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On Fast Domain Decomposition Solving Procedures for hp-Discretizations of 3D Elliptic Problems[1]

V. Korneev

St.-Petersburg State Technical University
Polytehnicheskaia 29, 195251, St.-Petersburg, Russia
E-mail: Vadim.Korneev@pobox.spbu.ru

U. Langer

Johannes Kepler University
Altenberger Str. 69, A-4040, Linz, Austria
E-mail: ulanger@numa.uni-linz.ac.at

L.S. Xanthis

Centre for Techno-Maths & Sci.Comput. Lab., University of Westminster
HA1 3TP, London, UK
E-mail: L.S.Xanthis@westminster.ac.uk

Abstract — We present a *DD* (Domain Decomposition) preconditioner for hierarchical hp discretizations of 3-d, 2-nd order, elliptic equations. Its arithmetical complexity behaves almost optimal in p. We adapt the wire basket substructuring technique to the hierarchical hp discretization, obtain a fast preconditioner-solver for faces by interpolation and show that for prolongations from faces a secondary iterative process may be efficiently used. The fast solver for local Dirichlet problems on subdomains of decomposition is based on our earlier derived finite-difference like preconditioner for the internal stiffness matrices of p-finite elements and fast solution procedures for systems with this preconditioner, which appeared recently. The relative condition number, provided by the *DD* preconditioner under consideration is $\mathcal{O}((1+\log p)^{3.5})$ and its total arithmetic cost is $\mathcal{O}((1+\log p)^{1.75}(p^3\mathcal{R}+p\mathcal{R}^2))$, where $\mathcal{R}$ is the number of finite elements. The term $p\mathcal{R}^2$ is due to the solver for the wire basket subsystem. We outline, how the cost of this component may be reduced to $\mathcal{O}(p\mathcal{R})$. The presented *DD* algorithms are highly parallelizable.

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1. Introduction

Suppose, $\mathbf{Av} = \mathbf{f}$ is a s.p.d. (symmetric positive definite) system of linear algebraic equations with a $n\times n$ matrix $\mathbf{A}$, where the dimension n of the system matrix is viewed as a parameter.

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In this paper, an algorithm for solving this system is termed *fast* or (asymptotically) *almost optimal*, if it requires $\mathcal{O}(n)$ or $\mathcal{O}((1+\log n)^k n)$ arithmetic operations with some fixed and not too large k. If $k=0$, then it is termed *optimal*. Often the solving procedure is the PCG (Preconditioned Conjugate Gradient) method with some preconditioner $\boldsymbol{\mathcal{A}}$. By the total arithmetical cost or complexity of a preconditioner, we understand the arithmetical cost for solving the system by the PCG method, except multiplications by the matrix $\mathbf{A}$ which are unavoidable at each PCG iterative step. That means, in distinction to some other papers, here the efficiency of a preconditioner is measured not only by the relative condition, but also by the cost of solving the system with the preconditioner for the matrix. Accordingly, the preconditioner is called fast, almost optimal or optimal, if it is such in total arithmetical complexity. When n depends on several parameters, similar terms are used with respect to any of these parameters.

This paper deals with systems of algebraic equations, resulting from hierarchical *hp* discretizations of 2-nd order elliptic equations. Specifically, our target is discretizations generated by means of the square or cubic reference elements with the coordinate functions defined by the tensor products of the integrated Legendre's polynomials. In recent years, a considerable progress has been achieved in developing efficient *DD* (Domain Decomposition) solvers for such systems. For the discretizations of 2-d elliptic problems, all basic components of *DD* solvers have been reasonably well studied and made indeed efficient. Due to [16], [1], [22], [12], [23] and [20] a number of Schur complement preconditioners has been designed, optimal or almost optimal in condition and requiring less, sometimes considerably, than $\mathcal{O}(p^2\mathcal{R})$ arithmetic operations, see, *e.g.,* [23]. Here $\mathcal{R}$ is the number of finite elements of the discretization. In derivation of some of them, an essential role was played by the fact that the reference element internal stiffness matrix, generated by the internal coordinate functions, has an optimal in total computational complexity finite-difference-like preconditioner, derived in [16], [22]. For the systems with this preconditioner, different types of almost optimal solvers were justified in the last 3 years, requiring sometimes subsidiary slightly modified preconditioners, better adapted to solvers of particular types. *DD* like solvers of [19] and [18], the latter applicable also to 3-d problems, multilevel solvers for the case of complete reference elements of [5], their extension to incomplete reference elements by [17] and [24] may be pointed out for the first breakthroughs in this area.

By the conditions of the generalized angular quasiuniformity, the same *DD* and multigrid like preconditioners-solvers with the same total arithmetical cost are applicable to local Dirichlet problems on subdomains of decomposition – otherwise the most time consuming subproblems in *DD* algorithms. Therefore, in such algorithms we may have primal and secondary domain decompositions, with the latter applied to the local problems on subdomain of the primal decomposition. For the obvious reasons of obtaining the most efficient algorithms and more clarity, as in many other papers, in this paper the domains of p elements are taken for the subdomains of the primal *DD*. It is also worth noting that the above mentioned local Dirichlet solvers for subdomains of the primal decomposition, *i.e., DD* as well as multigrid algorithms are such for the above mentioned finite-difference like preconditioner, specified on an artificial square domain. This domain is defined by the finite-difference (finite element with the first order finite elements) interpretation of the preconditioner. In the space of real *hp*-version unknowns and in the space of p-coordinate functions these solvers are some space decomposition and multispace methods with very specific decompositions in the subspaces, the numbers of which in the both cases are $\mathcal{O}(\log(p+1))$. At that, all the subspaces of coordinate functions are specified on the same subdomain of the primal *DD* or, equivalently, reference element, to which the solver is applied, see [19], [18], [5], [17] and [24].

Quite recently, [6] developed another type of fast solvers for the 2- and 3-d reference element stiffness matrices. It was derived as a wavelet multiresolution preconditioner for the matrix, obtained from the finite-difference like preconditioner of [22] by neglecting the minor term at the cost of $(1+\log p)^{d-1}$ in the condition. This term is the diagonal matrix of the diagonal predominance in the source preconditioner, deteriorating with the growth of p.

The above mentioned contributions made possible in [17] and [24] to design several *DD* preconditioners for 2-d elliptic problems, which are almost optimal in the total arithmetical complexity. In these works, similar in many aspects preconditioners were also justified for discretizations based on spectral elements. It is necessary to note that, unlike h-version with low order finite elements, the use of almost optimal (in the total computational cost) preconditioners does not mean that the *DD* solver in a whole will be fast. This is related to the fact that in the general case – variable coefficients in the elliptic equation, curvilinear elements – calculation on the global stiffness matrix, or unassembled element stiffness matrices, and multiplications of vectors by them at each *DD* iteration step may be rather costly. Stiffness matrices of finite elements may be completely filled in, and, therefore, *e.g.,* for 2-d problems one such multiplication may require $\mathcal{O}(\mathcal{R}p^4)$ arithmetic operations. However, analysis and improving efficiency of these operations, although being an interesting and promising topic, see, *e.g.*, [26], are out the scope of this paper. We only note, that there are many important practical cases where the total arithmetical costs of the *DD* preconditioner and the *DD* solver in a whole coincide without special sophisticated algorithms for the pointed out multiplications. One of them occur when finite elements are rectangular and coefficients of the elliptic equation are polynomials of a fixed, in practice not too large, order.

DD substructuring algorithms for hp discretizations of 3-d elliptic problems have been discussed in the literature to a much lesser extent. Existing publications are exclusively concerned with the wire basket preconditioning for the spectral element discretizations, see [27], [9] and [28]. At the same time neither for spectral nor for hierarchical hp discretizations, there is known a *DD* preconditioner, approaching in a whole optimality in the computational complexity. In this paper, we are suggesting such a *DD* preconditioner and show that it provides the condition number $\mathcal{O}((1+\log p)^{3.5})$ and requires $\mathcal{O}((1+\log p)^{1.75}[p^3\mathcal{R}+p\mathcal{R}^2])$ arithmetic operations. In order to derive *DD* preconditioner, first of all, we adapt the wire basket algorithm to the hierarchical hp discretization under consideration. We show that the wire-basket subsystem of equations may be solved efficiently. For prolongations from the wire-basket on faces, we consider two options. One is based on the iterative procedure with the efficient preconditioner for the face problem. Another is similar to that one used for spectral finite elements, although additionally it requires solving systems with the tridiagonal matrices. Iterative inexact solvers are also proved to be efficient for the prolongations inside elements from the interelement boundary. For solving local Dirichlet problems on subdomains of the primal decomposition, efficient solvers suggested in [18] and [6] may be used. In this paper, we describe shortly the latter, see Subsection 4.2. It is chosen for the reason, that similar solver may be obtained by the same technique for the mass matrix of the hierarchical reference element. In turn, this allows us to justify the efficient preconditioner-solver for the face subproblem by means of the operator K-interpolation technique. In the result, these components, when incorporated in the *DD* solver, define the *DD* preconditioner of almost optimal arithmetical complexity, pointed out above. The term $p\mathcal{R}^2$ in the given above estimate of the computational work is due to the solver for the wire-basket subsystem. This term is the same as for known algorithms in the same component of *DD* methods for spectral discretizations, see [27], [9] and [28]. We outline some alternative algorithms, which seem to allow us in the future to reduce the cost of this component to $\mathcal{O}(p\mathcal{R})$.

The use of inexact solvers for prolongations is a long story. It is well known, that convergence of outer *DD* iterations may be provided, even if local problems are solved by internal iterations rather crudely. At the use of an iterative solver for local problems in the procedures of restrictions-prolongations, the situation is different, and an instability of the outer iterations was often observed even at large numbers of inner iterations for prolongations. This is the reason for the growing attention to this problem (see [14] for the h-version). An interesting algebraic approach is being developed in [11], [30] with direct outcome to inexact solvers in *DD* algorithms, see [7]. In these works, some practical recommendations, well supported by numerical experiments, may be found. However, their conclusions cannot be used in our study, since they do not lead to estimates of the computational complexity with respect to h and p. Our approach is different and gives a strictly approved quantitative answers. Suppose, a spectrally equivalent preconditioner for local problems is used in the internal iterative solver, producing the prolongation. We show that in order to provide the spectral equivalence of *DD* preconditioner, the number of inner iterations should be $\mathcal{O}(1+\log p)$. Another conclusion, following from the analysis, is that in some range quality of the preconditioner for local problems, used in inner iterations, slightly influences the number of inner iterations, needed for a sufficiently good prolongation. For *DD* solvers of 2-d elliptic equations similar approach to prolongations and results were reported in [17], [18].

The paper is arranged as follows. In Section 2, we describe the 3-d model elliptic problem, its hierarchical hp discretization and introduce some notations. Section 3 is devoted to adaptation of the wire-basket algorithm to the hierarchical hp-version and improvement of some of its basic procedures. Namely, solving of the system of the wire-basket equations and prolongations from wire-basket to faces are discussed. Almost optimal preconditioners-solvers for internal and face subproblems are presented in Section 4. In Section 5, the *DD* algorithm in a whole and some of its modifications are described. It is also shown that inexact iterative solvers may be effectively used for prolongations inside elements from the interelement boundaries. Furthermore, questions of computational complexity are discussed.

2. Elliptic problem, discretization and some notations

For a model, we consider the Dirichlet problem: find $u \in H_0^1(\Omega)$

$$a_\Omega(u,v) = (f,v)_\Omega, \qquad \text{for all} \qquad v \in H_0^1(\Omega)\,, \tag{1}$$

where $H_0^1(\Omega)$ is the subspace of the functions from $H^1(\Omega)$, vanishing on the boundary $\partial\Omega$, and

$$a_\Omega(u,v) = \int_\Omega \varrho(x)\, \nabla u \cdot \nabla v \, dx\,, \qquad (f,v)_\Omega = \int_\Omega uv\, dx\,, \tag{2}$$

$\varrho \geqslant \mu_1 > 0$. It is assumed for simplicity that Ω coincides with the computational domain, *i.e.,* it is the domain

$$\overline{\Omega} = \bigcup_{r=1}^{\mathcal{R}} \overline{\tau}_r\,,$$

occupied by the assemblage of geometrically compatible and in general curvilinear finite elements τ_r. Finite elements τ_r are specified by nondegenerate mappings $x = \mathcal{X}^{(r)}(y) : \bar{\tau}_0 \to \bar{\tau}_r$ with positive Jacobian's, where τ_0 is the reference cube $\tau_0 = (-1,1)\times(-1,1)\times(-1,1)$. We assume that these mappings satisfy the conditions, called the *generalized conditions of the angular quasiuniformity.* If the mappings are trilinear, *i.e.,* all elements have straight edges, these conditions are equivalent to shape regularity. In a more general case, they imply that the inequalities in the first line of (4.3) in [22] are fulfilled with h specific for each element, but with $\alpha^{(1)}$ and θ independent of an element and h.

