutions of the biharmonic equation are obtained in this case (i.e., solutions invariant under rotations and translations). Since this PDE is also invariant under scaling transformations acting on the data space, it is obvious to extend our study to group of transformations acting also on the data space.

3 Equivalence transformations

Consider a one-parameter Lie group of transformations acting on an open set $\mathcal{D} \subset \Omega \times \mathcal{W} \times \mathcal{E}$, where $\mathcal{W}$ is the space of the data functions, and $\mathcal{E}$ is the space of the parameter functions, given by

$$\begin{cases} x^* &= x + \varepsilon\zeta(x,y,w,E) + \mathcal{O}(\varepsilon^2) \\ y^* &= y + \varepsilon\eta(x,y,w,E) + \mathcal{O}(\varepsilon^2) \\ w^* &= w + \varepsilon\phi(x,y,w,E) + \mathcal{O}(\varepsilon^2) \\ E^* &= E + \varepsilon\psi(x,y,w,E) + \mathcal{O}(\varepsilon^2), \end{cases} \tag{39}$$

where ε is the group parameter. Let

$$V = \zeta(x,y,w,E)\partial_x + \eta(x,y,w,E)\partial_y + \phi(x,y,w,E)\partial_w + \psi(x,y,w,E)\partial_E \tag{40}$$

be its associated general infinitesimal generator. The group of transformations (39) is called an *equivalence transformation* associated to the PDE (5) if this leaves the equation invariant, i.e., the form of the equation in the new coordinates remains unchanged, and the set of the analytical solutions is invariant under this transformation as well. The equivalence transformations are found by applying the classical Lie method to the equation (5), with E and w both considered as unknown functions (for more details see [10] and [21]). Following this procedure we obtain

$$\begin{cases} \zeta(x,y,w,E) &= k_1 + k_5 x - k_4 y \\ \eta(x,y,w,E) &= k_2 + k_4 x + k_5 y \\ \phi(x,y,w,E) &= k_3 + k_7 x + k_6 y + (4k_5 - k_8)w \\ \psi(x,y,w,E) &= k_8 E, \end{cases} \tag{41}$$

where k_i are real constants. Since the vector field (40) can be written as $V = \sum_{i=1}^{8} k_i V_i$, we have the following result

Proposition. *The equivalence transformations related to the PDE (5) are generated by the following infinitesimal generators*

$$\begin{gathered} V_1 = \partial_x, \quad V_2 = \partial y, \quad V_3 = \partial_w, \quad V_4 = -y\partial_x + x\partial_y, \quad V_5 = x\partial_x + y\partial_y + 4w\partial_w, \\ V_6 = y\partial_w, \quad V_7 = x\partial_w, \quad V_8 = -w\partial_w + E\partial_E. \end{gathered} \tag{42}$$

It follows that the equation is invariant under translations in the x-space, y-space, w-space, rotations in the space of the independent variables (x,y), scaling transformations in the (x,y,w)-space, Galilean transformations in the (y,w) and (x,w) spaces, and scaling transformations in the (w,E)-space, respectively.

Note that the conditional symmetries found in Section 2 are particular cases of these equivalence transformations. Since each one-parameter group of transformations generated by V_i is a symmetry group, if $w = G(x, y)$ and $E = F(x, y)$ is a known solution of the equation (5), so are the following

$$
\begin{array}{ll}
w^{(1)} = G(x - \varepsilon_1, y), & E^{(1)} = F(x - \varepsilon_1, y), \\
w^{(2)} = G(x, y - \varepsilon_2), & E^{(2)} = F(x, y - \varepsilon_2), \\
w^{(3)} = G(x, y) + \varepsilon_3, & E^{(3)} = F(x, y), \\
w^{(4)} = G(\tilde{x}, \tilde{y}), & E^{(4)} = F(\tilde{x}, \tilde{y}), \\
w^{(5)} = e^{4\varepsilon_5} G\left(e^{-\varepsilon_5} x, e^{-\varepsilon_5} y\right), & E^{(5)} = F\left(e^{-\varepsilon_5} x, e^{-\varepsilon_5} y\right) \\
w^{(6)} = G(x, y) + \varepsilon_6 y, & E^{(6)} = F(x, y), \\
w^{(7)} = G(x, y) + \varepsilon_7 x, & E^{(7)} = F(x, y), \\
w^{(8)} = e^{-\varepsilon_8} G(x, y), & E^{(8)} = e^{\varepsilon_8} F(x, y),
\end{array}
\tag{43}
$$

