Solution analysis of PDEs related to the Mumford-Shah functional with symbolic computation

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Abstract

We consider several partial differential equations frequently used in image and signal processing. These equations are derived from variational principles such as the Energy method. We transfer the equations into an appropriate weak formulation leading to systems of polynomial equations. By symbolic computation method we can analyze the parameter dependence of the solutions. In particular, we find that for certain parameter ranges uniqueness of the solutions is not given.

1 Introduction

In image and signal processing variational principles play an important role for extracting interesting information in the image or signal. In general a gray-scale image is modeled as a two-dimensional function $f(x, y)$. A given pixel-image is considered a discretization of this function. An important task in image processing is the segmentation problem, i.e., to split a given image into several disjoint regions where the image is homogeneous. Nonlinear partial differential equations and variational principle has been shown very successful for this task. We will analyze certain properties of the discretized equations derived from the Energy method (see Section 2) by symbolic computation.

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In the framework of signal processing we are interested in denoising and signal recovery. In this case a signal is described by a one-dimensional function $f(x)$ which values indicate the signal strength. Given real or simulated data, the signal is in general contaminated with noise. Considering the denoising problem, one is interested in reconstructing the unperturbed signals from the noise-contaminated one.

In many application the available signal has undergone several steps of filtering, for instance, the function $f(x)$ can be blurred. To extract physical relevant information, e.g. jumps, a process of deblurring has to be performed.

All these problems have in common, that the original information in the signal or image has to be reconstructed from perturbed data, hence they should be treated within the theory of inverse problems ([2, 6]).

We focus on the Energy method for denoising and signal recovery, that means, we want to reconstruct a given signal by minimizing an appropriate functional leading to a variational problem.

A extensive list of appropriate functionals are known (see, e.g., [1]). In particular, functionals involving the BV-norm play a central role, whenever one is interested in piecewise constant discontinuous solutions [13, 16]

Recently, the interest in certain nonlinear and nonconvex functionals has been increased, for they have shown to yield better results than the convex ones in many cases. We mention the well-known Mumford-Shah functional [12], which serves as a prototype for an image segmentation and deblurring tool [11].

Although the denoising problem is one of the simplest inverse problems, the Energy method is not restricted to this case. In fact, the Energy method has the potential to be generalized to nonlinear inverse problems. For instance, the Mumford-Shah functional has successfully been used as a regularization term in [15]. Hence, we consider the denoising and deblurring problem just a benchmark problem, with the hope that an analysis of this simple case can bring insight into the more complicated nonlinear one.

2 Energy method

Here we stick to the one-dimensional case, the two-dimensional case relevant for images is of course similar. By the Energy method we obtain an approximation to the unknown unperturbed signal by minimization of a functional

$$E(u) = \frac{1}{2}\int (u-u_0)^2 dx + \lambda' \int \phi(u_x^2) dx \tag{1}$$

over all $u(x) \in L^2(\Omega)$. Without loss of generality we may assume that $u(x)$ is defined on the unit interval $\Omega = [0, 1]$.

In (1) u_0 is the given perturbed signal and the reconstruction is sought as

$$u^\dagger := \min_{u \in L^2(\Omega)} E(u).$$

The parameter λ' has to be chosen appropriately, it controls the trade-off between the approximation term $\int (u-u_0)^2 dx$ and the regularization term $\int \phi(u_x^2)dx$. The latter is necessary to force the solution $u^\dagger$ to contain the interesting noise-free information.

The function ϕ determines the behavior of the solution $u^\dagger$. If the interest lies in non-smooth solutions, and in identifying jumps, frequently used functions are

$$\begin{aligned}\phi_1(s) &= \frac{1}{2}\log(1+\beta s)\\ \phi_2(s) &= \frac{1}{2}\frac{s}{1+\beta s},\end{aligned}$$

with a positive real parameter β.

