Weak Solutions for the Mean Curvature Flow of Planar Graphs

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Abstract

The aim of this paper is to develop a new notion of weak solutions for the planar mean curvature flow based on a variational formulation. This concept, generalizing the notion of viscosity solutions, is analyzed in detail in the case of the mean curvature flow of planar graphs.

For such flows, existence, uniqueness, and stability of weak solutions are shown under minimal assumptions on the initial value, which is assumed to be square-integrable only. For initial values of bounded variation, partial regularity of the solution is proved, as well as a result on the long-time asymptotic behavior.

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1 Introduction

The mean curvature flow of curves and surfaces has been a topic of major interest in differential geometry and applied mathematics over the last decades. In the planar case, the mean curvature flow of an evolving curve $\Gamma(t)$ is defined by a normal velocity

$$V_n = -\kappa = -\text{ div } n, \tag{1.1}$$

where n is the unit outward normal.

A lot of progress has been in made in the mathematical analysis of the mean curvature flow after the introduction of the *level set method* by Osher and Sethian [40], who used the implicit representation

$$\Gamma(t) = \{ \ (x,y) \in \mathbb{R}^2 \ \mid \ \phi(x,y,t) = 0 \ \} \tag{1.2}$$

of the evolving curve or surface in order to develop numerical schemes for several geometric motions, including motion with prescribed normal speed as well as the mean curvature flow. The level set formulation of the mean curvature flow is then given by

$$\phi_t = Q(\phi)\ \mathrm{div}\ \left(\frac{\nabla\phi}{Q(\phi)}\right), \qquad Q(\phi) = \sqrt{\phi_x^2 + \phi_y^2} \tag{1.3}$$

in a space time domain $\Omega_T := \Omega \times (0,T)$. Level set methods do not only lead to efficient computational schemes, but are also able to handle topological changes such as merging and splitting of connected components in an automatic way, which is impossible with classical methods based on curve parameterizations. Since the level set equation corresponding to such geometric motions are of Hamilton-Jacobi type, the natural mathematical tool to define generalized solutions is the theory of *viscosity solutions* (cf. [4, 10, 34] for a general overview and [9, 17, 18, 19, 20, 43] for the mean curvature flow).

A main ingredient of the theory of viscosity solutions are comparison principles, and therefore only bounded solutions are obtained with this concept. In some applications, such as imaging or inverse problems, the mean curvature flow or similar motions are applied for non-smooth initial values (cf. e.g. [28, 36]). In these problems, the aim is to compute regularized solutions of ill-posed operator equations of the form $A(u) = f$, with A being a continuous operator from L^2 into some Hilbert space H. If one uses standard regularization methods such as Tikhonov regularization in $L^2(\Omega)$ or $H^1(\Omega)$ (cf. [14, 46]), the discontinuites of the solution are smoothed out, which is an undesirable effect. In order to overcome this difficulty, other penalty terms added to the least-squares functional have been proposed in the last decades, such as the *total variation* of u (cf. [41]) and the Hausdorff-measure of the graph of u, called *curve* (respectively *surface*) *regularization* (cf. [37]). In order to minimize the arising functionals, alogorithms based on time evolution have been proposed leading to the evolution equations

$$u_t = Q(u)\left(\alpha\ \mathrm{div}\ \left(\frac{\nabla u}{Q(u)}\right) - A'(u)^*(A(u) - f)\right),$$

with $\alpha > 0$ being a regularization parameter, $Q(u) = |\nabla u|$ for total variation regularization (cf. [36]) and with $Q(u) = \sqrt{1 + |\nabla u|^2}$ for curve regularization (cf. [28]). Since the main motivation for these evolutions is the reconstruction of discontinuous solutions, this obviously raises the question whether solutions exist in a suitable sense even for initial values in $L^2(\Omega)$ or $BV(\Omega)$.

Moreover, it seems reasonable that a variational approach to the mean-curvature might be extendable to nonlocal terms such as $A'(u)^*(A(u)-f)$ in the above evolution (e.g., if A is some integral operator), though we will not treat this case explicitely in the present paper. The treatment of nonlocal terms is also a motivation for a variational approach to the analysis many moving boundary problems under non-equilibrium conditions such as dendritic growth (cf. e.g. [21, 24]) and level set methods for shape optimization (cf. e.g. [6, 39, 42]). We shall present some preliminary investigations of the non-parametric case in Section

Another motivation for such a concept of weak solutions arises from numerical computation. Recently, semi-implicit finite element methods have been constructed for the mean curvature flow (cf. [8, 11, 12, 30]). These methods use some kind of variational formulation and therefore do not fit very well to the standard concept of viscosity solutions (though it can be shown that they converge to a viscosity solution provided the data are sufficiently regular).