In this paper, we consider *hp* discretizations with the hierarchical reference element, which is equipped with the space $\mathcal{Q}_{p,x}$ of all polynomials of the order not greater p, $p \geqslant 1$, in each variable. Let us introduce the set $\mathcal{M}_{1,p} = (\mathcal{L}_i(s)\,,\ i = 0, 1, \ldots, p)$ of polynomials of one variable

$$\begin{gathered}\mathcal{L}_0(s) = \tfrac{1}{2}(1+s)\,, \qquad \mathcal{L}_1(s) = \tfrac{1}{2}(1-s)\,,\\ \mathcal{L}_i(s) := \beta_i \int_{-1}^{s} P_{i-1}(t)\,dt = \gamma_i[P_i(s) - P_{i-2}(s)]\,, \quad i \geqslant 2\,,\end{gathered}$$

where P_i are Legendre's polynomials and $\gamma_i = 0.5\sqrt{(2i-3)(2i+1)/(2i-1)}$. Thus, for $i \geqslant 2$, $\mathcal{L}_i$ are the *integrated Legendre's polynomials* up to the multipliers β_i. These multipliers are chosen such that $\|\mathcal{L}_i(s)\|_{0,(-1,1)} = 1$ for $i \geqslant 2$. For the hierarchical basis in $\mathcal{Q}_{p,x}$, it is accepted to use the set

$$\mathcal{M}_p = \big(L_\alpha(x) = \mathcal{L}_{\alpha_1}(x_1)\mathcal{L}_{\alpha_2}(x_2)\mathcal{L}_{\alpha_3}(x_3)\,,\ \alpha \in \omega\big)\ ,$$

where $\omega = (\alpha = (\alpha_1, \alpha_2, \alpha_3) : 0 \leqslant \alpha_1, \alpha_2, \alpha_3 \leqslant p)$. If we denote the coordinate functions of the element τ_r by $p_\alpha^{(r)}$, then they are defined by the equalities $p_\alpha^{(r)}(\mathcal{X}^{(r)}(y)) = L_\alpha(y)$. We write the system of finite element algebraic equations, resulting from the described discretization in the form

$$\mathbf{K}\mathbf{u} = \mathbf{f}\,. \tag{3}$$

At designing the *DD* solver, each p-element is treated as a subdomain of decomposition, typically for many other papers as well. There are several reasons for that. The main one is that, if we have an efficient solver for condensation of internal unknowns for each element, such method will be amongst the most efficient and provide the highest degree of parallelization. In order to clarify the presentation, the coefficient ϱ is accepted to be piecewise constant, so that $\varrho(x) = \varrho_r$ for $x \in \tau_r$.

In the 3-d case, it is natural to distinguish *internal, face, edge* and *vertex* degrees of freedom and respectively decompose the vector space

$$V = V_I \oplus V_F \oplus V_E \oplus V_V\,.$$

DD solvers or their parts are also often based on the decompositions

$$V = V_I \oplus V_F \oplus V_W\,, \qquad V = V_I \oplus V_B\,,$$

where $V_B = V_F \oplus V_E \oplus V_V$ and $V_W = V_E \oplus V_V$ are the subspaces of the (interelement) *boundary* and *wire basket* degrees of freedom. According to these subspaces, the finite element stiffness matrix may be represented in the block forms

$$\mathbf{K} = \begin{pmatrix} \mathbf{K}_I & \mathbf{K}_{IB} \\ \mathbf{K}_{BI} & \mathbf{K}_B \end{pmatrix} = \begin{pmatrix} \mathbf{K}_I & \mathbf{K}_{IF} & \mathbf{K}_{IW} \\ \mathbf{K}_{FI} & \mathbf{K}_F & \mathbf{K}_{FW} \\ \mathbf{K}_{EI} & \mathbf{K}_{EF} & \mathbf{K}_{EW} \end{pmatrix} = \begin{pmatrix} \mathbf{K}_I & \mathbf{K}_{IF} & \mathbf{K}_{IE} & \mathbf{K}_{IV} \\ \mathbf{K}_{FI} & \mathbf{K}_F & \mathbf{K}_{FE} & \mathbf{K}_{FV} \\ \mathbf{K}_{EI} & \mathbf{K}_{EF} & \mathbf{K}_E & \mathbf{K}_{EV} \\ \mathbf{K}_{VI} & \mathbf{K}_{VF} & \mathbf{K}_{VE} & \mathbf{K}_V \end{pmatrix} \tag{4}$$

For the corresponding spaces of the f.e. functions we use similar notations with V replaced by $\mathcal{V}$.

The restrictions of the introduced above spaces to the finite elements τ_r are supplied with additional upper index r, *e.g.*, $\mathcal{V}_B^{(r)}$ denotes the subspace of the boundary coordinate functions of the finite element τ_r with $r = 0$ reserved for the reference element. Similarly, $\mathbf{K}_F^{(r)}$ is the block of the stiffness matrix of an element τ_r, generated by the face coordinate functions. The spaces V and $\mathcal{V}$ for the reference element will be denoted U and $\mathcal{U} = \mathcal{Q}_{p,x}$, respectively, with the same indexation for subspaces.

Throughout the paper, we use boldface letters for matrices and vectors, except for multi-indices and vectors of the space variables. The notations $\mathbf{A} = \mathbf{K}^{(0)}$ and $\mathbf{M}$ are reserved for the reference element stiffness matrix, generated by Dirichlet integral, *i.e.*, with $\varrho \equiv 1$ in (2), and the mass matrix, respectively. The norms and quasinorms in Sobolev's spaces $H^k(\Omega)$ are denoted $\|\cdot\|_{k,\Omega}$, whereas the notations ${}_{00}\|\cdot\|_{1/2,\Theta}$ and ${}_{00}|\cdot|_{1/2,\Theta}$, with Θ being a subset of $\partial\Omega$, stand for the norm and the quasinorm in the space $H_{00}^{1/2}(\Theta)$. For their definitions, we refer to [13]. The signs $\prec$, $\succ$, $\asymp$ are used for the inequalities and equalities, fulfilled up to absolute constants.

3. Wire basket preconditioner for hierarchical *hp*-version

Subdivision of a structure to be analyzed numerically in substructures, direct elimination of the internal for substructures unknowns, and then solving the system with Schur complement for the matrix are still widely implemented in applied algorithms, generally called *substructuring algorithms*. This is an additional reason to discuss preconditioners for Schur complement $\mathbf{S}_B$, figuring in the factorization

$$\mathbf{K} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{K}_{BI}\mathbf{K}_I^{-1} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{K}_I & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_B \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{K}_I^{-1}\mathbf{K}_{IB} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}, \qquad \mathbf{S}_B = \mathbf{K}_B - \mathbf{K}_{BI}\mathbf{K}_I^{-1}\mathbf{K}_{IB}\,, \tag{1}$$

and governing the *interface problem* independently. In what follows, these preconditioners will be incorporated in the global *DD* solver. At first, we derive Schur complement preconditioner for the reference element stiffness matrix. Then we will obtain Schur complement preconditioner for each element by scaling the preconditioner for the reference element and obtain the global Schur complement preconditioner by subassembling.

For the reference element one may write similar to (1) factorization

$$\mathbf{A} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{A}_{BI}\mathbf{A}_I^{-1} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{A}_I & \mathbf{0} \\ \mathbf{0} & \mathbb{S}_B \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{A}_I^{-1}\mathbf{A}_{IB} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}, \qquad \mathbb{S}_B = \mathbf{A}_B - \mathbf{A}_{BI}\mathbf{A}_I^{-1}\mathbf{A}_{IB}\,, \tag{2}$$

Let us represent the Schur complement $\mathbb{S}_B$ in the block form

$$\mathbb{S}_B = \begin{pmatrix} \mathbb{S}_F & \mathbb{S}_{FW} \\ \mathbb{S}_{WF} & \mathbb{S}_W \end{pmatrix}.$$

The idea of the wire basket preconditioner for $\mathbb{S}_B$, which will be denoted by $\boldsymbol{\mathcal{S}}_B$, is usually explained by three following steps, see [28] .

i) Change the basis in the subspace V_W by means of the transformation matrix

$$\mathbb{C} := \begin{pmatrix} \mathbf{I} & -\mathbf{C} \\ \mathbf{0} & \mathbf{I} \end{pmatrix},$$

which is described below, so that the Schur complement is transformed to the matrix

$$\mathbb{S}_C = \mathbb{C}^T\mathbb{S}_B\mathbb{C} = \begin{pmatrix} \mathbb{S}_F & \text{nonzero} \\ \text{nonzero} & \widetilde{\mathbb{S}}_W \end{pmatrix}.$$

ii) Eliminate in $\mathbb{S}_C$ coupling between faces and wire basket and also eliminate the coupling between different faces. After this we have, for the tentative preconditioner of $\mathbb{S}_C$, the 2×2 block diagonal matrix

$$\widetilde{\mathbb{S}}_C = \begin{pmatrix} \widetilde{\mathbb{S}}_F & \mathbf{0} \\ \mathbf{0} & \widetilde{\mathbb{S}}_W \end{pmatrix},$$

in which the first block is 6×6 block diagonal matrix $\widetilde{\mathbb{S}}_F = \mathrm{diag}\,[\mathbb{S}_{F_1}, \mathbb{S}_{F_2}, .., \mathbb{S}_{F_6}]$ with each block related to one of the six faces F_k. Obviously, at the appropriate ordering of the face unknowns these blocks are identical.

iii) Replacethe blocks $\widetilde{\mathbb{S}}_F$ and $\widetilde{\mathbb{S}}_W$ by some scaled and simpler matrices $\widetilde{\boldsymbol{\mathcal{S}}}_F$ and $\widetilde{\boldsymbol{\mathcal{S}}}_W$, of which the former is block diagonal $\widetilde{\boldsymbol{\mathcal{S}}}_F = \mathrm{diag}\,[\boldsymbol{\mathcal{S}}_{0,1}, \boldsymbol{\mathcal{S}}_{0,2}, .., \boldsymbol{\mathcal{S}}_{0,6}]$.
Having completed these steps, we get the Schur complement preconditioner in a form

$$\widetilde{\boldsymbol{\mathcal{S}}}_B = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{C}^T & \mathbf{I} \end{pmatrix} \begin{pmatrix} \widetilde{\boldsymbol{\mathcal{S}}}_F & \mathbf{0} \\ \mathbf{0} & \widetilde{\boldsymbol{\mathcal{S}}}_W \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{C} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}. \tag{3}$$

A clear and helpful interpretation of these steps, repeated in many papers, follows from the factorized form of the Schur complement

$$\mathbb{S}_B = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbb{S}_{WF}\mathbb{S}_F^{-1} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbb{S}_F & \mathbf{0} \\ \mathbf{0} & \widehat{\mathbb{S}}_W \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbb{S}_F^{-1}\mathbb{S}_{FW} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}, \qquad \widehat{\mathbb{S}}_W = \mathbb{S}_W - \mathbb{S}_{WF}\mathbb{S}_F^{-1}\mathbb{S}_{FW}. \tag{4}$$

Therefore, in order to split faces and wire basket without considerable losses in condition, it is necessary to construct sufficiently good approximation $\mathbb{P}_a$ to the prolongation operator $\mathbb{P} : U_W \to U_B$, defined as

$$\mathbb{P}_a = \begin{pmatrix} -\mathbf{C} \\ \mathbf{I} \end{pmatrix}, \qquad \mathbb{P} = \begin{pmatrix} -\mathbb{S}_F^{-1}\mathbb{S}_{FW} \\ \mathbf{I} \end{pmatrix}. \tag{5}$$

Similarly to $\mathbb{P}$, the operator $\mathbb{P}_a$ should reproduce exactly constant vectors. It is also necessary that $\ker[\widetilde{\boldsymbol{\mathcal{S}}}_W] = \ker[\widehat{\mathbb{S}}_W]$, and evidently $\ker[\widehat{\mathbb{S}}_W]$ contains constant vectors. If these basic properties are violated, than $\ker[\boldsymbol{\mathcal{S}}_B] \neq \ker[\mathbb{S}_B]$, and therefore $\boldsymbol{\mathcal{S}}_B$ can't be a good preconditioner for $\mathbb{S}_B$. In our first approach, these properties will be automatically satisfied due to the following two reasons. One is the hierarchical basis of the reference element, in which constants belong to the subspace, spanned by the vertex coordinate functions. The definition of the prolongation operator by means of inexact solver for the face problems is another. This approach is justified by the following two propositions.

