Symbolic Differential Elimination for Symmetry Analysis

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Abstract

Differential problems are ubiquitous in mathematical modeling of physical and scientific problems. Algebraic analysis of differential systems can help in determining qualitative and quantitative properties of solutions of such systems. We describe several algebraic methods for investigating differential systems.

1 Introduction

The idea of an algebraic approach to differential equations (DEs) has a long history. In the 19th century Lie initiated the investigation of transformations, which leave a given differential equation invariant. Such transformations are commonly known as Lie symmetries. They form a group, a so-called Lie group. The basic idea here is to find a group of symmetries of a differential equations and then use this group to reduce the order or the number of variables appearing in the equation. Lie discovered that the knowledge of a one-parameter group of symmetries of an ordinary differential equation of order n allows us to reduce the problem of solving this equation to that of solving a new differential equation of order $n-1$ and integrating.

From the Riquier-Janet theory of PDEs at the beginning of the 20th century an algorithm emerged, the Janet bases algorithm, which is strikingly similar to the method of Gröbner bases for generating canonical systems for algebraic ideals as developed by Buchberger. By computing the Janet basis for the coefficients of the Lie symmetries of a differential equation, the determining system of these coefficients can be triangularized and ultimately solved. In fact, for linear systems of DEs we can directly apply Gröbner bases.

In symbolic treatment of DEs the ultimate goal should be a symbolic solution. However, that this is rarely achieved. But it is also of great importance to decide whether a system of DEs is solvable. If there are solutions, then we can derive differential systems in triangular form such that the solutions of the original system are the (non-singular) solutions of the output system. Deriving symmetries helps in verifying numerical schemes for solution approximation. In case the given system consists of differential algebraic equations (DAEs) we may get a complete overview of the algebraic relations which the solutions must satisfy.

The importance of computer algebra tools in this field is enormous. It can be demonstrated by comparing the impact made by symmetry analysis and differential Galois theory. The latter one is a hardly known theory studied by a few pure mathematicians. The former remained in the same state for many decades following Lie's original work. The main reason for this historical factum is definitely the tedious determination of the symmetry algebra.

As soon as computer algebra systems emerged, the first packages to set up at least the determining equations were written. An effective symbolic treatment of differential problems depends crucially on algorithms in differential elimination theory. While the algebraic theory of elimination is well developed, for differential ideals, there are still many open problems. For instance, the membership problem or the ideal inclusion problems for finitely generated differential ideals are still not solved (compare [2]).

The aim of this paper is the treatment of some aspects of differential elimination theory: differential Gröbner bases, involutive bases, characteristic sets, symmetry analysis.

2 Group Analysis of DEs

In this section, we present the method of group analysis of DEs by demonstrating its use in simplifying and integrating ODEs and PDEs.

We first introduce basic notions for symmetries of ODEs. These concepts extend to the case of partial differential equations, too.

2.1 Symmetries of ODEs

We introduce *transformation groups* and their *differential invariants*, which determine the *invariant equations* corresponding to the group. The differential invariants are solutions of a system of PDEs, called *system of differential invariants*.

2.1.1 Transformation Groups of Differential Equations

Introducing new variables into a given DE is a widely used method in order to facilitate the solution process. Usually this is done in an ad hoc manner without guaranteed success. In particular, there is no criterion to decide whether a certain class of transformations will lead to an integrable equation or not. A critical examination of these methods was the starting point for Lie's symmetry analysis. We will now have a look on the behavior of DEs under special kind of transformations.

Let an ODE of order n be given as

$$\omega(x, y, y', \ldots, y^{(n)}) = 0. \tag{1}$$

The general solution of such an equation is a set of curves in the x-y-plane depending on n parameters $C_1, \ldots, C_n$, given by

$$\Theta(x, y, C_1, \ldots, C_n) = 0. \tag{2}$$

Invertible analytic transformations between two planes with coordinates (x, y) and (u, v), respectively, that are of the form

$$u = \sigma(x, y), \quad v = \rho(x, y), \tag{3}$$

are called *point transformations*. We will encounter them in the form of *one-parameter groups of point transformations*

$$u = \sigma(x, y, \varepsilon), \quad v = \rho(x, y, \varepsilon). \tag{4}$$

Here the real parameter ε ranges over an open interval including 0, such that for any fixed value of ε, equation (4) represents a point transformation. In addition, there exists a real group composition Φ such that

$$\bar{x} = \sigma(x, y, \varepsilon), \quad \bar{y} = \rho(x, y, \varepsilon), \qquad \bar{\bar{x}} = \sigma(\bar{x}, \bar{y}, \bar{\varepsilon}), \quad \bar{\bar{y}} = \rho(\bar{x}, \bar{y}, \bar{\varepsilon})$$

$$\Longrightarrow \quad \bar{\bar{x}} = \sigma(x, y, \Phi(\varepsilon, \bar{\varepsilon})), \quad \bar{\bar{y}} = \rho(x, y, \Phi(\varepsilon, \bar{\varepsilon})).$$

Group transformations of this kind may be reparametrized such that we have $\Phi(\varepsilon, \bar{\varepsilon}) = \varepsilon + \bar{\varepsilon}$, and such that $\varepsilon = 0$ represents the identity element.