Formally deriving the first order necessary conditions we find that a solution has to satisfy the following nonlinear diffusion equations:

$$(u-u_0) = \lambda\nabla.\frac{\nabla u}{1+\beta|\nabla u|^2} \qquad \text{for } \phi=\phi_1 \tag{2}$$

$$(u-u_0) = \lambda\nabla.\frac{\nabla u}{(1+\beta|\nabla u|^2)^2} \qquad \text{for } \phi=\phi_2 \tag{3}$$

together with Neumann boundary conditions

$$u_x(0) = u_x(1) = 0$$

and $\lambda = \lambda'\beta$.

The diffusion operators in (2),(3) are sometimes called Perona-Malik-filters [14]. There is an interesting relation between the minimizers of the functional (1) and the Mumford-Shah functional mentioned in the introduction. In fact, it has been shown [4] that for certain choices of ϕ, solutions of the discretized equations approximate minimizers of the Mumford-Shah functional in the sense of Γ-convergence.

Note that solutions to these equations only exist formally, since the corresponding functionals are non-convex. The question of existence and uniqueness of such equations is rather involved and is not yet settled (see [1, 11], and [9] for the parabolic case).

In order to deal with equations for which existence can be proven, additional smoothing terms in the energy functional can be useful. For instance, in [5] the following energy functional was proposed:

$$E(u) = \frac{1}{2}\int (u-u_0)^2 dx + \lambda\int \phi_2(u_x^2)dx + \epsilon\int u_x^2(x)dx, \quad \epsilon > 0 \tag{4}$$

The corresponding equation for the first order necessary condition are similar to (3) but with an extra term $\epsilon\Delta u$ on the right hand side. In this equation an extra parameter ϵ has been introduced. From a practical point of view, ϵ should not be too large, since then the smoothing effects of the Laplace operator Δ dominates the equation and the solutions will be blurred. On the other hand, choosing ϵ too small might give rise to multiple solutions.

Hence, it seems to be important to investigate the interplay of the different parameters λ, β, ϵ and the discretization number. By symbolic computation, we will see, that the parameter dependence of the solutions can be shown by an implicit functions only related to one grid node. Moreover, for a certain range of parameters we can find branching points where uniqueness of the solution for the the discretized problem does not hold.

3 Discretization in the FE Space

Starting from Equations (2),(3) we derive a weak formulation. We then discretize this system using finite element methods, which leads to a system of polynomial equations in the unknowns. This system can be treated by algebraic methods, which yields us solutions, which are polynomials in the parameters λ, β.

In one dimension Equation (2) can be rewritten as

$$u - u_0 = \lambda \frac{u_{xx}(1 - \beta u_x^2)}{1 + \beta u_x^2}$$

or equivalently,

$$(u - u_0)(1 + \beta u_x^2) = \lambda \partial_x (u_x - \beta \frac{u_x^3}{3}).$$

Now we multiply the equation with a test function $v(x)$ and integrate by parts to find the following weak formulation:

$$A(u, v) := \int_{[0,1]} (u - u_0)(1 + \beta u_x^2) v dx + \lambda \int_{[0,1]} (u_x - \beta \frac{u_x^3}{3}) v_x dx = 0 \quad \forall v \quad (5)$$

The operator $A(u, v)$ is polynomial in u, u_x and linear in v, v_x. Now we approximate both $u(x), v(x)$ by usual finite element. Let $v_i(x)$ be the finite element shape functions, then we set

$$u(x) \sim u_h(x) = \sum_{i=1}^{N} u_i v_i(x) \qquad v(x) \sim v_h(x) = \sum_{i=1}^{N} v_i v_i(x).$$

This leads to a polynomial equations in the unknown u_i:

$$A(u_h, v_i) = 0 \quad i = 1, 2, ...N. \quad (6)$$

A discretization of (3) can be done in a similar manner: (3) can be rewritten as

$$(u - u_0)(1 + \beta u_x^2)^3 = \lambda \partial_x (u_x - \beta u_x^3).$$

Hence, the corresponding operator $A(u, v)$ is given by

$$A(u, v) = \int_{[0,1]} (u - u_0)(1 + \beta u_x^2)^3 v dx + \lambda \int_{[0,1]} (u_x - \beta u_x^3) v_x dx$$

Using finite elements we again arrive at a system of polynomial equations for the unknown $(u_i)_{i=1}^N$.