`Proposition 3.1` *Suppose the prolongation operator $\mathbb{P}_a$ satisfies the inequality*

$$\|\mathbb{P}_a \mathbf{v}_W\|_{\mathbb{S}_B} \leqslant c_{\mathbb{P}} \|\mathbf{v}_W\|_{\widehat{\mathbb{S}}_W} \tag{6}$$

with an absolute constant $c_{\mathbb{P}}$ and $\widetilde{\mathbb{S}}_F = \mathbb{S}_F$. Then the matrices $\mathbb{S}_C$ and $\widetilde{\mathbb{S}}_C$ are spectrally equivalent uniformly in p, and the same is true for the matrices $\widetilde{\mathbb{S}}_W$ and $\widehat{\mathbb{S}}_W$.

Proof. We give this proof, that is more or less known for similar situations (see, *e.g.,* [23]), only for completeness. The inequalities $\widehat{\mathbb{S}}_W \leqslant \widetilde{\mathbb{S}}_W \leqslant c_{\mathbb{P}}^2 \widehat{\mathbb{S}}_W$ are evident: the left one is the consequence of the minimization proprties whereas the right one is another form of (6). The inequality $\mathbb{S}_C \leqslant 2\widetilde{\mathbb{S}}_C$ is Cauchy inequality. For any vector $\mathbf{v}_B = \mathbf{v}_F + \mathbf{v}_W$ in the new basis, by means of the estimate of the square of the sum by the sum of the squares, (6) and the estimate $(\mathbf{v}_B, \mathbb{S}_\mathbb{C}\mathbf{v}_B) \geqslant (\mathbf{v}_W, \widehat{\mathbb{S}}_W\mathbf{v}_W)$, we have

$$
\begin{aligned}
&(\mathbf{v}_B, \widetilde{\mathbb{S}}_C\mathbf{v}_B) = (\mathbf{v}_F, \mathbb{S}_F\mathbf{v}_F) + (\mathbf{v}_W, \widetilde{\mathbb{S}}_W\mathbf{v}_W) \leqslant \\
&2(\mathbf{v}_B, \mathbb{S}_C\mathbf{v}_B) + 3(\mathbf{v}_W, \widetilde{\mathbb{S}}_W\mathbf{v}_W) \leqslant 2(\mathbf{v}_B, \mathbb{S}_C\mathbf{v}_B) + 3c_{\mathbb{P}}^2(\mathbf{v}_W, \widehat{\mathbb{S}}_W\mathbf{v}_W) \leqslant (2+3c_{\mathbb{P}}^2)(\mathbf{v}_B, \mathbb{S}_C\mathbf{v}_B)\,,
\end{aligned}
\tag{7}
$$

completing the proof. □

`Proposition 3.2` *Suppose* $\gamma_1\widetilde{\boldsymbol{\mathcal{S}}}_F \leqslant \mathbb{S}_F \leqslant \gamma_2\widetilde{\boldsymbol{\mathcal{S}}}_F$ *with positive* γ_k. *Suppose also that for any* $\mathbf{v}_W$ *the prolongation* $\mathbf{u} = \mathbb{P}_a\mathbf{v}_W$ *is defined in such a way that* $\mathbf{u}_W = \mathbf{v}_W$ *and the subvector* $\mathbf{u}_F = \mathbf{u}_F^{k_o}$ *is found from the iteration process*

$$
\mathbf{u}_F^{k+1} = \mathbf{u}_F^k - \sigma_{k+1}\widetilde{\boldsymbol{\mathcal{S}}}_F^{-1}(\mathbb{S}_F\mathbf{u}_F^k - \mathbb{S}_{FW}\mathbf{v}_W)\,, \qquad \mathbf{u}_F^0 = \mathbf{0}\,, \tag{8}
$$

with Chebyshev iteration parameters σ_k *for some fixed number* k_o *of iterations. Then at* $k_o \geqslant c(2\log p + \log(1+\log p))/(\log \varrho^{-1})$, *where* $\varrho = (1-\theta)/(1+\theta)$, $\theta = \sqrt{\gamma_1/\gamma_2}$, *the inequality (6) holds with the constant* $c_{\mathbb{P}}$ *independent of* p. *Let us note that, if* $\mathbf{v}_W$ *is a constant vector, then* $\mathbb{S}_{FW}\mathbf{v}_W = \mathbf{0}$ *and* $\mathbf{u}_F^k = \mathbf{0}$ *for* $k \geqslant 1$.

Proof. Let the vector $\boldsymbol{\varphi}_F$ be the solution of the system $\mathbb{S}_F\boldsymbol{\varphi}_F = \mathbb{S}_{FW}\mathbf{v}_W$ and $\boldsymbol{\varphi}_B = \boldsymbol{\varphi}_F + \mathbf{v}_W$. We have the convergence estimate

$$
\|\mathbf{u}_F^k - \boldsymbol{\varphi}_F\|_{\mathbb{S}_F} \leqslant \varrho^k\|\boldsymbol{\varphi}_F\|_{\mathbb{S}_F}\,, \tag{9}
$$

from which it follows

$$
\|\mathbf{u}_B^k - \boldsymbol{\varphi}_B\|_{\mathbb{S}_B} \leqslant \varrho^k(\|\boldsymbol{\varphi}_B\|_{\mathbb{S}_B} + \|\mathbf{v}_W\|_{\mathbb{S}_B})\,, \qquad \mathbf{u}_B^k = \mathbf{u}_F^k + \mathbf{v}_W\,. \tag{10}
$$

Let $v_W \in \mathcal{U}_W$ correspond to $\mathbf{v}_W$. Then by the definition of $\mathbb{S}_B$ as the stiffness matrix for discrete harmonic functions, by the continuation theorem in the space $\mathcal{Q}_p$ from $\partial\tau_0$ inside τ_0 and the multiplicative inequality, we have

$$
\|\mathbf{v}_W\|_{\mathbb{S}_B}^2 \equiv |v_W|_{1,\tau_0}^2 \leqslant c_1|v_W|_{1/2,\partial\tau_0}^2 \leqslant c_2\|v_W\|_{0,\partial\tau_0}\|v_W\|_{1,\partial\tau_0}\,, \tag{11}
$$

with $\|\cdot\|_{1,\partial\tau_0}$ understood as

$$
\|v\|_{1,\partial\tau_0}^2 := \left(\sum_{k=1}^{6}\|v\|_{1,F_k}^2\right)^{1/2}
$$

Taking into account definitions of the vertex and edge coordinate functions and Markov's type inequality, it may be shown that for any $v_W \in \mathcal{U}_W$ we have

$$
\|v_W\|_{1,F_k} \leqslant \|v_W\|_{1,\partial F_k} \leqslant p^2\|v_W\|_{0,\partial F_k}\,, \qquad \|v_W\|_{0,F_k} \leqslant \|v_W\|_{0,\partial F_k}\,.
$$

Now, substituting these inequalities in (11) and applying a Bramble-Hilbert lemma type argument, we get

$$
\|\mathbf{v}_W\|_{\mathbb{S}_B}^2 \leqslant p^2\inf_c\|v_W - c\|_{0,W}^2\,. \tag{12}
$$

[27], see Lemma 5.3, proved the following trace inequality: let $\overline{v}$ be the average of $v \in \mathcal{U}$ over the wire basket. Then,

$$||v - \overline{v}||^{\,2}_{\,0,W} \prec (1 + \log p)|v|^2_{1,\tau_o}\,. \tag{13}$$

Therefore, according to (12) and (13)

$$\|\mathbf{v}_W\|^{\,2}_{\,\mathbb{S}_B} \leqslant c_3 p^2(1 + \log p)\|\boldsymbol{\varphi}_B\|_{\,\mathbb{S}_B}\,, \tag{14}$$

and

$$\|\mathbf{u}^k_B\|_{\,\mathbb{S}_B} \leqslant (1 + \varrho^k + c_3\varrho^k p^2(1 + \log p))\|\mathbf{v}_W\|_{\,\widehat{\mathbb{S}}_W}\,. \tag{15}$$

At the last step, we combined (10) and (14) and used the equality $\|\boldsymbol{\varphi}_B\|_{\,\mathbb{S}_B} = \|\mathbf{v}_W\|_{\,\widehat{\mathbb{S}}_W}$. From (15), one concludes that, if $k_o \geqslant (2\log p + \log(1 + \log p) + \log c_3)/(\log \varrho^{-1})$, then

$$\|\mathbb{P}_a\mathbf{v}_W\|_{\,\mathbb{S}_B} \equiv \|\mathbf{u}^{k_o}_B\|_{\,\mathbb{S}_B} \leqslant 3\|\mathbf{v}_W\|_{\,\widehat{\mathbb{S}}_W}\,, \tag{16}$$

This inequality proves our proposition. □

A definition that is equivalent to that one presented in Proposition 3.2 is given by formula (5) with $\mathbf{C} = \widetilde{\mathbb{S}}^{-1}_{F,\mathrm{it}}\mathbb{S}_{FW}$ and

$$\widetilde{\mathbb{S}}^{-1}_{F,\mathrm{it}} = [\mathbf{I} - \prod_{k=1}^{k_o}(\mathbf{I} - \sigma_k\widetilde{\boldsymbol{\mathcal{S}}}_F^{-1}\mathbb{S}_F)]\,\mathbb{S}_F^{-1}\,. \tag{17}$$

Remark 3.1. *The value of ϱ does not have strong influence on the computational cost of the prolongation operator. If,* e.g., *$\gamma_2/\gamma_1 = p^s$, then according to Proposition 3.2 the corresponding factor in the number k_o of iterations will be $s\log p$.*

Remark 3.2. *In (8), (17) and in (3), two different preconditioners $\widetilde{\boldsymbol{\mathcal{S}}}_F$ may be used.*

Therefore, the prolongation operator may be chosen according to Proposition 3.2. However, the efficiency also depends on the preconditioner $\widetilde{\boldsymbol{\mathcal{S}}}_F$. As one of the candidates, in subsection 4.3 we consider a multilevel preconditioner, resulting from a multiscale wavelet analysis.

Let $\mathbb{S}_{00}$ be the matrix of the quadratic form ${}_{00}|v|^2_{1/2,F_k}$, $\forall v \in \mathcal{U}_{F_k}$, which is the square of the quasinorm in the space $H^{1/2}_{00}(F_k)$. Evidently, at the appropriate enumeration of coordinate functions of each face, this matrix is the same for all faces. Sufficiently good preconditioning may be achieved by setting $\widetilde{\boldsymbol{\mathcal{S}}}_F = \mathrm{diag}\,[\boldsymbol{\mathcal{S}}_{00}, \boldsymbol{\mathcal{S}}_{00}, .., \boldsymbol{\mathcal{S}}_{00}]$, when $\boldsymbol{\mathcal{S}}_{00}$ is a matrix close in the spectrum to $\mathbb{S}_{00}$, see, *e.g.,* [16], [27], [9] and [20]. By this reason, we assume, that for sufficiently good $\gamma_{F,k} > 0$

$$\gamma_{F,1}\boldsymbol{\mathcal{S}}_{00} \leqslant \mathbb{S}_{00} \leqslant \gamma_{F,2}\boldsymbol{\mathcal{S}}_{00}\,. \tag{18}$$

An alternative prolongation operator may be defined on the basis of the approach, similar in essence to one used for h-version and spectral element methods, see [10] and [9], but different algorithmically due to the difference in the bases. It provides a condition worse by the factor $(1 + \log p)$, but at a proper scaling the contributions to the total computational costs of the wire basket and *DD* algorithms are comparable to those from the prolongation operator, described above.