Finally, a concept of weak solutions for the mean curvature flow might help to define a similar concept for *motion by surface diffusion* (cf. [7]), i.e., the motion

$$V_n = \Delta_{\mathcal{S}} \kappa, \tag{1.4}$$

where $\Delta_{\mathcal{S}}$ denotes the surface Laplacian. In the case of a planar graph, motion by surface diffusion is equivalent to the equation (cf. [3])

$$u_t = -\left(\frac{1}{Q(u)}\left(\frac{u_x}{Q(u)}\right)_{xx}\right)_x, \qquad Q(u) = \sqrt{1+u_x^2}, \tag{1.5}$$

for the height u. Since this degenerate parabolic equation is fourth-order, maximum and comparison principles are not available and hence, it is impossible to define viscosity solutions in a similar way to first- and second-order equations. So far, only few theoretical results are available for surface diffusion (cf. [15, 35]), but a general theory is missing.

The aim of this paper is to realize a first step towards a resonable definition of weak solutions for the mean curvature flow based on a variational formulation. We start with the most simple case, namely the planar mean curvature flow of graphs, where the level set function can be written as $\phi(x,y,t) = u(x,t) - y$, resulting in the evolution equation

$$u_t = \sqrt{1+u_x^2}\left(\frac{u_x}{\sqrt{1+u_x^2}}\right)_x = \frac{u_{xx}}{1+u_x^2}, \tag{1.6}$$

in Ω_T with a domain $\Omega \subset \mathbb{R}^1$. For the construction of weak solutions we shall rewrite (1.6) in divergence form and then follow closely the approach of Lichnewsky and Temam [31] for the total variation flow of graphs (cf. also [22] for the general total variation flow). The result is a concept of weak solutions coinciding with the viscosity solution if the latter is defined.

The partial differential equation (1.6) is supplemented by the initialcondition

$$u(0) = u_0. \tag{1.7}$$

and suitable boundary conditions. For the sake of simplicity we restrict our attention to homogeneous Neumann boundary conditions, i.e.

$$u_x = 0 \qquad \text{on } \partial\Omega \times (0,T), \tag{1.8}$$

but we want to mention that a similar analysis is possible for Dirichlet boundary conditions, too.

The paper is organized as follows: in Section 2 we derive a variational formulation and introduce our new concept of weak solutions. Section 3 is devoted to the analysis of a viscous approximation, whose properties are essential for the analysis of the limit case, i.e., the planar mean curvature flow of graphs. Section 4 contains all main results of this paper, we prove existence and uniqueness of weak solutions as well as stability, partial regularity, and some long-time asymptotics. Finally, we discuss some extensions in Section 5.

Throughout the whole paper we use standard notation for partial derivatives (expressed by subscripts) and function spaces. In particular we use standard Lebesgue and Sobolev spaces (cf. [1, 32] for detailed definitions), the space of functions of bounded variation $BV(\Omega)$ (cf. [16, 25]), and vector-valued function spaces (cf. [33]).

2 Weak Solutions

In the following we derive a concept of weak solutions for the mean curvature flow of planar graphs (1.6) subject to the initial condition (1.7) and the boundary condition (1.8).

First assume that u is sufficiently regular, so that we can rewrite (1.6) as

$$u_t = \frac{u_{xx}}{1+u_x^2} = \left(F(u_x)\right)_x, \tag{2.1}$$

with $F(p) = \arctan p$. Consequently, for v sufficiently regular and $s \in [0,T]$ arbitrary, we obtain after integration by parts the identity

$$\int_0^s \int_\Omega (u_t(v-u) + F(u_x)(v_x - u_x)) \; dx \; ds = 0.$$

For the first term in the integral we obtain that

$$\begin{aligned} \int_0^s \int_\Omega (u_t(v-u)) \; dx \; dt &= \int_0^s \int_\Omega ((u-v)_t(v-u) + v_t(v-u)) \; dx \; dt \\ &= \frac{1}{2}\int_\Omega \left((u_0 - v(0))^2 - (u(s)-v(s))^2\right) \; dx + \\ &\qquad \int_0^s \int_\Omega v_t(v-u) \; dx \; dt. \end{aligned}$$

Due to the convexity of the function G defined by

$$G'(p) = F(p) = \arctan p, \qquad G(0) = 0, \tag{2.2}$$

we obtain that

$$F(s)(t-s) \leq G(t) - G(s), \qquad \forall \; s,t \in \mathbb{R}.$$

Hence, with the notation

$$J(v) := \int_\Omega G(v_x) \; dx, \tag{2.3}$$

we obtain the variational inequality

$$\begin{aligned} &\frac{1}{2}\int_\Omega \left((u(s)-v(s))^2 - (u_0 - v(0))^2\right) \; dx \leq \\ &\qquad \int_0^s (J(v) - J(u)) \; dt + \int_0^s \int_\Omega v_t(v-u) \; dx \; dt, \end{aligned} \tag{2.4}$$

for all $s \in [0,T]$ and all sufficiently regular test functions v.