where $\tilde{x} = x\cos(\varepsilon_4) + y\sin(\varepsilon_4)$, $\tilde{y} = -x\sin(\varepsilon_4) + y\cos(\varepsilon_4)$, and ε_i are real constants. Moreover, the general solution of (5) constructed from a known one is given by

$$
\begin{aligned}
w(x, y) &= e^{4\varepsilon_5 - \varepsilon_8} G\left(e^{-\varepsilon_5}(\tilde{x} - \tilde{k}_1), e^{-\varepsilon_5}(\tilde{y} - \tilde{k}_2)\right) + e^{4\varepsilon_5 - \varepsilon_8}\varepsilon_6 y + e^{4\varepsilon_5 - \varepsilon_8}\varepsilon_7 x + e^{4\varepsilon_5 - \varepsilon_8}\varepsilon_3 \\
E(x, y) &= e^{\varepsilon_8} F\left(e^{-\varepsilon_5}(\tilde{x} - \tilde{k}_1), e^{-\varepsilon_5}(\tilde{y} - \tilde{k}_2)\right),
\end{aligned}
$$

where $\tilde{k}_1 = \varepsilon_1\cos(\varepsilon_4) + \varepsilon_2\sin(\varepsilon_4)$, and $\tilde{k}_2 = \varepsilon_1\sin(\varepsilon_4) - \varepsilon_2\cos(\varepsilon_4)$.

The equivalence transformations form a Lie group $\mathcal{G}$ with a 8-dimensional associated Lie algebra $\mathcal{A}$. Using the adjoint representation of $\mathcal{G}$, one can find the optimal system of one-dimensional subalgebras of $\mathcal{A}$ (more details can be found in [18], pp. 203-209). This optimal system is spanned by the vector fields given in Table 1. Here denote z, I, and J the invariants related to the one-parameter group of transformations generated by each vector field V_i, F and G are arbitrary functions, (r, θ) are the polar coordinates, and a, b, c are non-zero constants. To reduce the order of the PDE (5) one can also integrates the first order PDE system

$$
\left\{
\begin{array}{lcl}
\zeta(x, y, w, E) w_x + \eta(x, y, w, E) w_y &=& \phi(x, y, w, E) \\
\zeta(x, y, w, E) E_x + \eta(x, y, w, E) E_y &=& \psi(x, y, w, E),
\end{array}
\right.
\tag{44}
$$

which are the characteristics of the vector field (40).

	Infintesimal Generator	Invariants	$w = w(x, y)$	$E = E(x, y)$	ODE
1.	V_1	$z = y$ $I = w$ $J = E$	$w = G(z)$	$E = F(z)$	(45)
2.	V_2	$z = x$ $I = w$ $J = E$	$w = G(z)$	$E = F(z)$	(45)
3.	V_4	$z = r$ $I = w$ $J = E$	$w = G(z)$	$E = F(z)$	(26)