An equation (1) is said to be *invariant* under the change of variables (3) where $v \equiv v(u)$, if it retains its form under this transformation, i.e. if the functional dependence of the transformed equation on u and v is the same as in the original equation (1). Such a transformation is called a *symmetry* of the DE. The same transformation acts on the curves (2). If it is a symmetry, the functional dependence of the transformed curves of u and v must be the same as in (2). This is not necessarily true for the parameters $C_1, \ldots, C_n$ because they do not occur in the DE itself. This means, the entirety of curves described by the two equations is the same, to any fixed values for the constants however may correspond a different curve in either set. In other words the solution curves are permuted among themselves by a symmetry transformation. It is fairly obvious that all symmetry transformations of a given DE form a group, the *symmetry group* of that equation.

2.1.2 Infinitesimal Generators and Prolongations

Let a curve in the $(x$-$y)$-plane described by $y = f(x)$ be transformed under a point transformation of the form (3) into $v = g(u)$. Now the question arises how the derivative $y' = \frac{df}{dx}$ corresponds to $v' = \frac{dg}{du}$ under this transformation. A simple calculation leads to the *first prolongation*

$$v' = \frac{dv}{du} = \frac{\rho_x + \rho_y y'}{\sigma_x + \sigma_y y'} \equiv \chi(x, y, y').$$

Note that the knowledge of (x, y, y') and the equations of the point transformation (3) already determine v' uniquely, the knowledge of the equation of the curve is not required. This may be expressed by saying that the line element (x, y, y') is transformed into the line element (u, v, v') under the action of a point transformation. Similarly, the transformation law for derivatives of second order is obtained as

$$v'' = \frac{dv'}{du} = \frac{\chi_x + \chi_y y' + \chi_{y'} y''}{\sigma_x + \sigma_y y'}.$$

For later applications it would be useful to express the second derivative v'' explicitly in terms of σ and ρ. We do not give this more lengthy formula here, but instead provide the prolongation formulas for one-parameter groups of point transformations of the form

$$u = \sigma(x, y, \varepsilon), \quad v = \rho(x, y, \varepsilon). \tag{5}$$

Here the transformation properties of the derivatives may be expressed in terms of the prolongation of the corresponding *infinitesimal generator*

$$X = \xi(x, y)\partial_x + \eta(x, y)\partial_y, \tag{6}$$

where

$$\xi(x, y) = \frac{d}{d\varepsilon}\sigma(x, y, \varepsilon)\bigg|_{\varepsilon=0}, \qquad \eta(x, y) = \frac{d}{d\varepsilon}\rho(x, y, \varepsilon)\bigg|_{\varepsilon=0}.$$

The n-th prolongation of (6) is now defined in terms of the operator of total differentiation w.r.t. x, denoted by $D = \partial_x + \sum_{k=1}^{\infty} y^{(k)}\partial_{y^{(k-1)}}$ as

$$\begin{aligned} X^{(n)} &= X + \sum_{k=1}^{n} \zeta^{(k)}\partial_{y^{(k)}}, \text{ where} \\ \zeta^{(1)} &= D(\eta) - y' D(\xi), \\ \zeta^{(k)} &= D(\zeta^{(k-1)}) - y^{(k)} D(\xi) \quad \text{for } k = 2, 3, \ldots \end{aligned}$$

We give the two lowest ζ's explicitly:

$$\begin{aligned} \zeta^{(1)} &= \eta_x + (\eta_y - \xi_x)y' - \xi_y y'^2, \\ \zeta^{(2)} &= \eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 \\ &\quad - \xi_{yy}y'^3 + (\eta_y - 2\xi_x)y'' - 3\xi_y y' y''. \end{aligned}$$

These two innocent looking expressions should not distract from the fact that the number of terms in $\zeta^{(k)}$ grows roughly as 2^k. But $\zeta^{(k)}$ is at least linear and homogeneous in $\xi(x, y)$ and $\eta(x, y)$ and its derivatives up to order k. For $k > 1$, $y^{(k)}$ occurs linearly and y' occurs with power $k + 1$ in $\zeta^{(k)}$.

2.1.3 Differential Invariants of Point Transformations

Any r-parameter Lie transformation group may be represented by r infinitesimal generators

$$X_i = \xi_i\partial_x + \eta_i\partial_y, \quad i = 1, \ldots, r. \tag{7}$$

Any ordinary DE of order m with this r-parameter Lie group as symmetry group has to vanish under all m-th prolongations of the generators (7) and vice versa, i.e. this DE $\Phi \equiv \Phi(x, y, y', y'', \ldots)$ is a solution of the following system of linear homogeneous first order partial differential equations:

$$X_i^{(m)}\Phi = 0, \quad i = 1, \ldots, r, \tag{8}$$

The system (8) is called *system of differential invariants*, its fundamental solutions are called the *differential invariants* of the respective Lie group. Lie has discussed these systems in detail, for a recent presentation see [20].

The group property guarantees that (8) is a complete system for Φ with $m+2-r$ solutions. It may be brought into Jacobian normal form, an analogon of the triangular form for matrices, before attempting to solve it. The dependencies of the fundamental solutions may then be chosen such that

$$\begin{aligned} \Phi_1 &\equiv \Phi_1(x, y, y', \ldots, y^{(r-1)}), \\ \Phi_2 &\equiv \Phi_2(x, y, y', \ldots, y^{(r)}), \\ &\vdots \\ \Phi_{m-r+2} &\equiv \Phi_{m-r+2}(x, y, y', \ldots, y^{(m)}). \end{aligned}$$

The invariants are linear in the highest derivative.

