Identification of a Temperature Dependent Heat Conductivity from Single Boundary Measurements

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Abstract

Considering the identification of a temperature dependent conductivity in a quasilinear elliptic heat equation from single boundary measurements, we proof uniqueness in dimensions $n \geq 2$. Taking noisy data into account we apply Tikhonov regularization in order to overcome the instabilities. By using a problem-adapted adjoint, we give convergence rates under substantially weaker and more realistic conditions than required by the general theory. Our theory is supported by numerical tests.

1 Introduction

The issue of parameter identification is to determine unknown parameters, appearing, e.g., in state equations, from indirect measurements related to the physical state. This inverse problem can be considered as a (mostly) nonlinear operator equation

$$F(q) = z,$$

where the forward operator F maps the parameter q onto the output z. As the physical state often cannot be observed exactly, one finds oneself in the situation of given noisy data z^δ instead of z. Now, as parameter identification problems are frequently ill-posed, the estimation of the parameter can be strongly influenced in a negative way by even only small data noise. Hence, for their stable numerical solution some type of regularization is required. Regularization techniques replace the ill-posed problem by a family of neighbouring well-posed problems, leading to a stable approximation of q, called the regularized solution. The probably most frequently used approach is Tikhonov regularization where the regularized solutions are sought as the minimizers of

$$q \to \|F(q) - z^\delta\|^2 + \beta\|q\|^2,$$

with some regularization parameter β.

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A careful mathematical analysis of the regularization method is needed in order to give useful guidance, under which conditions it will perform well, and confidence in its numerical results. Since for ill-posed problems, convergence of any numerical algorithm can be arbitrarily slow [21], conditions for convergence rates are of special theoretical interest. They are also practically relevant as they tell us for which problems fast convergence of numerical algorithms can be expected. But according to the general theory [4] such convergence rates can only be obtained under strong source conditions of the type

$$\exists w \;\; q - q^* = F'(q)^* w, \tag{1.1}$$

where q^* is an apriori guess for q, and $F'(q)^*$ is the adjoint of the Fréchet-derivative of F evaluated at q. This general theory has been applied to various inverse problems including parameter identification, see [14], [4] for elliptic problems and integral equations, and [19] for a parabolic equation. All these applications are for one-dimensional problems, since only there, the source condition (1.1) has a rather immediate explicit interpretation (usually requiring some additional smoothness and prescribed boundary behaviour for $q^\dagger - q^*$). In [16], condition (1.1) was weakened based on ideas from [8] and then fully interpreted for the identification of a nonlinearity $q(u)$ from distributed measurements of u in arbitrary dimensions.

But before taking data perturbations and convergence rates into account, one has to consider if the given data z at all contain enough information in order to identify the parameter q, i.e. if the mapping F from q onto z is injective. Often, a limit to the amount of available data is given by the setup of the experiment. Frequently (for example, in non-destructive testing) measurements cannot be done within the material Ω but only on (parts of) the boundary $\partial\Omega$, leading to data containing less information.

Considering the inverse conductivity problem, one is interested in finding the conductivity $q(x)$ in

$$\begin{aligned} -\nabla \cdot (q(x)\nabla u) &= 0 \;\text{ in }\; \Omega \\ u &= g \;\text{ on }\; \partial\Omega, \end{aligned} \tag{1.2}$$

given the additional boundary data

$$q(x)\frac{\partial u}{\partial n} = h \;\text{ on }\; \Gamma,$$

with $\Gamma \subset \partial\Omega$. For this case of single boundary measurements, the unique identifiability is widely investigated for parameters

$$q = 1 + \chi(D), \quad \bar{D} \subset \Omega,$$

where χ is the characteristic function of a unknown domain. Several partial results are given (see, e.g., [1], [10] and [13]), nonetheless a general uniqueness result is still missing.

Turning to inverse problems for nonlinear elliptic equations

$$-\Delta u + q(u) = 0 \text{ in } \Omega,$$

only local uniqueness results for (small) q are available, if in addition to the Dirichlet data g, Neumann data as well are prescribed on $\partial\Omega$. See [12] and the references given there.