And furthermore, a functional of the form (4) leads to the operators

$$A_\epsilon(u,v) = \int_{[0,1]} (u-u_0)(1+\beta u_x^2) v dx + \lambda \int_{[0,1]} ((1+\epsilon)u_x - \beta(1-\epsilon)\frac{u_x^3}{3}) v_x dx \tag{7}$$

for $\phi = \phi_1$, and

$$A(u,v) = \int_{[0,1]} (u-u_0)(1+\beta u_x^2)^3 v dx + \lambda \int_{[0,1]} p_\epsilon(u_x) v_x dx \tag{8}$$

for $\phi = \phi_2$ with

$$p_\epsilon(z) := (1+\epsilon)z + \beta(\epsilon-1)z^3 + \frac{3}{5}\epsilon\beta^2 z^5 + \frac{1}{7}\beta^3\epsilon z^7.$$

4 Solving the Discrete Solutions by Symbolic Computation

To solve for solutions to the discrete form (6), either Newton type methods or direct symbolic computation (since it is a system of polynomial equations) can be considered. The Newton type methods are very efficient if the discrete equation is elliptic, especially when it is parameter-free. However, based on the previous discussion, the specific differential equations which we investigated in this paper can not be proved to be elliptic, hence the finite element solutions to the discrete form will not be unique in many cases. We therefore prefer using the symbolic methods for a direct computation (like eliminations [17] or Groebner basis computation [18]), which is more promising to obtain all possible (parameter-dependent) solutions. And furthermore, the complexity of using those symbolic approaches will not be strongly affected by the existing number of parameters.

The symbolic computation for solving those finite element solutions can be carried out on certain computer algebra software (e.g. [8]). For instance, we can show one typical example for solving the discrete equation (6) for the case $\phi = \phi_1$.

Let the finite element space constructed by piecewise-linear sample function and a uniform domain partition on $[0,1]$ into 4 intervals, which produce 3 undeterminates u_1, u_2 and u_3 ($u_0 = u_1, u_3 = u_4$ according to the Neumann boundary condition).

We set the parameter $\beta = 1$ and by taking

$$u_0 = \begin{cases} 1, & \text{for } x \in [0,1/2], \\ -1, & \text{for } x \in (1/2,1], \end{cases}$$

a discontinuous initial signal. The discretized initial signal is obtained by piecewise-linear interpolation using the sample functions.

The discrete equation (6) turns out to be the following λ-parameter dependent form. It contains three polynomial equations:

$$
\begin{aligned}
P_1 := {} & 1/3u_1 + 1/24u_2 + 4u_2^2 - 8u_2u_1 + 4u_1^2 - 4u_2u_1^2 - 4\lambda u_2 + 64/3\lambda u_2^3 \\
& +4\lambda u_1 - 64/3\lambda u_1^3 - 64\lambda u_2^2u_1 + 64\lambda u_2u_1^2 + 4/3u_2^3 + 8/3u_1^3 - 128u_2^3u_1 \\
& +192u_2^2u_1^2 - 64/3u_2^4u_1 - 64/3u_2^3u_1^2 + 256/3u_2^2u_1^3 - 224/3u_2u_1^4 - 128u_2u_1^3 \\
& +3/8 + 32u_2^4 + 32u_1^4 + 32/3u_2^5 + 64/3u_1^5 = 0;
\end{aligned}
$$