Let us denote by $\mathcal{L}_{k,0}(s)$, $k=0,1$, the polynomials, obtained by the orthogonalization of polynomial $\mathcal{L}_k(s)$ to the internal polynomials $\mathcal{L}_i(s)$, $i=2,3,..,p$, in $L_2(-1,1)$. Evidently,

$$\mathcal{L}_{k,0}(s)=\mathcal{L}_k(s)+\sum_{i=2}^{p} c_{k,i}\mathcal{L}_i(s)\,. \tag{19}$$

Calculation of coefficients c_i requires $\mathcal{O}(p)$ operations, since the corresponding mass matrix is tridiagonal. First, we introduce some tentative prolongation operator. For definiteness, let us consider the face in the plane $x_3\equiv -1$. The restriction of $\mathcal{U}$ to this face is the space $\mathcal{U}_B$, spanned over polynomials $L_{i,j}(x)=\mathcal{L}_i(x_1)\mathcal{L}_j(x_2)$, $0\leqslant i,j\leqslant p$, whereas the restrictions to edges belong to the spaces spanned by the polynomials $\mathcal{L}_i(x_k)$, $0\leqslant i\leqslant p$. For the components of the vectors $\mathbf{v}_B$ and $\mathbf{v}_W$ for the face under consideration, we use the notations $v_{i,j}$. If the trace ϕ on the boundary of the face and the corresponding vector $\mathbf{v}_W$ are given, then the tentative prolongation $u=\widetilde{I}_{W\to B}\phi$ of ϕ is

$$\begin{aligned} u=\ &\textstyle\sum_{k,l=0}^{1}\mathcal{L}_{k,0}(x_1)\mathcal{L}_{l,0}(x_2)v_{k,l}+\sum_{k=0}^{1}\sum_{j=2}^{p}\mathcal{L}_{k,0}(x_1)\mathcal{L}_j(x_2)v_{k,j}+\\ &\textstyle\sum_{k=0}^{1}\sum_{i=2}^{p}\mathcal{L}_i(x_1)\mathcal{L}_{k,0}(x_2)v_{i,k}\end{aligned} \tag{20}$$

If $\widetilde{\mathbf{I}}_{W\to B}$ is the matrix representation of $\widetilde{I}_{W\to B}$, then from the expression (20) it follows that the prolongation $\widetilde{\mathbf{I}}_{W\to B}\mathbf{v}_W=\mathbf{v}_B$ requires $\mathcal{O}(p^2)$ arithmetic operations. Similarly, the prolongations to the other edges are defined and, therefore, we have defined the tentative prolongation operator from the wire basket to the boundary of the reference element, for which we retain the same notations.

In order to come to the prolongation operator by means of the tentative operator $\widetilde{I}_{W\to B}$, for wire basket algorithms, we typically define the function $\mathcal{F}=1-\widetilde{I}_{W\to B}\,1$, which vanishes at the wire basket. Evidently, it can be represented as

$$\mathcal{F}=\sum_{k=1}^{6}\mathcal{F}_k\,,$$

where each $\mathcal{F}_k$ is nonzero only on one face F_k. The prolongation operator $I_{W\to B}:\mathcal{U}_W\to\mathcal{U}_B$ is defined accordingly to

$$u=I_{W\to B}\,\phi=\widetilde{I}_{W\to B}\,\phi+\sum_{k=1}^{6}\mathcal{F}_k\,\overline{\phi}_{\partial F_k}\,, \tag{21}$$

where $\overline{\phi}_{\partial F_k}$ is the average value of ϕ on the boundary ∂F_k of the face F_k. We use the notations $\mathbb{P}_a\equiv\mathbf{I}_{W\to B}$ for the matrix of this prolongation operator.

Now we describe the preconditioner $\widetilde{\boldsymbol{\mathcal{S}}}_W$. Let $\mathbf{M}_W$ be the wire basket mass matrix for the reference element,

$$\mathbf{v}_W^T\mathbf{M}_W\mathbf{v}_W=||v||^2_{L_2(W)}\,,$$

and $\mathbf{z}_1$ be the vector of wire basket coefficients for the constant function equal to 1 and $|v|_{L_2(W)}=\inf_c||v-c||_{L_2(W)}$. Simple computation gives

$$|v_W|_{L_2(W)}=\int_W v_W^2\,ds-\frac{(\int_W v_W\,ds)^2}{\int_W\,ds}=\mathbf{v}_W^T\left(\mathbf{M}_W-\frac{(\mathbf{M}_W\mathbf{z}_1)(\mathbf{M}_W\mathbf{z}_1)^T}{\mathbf{z}_1^T\mathbf{M}_W\mathbf{z}_1}\right)\mathbf{v}_W \tag{22}$$

The preconditioner $\widetilde{\boldsymbol{\mathcal{S}}}_W$ for the reference element may be defined as the matrix generated by the quadratic form

$$b_W(v_W, v_W) = c_W |v_W|^2_{L_2(W)}\,, \tag{23}$$

where $c_W = c(1 + \log p)$ is a scaling factor, *i.e.*,

$$\widetilde{\boldsymbol{\mathcal{S}}}_W = c_W \left(\mathbf{M}_W - \frac{(\mathbf{M}_W \mathbf{z}_1)(\mathbf{M}_W \mathbf{z}_1)^T}{\mathbf{z}_1^T \mathbf{M}_W \mathbf{z}_1} \right) . \tag{24}$$

In distinction with wire basket algorithms for h-version and spectral element methods, the matrix $\mathbf{M}_W$ is not any more diagonal or spectrally equivalent to a diagonal matrix. Therefore, a modification of the solving procedure for the wire basket problem is required, which is described at the end of this section.

The preconditioners $\widetilde{\boldsymbol{\mathcal{S}}}_F$, $\widetilde{\boldsymbol{\mathcal{S}}}_W$ and the prolongation operator $\mathbb{P}_a$, introduced for the reference element, allow us to assemble the global preconditioners and the prolongation operator, which we denote $\boldsymbol{\mathcal{S}}_F$, $\boldsymbol{\mathcal{S}}_W$ and $\mathbf{P}_{\mathbf{a}}$, respectively. Therefore, the restriction of the global prolongation matrix $\mathbf{P}_a$ to each finite element τ_r (at the corresponding ordering of unknowns) is the prolongation matrix $\mathbb{P}_a$ for the reference element. Suppose, there are N_F faces inside of the computational domain. The global matrix $\boldsymbol{\mathcal{S}}_F$ is defined as a block diagonal matrix

$$\boldsymbol{\mathcal{S}}_F = \mathrm{diag}\,[\boldsymbol{\mathcal{S}}_{F_k}]\,_{k=1}^{N_F}\,, \qquad \boldsymbol{\mathcal{S}}_{F_k} = (h_{r_1(k)}\varrho_{r_1(k)} + h_{r_2(k)}\varrho_{r_2(k)})\boldsymbol{\mathcal{S}}_{00}\,, \tag{25}$$

with each block related to one face F_k. Here $r_1(k)$ and $r_2(k)$ are the numbers of two elements $\overline{\tau}_{r_1(k)}$ and $\overline{\tau}_{r_2(k)}$, sharing the face F_k. The global preconditioner $\boldsymbol{\mathcal{S}}_W$ is obtained by the assembly of the preconditioners for finite elements τ_r with each of them being $\boldsymbol{\mathcal{S}}_W^{(r)} = h_r\,\varrho_r \widetilde{\boldsymbol{\mathcal{S}}}_W$. Having $\boldsymbol{\mathcal{S}}_F$, $\boldsymbol{\mathcal{S}}_W$ and $\mathbf{P}_a$, we may define the global Schur complement preconditioner by the similar to (3) formula. The inverse to it, evidently, is

$$\boldsymbol{\mathcal{S}}_B^{-1} = \boldsymbol{\mathcal{S}}_F^{+} + \mathbf{P}_a \boldsymbol{\mathcal{S}}_W^{-1} \mathbf{P}_a^T\,. \tag{26}$$

At defining the global preconditioner $\boldsymbol{\mathcal{S}}_B$, we may proceed in an alternative way. For each element we define preconditioners $\boldsymbol{\mathcal{S}}_F^{(r)}, \boldsymbol{\mathcal{S}}_W^{(r)}$ and the prolongation operator $\mathbf{P}_a^{(r)}$ in a way similar to the reference element, but taking into account the geometrical form of the element. The corresponding global matrices result from the described above procedure of assembling (which for the practical purposes are, indeed, not needed). This may result in a better *DD* preconditioner with respect to relative condition number. However, in order to make the presentation more clear, in what follows, we assume more simple definitions as before. Apart from this, in our opinion the described above preconditioner $\boldsymbol{\mathcal{S}}_B$ results in a more efficient algorithm.

The system $\boldsymbol{\mathcal{S}}_W \mathbf{v} = \mathbf{f}$ has $\mathcal{O}(Rp)$ unknowns. Optimal solution procedures for it have not been obtained yet. However, optimization of computational work with respect to p may be achieved by necessary modification of the traditional wire basket algorithms. This system is equivalent to the problem of minimizing the functional

$$\frac{1}{2} \min_{\mathbf{v}} \left(\sum_r \min_{c_r} (\mathbf{v}^{(r)} - \mathbf{z}_1 c_r)^T \mathbf{M}_W^{(r)} (\mathbf{v}^{(r)} - \mathbf{z}_1 c_r) - \mathbf{v}^T \mathbf{f} \right) . \tag{27}$$

with $\mathbf{v}^{(r)}$ being the restriction of $\mathbf{v}$ to the element $\overline{\tau}_r$ and $\mathbf{M}_W^{(r)} = h_r \varrho_r \mathbf{M}_W$. Let $\mathbb{M}_W$ be the matrix obtained by the assembly of the matrices $\mathbf{M}_W^{(r)}$. From (27) we come to the system

$$\mathbf{z}_1^T \mathbf{M}_W^{(r)} (\mathbf{v}^{(r)} - \mathbf{z}_1 c_r) = 0\,, \qquad \mathbb{M}_W \mathbf{v} - \sum_r \mathbf{M}_W^{(r)} \mathbf{z}_1 c_r = \mathbf{f}\,. \tag{28}$$

For the first step at solving it, we express $\mathbf{v}$ from the second subsystem and substitute in the first subsystem. This gives the subsystem for c_k's

$$(\mathbf{z}_1^T \mathbf{M}_W^{(r)} \mathbf{z}_1) c_r - \mathbf{z}_1^T \mathbf{M}_W^{(r)} \mathbb{M}_W^{-1} \sum_i \mathbf{M}_W^{(i)} \mathbf{z}_1 c_i = \mathbf{z}_1^T \mathbf{M}_W^{(r)} \mathbf{M}_W^{-1} \mathbf{f} \,. \tag{29}$$

Having it solved, we return to the second subsystem (28) in order to find $\mathbf{v}$. For convenience, we consider vectors $\mathbf{M}_W^{(r)} \mathbf{z}_1$ in (28) and (29) as expanded by zero entries to match the dimension of $\mathbf{v}$.