We are going to seek a weak solution in the space

$$\mathcal{U} := L^1(0,T;BV(\Omega)) \cap C(0,T;L^2(\Omega)), \tag{2.5}$$

equipped with the usual norm being the sum of the individual norms (cf. [33]). The test functions v shall be chosen from the space

$$\mathcal{V} := L^1(0,T;BV(\Omega)) \cap H^1(0,T;L^2(\Omega)), \tag{2.6}$$

again equipped with the usual norm. Note that due to the continuous embedding $H^1(0,T;L^2(\Omega)) \hookrightarrow C(0,T;L^2(\Omega))$ (cf. [33]), the space $\mathcal{U}$ is larger than $\mathcal{V}$ and the initial value of $v \in \mathcal{V}$ is well-defined. This setting now yields the following notion of a weak solution:

Definition 2.1. A function $u \in \mathcal{U}$ is called a weak solution of the mean curvature flow of planar graphs (1.6), if the inequality (2.4) holds for all $v \in \mathcal{V}$.

Note that in the above definition of a weak solution it suffices to have an initial value in $L^2(\Omega)$, we shall see below that we obtain existence and uniqueness of a solution even under this almost minimal regularity assumption. Moreover, we shall see below that the regularity of the solution increases with the regularity of the initial value. E.g., if $u_0 \in BV(\Omega)$, we obtain that $u \in L^\infty(0,T;BV(\Omega)$.

Another important observation concerns the functions F and G introduced above. Since in the level set perspective we have $\phi_x = u_x$ and $\phi_y = -1$, we obtain that the angle between the x-axis and the front (i.e., the zero level set of ϕ) is given by

$$\theta = \arctan\left(\frac{\phi_x}{\phi_y}\right) = -\arctan u_x = -F(u_x).$$

Hence, the value $F(u_x)$ has a clear geometric interpretation, in terms of the angle θ defined above, G is given by

$$G(s) = \tilde{G}(\theta) = \theta\ \tan\theta + \ln(\cos\theta). \tag{2.7}$$

Note that the use of the angle θ corresponds very well to several presentations of evolving phase boundaries in materials science (cf. e.g. [26]), which are given partly in terms of the angle θ.

Our final aim in this section is to provide an estimate between the seminorm in $BV(\Omega)$ and the functional J:

Proposition 2.2. *Let $w \in BV(\Omega)$, then the estimate*

$$\frac{2}{\pi}J(w) \leq |w|_{BV(\Omega)} \leq \frac{4}{\pi}J(w) + |\Omega|. \tag{2.8}$$

holds for the functional J defined as the lower semicontinuous extension (with respect to L^1-convergence) of (2.3).

Proof. It suffices to verify the identity for $w \in C^1(\Omega)$, then the estimate can be extended to $BV(\Omega)$ by lower semicontinuity of the seminorm and the convex functional J. For $w \in C^1(\Omega)$, we have that

$$J(w) = \int_\Omega G(w_x)\ dx \leq \sup_{s\in\mathbb{R}} |G'(s)| \int_\Omega |w_x|\ dx,$$

and since $|G'(s)| = |\arctan s| \leq \frac{\pi}{2}$ we obtain the first estimate in (2.8).

In order to obtain the estimate on the right-hand side we use the convexity of G to deduce

$$\begin{aligned}
J(w) &= \int_\Omega (G(w_x) - G(0))\ dx \\
&\geq \int_\Omega G'(w_x)(w_x - 0)\ dx = \int_\Omega |\arctan w_x||w_x|\ dx \\
&\geq \frac{\pi}{4} \int_{\{4|\arctan w_x| \geq \pi\}} |w_x|\ dx \\
&= \frac{\pi}{4} \int_\Omega |w_x|\ dx - \frac{\pi}{4} \int_{\{4|\arctan w_x| \leq \pi\}} |w_x|\ dx \\
&\geq \frac{\pi}{4} |w|_{BV(\Omega)} - \frac{\pi}{4} \int_\Omega (\tan \frac{\pi}{4})\ dx,
\end{aligned}$$

which implies the assertion. $\square$

3 Viscous Approximation

In this section we investigate a viscous approximation to (1.6), i.e., we consider the strongly parabolic equation

$$u_t = \left(\epsilon + \frac{1}{1 + u_x^2}\right) u_{xx}, \tag{3.1}$$

for positive real ϵ subject to the homogeneous Neumann boundary condition (1.8) (called Neumann-problem for (3.1) in the following). The corresponding initial values $u_0^\epsilon \in C^\infty(\Omega)$ are chosen such that $u_0^\epsilon \to u$ in $L^2(\Omega)$ as $\epsilon \to 0$. From the standard theory of parabolic equations (cf. [23, 29]) we obtain the following existence and uniqueness result concerning a classical solution:

Theorem 3.1. *Let $\epsilon > 0$ and let $u_0^\epsilon \in C^\infty(\Omega)$. Then the Neumann-problem for (3.1) has a unique classical solution $u^\epsilon \in C^\infty(\Omega_T)$. under the initial condition $u^\epsilon(0) = u_0^\epsilon$.*

In the way we derived the weak formulation for (1.6) above, we can now derive a notion of weak solutions for (3.1), which also includes classical solutions. A weak solution in this case is defined as a function $u \in \mathcal{U} \cap$