$$
\begin{aligned}
P_2 := {} & 1/24u_1 + 1/6u_2 + 1/24u_3 - 8u_2u_1 + 4u_1^2 - 4u_3^2 + 8u_3u_2 - 4u_2^2u_1 \\
& +8\lambda u_2 - 128/3\lambda u_2^3 - 4\lambda u_1 + 64/3\lambda u_1^3 - 4u_3u_2^2 - 4\lambda u_3 + 64/3\lambda u_3^3 \\
& +64\lambda u_2^2u_1 - 64\lambda u_2u_1^2 - 64\lambda u_3^2u_2 + 64\lambda u_3u_2^2 \\
& +16/3u_2^3 + 4/3u_1^3 + 4/3u_3^3 - 128u_2^3u_1 + 192u_2^2u_1^2 - 128u_2u_1^3 \\
& -224/3u_2^4u_1 + 256/3u_2^3u_1^2 - 64/3u_2^2u_1^3 - 64/3u_2u_1^4 + 32u_1^4 + 128/3u_2^5 \\
& +32/3u_1^5 - 32u_3^4 + 32/3u_3^5 + 128u_3^3u_2 - 192u_3^2u_2^2 + 128u_3u_2^3 - 64/3u_3^4u_2 \\
& -64/3u_3^3u_2^2 + 256/3u_3^2u_2^3 - 224/3u_3u_2^4 = 0;
\end{aligned}
$$

$$
\begin{aligned}
P_3 := {} & 1/24u_2 + 1/3u_3 - 4u_2^2 - 4u_3^2 + 8u_3u_2 - 4\lambda u_2 + 64/3\lambda u_2^3 - 4u_3^2u_2 \\
& +4\lambda u_3 - 64/3\lambda u_3^3 + 64\lambda u_3^2u_2 - 64\lambda u_3u_2^2 + 4/3u_2^3 \\
& +8/3u_3^3 - 32u_2^4 + 32/3u_2^5 - 32u_3^4 + 64/3u_3^5 + 128u_3^3u_2 - 192u_3^2u_2^2 \\
& +128u_3u_2^3 - 224/3u_3^4u_2 + 256/3u_3^3u_2^2 - 64/3u_3^2u_2^3 - 64/3u_3u_2^4 - 3/8 = 0.
\end{aligned}
$$

We now obtain various results for $u_i, (i = 1, 2, 3)$, depending on the evaluation of the parameter λ. For the case $\lambda \in \{-1, \frac{1}{2}\}$ we obtain unique solutions:

$$
\begin{array}{llll}
\lambda = -1 & u_1 = 0.7771972598, & u_2 = 0, & u_3 = -0.7771972598 \\
\lambda = \frac{1}{2} & u_1 = 1.7560008097, & u_2 = 0, & u_3 = -1.7560008097.
\end{array}
$$

However, for the value $\lambda = 1$ we find that the discrete equation (6) contains more than one solutions according to the symbolic computation: The values of u_1, u_2, u_3 can be various combinations of the following results.

$$
\begin{aligned}
u_1 & \in \{.06678774, .1011679, .2215499, .2690963, \\
& \quad 1.957513, 1.957947, 1.967635, \} \\
u_2 & \in \{-.04065422, -.03888544, 0, .03888544, .04065422, \} \\
u_3 & \in \{-1.967635, -1.957947, -1.957513, -.2690963, \\
& \quad -.2215499, -.1011679, -.06678774\},
\end{aligned}
$$

Note that all those possible results are not easy to allocated by usual Newton type methods.

Obviously, the same solving procedure of the symbolic methods also works in other cases, e.g., if $\phi = \phi_2$ or if we change the construction way of the finite element space.

5 Investigation of Parameter-dependent Solutions

We may also obtain the λ-parameter solution $u_i, (i = 1, 2...N)$ directly by symbolic computation, and the relation of those 2 variables (u_i and λ) can be shown clearly by implicit function graphs.

For instance, based on the example we discussed in the last section, let us set up an uniform partition of the domain $[0, 1]$ yielding $N = 2$ and let us use the same construction of the finite element space.

We define the initial signal as the following discontinuous function

$$u_0 = \begin{cases} -1, & \text{for } x \in [0, 1/3], \\ 0, & \text{for } x \in (1/3, 2/3], \\ 1, & \text{for } x \in (2/3, 1], \end{cases}$$

then the relation between the parameter λ and one certain indeterminate (e.g. u_2) can be solved out easily by the symbolic computation, which will be the following equation with only two variables:

$$1296\lambda u_2^3 - 108\lambda u_2 - 1296u_2^5 - 72u_2^3 - 7u_2 + 6 = 0.$$

We can now draw the function by the computer algebra software on Maple. Let us take λ corresponding to the x-coordinate and u_2 to the y-coordinate, then the function graph appears as Figure 1.