Example: We consider the following transformation group that acts on the (x, y)-plane which is represented by

$$g = \{\partial_x, 2x\partial_x + y\partial_y, x^2\partial_x + xy\partial_y\}.$$

Prolongation of its three generators up to the third order yields the following system of differential invariants (8):

$$\begin{gathered} \Phi_x = 0, \\ 2x\Phi_x + y\Phi_y - y'\Phi_{y'} - 3y''\Phi_{y''} - 5y'''\Phi_{y'''} = 0, \\ x^2\Phi_x + xy\Phi_y - (y'x - y)\Phi_{y'} - 3y''x\Phi_{y''} - (5y'''x + 3y'')\Phi_{y''} = 0. \end{gathered}$$

Using some strategy for solving systems of linear PDEs, for example, iterated narrowing transformations or *elimination*, we may arrive at the following two fundamental solutions:

$$\Phi_1 \equiv y''y^3, \quad \Phi_2 \equiv y'''y^5 + 3y''y'y^4.$$

The DEs of order not higher than three that have the respective Lie group g as symmetry group have the general form $\omega(\Phi_1, \Phi_2)$ for some differentiable function ω.

2.2 Symmetries of PDEs

Finding differential invariants is accomplished in analogy to the ordinary case: the group generators have to be prolongated to the desired order; the prolongations are then interpreted as a system of linear PDEs whose fundamental solutions provide a basis of differential invariants.

We introduced the prolongation formulas that apply to the case of partial differential equations with one dependent variable u and n independent variables $x = x_1, \ldots, x_n$ (compare [1]). Partial derivatives $\partial_{x_{i_1}} \cdots \partial_{x_{i_k}} u$ are represented by formal variables $u_{i_1 \ldots i_k}$, called *differential indeterminates*. They are symmetric in their indices. The differential variables of order k are denoted by $u^{(k)}$. We also use the convention to sum over the range of multiply occurring indices in products, e.g. $(D_i\xi_j)u_j = \sum_{j=1}^n (D_i\xi_j)u_j$.

The one-parameter Lie group of transformations in the parameter ε

$$x_i^* = X_i(x, u; \varepsilon) = x_i + \varepsilon \xi_i(x, u) + O(\varepsilon^2), \tag{9}$$

$$u^* = U(x, u; \varepsilon) = u + \varepsilon \eta(x, u) + O(\varepsilon^2), \tag{10}$$

$i = 1, 2, \ldots, n$, acting on (x, u)-space has as its infinitesimal generator

$$X = \xi_i(x, u)\partial_{x_i} + \eta(x, u)\partial_u.$$

The k-th extension of (9), (10), given by

$$\begin{aligned}
x_i^* &= X_i(x, u; \varepsilon) = x_i + \varepsilon \xi_i(x, u) + O(\varepsilon^2), \\
u^* &= U(x, u; \varepsilon) = u + \varepsilon \eta(x, u) + O(\varepsilon^2), \\
&\vdots \\
u^*_{i_1 i_2 \ldots i_k} &= U_{i_1 i_2 \ldots i_k}(x, u, u^{(1)}, \ldots, u^{(k)}; \varepsilon) \\
&= u_{i_1 i_2 \ldots i_k} + \varepsilon \eta^{(k)}_{i_1 i_2 \ldots i_k}(x, u, u^{(1)}, \ldots, u^{(k)}) + O(\varepsilon^2),
\end{aligned}$$

where $i = 1, 2, \ldots, n$ and $i_l = 1, 2, \ldots, n$ for $l = 1, 2, \ldots, k$ with $k = 1, 2, \ldots$, has as its k-th extended infinitesimal generator

$$\begin{aligned}
X^{(k)} = {} & \xi_i(x, u)\partial_{x_i} + \eta(x, u)\partial_u + \eta_i^{(1)}(x, u, u^{(1)})\partial_{u_i} + \ldots \\
& + \eta^{(k)}_{i_1 i_2 \ldots i_k} \partial_{u_{i_1 i_2 \ldots i_k}},
\end{aligned}$$

$k = 1, 2, \ldots$; explicit formulas for the extended infinitesimals $\{\eta^{(k)}\}$ are given recursively by

$$\eta_i^{(1)} = D_i \eta - (D_i \xi_j) u_j, \quad i = 1, 2, \ldots, n, \tag{11}$$

$$\eta^{(k)}_{i_1 i_2 \ldots i_k} = D_{i_k} \eta^{(k-1)}_{i_1 i_2 \ldots i_k} - (D_{i_k} \xi_j) u_{i_1 i_2 \ldots i_{k-1} j}, \tag{12}$$

$i_l = 1, 2, \ldots, n$ for $l = 1, 2, \ldots, k$ with $k \geq 2$.

3 Examples

The two following examples demonstating the use of symmetries on ODEs are taken from [9].

3.1 A First order ODE

We demonstrate how to reduce the order of a first-order ODE with the help of a symmetry. This results in integration. We use the method of canonical coordinates.