	Infintesimal Generator	Invariants	$w = w(x,y)$	$E = E(x,y)$	ODE
4.	V_5	$z = \frac{y}{x}$ $I = x^{-4}w$ $J = E$	$w = x^4 G(z)$	$E = F(z)$	(47)
5.	$cV_3 + V_4$	$z = r$ $I = w - c\theta$ $J = E$	$w = c\theta + G(z)$	$E = F(z)$	(26)
6.	$V_5 + cV_8$	$z = \frac{y}{x}$ $I = x^{c-4}w$ $J = x^{-c}E$	$w = x^{4-c}G(z)$	$E = x^c F(z)$	(48)
7.	$V_4 + cV_8$	$z = r$ $I = e^{c\theta}w$ $J = e^{-c\theta}E$	$w = e^{-c\theta}G(z)$	$E = e^{c\theta}F(z)$	(49)
8.	$V_4 + cV_5$	$z = re^{-c\theta}$ $I = r^{-4}w$ $J = E$	$w = r^4 G(z)$	$E = F(z)$	(51)
9.	$V_4 + aX_5 + bV_8$	$z = re^{-a\theta}$ $I = r^{\frac{b}{a}-4}w$ $J = r^{-\frac{b}{a}}E$	$w = r^{4-\frac{b}{a}}G(z)$	$E = r^{\frac{b}{a}}F(z)$	(52)
10.	$V_1 + cV_6$	$z = y$ $I = w - cxy$ $J = E$	$w = cxy + G(z)$	$E = F(z)$	(45)
11.	$V_2 + cV_7$	$z = x$ $I = w - cxy$ $J = E$	$w = cxy + G(z)$	$E = F(z)$	(45)
12.	$V_1 + cV_8$	$z = y$ $I = e^{cx}w$ $J = e^{-cx}E$	$w = e^{-cx}G(z)$	$E = e^{cx}F(z)$	(53)
13.	$V_2 + cV_8$	$z = x$ $I = e^{cy}w$ $J = e^{-cy}E$	$w = e^{-cy}G(z)$	$E = e^{cy}F(z)$	(53)

Table 1

The reduced ODEs are determined by integrating the associated systems (44) obtained for each of the vector fields given in Table 1. Thus, the invariance of the equation (5) under the translations described by V_1, V_2 and $V_1 + V_6$ implies the following ODE

$$F''(z)G''(z) + 2F'(z)G'''(z) + F(z)G''''(z) = 1, \tag{45}$$

with its general solution

$$F(z) = \frac{z^2 + C_1 z + C_2}{2G''(z)}. \tag{46}$$

The invariance of the PDE (5) under the scaling transformation generated by the vector field V_5

yields the reduced ODE

$$
\begin{aligned}
&\left[G''\left(z^2+1\right)^2-6z(z^2+1)G'+12(z^2+\nu)G\right]F''\\
&\quad+2\left[\left(z^2+1\right)^2G'''-5z(z^2+1)G''+3(4z^2+\nu+1)G'-12zG\right]F'\\
&\quad+\left[\left(z^2+1\right)^2G''''-4z(z^2+1)G'''+4(3z^2+1)G''-24zG'+24G\right]F=1.
\end{aligned}
\tag{47}
$$

In the case 6, the reduced ODE is the following

$$
\begin{aligned}
&\left[\left(z^2+1\right)^2G''+2(c-3)z(z^2+1)G'+(c-3)(c-4)(z^2+\nu)G\right]F''\\
&+\left\{2\left(z^2+1\right)^2G'''+2(2c-5)z(z^2+1)G''+2(c-3)[z^2(c-4)+\nu(c-1)-1]G'\right.\\
&\left.-2(c-3)(c-4)zG\right\}F'+\left\{\left(z^2+1\right)^2G''''+2(c-2)z(z^2+1)G'''+\left[(c-3)(c-4)z^2\right.\right.\\
&\left.\left.-2(c-2)-\nu c(c-1)\right]G''-2(c-4)(c-3)zG'+2(c-4)(c-3)G\right\}F=1.
\end{aligned}
\tag{48}
$$

The case 7 yields the reduced equation

$$
\begin{aligned}
&\left[G''+\frac{\nu}{r}G'+\frac{\nu c^2}{r^2}G\right]F''+\left[2G'''+\frac{\nu+2}{r}G''+\frac{2\nu c^2-1}{r^2}G'-\frac{c^2(1+2\nu)}{r^3}G\right]F'\\
&\quad+\left[G''''+\frac{2}{r}G'''+\frac{c^2\nu-1}{r^2}G''+\frac{1-2c^2(\nu+1)}{r^3}G'+\frac{2c^2(\nu+1)}{r^4}G\right]F=1,
\end{aligned}
\tag{49}
$$

which turns into the first order ODE

$$
\left(G''+\frac{\nu}{r}G'+\frac{\nu}{r^2}G\right)F'+\left(G'''+\frac{1}{r}G''+\frac{c^2\nu-1}{r^2}G'-\frac{1+c^2\nu}{r^3}G\right)=\frac{r}{2}.
\tag{50}
$$