The global mass matrix $\mathbb{M}_W$ may be viewed as assembled from element matrices $\mathbf{M}_{E_j}$ with an element E_j being the edge "j" together with its vertices. Evidently,

$$\mathbf{M}_{E_j} = c_W \left(\sum_i h_{r_i(j)} \varrho_{r_i(j)} \right) \mathbf{M}_{E,0} \,,$$

where the matrix $\mathbf{M}_{E,0}$ is induced by the quadratic form $\|\cdot\|^2_{L^2(-1,1)}$ and $r_i(j)$ are the numbers of elements $\overline{\tau}_r$, containing the edge E_j. The block, obtained from $\mathbf{M}_{E,0}$ by deleting two rows and two columns related to the vertices, is a tridiagonal matrix, see Subsection 4.1, and only two of the internal degrees of freedom for the edge are coupled with the vertices of the edge. Therefore, for solving subsystems with the matrix $\mathbb{M}_W$ the following procedure may be used. First, for each 1-d element E_j in parallel, condense the internal (edge) unknowns and assemble the subsystem with respect to the subvector $\mathbf{v}_V$ of the vertex components of the vector $\mathbf{v} = \mathbf{v}_W$. It is easy to see, that the matrix of this system is spectrally equivalent to its diagonal. Thus, it may be solved by the PCG with the diagonal preconditioner. Therefore, elimination of $\mathbf{v}$ for obtaining (29) requires $\mathcal{O}(\mathcal{R}^2 p)$ operations and is the same as in the case of spectral elements. For hierarchical elements, some other operations are simplified. For instance, vectors $\mathbf{M}_W^{(r)} \mathbf{z}$ contain a fixed number of nonzero entries that is independent of p. However, the total asymptotic cost of this algorithm seems to be the same as for the spectral elements, because the complexity of solving the subsystem (29) is the same in the both cases.

Let us turn to an alternative way of solving $\boldsymbol{\mathcal{S}}_W \mathbf{v} = \mathbf{f}$. If we define the c_r's from the first subsystem (28) and substitute them in the second subsystem, we arrive at the system

$$(\mathbb{M}_W - \mathbb{M}_C)\mathbf{v} = \mathbf{f} \,, \qquad \mathbb{M}_C = \frac{1}{24} \sum_r \mathbf{M}_W^{(r)} \mathbf{z}_1 \mathbf{z}_1^T \mathbf{M}_W^{(r)} \,, \tag{30}$$

with respect to $\mathbf{v}$. Since the vector $\mathbf{z}_1$ has only eight nonzero coefficients, related to vertices and equal to 1, and the matrices $\mathbf{M}_W^{(r)}$ have in each row the number of nonzero entries independent of p, the cost of the matrix $\mathbb{M}_C$ is $\mathcal{O}(\mathcal{R})$. Therefore, it is sufficient to derive a sufficiently fast solver for (30), *e.g.*, requiring not more than $\mathcal{O}(\mathcal{R}p^3)$ arithmetic operations, in order to provide the opportunity of coming to a fast *DD* global solver. This task seems feasible, but in this paper we restrict ourselves only to several remarks. It is easy to note that the only nonzero off-diagonal entries of the matrix $\mathbb{M}_C$ are those coupling the vertices of each finite element and that the values of them do not depend on p. Due to this structure of $\mathbb{M}_C$, the arithmetical cost of elimination of $\mathbf{v}_e$ from (30) and from the second subsystem (28) is the same, *i.e.*, $\mathcal{O}(\mathcal{R}p)$. Therefore, it is left to solve efficiently only the subsystem for $\mathbf{v}_V$ that results from (30). For instance, if the finite element mesh is topologically equivalent to orthogonal and $\varrho = const$, this subsystem will be regular with the 27-point stencil and some fast solver can most probably be derived. Let us also mention another property of (30). At sufficiently small sizes of elements, the matrices $(\mathbb{M}_W - \mathbb{M}_C)$ and $\mathbb{M}_W$ became spectrally equivalent. This is for the reason that with diminishing of the maximal size h of finite elements the matrices ($\mathbb{M}_W$ and $\mathbb{M}_C$) deteriorate as h and h^2, respectively. Therefore, we can apply PCG to (30) with the preconditioner $\mathbb{M}_W$, fast solver for which was described above. However, such sizes of elements can be out of practical interest, in particular, due to their dependence on p.

4. Fast solvers of the internal and the face problems on the reference element

4.1. Finite-difference like preconditioner for the reference element stiffness and mass matrices

We will need preconditioners for 2-d and 3-d reference element stiffness and some other matrices. As a starting point for their derivation, we will introduce tentative finite-difference like preconditioners. Our final preconditioners are indeed based on the fast solvers for tentative preconditioners, derived with the use of the finite-difference or finite element (with the first order elements) analogy.

`Finite-difference like preconditioners for the square reference element.` By means of two $(p-1)\times(p-1)$ matrices

$$\boldsymbol{\mathcal{D}} = \operatorname{diag}\left[i^2\right] , \qquad \overline{\boldsymbol{\Delta}} = \frac{1}{2}\begin{pmatrix} 2 & 0 & -1 & & & & \\ & 2 & 0 & -1 & & \Large 0 & \\ & & \cdots & \cdots & & & \\ & & & \cdots & \cdots & & \\ & \text{SYM} & & & 2 & 0 & -1 \\ & & & & & 2 & 0 \\ & & & & & & 2 \end{pmatrix}$$

we may introduce the $(p-1)^2 \times (p-1)^2$ matrices

$$\boldsymbol{\Lambda}_I = \boldsymbol{\mathcal{D}} \otimes (\overline{\boldsymbol{\Delta}} + \boldsymbol{\mathcal{D}}^{-1}) + (\overline{\boldsymbol{\Delta}} + \boldsymbol{\mathcal{D}}^{-1}) \otimes \boldsymbol{\mathcal{D}}\,, \qquad \Xi_I = (\overline{\boldsymbol{\Delta}} + \boldsymbol{\mathcal{D}}^{-1}) \otimes (\overline{\boldsymbol{\Delta}} + \boldsymbol{\mathcal{D}}^{-1})\,.$$

Lemma 4.1. *Let the conditions of the generalized angular quasiuniformity be fulfilled. Then*

$$\boldsymbol{\Lambda}_I \prec \mathbf{A}_I \prec \boldsymbol{\Lambda}_I\,, \qquad \Xi_I \prec \mathbf{M}_I \prec \Xi_I\,, \tag{1}$$

Proof. Inequalities (1) were proved in [16], see also Lemma 4.2 and Remark 5.1 in [22]. □

The preconditioner takes a simpler form, if we rearrange rows and columns in the order, corresponding to separation of basis polynomials in four groups: 1-st containing $L_{i,j}$ with all i and j even, 2-nd – with i even and j odd, 3-rd – with i odd and j even and 4-h – with all i and j odd. This splits the matrices $\boldsymbol{\Lambda}_1$, $\mathbf{A}_I$, Ξ_I and $\mathbf{M}_I$ in four independent blocks. Without loss of generality we may assume $p = 2N+1$. After applying these perturbations, the preconditioners $\boldsymbol{\Lambda}_I$ and Ξ_I take forms

$$\boldsymbol{\Lambda}_I = \operatorname{diag}\left[\boldsymbol{\Lambda}_{e,e}, \boldsymbol{\Lambda}_{e,o}, \boldsymbol{\Lambda}_{o,e}, \boldsymbol{\Lambda}_{o,o}\right] , \qquad \Xi_I = \operatorname{diag}\left[\Xi_{e,e}, \Xi_{e,o}, \Xi_{o,e}, \Xi_{o,o}\right] , \tag{2}$$

each with four $N^2 \times N^2$ independent blocks

$$\begin{aligned} \mathbf{\Lambda}_{a,b} &= \boldsymbol{\mathcal{D}}_a \otimes (\mathbf{\Delta} + \boldsymbol{\mathcal{D}}_b^{-1}) + (\mathbf{\Delta} + \boldsymbol{\mathcal{D}}_a^{-1}) \otimes \boldsymbol{\mathcal{D}}_b \;, \\ \Xi_{a,b} &= (\mathbf{\Delta} + \boldsymbol{\mathcal{D}}_a^{-1}) \otimes (\mathbf{\Delta} + \boldsymbol{\mathcal{D}}_b^{-1}) \,, \qquad\qquad a, b = e, o \;, \end{aligned} \tag{3}$$

where

$$\boldsymbol{\mathcal{D}}_e = \mathrm{diag}[4i^2] \;, \qquad \boldsymbol{\mathcal{D}}_0 = \mathrm{diag}[4i^2 + 4i + 1] \;, \qquad i = 1, 2, \ldots, N \;,$$

$$\mathbf{\Delta} = \frac{1}{2} \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & 0 & \\ & & \cdots & \cdots & & \\ & 0 & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{pmatrix} .$$

Moreover, blocks of the matrix $\mathbf{\Lambda}_I$ are spectrally equivalent as well as the blocks of Ξ_I, *e.g.*, for the matrices $\mathbf{\Lambda} = \mathrm{diag}\,[\mathbf{\Lambda}_{e,e}, \mathbf{\Lambda}_{e,e}, \mathbf{\Lambda}_{e,e}, \mathbf{\Lambda}_{e,e}]$ and $\Xi = \mathrm{diag}\,[\Xi_{e,e}, \Xi_{e,e}, \Xi_{e,e}, \Xi_{e,e}]$ we have

$$\mathbf{\Lambda} \prec \mathbf{\Lambda}_I \prec \mathbf{\Lambda} \;, \qquad\qquad \Xi \prec \Xi_I \prec \Xi \,. \tag{4}$$

Therefore, if one has an iterative solver, say, for $\mathbf{\Lambda}_{e,e}$, similar solvers would be applicable to the other blocks of $\mathbf{\Lambda}_I$. Alternatively, one may use the preconditioner $\mathbf{\Lambda}$ with the same solver for all four blocks. In the both cases, it is sufficient to derive a fast solver for one block $\mathbf{\Lambda}_{e,e}$.

Remark 4.1. *The reference element internal stiffness matrix* $\mathbf{A}_I$ *and the mass matrix* $\mathbf{M}_I$ *have the same pattern of nonzero entries with the matrices* $\mathbf{\Lambda}_I$ *and* Ξ_I*, respectively. In particular, at the pointed out special ordering of unknowns,* $\mathbf{A}_I = \mathrm{diag}[\mathbf{A}_{e,e}, \mathbf{A}_{o,e}, \mathbf{A}_{e,o}, \mathbf{A}_{o,o}]$ *and* $\mathbf{M}_I = \mathrm{diag}[\mathbf{M}_{e,e}, \mathbf{M}_{o,e}, \mathbf{M}_{e,o}, \mathbf{M}_{o,o}]$ *with the blocks defined similarly to (3), but with differently defined diagonal and tridiagonal matrices, participating in Kronecker's products. For the explicit expressions see [16] and [22].*

`Finite-difference like preconditioners for the cubic reference element.` In the 3-d case, we assume an ordering of the basis polynomials that is similar to that one described for the square reference element. This results in the preconditioner

$$\mathbf{\Lambda}_I = \mathrm{diag}\,[\mathbf{\Lambda}_{e,e,e}, \mathbf{\Lambda}_{e,e,o}, \mathbf{\Lambda}_{e,o,e}, ..., \mathbf{\Lambda}_{o,o,e}, \mathbf{\Lambda}_{o,o,o}] \;, \tag{5}$$

with eight independent $N^3 \times N^3$ blocks of evident definitions

$$\mathbf{\Lambda}_{a,b,c} = \boldsymbol{\mathcal{D}}_a \otimes \mathbf{\Delta}_a \otimes \mathbf{\Delta}_c + \mathbf{\Delta}_a \otimes \boldsymbol{\mathcal{D}}_b \otimes \mathbf{\Delta}_c + \mathbf{\Delta}_a \otimes \mathbf{\Delta}_b \otimes \boldsymbol{\mathcal{D}}_c \;, \tag{6}$$

where $\mathbf{\Delta}_a = \mathbf{\Delta} + \boldsymbol{\mathcal{D}}_a^{-1}$ and $a, b, c = e, o$. Estimates (1) and (4) remain where

$$\mathbf{\Lambda} = \mathrm{diag}\,[\mathbf{\Lambda}_{e,e,e}, \mathbf{\Lambda}_{e,e,e}, \mathbf{\Lambda}_{e,e,e}, ..., \mathbf{\Lambda}_{e,e,e}, \mathbf{\Lambda}_{e,e,e}] \;, \tag{7}$$