$L^2(0,T;H^1(\Omega))$ satisfying

$$\frac{1}{2}\int_\Omega \left((u(s)-v(s))^2-(u_0-v(0))^2\right)\,dx \le \int_0^s (J^\epsilon(v)-J^\epsilon(u))\,dt+\int_0^s\int_\Omega v_t(u-u)\,dx\,dt, \quad (3.2)$$

for all $s\in[0,T]$ and all test functions $v\in\mathcal{V}\cap L^2(0,T;H^1(\Omega))$, where

$$J^\epsilon(u_x):=\int_\Omega \left(G(u_x)+\frac{\epsilon}{2}|u_x|^2\right)\,dx.$$

By choosing the special test function $v\equiv 0$ and using the nonnegativity of $J^\epsilon(u)-J(u)$, we immediately obtain an a-priori estimate independent of ϵ. We summarize these results in the following lemma:

Lemma 3.2. *Let $\epsilon>0$ and let $u_0^\epsilon\in C^\infty(\Omega)$. Then the unique classical solution u^ϵ of the Neumann-problem for (3.1) with initial value u_0^ϵ satisfies (3.2) for all $s\in[0,T]$ and all $v\in\mathcal{V}\cap L^2(0,T;H^1(\Omega))$. Moreover, the estimate*

$$\begin{aligned}\|u^\epsilon(s)\|^2_{L^2(\Omega)}+\int_0^s\left(2J(u^\epsilon)+\epsilon\|u_x^\epsilon\|^2_{L^2(\Omega)}\right)\,dt &\le \|u_0^\epsilon\|^2_{L^2(\Omega)}\\ &\le \|u_0\|^2_{L^2(\Omega)}+R(\epsilon) \quad (3.3)\end{aligned}$$

holds for all $s\in[0,T]$, where $R(\epsilon)$ is a constant independent of s, which tends to zero as $\epsilon\to 0$.

Lemma 3.2 will be the main ingredient of our existence proof for weak solutions of (1.6). From the a-priori estimate (3.3) we immediately obtain the existence of a convergent subsequence in the weak-star topology of $L^\infty(0,T;L^2(\Omega))$. Moreover, we obtain the uniform boundedness of the regularization term $\sqrt{\epsilon}\|u_x^\epsilon\|_{L^2(\Omega_T)}$. From the uniform bound on $J(u^\epsilon)$ we can deduce uniform boundedness in $L^1(0,T;BV(\Omega)$ due to Proposition 2.2, which will be used with the weak lower semicontinuity of the functional J to pass to the limit in the variational inequality (3.2). One observes that by testing (3.1) by an arbitrary function $v\in L^2(0,T;H^1(\Omega))$ we obtain the estimate

$$\begin{aligned}\|u_t^\epsilon\|_{L^2(0,T;H^{-1}(\Omega))} &= \sup_{v\in L^2(0,T;H^1(\Omega)),\|v\|=1}\int_0^T\int_\Omega u_t^\epsilon v\,dx\,dt\\ &= \int_0^T\int_\Omega (F(u_x^\epsilon)v_x+\epsilon u_x^\epsilon v_x)\,dx\,dt\\ &\le \frac{\pi}{2}\sqrt{T|\Omega|}+\epsilon\|u_x^\epsilon\|_{L^2(\Omega_T)}, \quad (3.4)\end{aligned}$$

which implies a uniform bound for $\|u_t^\epsilon\|_{L^2(0,T;H^{-1}(\Omega))}$ due to (3.3). Moreover, by testing (1.6) with tu_t^ϵ and simple manipulations we obtain that

$$t\int_\Omega (u_t^\epsilon)^2\,dx + \frac{d}{dt}(tJ(u^\epsilon)) + \frac{\epsilon}{2}\frac{d}{dt}\int_\Omega t(u_x^\epsilon)^2\,dx = J(u^\epsilon) + \frac{\epsilon}{2}\int_\Omega (u_x^\epsilon)^2\,dx$$

and together with (3.3) and (3.4)

$$\int_0^s\int_\Omega (\sqrt{t}\,u_t^\epsilon)^2\,dx\,dt + sJ(u^\epsilon(s)) + \frac{\epsilon}{2}\int_\Omega (\sqrt{s}\,u_x^\epsilon(s))^2\,dx \le C \tag{3.5}$$

for a constant C independent of ϵ.

Finally, we shall prove some further energy estimates useful for proving further regularity of the weak solution. The first one is the natural energy estimate (cf. [2, 12, 44]) associated with the mean curvature flow concerning the energy functional

$$\mathcal{E}(t) = \frac{1}{2}\int_0^t\int_{\Gamma(s)} V_n^2\,d\sigma\,ds + \int_{\Gamma(t)} 1\,d\sigma.$$

Lemma 3.3. *Let $u^\epsilon \in C^\infty(\Omega_T)$ be the unique solution of the Neumann-problem for (3.1). Then u^ϵ satisfies*

$$\int_0^s\int_\Omega \frac{(u_t^\epsilon)^2}{\sqrt{1+(u_x^\epsilon)^2}}\,dx\,dt + \int_\Omega\left(\sqrt{1+u_x^\epsilon(s)^2} + \epsilon H(u_x^\epsilon(s))\right)\,dx \le$$
$$\int_\Omega\left(\sqrt{1+((u_0^\epsilon)_x)^2} + \epsilon H((u_0^\epsilon)_x)\right)\,dx, \tag{3.6}$$

where H is the convex, nonnegative function defined by $H'(s) = \arcsin s$ and $H(0) = 0$.