The change of the variable $z=\exp(t)$ in the cases 8 and 9, yield the following ODEs

$$
\begin{aligned}
&\left\{(c^2+1)^2G''+\left(\frac{1}{c}+c\right)[c(\nu+7)-2b]G'+\left(\frac{4}{c}-b\right)[c^3(1+3\nu)-c^2\nu b\right.\\
&\left.+c(\nu+3)-b]G\right\}F''+\left\{2(c^2+1)^2G'''+\left(\frac{1}{c}+c\right)[c(\nu+19)-4b]\right.\\
&+\left[\frac{b^2(\nu+1)}{c^2}+c^2(3\nu+13)-4bc(\nu+1)-12\frac{b}{c}+\nu+29\right]G'\\
&\left.+\left(4-\frac{b}{c}\right)[2c(\nu+7)+b(\nu-5)]G\right\}F'+\left\{(c^2+1)^2G''''\right.\\
&+2\left(c+\frac{1}{c}\right)(6c-b)G'''+\left[\frac{b^2}{c^2}+\frac{b}{c}(\nu-17)+20c^2-bc(\nu+7)+\nu b^2+52\right]G''\\
&\left.+\left(\frac{6}{c}-\frac{b}{c^2}\right)[16c+b(\nu-5)]G'+2\left(\frac{4}{c}-\frac{b}{c^2}\right)[8c+b(\nu-3)]\right\}F=1,
\end{aligned}
\tag{51}
$$

and respectively,

$$
\begin{aligned}
&\left\{(c^2+1)G''+(c^2+1)(\nu+7)G'+4\left[\nu(3c^2+1)+c^2+3\right]G\right\}F''\\
&+\left\{2(c^2+1)^2G'''+(c^2+1)(\nu+19)G''+(6\nu+26)(c^2+1)G'+8(\nu+7)G\right\}F'\\
&+\left\{(c^2+1)^2G''''+12(c^2+1)G'''+4(5c^2+13)G''+96G'+64G\right\}F=1.
\end{aligned}
\tag{52}
$$

In the last two cases, we get the same ODE

$$\left(G'' + \nu c^2 G\right) F'' + 2\left(G''' + \nu c^2 G'\right) F' + \left(G'''' + \nu c^2 G''\right) F = 1, \tag{53}$$

with the general solution given by

$$F(z) = \frac{z^2 + C_1 z + C_2}{G''(z) + \nu c^2 G(z)}, \tag{54}$$

where C_1 and C_2 are real arbitrary constants.

4 Conclusions

The target shapes and the Young modulus respectively can be expressed in terms of the invariants z, I and J of the group action, as

$$I = G(z) \quad \text{and} \quad J = F(z),$$

where the data w appears implicitly in I, and the parameter E in J, respectively. So far the technique of reducing the PDE (5) to an ODE has been applied only in the case of radial functions (see for example [15] and [22]). Indeed, this is the case 3 in Table 1. Moreover, other target shapes might be considered in the polar coordinates, see cases 5, 7, 8 and 9 in Table 1. If we consider the inverse problem on a circular domain, and we require the boundary conditions (3) and (4) to be satisfied, then, for example, we might consider also the target shapes of the form

$$w(r, \theta) = e^{-c\theta} G(r).$$

Scaling transformations (cases 4 and 6 in Table 1) are also interesting symmetry reductions of the model. If in the rotational case the boundary conditions do not cause problems, in the second case, the required boundary conditions are not all fulfilled, as we shall see in that follows. Remark that the first two cases and the last four cases listed in Table 1 might not be of interest at the first sight, but using (43) we can construct other data. However, when looking for examples of data, we have to keep in mind that the function that models the target shape of the car windshield must obey more properties, such as a positive curvature graph at least in the center of the considered domain.