Still further simplifications are possible. For instance, the terms like $\boldsymbol{\mathcal{D}}_a \otimes \boldsymbol{\mathcal{D}}_b^{-1}$ and $\boldsymbol{\mathcal{D}}_a^{-1} \otimes \boldsymbol{\mathcal{D}}_b^{-1}$ in the 2-d stiffness and mass matrices and terms $\boldsymbol{\mathcal{D}}_a \otimes \boldsymbol{\mathcal{D}}_b \otimes \boldsymbol{\mathcal{D}}_c^{-1}$ in the 3-d stiffness matrices may be omitted at some cost in the condition. Namely, if we replace (6) by

$$\mathbf{\Lambda}_{a,b,c} = \boldsymbol{\mathcal{D}}_a \otimes \mathbf{\Delta} \otimes \mathbf{\Delta} + \mathbf{\Delta} \otimes \boldsymbol{\mathcal{D}}_b \otimes \mathbf{\Delta} + \mathbf{\Delta} \otimes \mathbf{\Delta} \otimes \boldsymbol{\mathcal{D}}_c \,, \tag{8}$$

then

$$\mathbf{\Lambda}_I \prec \mathbf{A}_I \prec (1 + \log p)^{d-1} \mathbf{\Lambda}_I \,, \qquad\qquad \mathbf{\Lambda} \prec \mathbf{A}_I \prec (1 + \log p)^{d-1} \mathbf{\Lambda} \,. \tag{9}$$

These estimates follow from Lemma 4.1, (4) and the following inequality (10), see [5],

$$\boldsymbol{\Delta} \prec \boldsymbol{\Delta}_a \prec (\log p)\boldsymbol{\Delta}\,. \tag{10}$$

The preconditioner $\boldsymbol{\Lambda}_I$ served also as a starting point for other slightly different spectrally equivalent or almost spectrally equivalent preconditioners, more suitable for designing fast solvers of several specific types. These modified preconditioners are obtained in three ways: by replacing the coefficients in the finite-difference operator by piecewise constant coefficients, by omitting the minor terms as in (9) at some cost in the condition and by replacing the finite-difference matrix by the finite element one, induced by triangular, square or cubic finite elements. For the discrete problems with the matrices $\boldsymbol{\Lambda}$, $\boldsymbol{\Lambda}_I$, now we can use *DD* like solvers of [19], [18], multigrid solvers of [5], see also [24] for the case of incomplete elements, and multiresolution wavelet solvers [6]. Multigrid solvers have been justified only for the 2-d case. In what follows, we use the solver from the last preprint for the main reason that on its basis we are able to design a fast solver for the face problem by the K-method of interpolation.

4.2. Multiresolution wavelet solver for the internal Dirichlet problems by Beuchler-Schneider-Schwab

In order to simplify our notations, in the following we use $\boldsymbol{\Lambda}_e$ instead of $\boldsymbol{\Lambda}_{e,e}$ and $\boldsymbol{\Lambda}_{e,e,e}$ and similarly for the corresponding blocks of the stiffness and mass matrices of the reference element.

The representation of the preconditioner $\boldsymbol{\Lambda}_e$ by the Kronecker products of two matrices, namely by the stiffness and mass matrices, induced by 1-d bilinear forms, suggests the following approach to solve of the discrete Dirichlet problems on the reference element. Suppose that we are able to find a transformation to the basis, in which both matrices became simultaneously spectrally equivalent to their diagonals uniformly in p. Then, by the properties of the Kronecker product, we get the transformation of $\boldsymbol{\Lambda}_e$ to the matrix spectrally equivalent to its diagonal. Additionally, the matrix, obtained by the backward transformation of this diagonal, may be used as a preconditioner spectrally equivalent to $\boldsymbol{\Lambda}_e$. The computational cost of the preconditioner depends on the cost of the transformation. The existence of such a *multiscale wavelet basis* with transformations which are optimal with respect to the operation count was established in [6]. Here, we very briefly present one of their results.

For simplicity, it is assumed $p = 2N + 1$ and $N + 1 = 2^{l_0}$. For each $l = 1, 2, ..., l_0$, we introduce on $[0, 1]$ the uniform mesh of the size $\hbar_l = 2^{-l}$ and the space $\mathcal{V}_l(0,1)$ of the continuous piecewise linear functions, vanishing at the ends of the interval (0,1). The dimension of $\mathcal{V}_l(0,1)$ is $N_l = 2^l - 1$ with $N_{l_0} = N$. Let $\phi_i^l \in \mathcal{V}_l(0,1)$ be the the nodal basis function for the node x_i, so that $\phi_i^l(x_j) = \delta_{i,j}$ and $\mathcal{V}_l(0,1) = \text{span}\,\big(\phi_i^l\big)_{i=1}^{N_l}$. We consider the matrices

$$\boldsymbol{\Delta}_l = N_l^{-1}\left(\langle(\phi_i^l)', (\phi_j^l)'\rangle_{\omega=1}\right)_{i,j=1}^{N_l}\,,\quad \boldsymbol{\mathcal{M}}_l = N_l\left(\langle\phi_i^l, \phi_j^l\rangle_{\omega=x}\right)_{i,j=1}^{N_l}\,,\quad \langle u, v\rangle_\omega = \int_0^1 \omega^2 u(x)v(x)\,dx\,.$$

We have $\boldsymbol{\Delta}_{l_0} = \boldsymbol{\Delta}$, and it is easy to establish that $\boldsymbol{\mathcal{M}} = \boldsymbol{\mathcal{M}}_{l_0}$ is spectrally equivalent to $\boldsymbol{\mathcal{D}}_e^{-1}$ uniformly in N. Now, we may replace the preconditioner for $\boldsymbol{\Lambda}_e$ by

$$\boldsymbol{\mathcal{A}}_e = \begin{cases} \boldsymbol{\mathcal{M}} \otimes \boldsymbol{\Delta} + \boldsymbol{\Delta} \otimes \boldsymbol{\mathcal{M}}\,, & d = 2\,,\\ \boldsymbol{\mathcal{M}} \otimes \boldsymbol{\Delta} \otimes \boldsymbol{\Delta} + \boldsymbol{\Delta} \otimes \boldsymbol{\mathcal{M}} \otimes \boldsymbol{\Delta} + \boldsymbol{\Delta} \otimes \boldsymbol{\Delta} \otimes \boldsymbol{\mathcal{M}}\,, & d = 3\,, \end{cases} \tag{11}$$

and from (1), (4) and (10) it follows that

$$c_1\boldsymbol{\mathcal{A}}_e \leqslant \mathbf{A}_e \leqslant c_2(1+\log p)^{d-1}\boldsymbol{\mathcal{A}}_e\,. \tag{12}$$

For the reason that $\boldsymbol{\mathcal{M}}$ is a finite element matrix, as is $\boldsymbol{\Delta}$, it is more convenient for the multiresolution analysis.

The representation of each $\mathcal{V}_l$ by the sum $\mathcal{V}_l = \mathcal{V}_{l-1}\oplus\mathcal{W}_{l-1}$ results in the decomposition

$$\mathcal{V} = \mathcal{W}_{-1}\oplus\mathcal{W}_0\oplus ..\oplus\mathcal{W}_{l_0-1}$$

with the notations $\mathcal{V} = \mathcal{V}_{l_0}$ and $\mathcal{W}_{-1} = \mathcal{V}_0$. By $(\psi_k^l)_{k=1,l=-1}^{N_l,l_0}$ is denoted the *multiscale wavelet basis*, composed of *single scale* bases $(\psi_k^l)_{k=1,2,..,N_l}$. The multiscale wavelet basis is, evidently, the Riesz basis in $\mathcal{V}$ and it induces the matrices

$$\begin{array}{ll} \boldsymbol{\Delta}_{\text{wavelet}} = \big(\langle(\psi_i^k)',(\psi_j^l)'\rangle_\omega\big)_{i,j=1,kl=-1}^{N_l,l_0}\,, & \boldsymbol{\mathcal{M}}_{\text{wavelet}} = \big(\langle\psi_i^k,\psi_j^l\rangle_\omega\big)_{i,j=1,kl=-1}^{N_l,l_0}\,, \\ \mathbb{D}_1 = \operatorname{diag}[\langle(\psi_i^l)',(\psi_i^l)',\rangle]_{i=1,l=-1}^{N_l,l_0}\,, & \mathbb{D}_0 = \operatorname{diag}[\langle\psi_i^l,\psi_i^l,\rangle]_{i=1,l=-1}^{N_l,l_0}\,. \end{array} \tag{13}$$

By $\mathbf{Q}$ is denoted the transformation matrix from the wavelet basis to the basis $(\phi_k^{l_0})_{k=1}^N$. If $\mathbf{v}$ and $\mathbf{v}_{\text{wavelet}}$ are the vectors of the coefficients of a function from $\mathcal{V}_l(0,1)$ in the nodal and wavelet bases, respectively, then $\mathbf{Q}\,\mathbf{v} = \mathbf{v}_{\text{wavelet}}$. The result of the multiresolution analysis of [6], which they used for hp-version preconditioning, may be formulated as follows.

Theorem 4.1. *There exist multiresolution wavelet bases such that the matrices* $\boldsymbol{\Delta}_{\text{wavelet}}$ *and* $\boldsymbol{\mathcal{M}}_{\text{wavelet}}$ *are simultaneously spectrally equivalent to their diagonals* $\mathbb{D}_1$ *and* $\mathbb{D}_0$*, respectively, uniformly in* N*, and that the transformations* $\mathbf{Q}\,\mathbf{v}$ *and* $\mathbf{Q}^T\,\mathbf{v}_{\text{wavelet}}$ *require* $\mathcal{O}(N)$ *arithmetic operations.*

The next result follows immediately from Theorem 4.1 by the spectral equivalence of the matrices $\mathbf{M}$ and $\boldsymbol{\mathcal{D}}_e^{-1}$ uniform in N, which result, in particular, in the relationship

$$\boldsymbol{\mathcal{A}}_{n\leftarrow w} \asymp \boldsymbol{\mathcal{A}}_e\,, \tag{14}$$

and the properties of the Kronecker product.

Theorem 4.2. *Let*

$$\boldsymbol{\mathcal{A}}_{n\leftarrow w}^{-1} = \left\{\begin{array}{ll} (\mathbf{Q}^T\otimes\mathbf{Q}^T)[\,\mathbb{D}_0\otimes\mathbb{D}_1+\mathbb{D}_1\otimes\mathbb{D}_0\,]^{-1}\,(\mathbf{Q}\otimes\mathbf{Q})\,, & d=2\,, \\ (\mathbf{Q}^T\otimes\mathbf{Q}^T\otimes\mathbf{Q}^T)\,[\,\mathbb{D}_0\otimes\mathbb{D}_1\otimes\mathbb{D}_1+\mathbb{D}_1\otimes\mathbb{D}_0\otimes\mathbb{D}_1+ & \\ \qquad\qquad \mathbb{D}_1\otimes\mathbb{D}_1\otimes\mathbb{D}_0\,]^{-1}\,(\mathbf{Q}\otimes\mathbf{Q}\otimes\mathbf{Q})\,, & d=3\,, \end{array}\right. \tag{15}$$

then $\operatorname{cond}[\boldsymbol{\mathcal{A}}_{n\leftarrow w}^{-1}\mathbf{A}_e] \prec (1+\log p)^{d-1}$.

In support of their analysis, [6] reported results of numerical experiments with some specific wavelet bases, used for preconditioning 2-d reference element stiffness matrix.

4.3. Multiresolution wavelet solver for the face problems

We use here the result of Theorem 4.1 for deriving a fast solver for the face problem. Namely, similarly with (15) we define a multilevel wavelet preconditioner for the mass matrix of 2-d reference element, then we use the K-method of interpolation for obtaining the matrix equivalent to the quadratic form ${}_{00}|\cdot|^2_{1/2,F_k}$, and then transform this matrix from the wavelet basis to the initial one.