Proof. We divide (3.1) by $\sqrt{1+(u_x^\epsilon)^2}$, multiply by u_t^ϵ and integrate with respect to x over Ω and with respect to t from 0 to s. Using integration by parts with respect to x we obtain that

$$\int_0^s\int_\Omega\left(\frac{(u_t^\epsilon)^2}{\sqrt{1+(u_x^\epsilon)^2}} + \frac{u_x^\epsilon u_{xt}^\epsilon}{\sqrt{1+(u_x^\epsilon)^2}} + \epsilon\,\operatorname{arcsinh}(u_x^\epsilon)u_{xt}^\epsilon\right)\,dx\,dt = 0,$$

and hence, we obtain (3.6) by integration with respect to x. □

The second energy estimate concerns the evolution of the functional J defined above:

Lemma 3.4. *Let u^ϵ be the unique solution of the Neumann-problem of* (3.1). *Then u^ϵ satisfies*

$$\int_0^s \int_\Omega (u_t^\epsilon)^2 \ dx \ dt + \int_\Omega \left(G(u_x^\epsilon(s)) + \frac{\epsilon}{2}(u_x^\epsilon(s))^2 \right) \ dx \leq \int_\Omega \left(G((u_0^\epsilon)_x) + \frac{\epsilon}{2}((u_0^\epsilon)_x)^2 \right) \ dx, \tag{3.7}$$

and

$$\int_0^s \int_\Omega \left((F(u_x^\epsilon)_x)^2 + \epsilon \frac{(u_{xx}^\epsilon)^2}{1+(u_x^\epsilon)^2} \right) \ dx \ dt \leq J(u_0^\epsilon) - J(u^\epsilon(s)) \tag{3.8}$$

for all $s \in [0,T]$.

Proof. The estimate (3.7) can be obtained by multiplying (3.1) with u_t^ϵ and integrating with respect to x over Ω and with respect to t from 0 to s.

The second estimate (3.8) is obtained by multiplying (3.1) with $F(u_x^\epsilon)_x = \frac{u_{xx}^\epsilon}{1+(u_x^\epsilon)^2}$ and integrating with respect to x over Ω and with respect to t from 0 to s. ☐

4 Analysis of Weak Solutions

Now we turn our attention to the existence and uniqueness of weak solutions for (1.6). Moreover, we shall provide a stability estimate for weak solutions of (1.6) with different initial values. The following theorem is the central result in a theory of weak solutions for the planar mean curvature flow. Note that its proof follows closely the proof of Theorem 1.1 in [22].

Theorem 4.1. *For each initial value u_0 in $L^2(\Omega)$ there exists a unique weak solution $u \in \mathcal{U}$ of* (1.6). *If u and $\tilde{u}$ denote the weak solutions for the initial values u_0 and $\tilde{u}_0$ respectively, then the estimate*

$$\int_\Omega (u(s) - \tilde{u}(s))^2 \ dx \leq \int_\Omega (u_0 - \tilde{u}_0)^2 \ dx \tag{4.1}$$

holds.

Proof. In order to prove the existence of a weak solution, consider the viscous approximation (3.1). Using Proposition 2.2 and the a-priori estimates (3.3) - (3.5), we can apply analogous reasoning as in [22] to deduce the existence of a function $u \in L^1(0,T;BV(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$ such that there exists a subsequence u^{ϵ_k} with

$$\begin{aligned} u^{\epsilon_k} \to u \quad & \text{weakly-* in } L^\infty(0,T;L^2(\Omega)) \\ & \text{weakly in } L^2(\Omega_T) \qquad (4.2) \\ & \text{strongly in } L^1(0,T;L^p(\Omega)), p \in [1,\infty) \end{aligned}$$

$$\sqrt{t}u^{\epsilon_k}(t) \to \sqrt{t}u(t) \quad \text{strongly in } L^p(\Omega), p \in [1,\infty) \text{ for a.e. } t \in (0,T] \qquad (4.3)$$

$$u_t^{\epsilon_k} \to u_t \quad \text{weakly in } L^2(0,T;H^{-1}(\Omega) \qquad (4.4)$$

$$\sqrt{t}u_t^{\epsilon_k} \to \sqrt{t}u_t \quad \text{weakly in } L^2(\Omega_T). \qquad (4.5)$$

The remaining steps to show existence are exactly the same as in [22], where we use the lower semicontinuity of the convex functional J in $L^1(0,T;L^p(\Omega))$ to pass to the limit in the weak formulation.