Example (Canonical Coordinates) We consider the Riccati equation

$$y' + y^2 - \frac{2}{x^2} = 0. \tag{13}$$

It is invariant under the group of transformations

$$\overline{x} = xe^{\varepsilon}, \quad \overline{y} = ye^{-\varepsilon} \quad (, \overline{y}' = y'e^{-2\varepsilon}). \tag{14}$$

Its infinitesimals $\left(\frac{d}{d\epsilon}\overline{x}, \frac{d}{d\epsilon}\overline{y}\right)_{\epsilon=0} = (x, -y)$ determine the infinitesimal symmetry

$$X = x\partial_x - y\partial_y.$$

Canonical coordinates t, u for (13) are obtained by solving $X(t) = 1$, $X(u) = 0$ and have the form

$$t = \ln|x|, \quad u = xy.$$

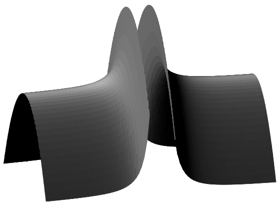
In these coordinates, the inhomogeneous stretchings (14) are replaced by the translation group

$$\overline{t} = t + \varepsilon, \quad \overline{u} = u, \quad \overline{u}' = u'$$

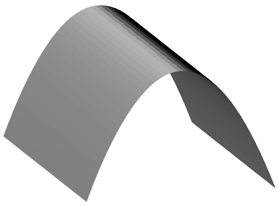
and (13) takes the integrable form

$$u' + u^2 - u - 2 = 0. \tag{15}$$

Geometrically, the frame of (15) is now a "straightened out" parabolic cylinder. In general, the frame of a first order ODE $y' = f(x, y)$ is the surface in the space of three independent variables, x, y, and p, given by $p = f(x, y)$.



frame of Riccati's equation



and its transform

Analytically, we note that (15) does not depend on t explicitly. Integrating (15) gives

$$\ln\left|\frac{u+1}{u-2}\right| - 3t = \text{const.},$$

provided that $u + 1 \neq 0$ and $u - 2 \neq 0$. Substituting the expressions for t and u in terms of x and y, one arrives at the solution

$$y = \frac{2x^3 + C}{x(x^3 - C)}, \quad C = \text{const.},$$

provided that $xy - 2 \neq 0$ and $xy + 1 \neq 0$. In case these expressions are zero, one arrives at $y_1 = \frac{2}{x}$ and $y_2 = -\frac{1}{x}$, respectively.

3.2 A Second Order ODE

If a second order ODE $y'' = f(x, y, y')$ admits one symmetry, its order may be reduced by one. In case it admits two symmetries, integration can be achieved. Reduction of order and successive integration are applicable to higher order equations as well. The restriction to second order is essential, however, for the method of integration using canonical forms of two-dimensional Lie algebras, see [9]. These canonical forms and their invariant second-order equations are presented in the following table. For $X_i = \xi_i\partial_x + \eta_i\partial_y$, we denote the wedge-product of X_1, X_2 by $X_1 \vee X_2 := \xi_1\eta_2 - \xi_2\eta_1$.

Type	L_2 structure		Basis of L_2		Invariant Equation
	$[X_1, X_2]$	$X_1 \vee X_2$	X_1	X_2	
I	0	$\neq 0$	∂_x	∂_y	$y'' = f(y')$
II	0	0	∂_y	$x\partial_y$	$y'' = f(x)$
III	X_1	$\neq 0$	∂_y	$x\partial_x + y\partial_y$	$y'' = \frac{1}{x}f(y')$
IV	X_1	0	∂_y	$y\partial_y$	$y'' = f(x)y'$

Based on this classification, we sketch *Lie's integration algorithm* for integrating second-order ODEs that admit at a two-dimensional Lie algebra.

Step	Action	Result
1.	Compute admitted Lie Algebra L_r.	basis $X_1, \ldots, X_r$.
2.	If $r = 2$ go to step 3. If $r > 2$ distinguish any 2-dimensional subalgebra L_2 of L_r.	basis X_1, X_2 for L_2.
3.	Determine type of L_2 according to table; eventually choose a new basis X_1', X_2'.	canonical form.
4.	Go over to canonical variables x, y. Rewrite equation in these variables and integrate it.	change of variables.
5.	Rewrite solution in terms of original variables.	solution.

Example (Lie's integration algorithm) We consider the second order ODE

$$y'' = \frac{y'}{y^2} - \frac{1}{xy}. \tag{16}$$

Step 1. The *calculation of its admissible Lie algebra* is demonstrated in Subsection 4, yielding two linearly independent operators.

$$X_1 = x^2\partial_x + xy\partial_y, \qquad X_2 = x\partial_x + \frac{y}{2}\partial_y. \tag{17}$$

According to the algorithm, we advance directly to the third step.

Step 3. To *determine the type of the Lie algebra*, we consider

$$[X_1, X_2] = -X_1, \qquad X_1 \vee X_2 = -\frac{x^2 y}{2} \neq 0.$$

After merely changing the sign of X_2, the basis has exactly the structure of type III in the canonical form table.