We consider the mass matrix $\mathbf{M}_I$ for the 2-d reference element. From Lemma 4.1, (4) and (10), it follows that the matrix

$$\mathbb{M}_\triangle = \operatorname{diag}[\mathbb{M}_e, \mathbb{M}_e, \mathbb{M}_e, \mathbb{M}_e], \qquad \mathbb{M}_e = \boldsymbol{\Delta} \otimes \boldsymbol{\Delta},$$

is a sufficiently good preconditioner for $\mathbf{M}_I$, which satisfies the inequalities

$$\mathbb{M}_\triangle \prec \mathbf{M}_I \prec (1+\log p)^2 \mathbb{M}_\triangle . \tag{16}$$

Further, in the same way as Theorem 4.2, it is proved that

$$\mathbb{M}_e \asymp (\mathbf{Q}^T \otimes \mathbf{Q}^T)\, \mathbb{D}_1 \otimes \mathbb{D}_1\, (\mathbf{Q} \otimes \mathbf{Q}) . \tag{17}$$

Let us denote the entries of the diagonal matrices $\mathbb{D}_0$ and $\mathbb{D}_1$ via $d_{0,i}$ and $d_{1,i}$, respectively. The diagonal entries of the matrix $\mathbb{D}_{\Delta,e} := \mathbb{D}_0 \otimes \mathbb{D}_1 + \mathbb{D}_1 \otimes \mathbb{D}_0$ are $d^{(1)}_{i,j} = d_{1,j} d_{0,i} + d_{1,i} d_{0,j}$. Similarly, the diagonal matrix $\mathbb{D}_{M,e} := \mathbb{D}_1 \otimes \mathbb{D}_1$ has the diagonal entries $d^{(o)}_{i,j} = d_{1,i} d_{1,j}$. Now, we define the diagonal matrix $\mathbb{D}_{1/2}$ with the entries

$$d^{(1/2)}_{i,j} = d_{1,i} d_{1,j} \sqrt{\frac{d_{0,i}}{d_{1,i}} + \frac{d_{0,j}}{d_{1,j}}} \tag{18}$$

and the preconditioner

$$\boldsymbol{\mathcal{S}}_{00} = \operatorname{diag}[\boldsymbol{\mathcal{S}}_{n\leftarrow w}, \boldsymbol{\mathcal{S}}_{n\leftarrow w}, \boldsymbol{\mathcal{S}}_{n\leftarrow w}, \boldsymbol{\mathcal{S}}_{n\leftarrow w}], \qquad \boldsymbol{\mathcal{S}}^{-1}_{n\leftarrow w} = (\mathbf{Q}^T \otimes \mathbf{Q}^T)\, \mathbb{D}^{-1}_{1/2}\, (\mathbf{Q} \otimes \mathbf{Q}) . \tag{19}$$

Theorem 4.3. *The inequalities (18) hold with $\gamma_{F,1} \succ 1$, $\gamma_{F,2} \prec (1+\log p)^{3/2}$, and therefore* $\operatorname{cond}[\boldsymbol{\mathcal{S}}^{-1}_{00} \mathbb{S}_{00}] \prec (1+\log p)^{3/2}$.

Proof. Let us consider any $v_F \in \mathcal{U}_F$, the corresponding vector $\mathbf{v}_F \in U_F$ and $\mathbf{v}_{F,w} := (\mathbf{Q} \otimes \mathbf{Q})\, \mathbf{v}_F$. According to (12),(14),(15), (16) and (17) we have the estimates

$$\begin{aligned} (\mathbf{v}_{F,w}, \mathbb{D}_M \mathbf{v}_{F,w}) \prec \|v_F\|_{0,\tau_0} \prec (1+\log p)^2 (\mathbf{v}_{F,w}, \mathbb{D}_M \mathbf{v}_{F,w}) , \\ (\mathbf{v}_{F,w}, \mathbb{D}_\Delta \mathbf{v}_{F,w}) \prec |v_F|_{1,\tau_0} \prec (1+\log p) (\mathbf{v}_{F,w}, \mathbb{D}_\Delta \mathbf{v}_{F,w}) , \end{aligned} \tag{20}$$

where

$$\mathbb{D}_M = \operatorname{diag}[\mathbb{D}_{M,e}, .., \mathbb{D}_{M,e}], \qquad \mathbb{D}_\Delta = \operatorname{diag}[\mathbb{D}_{\Delta,e}, ..., \mathbb{D}_{\Delta,e}] .$$

For any face F_k, we have ${}_{00}\|v_F\|_{1/2,F_k} \asymp (\mathbf{v}_{F_k}, \mathbb{S}_{00} \mathbf{v}_{F_k})$, and besides ${}_{00}\|\cdot\|_{1/2,F_k}$ may be defined by the interpolation beteen $\|\cdot\|_{0,F_k}$ and $\|\cdot\|_{1,F_k}$ for the finctions from the space $H^1(F_k)$, vanishing on ∂F_k. Therefore, we obtain the result by the interpolation between inequalities (20) and the transformation to the initial basis. $\square$

5. Iterative Schur complement and DD algorithms and their complexity

5.1. Schur complement algorithms

First we give a brief summary of the algorithm, based on the components described in the preceding sections. We assume that the system

$$\mathbf{S}_B \mathbf{v}_B = \mathbf{f}$$

is solved by the method of PCG with the preconditioner $\boldsymbol{\mathcal{S}}_B$ and, therefore, on each PCG iteration it is necessary to complete the multiplication $\boldsymbol{\mathcal{S}}_B^{-1}\mathbf{v}_B =: \mathbf{w}_B$, where the preconditioner $\boldsymbol{\mathcal{S}}_B$ was described in Section 3. The sequence of the procedures realizing this multiplication is the following.

1. Find $\mathbf{u}_F = \mathbf{u}_F^{k_o}$ from the iteration process
$$\mathbf{u}_F^{k+1} = \mathbf{u}_F^k - \sigma_{k+1}\boldsymbol{\mathcal{S}}_F^{-1}(\mathbf{S}_F\mathbf{u}_F^k - \mathbf{v}_F)\,, \qquad \mathbf{u}_F^0 = \mathbf{0}\,. \tag{1}$$
2. Find $\mathbf{u}_W := \mathbf{v}_W - \mathbf{S}_{WF}\mathbf{u}_F$ and solve the wire basket subsystem $\boldsymbol{\mathcal{S}}_W\mathbf{w}_W = \mathbf{u}_W$.
3. Find $\mathbf{u}_{FW} = \mathbf{u}_F^{k_o}$ from the iteration process (1) at $\mathbf{v}_F = -\mathbf{S}_{FW}\mathbf{w}_W$.
4. For the wire basket and face subvectors of $\mathbf{w}$, set $\mathbf{w}_W$, found in 2, and $\mathbf{w}_F = \mathbf{u}_F + \mathbf{u}_{FW}$, respectively.

It is assumed that at solving the wire basket subsystem we produce some fixed number of iterations, which allow to reduce the relative error in the norm $\|\cdot\|_{\boldsymbol{\mathcal{S}}_W}$, *e.g.,* in half.

Theorem 5.1. *Let the preconditioner* $\boldsymbol{\mathcal{S}}_{00}$ *be defined as in Th. 4.3 and let* $k_o \succ (1+\log p)^{2.75}$. *Then* $\operatorname{cond}[\boldsymbol{\mathcal{S}}_B^{-1}\mathbf{S}_B] \leqslant c(1+\log p)$ *with the constant depending only* $\alpha^{(1)}$ *and* θ *from the generalized angular quasiuniformity conditions.*

Proof. We restrict the proof to the case of the refernce element Schur complement matrix. To the general case of hp-discretization, it is expanded by means of assembling and generalized conditions of angular quasiuniformity. Let $\mathbb{S}_{F,00} := \operatorname{diag}[\mathbb{S}_{00},...,\mathbb{S}_{00}]$. The left inequality of
$$\mathbb{S}_F \prec \mathbb{S}_{F,00} \prec (1+\log p)^2\mathbb{S}_F$$
is the consequence of the Cauchy inequality and $\mathbb{S}_{F_k} \prec \mathbb{S}_{00}$. The right one is a direct consequence of Lemma 5.9 in [27]. Now, on the basis of Proposition 3.2, inequality (16) and Th. 4.3, we conclude that for $k_0 \geqslant 2.75$ inequality (6) holds with $c_{\mathbb{P}} \asymp 1$.

Let us note, that the algorithm, described above, uses the preconditioner, which for the reference element has the form
$$\widetilde{\boldsymbol{\mathcal{S}}}_B = \begin{pmatrix}\mathbf{I} & \mathbf{0}\\ \mathbf{C}^T & \mathbf{I}\end{pmatrix}\begin{pmatrix}\widetilde{\mathbb{S}}_{F,\mathrm{it}} & \mathbf{0}\\ \mathbf{0} & \widetilde{\boldsymbol{\mathcal{S}}}_W\end{pmatrix}\begin{pmatrix}\mathbf{I} & \mathbf{C}\\ \mathbf{0} & \mathbf{I}\end{pmatrix}, \qquad \mathbf{C} = \widetilde{\mathbb{S}}_{F,\mathrm{it}}^{-1}\mathbb{S}_{FW}\,. \tag{2}$$

In order to conclude the proof, we turn to the Proposition 1, from which it follows that
$$\gamma_{1,B}\widetilde{\boldsymbol{\mathcal{S}}}_B \leqslant \mathbb{S}_B \leqslant \gamma_{2,B}\widetilde{\boldsymbol{\mathcal{S}}}_B$$
with $\gamma_{1,B} = \min(\gamma_{1,F},\gamma_{1,W})$ and $\gamma_{2,B} = \max(\gamma_{2,F},\gamma_{2,W})$, where $\gamma_{k,F},\gamma_{k,W}$ are taken from the inequalities
$$\gamma_{1,F}\widetilde{\mathbb{S}}_{F\mathrm{it}} \leqslant \mathbb{S}_F \leqslant \gamma_{2,F}\widetilde{\mathbb{S}}_{F\mathrm{it}}\,, \qquad \gamma_{1,W}\widetilde{\boldsymbol{\mathcal{S}}}_W \leqslant \widehat{\mathbb{S}}_W \leqslant \gamma_{2,W}\widetilde{\boldsymbol{\mathcal{S}}}_W\,.$$
Standard arguments for the use of the iterative preconditioners from Proposition 3.2 leads to
$$\widetilde{\mathbb{S}}_{F,\mathrm{it}} \asymp \mathbb{S}_F\,.$$
Also, we have the inequality
$$(\mathbf{v}_W, \widehat{\mathbb{S}}_W\mathbf{v}_W) \prec ||v-\overline{v}||^2_{0,W}\,, \tag{3}$$

the easiest proof of which may be obtained by means of the simple prolongation operator, used in [9], see Lemma 9. Since the prolongation operator $\mathbb{P}_a$ satisfies (6), it follows directly from the cited Lemma and the definition of the matrix $\widehat{\mathbb{S}}_W$. If we take into account the definition of the preconditioner $\widetilde{\boldsymbol{\mathcal{S}}}_W$, inequalities (3) and (13) mean that $\gamma_{1,W} \succ 1/(1+p)$ and $\gamma_{2,W} \asymp 1$, which completes the proof. $\square$

According to Theorems 4.1 and 5.1, the total arithmetical cost of the preconditioner[2] $\mathcal{S}_B$ is $\mathcal{O}((1+\log p)^{3.25}p^2)$.

Another Schur complement solver may be based on the preconditioner, having for the reference element the form

$$\widetilde{\boldsymbol{\mathcal{S}}}_B = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{C}^T & \mathbf{I} \end{pmatrix} \begin{pmatrix} \widetilde{\boldsymbol{\mathcal{S}}}_F & \mathbf{0} \\ \mathbf{0} & \widetilde{\boldsymbol{\mathcal{S}}}_W \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathbf{C} \\ \mathbf{0} & \mathbf{I} \end{pmatrix},$$

with $\mathbb{P}_a \equiv \mathbf{I}_{W\to B}$ defined by the expressions (5),(20) and (21).