Since the uniqueness of the weak solution obviously follows from (4.1), it suffices to prove this stability estimate. In order to prove this estimate, we use special test functions in the variational inequalities defining u and $\tilde{u}$. In particular we define w and w^δ by the relations

$$w = \frac{u+\tilde{u}}{2}, \quad w(0) = w^\delta(0) = \frac{u_0+\tilde{u}_0}{2},$$

and

$$\delta w_t^\delta + w_t^\delta = w \qquad \text{in } (0,T).$$

One can show that (cf. [22, p.11])

$$w^\delta(s) \to w(s) \text{ strictly in } L^2(\Omega), \quad \forall\, s \in [0,T]$$

and

$$w^\delta \to w \text{ strictly in } L^1(0,T;BV(\Omega))$$

as $\delta \to 0$.

By choosing $v = w^\delta$ in (2.4) with initial value u_0 and $\tilde{u}_0$, respectively, we obtain that

$$\frac{1}{2}\int_\Omega \left((u(s)-w^\delta(s))^2 - \frac{1}{4}(u_0-\tilde{u}_0)^2 \right) dx \le \int_0^s (J(w^\delta) - J(u))\, dt + \frac{1}{\delta}\int_0^s \int_\Omega (u-w^\delta)(\tilde{u}-u)\, dx\, dt,$$

and

$$\frac{1}{2}\int_\Omega \left((\tilde{u}(s)-w^\delta(s))^2 - \frac{1}{4}(u_0-\tilde{u}_0)^2 \right) dx \le \int_0^s (J(w^\delta) - J(\tilde{u}))\, dt + \frac{1}{\delta}\int_0^s \int_\Omega (\tilde{u}-w^\delta)(u-\tilde{u})\, dx\, dt.$$

Adding these inequalities and using the convexity of J we deduce

$$\begin{aligned}\frac{1}{4}\int_\Omega &\left((u(s)-\tilde{u}(s))^2-(u_0-\tilde{u}_0)^2\right)\,dx \leq \\ &\frac{1}{4}\int_\Omega \Big((u(s)-\tilde{u}(s))^2-2(u(s)-w^\delta(s))^2-2(\tilde{u}(s)-w^\delta(s))^2\Big)\,dx \\ &\quad + \quad 2\int_0^s (J(w^\delta)-J(w))\,dt.\end{aligned}$$

Due to the above convergence properties of w^δ and the lower semicontinuity of J, the terms on the right-hand side tend to zero, and thus, we obtain (2.4) in the limit. □

We want to mention that stability results like (4.1) have been obtained in the Supremum-norm or in Hölder norms for viscosity solutions (cf. [27]), to our knowledge this is the first result in the L^2-norm.

In the above result we have used the convergence of solutions of the viscous approximation (3.1) as $\epsilon \to 0$ in order to prove existence. Since the solutions of the viscous approximation always converge to the unique viscosity solution, if the latter exists, we immediately obtain the following coherence result for our notion of weak solutions:

Corollary 4.2. *Let the initial value u_0 be such that a unique viscosity solution of* (1.6) *exists. Then the weak solution and the viscosity solution are equal.*

4.1 Regularity

The weakest type of regularity we are interested in is of bounded variation type. In the following we show that this type of regularity is maintained during the main-curvature flow and obtain some Lyapunov-functionals. An obvious one is J, another is the lower semicontinuous extension of

$$\Lambda(w) := \int_\Omega \sqrt{1+w_x^2}\,dx \tag{4.6}$$

to $BV(\Omega)$, which we shall denote again by Λ.

Theorem 4.3. *Let $u_0 \in L^2(\Omega)\cap BV(\Omega)$. Then,*

- *$u(s) \in BV(\Omega)$ for all $s \in [0,T]$ and the functional $s \mapsto \Lambda(u(s))$ is monotonically nonincreasing.*

- *$J(u(s)) < \infty$ for all $s \in [0,T]$ and the functional $s \mapsto J(u(s))$ is monotonically non-increasing.*

Moreover, $u_t \in L^2(\Omega_T)$ and $\theta = -F(u_x) \in L^2(0,T;H^1(\Omega))$.

Proof. We can find approximations u_0^ϵ of the initial value u_0 such that

$$\epsilon \int_\Omega H((u_0^\epsilon)_x)\ dx \to 0 \qquad \text{and} \qquad \epsilon \int_\Omega ((u_0^\epsilon)_x)^2\ dx \to 0,$$

as $\epsilon \to 0$, respectively. These identities can be used to pass to the limit in (3.6), (3.7), and (3.8), which imply the assertions. □

As a consequence of this regularity result we obtain by the continuous embedding $H^1(\Omega) \to C(\Omega)$ that $\theta \in L^2(0,T;C(\Omega))$ if the total variation of the initial value u_0 is bounded.

4.2 Long-Time Asymptotics

Finally, we derive some results on the long-time asymptotics of weak solutions. The asymptotic behavior obtained for the mean curvature flow (namely convergence to a flat curve) is very well known for viscosity solutions, we generalize the result to weak solutions in the class $L^\infty(0,T;BV(\Omega))$.