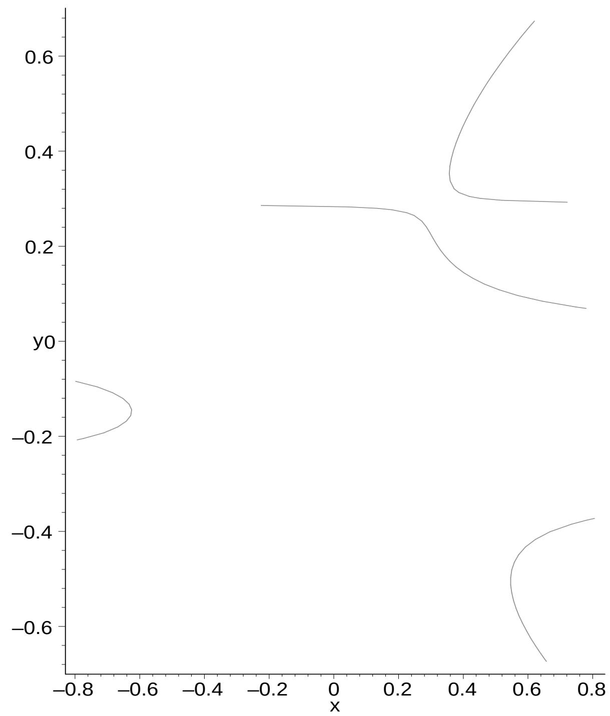

Figure 1:

It is clearly shown from Figure 1 that only if we chose the value of parameter λ within a small interval (roughly in $(-0.52, 0.3)$), we can solve the discrete equation uniquely, and the value around -0.52, 0.3, 0.48 etc. are the critical points where the number of the discrete solutions changes.

(The graph is completed by a "pacPlot" function of computer algebra software , which can make the output quality much better than the normal implicit plot Maple function.) Analyze the relation between λ and the other indeterminate u_i is now trivial.

We can consequently analyze the ϵ-dependent solution to (7) based on the previous results. From the graph we know there will be unique discrete solution if we chose $\lambda = 1/5$. Let us compute (7) by fixing the parameter $\lambda = 1/5$, we will get the function graph in Figure 2 which shows the relation between parameter ϵ and grid value of u_2 on grid node $2/3$. The associated implicit function is:

$$1296\epsilon u_2^3 + 108u_2\epsilon - 936u_2^3 + 6480u_2^5 + 143u_2 - 30 = 0.$$

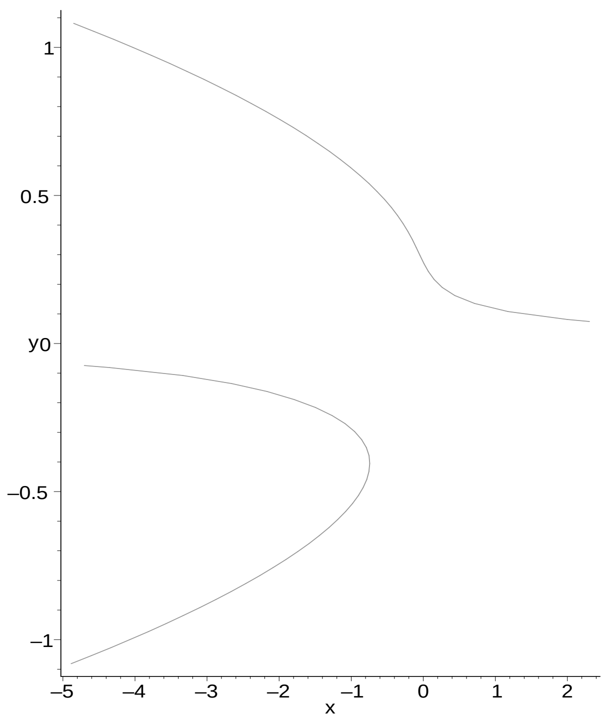

Figure 2:

From Figure 2 we see that for positive ϵ, $(x = \epsilon)$ the discrete solution is unique. This picture fits into the theory, since the term involving ϵ was introduced in (4) in order to obtain a unique solution.