Step 4. To *determine an integrating change of variables*, we first introduce canonical variables for X_1 as the solutions of $X_1(t) = 1$ and $X_1(u) = 0$. They are given by

$$t = \frac{y}{x}, \qquad u = -\frac{1}{x},$$

transforming the operators to

$$X_1 = \partial_u, \qquad X_2 = \frac{t}{2}\partial_t + u\partial_u.$$

This is basically still type III, the factor $\frac{1}{2}$ in X_2 does not hinder integration. Excluding the solution $y = Kx$, the equation written in the new variables is

$$u'' + \frac{1}{t^2}u'^2 = 0.$$

Integrating once, we get $u' = t/(C_1 t - 1)$. Hence

$$u = -\frac{t^2}{2} + C \text{ for } C_1 = 0, \text{ or}$$
$$u = \frac{t}{C_1} + \frac{1}{C_1^2} \ln|C_1 t - 1| + C_2 \text{ for } C_1 \neq 0.$$

Step 5. The solutions in the original variables are then

$$y = Kx, \quad y = \pm\sqrt{2x + Cx^2},$$
$$0 = C_1 y + C_2 x + x \ln\left|C_1 \frac{y}{x} - 1\right| + C_1^2.$$

3.3 Two Second Order PDEs

In this subsection we present the calculation of symmetries and their use in finding invariant solutions of second order PDEs. The following two examples can be found in [1].

3.3.1 The Heat Equation

The heat equation

$$z_{xx} - z_y = 0 \tag{18}$$

is an example of a second order PDE by which we demonstrate the computation of symmetry generators and their use in finding invariant solutions. In analogy to Subsection 2.1.3, a necessary and sufficient condition for an infinitesimal generator

$$X = \xi_1(x,y,z)\partial_x + \xi_2(x,y,z)\partial_y + \eta(x,y,z)\partial_z \tag{19}$$

to be admitted by (18) is

$$X^{(2)}(z_{xx} - z_y) = 0 \bmod z_{xx} = z_y, \tag{20}$$

where we replace any occurrence of z_{xx} by z_y after application of the operator $X^{(2)}$. The operator $X^{(2)}$ is the second order prolongation of X and given by

$$X^{(2)} = \xi_1\partial_x + \xi_2\partial_y + \eta\partial_z + \eta_1^{(1)}\partial_{z_x} + \eta_2^{(1)}\partial_{z_y} + \eta_{11}^{(2)}\partial_{z_{xx}} + \eta_{12}^{(2)}\partial_{z_{xy}} + \eta_{22}^{(2)}\partial_{z_{yy}},$$

where $\eta_1^{(1)}, \eta_2^{(1)}, \eta_{11}^{(2)}, \eta_{12}^{(2)}, \eta_{22}^{(2)}$ are defined as in subsection 2.1.3. The determining equation for (18) is

$$\eta_{11}^{(2)} - \eta_2^{(1)} = 0 \bmod z_{xx} = z_y. \tag{21}$$

We treat (21) as $\eta_{11}^{(2)} - \eta_2^{(1)} = 0$, where every occurrence of z_{xx} is replaced by z_y. This equation is polynomial in $z_x, z_y, z_{xx}, z_{xy}, z_{yy}$, and since ξ_1, ξ_2, η only depend on x, y, z, we may equate the coefficients of $z_x, z_y, z_{xx}, z_{xy}, z_{yy}$ (and their powers) in (20) to zero. The result is an overdetermined system of linear homogeneous equations in ξ_1, ξ_2, η and their partial derivatives up to order two, called *determining system*.

The procedure outlined above holds in general. We demonstarte how to solve such a system in the next example. The solution gives the Lie algebra spanned by the following six generators, each of which corresponds to a one-parameter group:

$$X_1 = \partial_x, \quad X_2 = \partial_y, \quad X_3 = x\partial_x + 2y\partial_y,$$
$$X_4 = 4xy\partial_x + 4y^2\partial_y - (x^2 + 2y)z\partial_z, \quad X_5 = 2y\partial_x - xz\partial_z, \quad X_6 = z\partial_z.$$

Let us consider the infinitesimal generator X_4, which corresponds to the parameter c_1. The one-parameter Lie group of transformations

$$\bar{x}(x, y, z, \epsilon), \quad \bar{y}(x, y, z, \epsilon), \quad \bar{z}(x, y, z, \epsilon) \tag{22}$$

corresponding to $X_4 = 4xy\partial_x + 4y^2\partial_y - (x^2 + 2y)z\partial_z$ is obtained by solving the initial value problem

$$(\bar{x}, \bar{y}, \bar{z})[\epsilon = 0] = (x, y, z) \tag{23}$$

for the following first order system of ODEs:

$$\frac{d\bar{x}}{d\varepsilon} = 4\bar{x}\bar{y}, \tag{24}$$
$$\frac{d\bar{y}}{d\varepsilon} = 4\bar{y}^2, \tag{25}$$
$$\frac{d\bar{z}}{d\varepsilon} = -(\bar{x}^2 + 2\bar{y})\bar{z}. \tag{26}$$