Theorem 5.2. *Let the preconditioner $\boldsymbol{\mathcal{S}}_{00}$ be defined as in Th. 4.3, and the prolongation operators for finite elements be $\mathbb{P}_a \equiv \mathbf{I}_{W\to B}$. Then* $\operatorname{cond}[\boldsymbol{\mathcal{S}}_B^{-1}\mathbf{S}_B] \leqslant c(1+\log p)^{3.5}$ *with the constant depending only on $\alpha^{(1)}$ and θ from the generalized angular quasiuniformity conditions.*

Proof. The proof follows the same lines as the one of Theorem 5.1. However, instead of (3) we have now

$$(\mathbf{v}_W, \widehat{\mathbb{S}}_W \mathbf{v}_W) \prec (1+\log p)||v-\overline{v}||^{\,2}_{0,W}\,. \tag{4}$$

For the proof of (4), one may use the results of [9]. It may be shown that our prolongation, though different from the one used in this paper, is close to it and is bounded in the same energy norm uniformly in p. Then, Lemma 9 of [9] may be applied to estimate $\widetilde{\mathbf{I}}_{W\to B}$. $\square$

5.2. DD algorithm with the inexact solvers for basic components

The drawback of the Schur complement iterative method presented above is that it assumes the elimination of the internal unknowns and the multiplication by the Schur complement $\boldsymbol{\mathcal{S}}_B$ at each iteration. The both may be rather expensive. The matrix $\boldsymbol{\mathcal{S}}_B$ may be almost completely filled in even in the case on the square finite element mesh and, therefore, multiplication of it by a vector may require $\mathcal{O}(\mathcal{R}^2p^4)$ operations.

Multiplication by the global stiffness matrix may be the most costly operation of a *DD* algorithm. In other to avoid additional costly multiplications by the internal element stiffness matrices in operations of iterative restriction to and prolongation from the interface, we construct *DD* preconditioner completely on the basis of the reference element.

Since we assume that the *DD* preconditioner is defined by assembly of the properly scaled preconditioners and prolongation operators for the reference element, again it is sufficient to describe the master preconditioner for the reference element. It is defined in the factorized form

$$\boldsymbol{\mathcal{A}} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathrm{G}^T & \mathbf{I} \end{pmatrix} \begin{pmatrix} \boldsymbol{\mathcal{B}}_I & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\mathcal{B}}_B \end{pmatrix} \begin{pmatrix} \mathbf{I} & \mathrm{G} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \tag{5}$$

[2]Let us remember that we do not include in the cost of preconditioning operations the multiplications by matrices of the systems to be solved.

with the simplest choice $\boldsymbol{\mathcal{B}}_I = \boldsymbol{\mathcal{A}}_{n\leftarrow w}$ as in Th. 4.2 and $\boldsymbol{\mathcal{B}}_B = \widetilde{\boldsymbol{\mathcal{S}}}_B$ as in Th. 5.2. However, since we define the prolongation operator $\mathbb{P}_{U_B\to U}$ by the iterative process, it is necessary to adjust the preconditioners $\boldsymbol{\mathcal{B}}_I$ and $\boldsymbol{\mathcal{B}}_B$ appropriately.

Proposition 5.1 *Let us assume that the spectral equivalence inequilities $\gamma_{I,1}\boldsymbol{\mathcal{A}}_I \leqslant \mathbf{A}_I \leqslant \gamma_{I,2}\boldsymbol{\mathcal{A}}_I$ are valid with positive spectral equivalence constants. Further, we also suppose that, for any $\mathbf{v}_B \in U_B$, the prolongation $\mathbf{u} = \mathbb{P}_{U_B\to U}\mathbf{v}_B$ is defined in such a way that $\mathbf{u}_B = \mathbf{v}_B$ and the subvector $\mathbf{u}_I = \mathbf{u}_I^{l_o}$ is found from the iteration process*

$$\mathbf{u}_I^{k+1} = \mathbf{u}_I^k - \sigma_{k+1}\boldsymbol{\mathcal{A}}_I^{-1}(\mathbf{A}_I\mathbf{u}_I^k - \mathbf{A}_{IB}\mathbf{v}_B)\,, \qquad \mathbf{u}_F^0 = \mathbf{0}\,, \tag{6}$$

with Chebyshev iteration parameters σ_k for some fixed number l_o of iterations. Then at $l_o \geqslant c(1+\log p))/(\log\varrho^{-1})$, where $\varrho = (1-\theta)/(1+\theta)$, $\theta = \sqrt{\gamma_{I,1}/\gamma_{I,2}}$, the inequality

$$\|\mathbb{P}_{U_B\to U}\mathbf{v}_B\|_{\mathbf{A}} \leqslant c_{\mathbb{P}}\|\mathbf{v}_B\|_{\mathbb{S}_B} \tag{7}$$

holds with the constant $c_{\mathbb{P}}$ independent of p.

Let us note that, the inequality (7) is equivalent to

$$|u|_{1,\tau_0} \prec c_{\mathbb{P}}|v_B|_{1/2,\partial\tau_0} \tag{8}$$

where $u \leftrightarrow \mathbf{u}$ and $v_B \leftrightarrow \mathbf{v}_B$. If $\mathbf{v}_B$ is a constant vector, then $\mathbf{A}_{IB}\mathbf{v}_B = \mathbf{0}$ and $\mathbf{u}_I^k = \mathbf{0}$ for $k \geqslant 1$.

Proof. The proof is simpler than the proof of Proposition 3.2, but deals with different norms. Let the vector $\boldsymbol{\varphi}_I$ be the solution of the system $\mathbf{A}_I\boldsymbol{\varphi}_I = \mathbf{A}_{IB}\mathbf{v}_B$ and $\boldsymbol{\varphi}$ has for the subvectors $\boldsymbol{\varphi}_I$ and $\mathbf{v}_B$. We have the convergence estimate

$$\|\mathbf{u}_I^k - \boldsymbol{\varphi}_I\|_{\mathbf{A}_I} \leqslant \varrho^k\|\boldsymbol{\varphi}_I\|_{\mathbf{A}_I}\,, \tag{9}$$

from which follows

$$\|\mathbf{u}^k - \boldsymbol{\varphi}\|_{\mathbf{A}} \leqslant \varrho^k(\|\boldsymbol{\varphi}\|_{\mathbf{A}} + \|\mathbf{v}_B\|_{\mathbf{A}_B})\,, \qquad \mathbf{u}^k = \mathbf{u}_I^k + \mathbf{v}_B\,, \tag{10}$$

Simple estimates, Markov's inequality and trace continuation theorems in polynomial spaces, equipped with Sobolev's norms, see [3] and [4], allow us to obtain

$$\|\mathbf{v}_B\|_{\mathbf{A}_B} \equiv |v_B|_{1,\tau_0} \prec |v_B|_{1,\partial\tau_0} \prec p^2|v_B|_{1/2,\partial\tau_0} \prec p^2\|\mathbf{v}_B\|_{\mathbb{S}_B}\,. \tag{11}$$

and

$$\|\boldsymbol{\varphi}\|_{\mathbf{A}} \equiv \|\mathbf{v}_B\|_{\mathbb{S}_B} \prec |v_B|_{1/2,\partial\tau_0} \tag{12}$$

Combining (10),(11) and (12), we get

$$\|\mathbf{u}^k\|_{\mathbf{A}} \prec (1+\varrho^k(1+p^2)\|\mathbf{v}_B\|_{\mathbb{S}_B}\,,$$

from where the proposition immediately follows. □

Corollary 5.1. *Let $\boldsymbol{\mathcal{A}}_I = \boldsymbol{\mathcal{A}}_{n\leftarrow w}$, then $l_0 \asymp (1+\log p)^2$.*

Formally, the prolongation operator, defined in Proposition 5.1 and Corollary 5.1, is

$$\mathbb{P}_{U_B\to U} = \begin{pmatrix} -\mathbf{G} \\ I \end{pmatrix}\,, \quad \mathbf{G} = \boldsymbol{\mathcal{A}}_{I,\mathrm{it}}^{-1}\mathbf{A}_{IB}\,, \quad \boldsymbol{\mathcal{A}}_{I,\mathrm{it}}^{-1} = [\mathbf{I} - \prod_{k=1}^{l_o}(\mathbf{I} - \sigma_k\boldsymbol{\mathcal{A}}_{n\leftarrow w}^{-1}\mathbf{A}_I)]\,\mathbf{A}_I^{-1}\,. \tag{13}$$

Assuming that the matrix $\mathbf{G}$ is defined according to (13), it is reasonable to set $\boldsymbol{\mathcal{B}}_I = \boldsymbol{\mathcal{A}}_{I,\mathrm{it}}$.

Let the master preconditioner (5) be given. By $\boldsymbol{\mathcal{K}}$ we denote the *DD* preconditioner for the global stiffness matrix $\mathbf{K}$ and may define it in the similar to (5) factorized form or by the equivalent formula

$$\boldsymbol{\mathcal{K}}^{-1} = \boldsymbol{\mathcal{K}}_I^{+} + \mathbf{P}_{U_B\to U}\boldsymbol{\mathcal{S}}_B^{-1}\mathbf{P}_{U_B\to U}^{T}\,. \tag{14}$$

For the matrix $\boldsymbol{\mathcal{K}}_I$, we take the block diagonal matrix $\boldsymbol{\mathcal{K}}_I = \mathrm{diag}[\boldsymbol{\mathcal{K}}_{I,r}]_{r=1}^{\mathcal{R}}$ with each block $\boldsymbol{\mathcal{K}}_{I,r} = h_r\varrho_r\boldsymbol{\mathcal{B}}_I$. The prolongation matrix $\mathbf{P}_{U_B\to U}$ may be assumed such, that its restriction to each element (at the local ordering of the degrees of freedom of an element corresponding to the ordering of degrees of freedom of the reference element) is $\mathbb{P}_{U_B\to U}$. We set $\boldsymbol{\mathcal{B}}_B = \boldsymbol{\mathcal{S}}_B$, assuming the way of assembling of the matrix $\boldsymbol{\mathcal{S}}_B$ which was described earlier.

Theorem 5.3. *Let the DD preconditioner $\boldsymbol{\mathcal{K}}$ is assembled from the master preconditioner (5) with the prolongation operator (13), $\boldsymbol{\mathcal{B}}_I = \boldsymbol{\mathcal{A}}_{I,\mathrm{it}}$ and $\boldsymbol{\mathcal{B}}_B = \widetilde{\boldsymbol{\mathcal{S}}}_B$. Then* $\mathrm{cond}\,[\boldsymbol{\mathcal{K}}^{-1}\mathbf{K}] \leqslant c(1+\log p)^{3.5}$

Proof. Let us consider the inequalities

$$\gamma_1\boldsymbol{\mathcal{A}} \leqslant \mathbf{A} \leqslant \gamma_2\boldsymbol{\mathcal{A}}\,, \quad \gamma_{I,1}\boldsymbol{\mathcal{A}}_I \leqslant \mathbf{A}_{I,\mathrm{it}} \leqslant \gamma_{I,2}\boldsymbol{\mathcal{A}}_I\,, \quad \gamma_{B,1}\boldsymbol{\mathcal{S}}_B \leqslant \mathbb{S}_B \leqslant \gamma_{B,2}\boldsymbol{\mathcal{S}}_B \tag{15}$$

The proof of the fact that $\gamma_1 = \min(\gamma_{I,1},\gamma_{B,1})$ and $\gamma_2 = \max(\gamma_{I,2},\gamma_{B,2})$ with the use of Proposition 5.1 and Corollary 5.1 is the same as of the Proposition 3.1. Similarly to Proposition 5.1, it may be shown that $\gamma_{I,k}$ are constants, whereas $\gamma_{B,k}$ have been estimated in the proof of Th. 5.2. Therefore, $\mathrm{cond}\,[\boldsymbol{\mathcal{A}}^{-1}\mathbf{A}] \leqslant c(1+\log p)^{3.5}$. The proof of the theorem follows by assembling and use of the generalized conditions of the angular quasiuniformity. $\square$

The asymptotical cost of the presented *DD* preconditioner is $(1+\log p)^{1.75}[p^3\mathcal{R}+p\mathcal{R}^2]$ with the terms in the square brackets corresponding to solving the internal and edge problems, respectively.

In closing, let us underline two features of the presented algorithms. The main bulk of computations may be done in parallel, *e.g.,* for all interiors of finite elements, faces, edges and vertex problem. If for the Dirichlet problems on finite elements secondary *DD*-type or wavelet based multilevel inexact solvers are implemented, a deeper parallelization is easily arranged. In part due to this feature, our *DD* solvers are convenient for adaptive computations. This is for the reason that they are fully applicable, when for the interior of each finite element, each face and edge the orders the polynomial subspaces (on the respective reference configuration) are different.

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