The natural tool for analyzing long-time asymptotics of weak solutions are the decay properties of the functional $\psi : s \mapsto J(u(s))$. From the above analysis, we may conclude that $\psi \in L^1([0,T])$ for any $T \in \mathbb{R}$ (uniformly bounded with respect to T) and that ψ is monotonically non-increasing. Hence, we obtain

$$\psi(s) \leq \frac{1}{s} \int_0^s \psi(t)\ dt \leq \frac{C}{s},$$

where C is a constant dependent on u_0 only (in particular independent of s). Thus, $\psi(s)$ decays to zero at least of order $\frac{1}{s}$. This decay property leads to the following result on the long-time asymptotics:

Theorem 4.4. *Let $u_0 \in BV(\Omega)$ and let $u \in \mathcal{U}$ be the unique weak solution defined by* (2.4)*. Then*

$$u(s) \to \frac{1}{\Omega} \int_\Omega u_0\ dx \qquad \textit{strongly in } L^p(\Omega), p \in [1,\infty)$$

as $s \to \infty$.

Proof. If we use the test function $v \equiv 0$ in the weak form (2.4), then we obtain a uniform bound for the L^2-norm of $u(s)$, which is independent of s. Moreover, the decay of functional $\psi(s) = J(u(s))$ together with (2.2) implies the uniform boundedness of $|u(s)|_{BV(\Omega)}$ with respect to s. Thus, $u(s)$ is uniformly bounded in $BV(\Omega)$ and due to compact embedding we may conclude that each sequence $s_k \to 0$ has a subsequence (again denoted by s_k) such that $u(s_k) \to u^*$ for some $u^* \in L^2(\Omega)$. Due to the lower semicontinuity of J, we may conclude that $J(u^*) = 0$ and hence, u^* is constant. Because $\int_\Omega u_t(s)\ dx = 0$, we obtain that

$$u^* = \frac{1}{\Omega} \int_\Omega u_0\ dx.$$

The uniqueness of the limit finally implies the convergence $u(s) \to u^*$ by standard arguments. □

5 Extensions and Open Problems

Finally we discuss some extensions of the mean curvature flow, in particular we provide an analysis of weak solutions for anisotropic mean curvature flows of planar graphs. In the case of the mean curvature flow with an additional forcing term as well as for the planar mean curvature flow of level sets we provide some preliminary ideas and raise open questions.

5.1 Anisotropic Mean Curvature Flow

The anisotropic mean curvature flow of a planar graph can be written as (cf. [26])

$$V_n = -g(\theta)\kappa = -g(\theta)\ \text{div}\ n, \tag{5.1}$$

where θ denotes again the angle between the x-axis and the normal. Thus, in the graph case, anisotropic motion by mean curvature can be written as

$$u_t = g(\arctan u_x)\frac{u_{xx}}{1+u_x^2} = F_a(\arctan u_x)_x, \tag{5.2}$$

where the function F_a is defined by $F_a(0) = 0$, $F_a'(p) = g(p)$. A weak solution can be defined in this case by (2.4) with the functional J being the lower semicontinuous extension of

$$J(w) = \int_\Omega G_a(w_x)\ dx, \qquad G_a(p) := \int_0^p F_a(\arctan q)\ dq \tag{5.3}$$

to $BV(\Omega)$. For nonnegative g, the functional J is still convex, one of the main properties needed for an existence proof. The second key ingredient for such a proof is again an estimate between the functional J and the BV-seminorm, which can be obtained if g is bounded from below and above:

Proposition 5.1. *Let g be defined in $[-\frac{\pi}{2}, \frac{\pi}{2}]$ and let*

$$0 < \gamma \leq g(\theta) \leq \Gamma, \qquad \text{for all } \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}], \tag{5.4}$$

where γ and Γ are positive real constants. Then the estimate

$$\frac{2}{\pi\Gamma} J(w) \leq |w|_{BV(\Omega)} \leq \frac{4}{\pi\gamma} J(w) + |\Omega|. \tag{5.5}$$

holds for all $w \in BV(\Omega)$.

Proof. As in the proof of Proposition (2.2), we may restrict our attention to functions $w \in C^1(\Omega)$. For such functions we may estimate

$$J(w) = \int_\Omega G_a(w_x)\ dx \leq \sup_{s\in\mathbb{R}} |G_a'(s)| \int_\Omega |w_x|\ dx,$$

and since

$$|G'(s)| = |F_a(\arctan s)| = |\int_0^{\arctan s} g(\theta)\ d\theta| \leq \frac{\pi}{2}\Gamma$$

we obtain the first estimate in (5.5).

In order to obtain the estimate on the right-hand side we use the convexity of G to deduce

$$\begin{aligned} J(w) &= \int_\Omega (G_a(w_x) - G(0))\ dx \\ &\geq \int_\Omega |F_a(\arctan w_x)||w_x|\ dx \\ &\geq \frac{\pi\gamma}{4} \int_{\{4|\arctan w_x| \geq \pi\}} |w_x|\ dx \\ &\geq \frac{\pi\gamma}{4} |w|_{BV(\Omega)} - \frac{\pi\gamma}{4} \int_\Omega (\tan\frac{\pi}{4})\ dx, \end{aligned}$$

which implies the assertion. □

All steps of the proof of Theorem 4.1 can now be carried out just as above and therefore we immediately obtain that the assertions of Theorem 4.1 and Corollary 4.2 hold for the anisotropic mean curvature flow of planar graphs (5.2) if g satisfies (5.4).