The solution of (25) is $\bar{y} = \frac{1}{C - 4\epsilon}$, and by (23) we obtain

$$\bar{y}(x, y, z, \epsilon) = \frac{y}{1 - 4\epsilon y}. \tag{27}$$

By this and (24) we get $\bar{x} = \frac{C}{1 - 4\epsilon y}$, and by (23) we obtain

$$\bar{x}(x, y, z, \epsilon) = \frac{x}{1 - 4\epsilon y}. \tag{28}$$

Similarly, by (28, 27, 26) and (23) we obtain

$$\bar{z}(x, y, z, \epsilon) = z\sqrt{1 - 4\epsilon y}\exp\left(-\frac{\epsilon x^2}{1 - 4\epsilon y}\right). \tag{29}$$

Every invariant solution $z = \Phi(x, y)$ of (18) corresponding to X_4 satisfies

$$X_4(z - \Phi(x, y)) = 0 \text{ when } z - \Phi(x, y),$$

i.e.

$$4xy\frac{\partial \Phi}{\partial x} + 4y^2\frac{\partial \Phi}{\partial y} = -(x^2 + 2y)\Theta. \tag{30}$$

We solve (30) by solving the corresponding characteristic equation

$$\frac{dx}{4xy} = \frac{dy}{4y^2} = \frac{dz}{-(x^2 + 2y)z}$$

which has the two invariants

$$\frac{x}{y} \text{ and } z\sqrt{y}e^{x^2/4y}.$$

The solution of (18) is now defined by the invariant form

$$z\sqrt{y}e^{x^2/4y} = \phi(\frac{x}{y}),$$

or, in explicit form,

$$z = \Phi(x,y) = \frac{1}{\sqrt{y}}e^{-x^2/4y}\phi(\zeta), \tag{31}$$

where $\zeta = \frac{x}{y}$ is the similarity variable. Substitution of (31) into (18) leads to $\phi''(\zeta) = 0$. Hence, invariant solutions of (18) resulting from X_4 are of the form

$$z = \Phi(x,y) = \frac{1}{\sqrt{y}}e^{-x^2/4y}\{C_1 + C_2\frac{x}{y}\}.$$

For any solution $z = \Phi(x,y)$ of (18), that is not invariant under X_4, we find a one-parameter family of solutions $z = \phi(x,y,\epsilon)$ generated by X_4: Let

$$\begin{aligned} x^* &= \bar{x}(x,y,z,\epsilon) = \frac{x}{1-4\epsilon y}, \\ y^* &= \bar{y}(x,y,z,\epsilon) = \frac{y}{1-4\epsilon y}, \\ z^* &= \Phi(\bar{x},\bar{y}). \end{aligned}$$

By $\bar{z}(\cdot,\cdot,\cdot,-\epsilon)$ we denote the third component of the inverse transformation corresponding to X_4. Then $z = \phi(x,y,\epsilon) = \bar{z}(x^*,y^*,z^*,-\epsilon) =$

$$\Phi(\frac{x}{1-4\epsilon y}, \frac{y}{1-4\epsilon y})\frac{1}{\sqrt{1-4\epsilon y}}\exp(\frac{\epsilon x^2}{1-4\epsilon y}).$$

3.3.2 Wave Equation for an Inhomogeneous Medium

We consider the wave equation for a variable wave speed $c(x)$:

$$z_{yy} = c(x)^2 z_{xx}. \tag{32}$$

It is a linear PDE and hence (see [16, Sec. 27]) can only admit infinitesimal generators of the form

$$X = \xi_1(x,y)\partial_x + \xi_2(x,y)\partial_y + f(x,y)z\partial_z.$$

In analogy to the previous example we obtain the invariance condition

$$\eta_{22}^{(2)} = c(x)^2\eta_{11}^{(2)} + 2c(x)c'(x)\xi_1 z_{xx} \text{ when (32)}.$$

The resulting determining system is

$$(\xi_1)_y - c(x)^2(\xi_2)_x = 0, \tag{33}$$
$$c(x)[(\xi_2)_y - (\xi_1)_x] + c'(x)\xi_1 = 0, \tag{34}$$
$$(\xi_2)_{yy} - c(x)^2(\xi_2)_{xx} - 2f_y = 0, \tag{35}$$
$$(\xi_1)_{yy} + c(x)^2[2f_x - (\xi_1)_{xx}] = 0, \tag{36}$$
$$f_{yy} - c(x)^2 f_{xx} = 0. \tag{37}$$

Solving (33) for $(\xi_2)_x$ and (34) for $(\xi_2)_y$ and setting $(\xi_2)_{xy} = (\xi_2)_{yx}$ we find

$$(\xi_1)_{xx} - (\xi_1)_{yy}/c(x)^2 - (\xi_1 H(x))_x = 0, \tag{38}$$

where $H(x) = c'(x)/c(x)$. Solving (38) and (36) leads to

$$f(x,y) = \frac{1}{2}H(x)\xi_1(x,y) + S(y), \tag{39}$$

where $S(y)$ is an arbitrary function of y. Substituting (39) into (35) and then solving (33) for $(\xi_1)_y$ and (34) for $(\xi_1)_x$ and setting $(\xi_1)_{xy} = (\xi_1)_{yx}$, we find that $S(y) = const = s$, so that $f = \frac{1}{2}H\xi_1 + s$. Substituting f in (37) and using (36) we get $H''\xi_1 + 2H'(\xi_1)_x + H(H\xi_1)_x = 0$ or, equivalently,

$$[(2H' + H^2)(\xi_1)^2]_x = 0.$$

We now only consider the case $2H' + H^2 = 0$. Then

$$c(x) = (Ax + B)^2,$$

where A, B are arbitrary constants. Then $H(x) = \frac{2A}{Ax+B}$. For any solution $\xi_1(x,y)$ of equation (38), one finds that $\xi_2(x,y)$, $f(x,y)$ solving (33-37) are given by:

$$\xi_2(x,y) = \int [(\xi_1)_x - H\xi_1]dy,$$
$$f(x,y) = \frac{A\xi_1(x,y)}{Ax + B}.$$