If we change the definition of the initial signal

$$u_0 = \begin{cases} 1, & \text{for } x \in [0, 1/3], \\ 0, & \text{for } x \in (1/3, 1], \end{cases}$$

we obtain a relation between the parameter λ and u_2 by the following implicit equation, which looks a little more complicated:

$$\begin{aligned}
&-75/8-4310577/4\lambda^2u_2^5-5609655/8\lambda^3u_2-23085/4\lambda^2+1594323/4\lambda^4\\
&+338985/8u_2^2\lambda^2-20962395/8\lambda^3u_2^3+4782969/4\lambda^4u_2^3-33480783/2\lambda^5u_2\\
&-46943955/8\lambda^4u_2+387420489\lambda^5u_2^3-38263752\lambda^3u_2^5-387420489u_2^5\lambda^4\\
&-203391/2\lambda^3-16081011/8\lambda^2u_2^4+56549259/16u_2^2\lambda^3+20726199/2\lambda^5\\
&+16474671/2u_2^2\lambda^4+5314410u_2^4\lambda^3-172186884\lambda^5u_2^2+172186884\lambda^4u_2^4\\
&-1161/8\lambda-153/2u_2-2241/32u_2^2-42039/16\lambda u_2^2-229635/2u_2^5\lambda\\
&-21627/16u_2^4-5427/8u[2]^3-1763451/64u_2^5-618921/8\lambda u_2^4\\
&+1764909/16\lambda^2u_2^3-201933/4\lambda^2u_2-43821/16\lambda u_2+1317303/32\lambda u_2^3\\
&=0.
\end{aligned}$$

However, we can still draw the above function graph so that the numbers of the discrete solution can be seen clearly with respect to parameter λ:

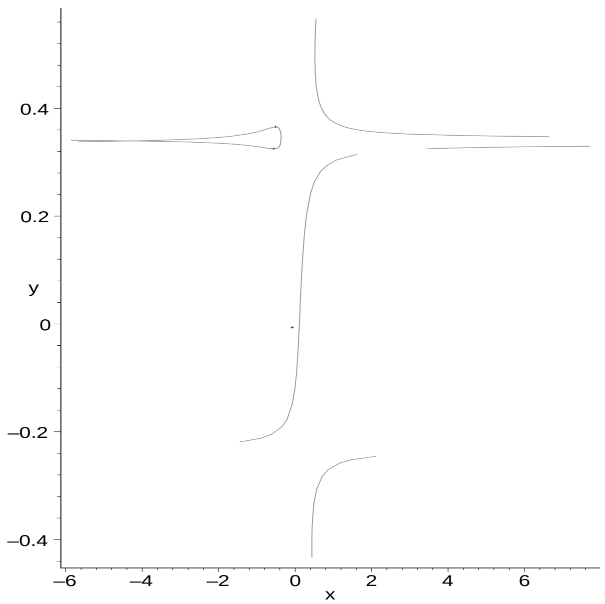

Figure 3:

Once again (from Figure 3) we find, only if we choose the value of parameter λ close to 0, the discrete solution is unique.

If the numbers of parameters is less than 2, we can always compute an implicit representation form for the unknowns $u_i, i = 1, 2...N$ depending on the parameters. The complexity of the symbolic computation only increases slightly.

6 Conclusion and Remarks

We have seen that symbolic computation can successfully be applied to the above partial differential equations, leading to qualitative and quantitative information on the solutions. In particular, the set of the solutions for certain parameter ranges can be found easily, in contrast to numerical methods. A combination of symbolic and numerical method by a two-grid algorithm, where the results of symbolic computation on a coarse grid are used as a starting guess for a Newton type method on the fine grid seems to be an interesting perspective for the future. Such an algorithm has been used for approximating minimal surfaces [7]. Recently, equations similar to the minimal surface equation have become important in the field of *Level-Set* methods [3] and for inverse problems with discontinuous solutions [10]. We expect that a symbolic solution technique can significantly improve such algorithms.

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