5.2 Mean Curvature Flow with Forcing

The mean curvature flow with forcing is a geometric motion of the form

$$V_n = -\kappa + f = -\operatorname{div} n + f, \tag{5.6}$$

with f a given function of location and time, and possibly of the normal n. For a planar graph, an additional forcing term means that (1.6) has to be modified to

$$u_t = \sqrt{1+u_x^2}\left(\left(\frac{u_x}{\sqrt{1+u_x^2}}\right)_x + f\right) = \frac{u_{xx}}{1+u_x^2} + f\sqrt{1+u_x^2}, \tag{5.7}$$

in Ω_T. In this case, f can be interpreted as a function of time, location, and of the angle $-\theta = \arctan u_x$.

If we apply similar reasoning as in the derivation of (2.4), then we obtain the variational inequality

$$\begin{aligned}\frac{1}{2}\int_\Omega \left((u(s)-v(s))^2 - (u_0 - v(0))^2\right)\, dx + \int_0^s (J(u) - J(v))\, dt \leq \\ \int_0^s \int_\Omega v_t(v-u)\, dx\, dt + \int_0^s \int_\Omega f\sqrt{1+v_x^2}(v-u)\, dx\, dt.\end{aligned}$$

The last term on the right-hand side arising due to the forcing term creates a fundamental difficulty, since even in the simplest case of $f \equiv 1$ it can not be well-defined for $u \in \mathcal{U}$ and $v \in \mathcal{V}$. In order to obtain well-definedness of an integral functional of the form

$$\mathcal{I} : w \mapsto \int_\Omega f\ \sqrt{1+w_x^2}\ \psi\ dx \tag{5.8}$$

for $w \in BV(\Omega)$ one needs at least that the test function satisfies $\psi \in C(\Omega)$. However, in the setup of (5.8) we need $\psi \in BV(\Omega)$, which does not imply continuity. A possible way to circumvent this problem is to find a convex functional, whose directional derivative in direction ψ is given by (5.8) - the same strategy we applied by introducing the functional J in the weak formulation (2.4). From standard nonlinear functional analysis one knows that $\mathcal{I}$ is the derivative of a functional, if and only if the derivative

$$\mathcal{I}'(w)\varphi = \int_\Omega f\ \frac{\varphi_x w_x}{\sqrt{1+w_x^2}}\ \psi\ dx \tag{5.9}$$

defines a symmetric bilinear form (with respect to φ and ψ) for each w. In the particular case of a linear function $w(x) = x$, this bilinear form is given by

$$B(\varphi, \psi) = \int_\Omega f \, \frac{\varphi_x}{\sqrt{2}} \, \psi \; dx,$$

which is not symmetric for f different from zero.

The above argumentation shows that our approach to weak solutions cannot be carried over in a straight-forward way to the case of an additional forcing term. However, the fact that some of the energy estimates like (3.7) can be obtained in a similar in the presence of a forcing term raises the hope to obtain at least weak solutions at least for initial values $u_0 \in BV(\Omega)$. The definition and analysis of weak solutions for the mean curvature flow with forcing remains open as a challenging problem for future research.

5.3 Non-Parametric Planar Mean Curvature Flow

In the following we turn our attention to the planar mean curvature flow of level sets (1.3). The natural extension of the graph case is to define the angle θ by the polar decomposition of the gradient

$$\nabla \phi = (\phi_x, \phi_y) = Q(\phi)(\sin\theta, \cos\theta), \tag{5.10}$$

which allows to rewrite (1.3) as

$$\phi_t = Q(\phi)((\cos\theta) - (\sin\theta)\theta_y) = \theta_x \phi_y - \theta_y \phi_x. \tag{5.11}$$

Multiplication of (5.11) with an arbitrary test function ψ and integration by parts yields the identity

$$\int_0^T \int_\Omega (\psi \phi_t + \theta(\psi_x \phi_y - \psi_y \phi_x)) \; dx \; dt = 0, \tag{5.12}$$

i.e., apart from the term including the time derivative, we obtain a weak form depending only on $\nabla\phi$ and $\nabla\psi$. Unfortunately, the weak form

$$B(\nabla\phi, \nabla\psi) := \int_0^T \int_\Omega \theta(\psi_x \phi_y - \psi_y \phi_x) \; dx \; dt$$

does not correspond to the derivative of a functional depending on $\nabla\phi$, which can be seen by inspecting the compatibility conditions given by

$$\frac{\partial}{\partial \phi_y}(\theta \phi_y) = -\frac{\partial}{\partial \phi_x}(\theta \phi_x).$$

A simple calculation shows that this condition is equivalent to $\theta = 0$ and hence, it is violated for almost all values of θ and of $\nabla\phi$, respectively.

Nonetheless, the use of the weak form B for the derivation of a concept of weak solutions for the planar mean curvature flow seems to be a promising approach to be investigated in future research.

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