So $\{\xi_1, \xi_2, f\}$ determine a non-trivial infinite-parameter Lie group for

$$z_{yy} = (Ax + B)^4 z_{xx}. \tag{40}$$

If $A \neq 0$ this equation can be transformed to the wave equation

$$\bar{z}_{\bar{x}\bar{y}} = 0$$

by the point transformation

$$\bar{x} = (Ax + B)^{-1} + Ay,$$
$$\bar{y} = (Ax + B)^{-1} - Ay,$$
$$\bar{z} = (Ax + B)^{-1}z.$$

The general solution of PDE (40) is then

$$z = (Ax + B)[F(\bar{x}) + G(\bar{y})],$$

where F, G are twice differentiable functions.

3.4 Literature and Implementations

The most complete work on *group analysis of ordinary differential* equations is still [12]. A very broad introduction and comprehensive reference for group analysis of differential equations in general is [8]. In handbook style, this series presents *newly developed theoretical and computational methods*, meeting the needs of the applied reader as well as those of the researcher. In Chapter 13, 14 in volume 3, the reader finds an account on *symbolic software for calculating symmetries* by Hereman. The following table is taken from [6].

Scope of Lie symmetry programs

Name	System	Developer(s)	Point	Gen.	Non-class.	Solves Det. Eqs.
CRACK	REDUCE	Wolf & Brand	–	–	–	Yes
DELiA	Pascal	Bocharov et al.	Yes	Yes	No	Yes
DIFFGROB2	Maple	Mansfield	–	–	–	Reduction
DIMSYM	REDUCE	Sherring	Yes	Yes	No	Yes
LIE	REDUCE	Eliseev et al.	Yes	Yes	No	No
LIE	muMATH	Head	Yes	Yes	Yes	Yes
Lie	Mathematica	Baumann	Yes	No	Yes	Yes
LieBaecklund	Mathematica	Baumann	No	Yes	No	Interactive
LIEDF/INFSYM	REDUCE	Gragert &	Yes	Yes	No	Interactive
LIEPDE	REDUCE	Wolf & Brand	Yes	Yes	No	Yes
Liesymm	Maple	Carminati et al.	Yes	No	No	Interactive
MathSym	Mathematica	Herod	Yes	No	Yes	Reduction
NUSY	REDUCE	Nucci	Yes	Yes	Yes	Interactive
PDELIE	MACSYMA	Vafeades	Yes	Yes	No	Yes
SPDE	REDUCE	Schwarz	Yes	No	No	Yes
SYMCAL	Maple/ MACSYMA	Reid & Wittkopf	–	–	–	Reduction
SYM_DE	MACSYMA	Steinberg	Yes	No	No	Partially
symgroup.c	Mathematica	Bérubé & de Montigny	Yes	No	No	No
SYMMGRP.MAX	MACSYMA	Champagne et al.	Yes	No	Yes	Interactive
SYMSIZE	REDUCE	Schwarz	–	–	–	Reduction

The last four columns in this table indicate the scope of the programs: point symmetries, generalized symmetries, non-classical symmetries and whether the determining system can be solved automatically. Recent MAPLE programs for generating classical symmetries are DESOLV by Carminati and Vu [3], RIF by Reid and Wittkopf and SYMMETRIE by Hickman.

Finally, some text books for the more applied reader are [22, 23]. At RISC, the first author contributed to the symmetry classification problem for a special class of PDEs [7]. This work was inspired by Fritz Schwarz, whose expertise in the algorithmic aspects of the field is reflected in [20].

4 Differential Elimination

Several methods in polynomial elimination theory can be reformulated to also apply to ideals of differential polynomials, or they have first been defined for differential polynomials but have found successful application to algebraic polynomials.

Differential Gröbner bases appeared first in [4] with further developments in [14] and [13]. Unfortunately, differential Gröbner bases are generally infinite, so they do not provide a general solution of the differential ideal membership problem. It is even known that the general membership problem is undecidable [5]. If, however, a finite differential Gröbner basis is known, ideal membership can be tested effectively. Carrá-Ferro could show that differential ideals that are generated by finitely many linear differential polynomials have a finite differential Gröbner basis with respect to an orderly ranking.

For linear PDEs with polynomial coefficients it is also possible to use an extension of the ordinary polynomial Gröbner bases theory to Weyl algebras in order to simplify overdetermined systems. Here the system is saturated by all integrability conditions.