Let us make the remark: if the domain Ω is such that $\partial\Omega = \{(x, y)|\ z(x, y) = k\}$ is the k-level set of the function z (here k being a non-zero constant) and $||\nabla z|| > 0$ on $\bar{\Omega}$, then the normal unit vector to $\partial\Omega$ is given by $n = \frac{\nabla u}{||\nabla u||}|_{\partial\Omega}$. Moreover, if the target shape is given by $w(x, y) = a(x, y)G(z(x, y))$, where $a(x, y)$ is a suitable function according to Table 1, it follows that the boundary conditions (3) and (4) can be reduced to

$$G(k) = 0, \quad G'(k) = 0, \tag{55}$$

so that, *the clamped plate problem is satisfied for those functions $G = G(z)$ with at least two zeros on $\partial\Omega$*. Note that the target shapes given by homogeneous functions, in particular homogeneous polynomials, can be derived from the invariance of the equation with respect to scaling transformations. Even though they fulfil the boundary condition (3) only, these can be considered defined on elliptical domains or on rounded corners frames.

Example 1. Particular target shapes on rounded square domains. Consider the family of target shapes of the form

$$w(x, y) = \left(x^{2n} + y^{2n}\right)^m - 1, \tag{56}$$

defined on the square domain with rounded corners $\Omega = \{(x, y)|\ x^{2n} + y^{2n} \leq 1\}$, here $m \geq 1$ and $n \geq 2$ are natural numbers. The target shapes defined by (56) are invariant with respect the vector field $V_5 + cV_8 + (4-c)V_3$, where $c = 4 - 2mn$. In this case, the equation (5) is elliptical on $\Omega - \{(0,0)\}$ and parabolic in $(0,0)$. Writing the functions (56) as follows

$$w(x,y) = x^{2mn}G(z) - 1, \quad G(z) = \left(1 + z^{2n}\right)^m, \quad z = \frac{y}{x},$$

for $x > 0$ or $x < 0$, it results that the associated Young modulus is given by

$$E(x,y) = x^{4-2mn}F(z), \quad z = \frac{y}{x}.$$

Example 2. Particular target shapes on elliptical domains. Consider the class of data

$$w(x,y) = \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^m - 1, \tag{57}$$

on the elliptical domain $\Omega = \{(x,y)|\ \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1\}$, where $m \geq 1$ is a natural number. The functions (57) are obtained from the invariance of the equation (5) with respect to $V_5 + cV_8 + (4-c)V_3$, where $c = 4 - 2m$. The PDE (5) is elliptical on $\Omega - \{(0,0)\}$ and parabolic in the origin. To reduce the equation to an ODE, we restrict ourselves to the case $x > 0$ or $x < 0$, for which the functions (57) are written as

$$w(x,y) = x^{2m}G(z) - 1, \quad G(z) = \left(\frac{1}{a^2} + \frac{z^2}{b^2}\right)^m, \quad z = \frac{y}{x}.$$

In this case, we are looking for solutions to (5) of the form

$$E(x,y) = x^{4-2m}F(z), \quad z = \frac{y}{x}.$$

Remark. Suppose we have given a target shape w on a domain Ω. In order to verify if w is invariant with respect to one of the symmetry reductions given in Table 1, we can easily check if that is a solution of the first equation of the system (44), with ζ, η and ϕ are given by (41). This is equivalent to the existence of certain non-zero constants k_i such that the resulting relations hold. Afterwards, if this condition is satisfied, by integrating the second PDE in (44) we can determine the form of the parameter. Next step is to write the associated reduced ODE.

The geometrical significance of the nonlinearity occurring between data and parameter in the inverse problem (1) is reflected by the group analysis tools. Investigating special group of transformations connected to this equation, the order of the model can be reduced. The equation will be then written in terms of the invariants of the group actions. Another advantage of this approach is that of relating the direct and the inverse problems through these symmetry reductions. It might be interesting for a future work to link these results to the common approach of the inverse problems theory, especially in finding new regularization methods based on the invariance of data under certain symmetries. For other target shapes defined by functions which are not invariant under the symmetry reductions presented in this paper, the classical theory of linear second order PDE can be applied, but this might be quite difficult due to the form of the discriminant.

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