Take, for example, the equation (16). In order to determine the Lie symmetry algebra, one starts with undetermined functions $\xi(x, y)$ and $\eta(x, y)$ for the infinitesimal generator

$$X := \xi\partial_x + \eta\partial_y$$

and first sets up the determining system, as described, for example, in [15]. Basically ξ and η have to satisfy an equation (identically for all x and y satisfying (16)) that is obtained by applying the second prolongation $X^{(2)}$ of X to the original equation (16). Equating coefficients of higher order derivatives leads to the following equations for ξ and η.

$$\begin{aligned}
\frac{\partial^2\xi}{\partial y^2} &= 0\\
y^2\frac{\partial^2\eta}{\partial y^2} - 2y^2\frac{\partial^2\xi}{\partial x\partial y} - 2\frac{\partial\xi}{\partial y} &= 0\\
2xy^3\frac{\partial^2\eta}{\partial x\partial y} - xy^3\frac{\partial^2\xi}{\partial x^2} - xy\frac{\partial\xi}{\partial x} + 3y^2\frac{\partial\xi}{\partial y} + 2x\eta &= 0\\
x^2y^2\frac{\partial^2\eta}{\partial x^2} + 2xy\frac{\partial\xi}{\partial x} - x^2\frac{\partial\eta}{\partial x} - xy\frac{\partial\eta}{\partial y} - y\xi - x\eta &= 0
\end{aligned}$$

This is a system of *linear* PDEs. A computation of a Gröbner basis (with respect to an appropriate elimination ranking) in the algebra of linear differential operators leads to the triangular system

$$\begin{aligned}
\frac{\partial^2\eta}{\partial y^2} &= 0\\
y\frac{\partial\eta}{\partial y} - \eta &= 0\\
x^2\frac{\partial^2\eta}{\partial x^2} &= 0\\
\xi + x^2\frac{\partial^2\eta}{\partial x\partial y} - 2x\frac{\partial\eta}{\partial y} &= 0
\end{aligned}$$

which is much easier to solve than the original system of determining equations. As a general solution we get

$$\xi = C_1x^2 + 2C_2x, \qquad \eta = (C_1x + C_2)y$$

from which the independent operators in (17) are derived.

Usually, the system of determining equations contains a huge number of equations. Take, for example, the Boussinesq equation

$$\frac{\partial^2 u}{\partial t^2} + u\left(\frac{\partial^2 u}{\partial x^2}\right)^2 + \frac{\partial^4 u}{\partial x^4} = 0.$$

For this fourth order equation we set up the equations in order to determine the coefficients ξ_1, ξ_2, and η of the general symmetry generator

$$X := \xi_1(x,t,u)\partial_x + \xi_2(x,t,u)\partial_t + \eta(x,t,u)\partial_u.$$

In analogy to the previous example we have to compute the fourth prolongation of X. It leads to a system of 47 equations which can be generated automatically, for example, by the Maple package `DESOLV_R5` by K. T. Vu and J. Carminati (cf. [3]) in the following way.

```
read("Desolv_r5"):
infolevel[gendef] := 10:
bq := D[1,1,1,1](u)(x,t) + D[1](u)(x,t)^2 +
      u(x,t)*D[1,1](u)(x,t) + D[2,2](u)(x,t);
deteqs:=gendef([bq],[u],[x,t]):
nops(deteqs[1]);
```

The package immediately applies some simplifications to reduce the number of equations to 12 of order 4. The question arises whether or not this system is consistent, i.e., whether there are solutions at all. In the linear case, Gröbner bases are one tool to decide this problem. The computation of a Gröbner basis of the determining equations of the Boussinesq equation with respect to an appropriate ranking leads to an easily solvable system of 10 equations of order 2. We find that the symmetry algebra is spanned by the three elements

$$v_1 = \partial_x \qquad v_2 = \partial_t \qquad v_3 = x\partial_x + 2t\partial_t - 2u\partial_u.$$

Gröbner bases are not the only tool for decisions and computations in differential elimination theory. The theory of *involutive bases* has its foundation in the theory of PDEs given by Riquier [18] and Janet [10, 11] at the beginning of the 20th century. From the observation that a closed form solution of any system of partial differential equations may only be obtained for exceptional cases they focused their study to restricted questions of whether a solution exists at all or how one could find its degree of arbitrariness. Their constructive approach to algebraic analysis of PDEs was later followed by Thomas [24] and more recently by Pommaret [17]. The main idea of the approach is rewriting the initial differential system into another, so-called involutive form so that all its integrability conditions are satisfied. In contrast to differential Gröbner bases, involutive bases are finite. Since an involutive basis has all integrability conditions included it is possible to compute a Taylor series expansion of an analytic solution in a straightforward way. From an involutive basis one can immediately read off the degree of arbitrariness of the solution, cf. [21].

Characteristic sets are due to Ritt [19] and have further been adapted to algebraic polynomials by Wu [25]. The main idea is to transform the equations into triangular form in such a way that the solutions stay the same. However, the ideal is not preserved in general, multiplicities of solutions can change.

5 Conclusion

As we have seen above, current computer algebra techniques provide a computational algebraic approach to the analysis of systems of differential equations and sometimes also to their solution. But despite all the success of symbolic methods in differential equations (Lie symmetries, differential Galois theory, Janet bases, differential Gröbner bases, etc.), these theories are not and probably never will be able to solve the majority of differential problems in engineering. However, with further research into this area we might be able to tackle simplified problems. Toy models that can be solved analytically are important for obtaining a deeper understanding of the underlying structures. A deeper understanding of such simplified problems may well lead to more efficient numerical algorithms for large problems.

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