In order to enhance the chances of identifiability one often resorts to many boundary measurements: For any Dirichlet data g in (1.2) one is given Neumann data h, in other words, the results of all possible boundary measurements are known. Then, the information to identify the parameter is contained in the so-called Dirichlet to Neumann map

$$\Lambda : g \to h.$$

Based on these multiple boundary measurements, the aim of impedance tomography is the reconstruction of the conductivity $q(x)$ or $q(x, u)$ in (1.2) within the human body or some material. For the linear case, i.e., $q = q(x)$, global uniqueness was proven in [24] for dimensions $n \geq 3$ and in [18] for two dimensions. The uniqueness result for the quasilinear case $q = q(x, u)$ can be found in [23] for dimensions $n \geq 2$. There, also the anisotropic case, i.e., q is a matrix, is investigated.

We now specify the inverse problem we are looking at in this paper. Our goal is to identify the temperature dependent heat conductivity q in

$$-\nabla \cdot (q(u)\nabla u) = f \text{ in } \Omega \tag{1.3}$$

from only single boundary measurements of the temperature u. Note that not only the inverse but also the forward problem is nonlinear. We show that the parameter is uniquely identifiable on the temperature range as a function of one variable. Besides, further developments of our Tikhonov regularization analysis from [16] allow us to provide a fully interpretable weak source condition for the convergence rate of the regularized solutions.

In Section 2 we briefly discuss the nonlinear direct problem (1.3) with mixed boundary conditions for u. For a positive parameter q of H^1-regularity, we guarantee the existence of a unique weak solution $u \in H^1(\Omega)$. Furthermore, we give an estimate of the temperature range governed by u in Ω, which is equivalent to the one dimensional domain the parameter q lives on. Together with the temperature data on the boundary, the a-priori unknown interval, on which the the parameter can be recovered, can then be estimated.

In Sections 3 and 4 we investigate the inverse problem. First we show that the parameter is uniquely determined by single boundary temperature measurements. From [23], the uniqueness only follows for the case of multiple measurements. Afterwards, we give a stability analysis for Tikhonov regularization and prove convergence rates under much weaker assumptions than required in the general theory by taking advantage of a special adjoint approach. This kind of approach was first introduced in [8] for the identification of a space-dependent heat conductivity $q(x)$ from distributed temperature measurements.

Section 5 contains a detailed interpretation of the source condition needed for the convergence rate proof both in two and three dimensions. This is different to [8] where a full interpretation could only be given in the one dimensional case.

Section 6 sketches variants in the setup for the inverse problem. In Section 7 we present results of numerical tests which support our theory.

2 The Direct Problem

In many applications modelled by the heat equation, for example in the context of steel production (see [11], [6], [7]), the heat conductivity q does not vary spatially but rather depends on the temperature u itself. Considering the stationary case, the heat distribution is described by the nonlinear elliptic equation

$$-\nabla \cdot (q(u)\nabla u) = f \quad \text{in} \quad \Omega \tag{2.1}$$

with, e.g., the boundary conditions

$$q(u)\frac{\partial u}{\partial n} = h \quad \text{on} \quad \Gamma_1 \tag{2.2}$$

and

$$u(x) = g \quad \text{on} \quad \Gamma_2. \tag{2.3}$$

Here, Ω is an open bounded connected domain in $\mathbb{R}^n$, $n \geq 2$, with boundary $\partial\Omega \in C^2$. We assume $\Gamma_2 \subset \partial\Omega$ to be connected and to have positive measure and set $\Gamma_1 = \partial\Omega \backslash \Gamma_2$. f is a given heat source density, h is a given temperature flux and g is a prescribed (boundary) temperature.

We set

$$V = \{v \in H^1(\Omega) \mid v|_{\Gamma_2} = 0\}$$

and assume - already with respect to the inverse problem - that

$$g \text{ is constant, } h \in C(\Gamma_1) \text{ and } f \in C(\Omega).$$

By the trace theorem we then get a $\tilde{g} \in H^1(\Omega)$ such that $\tilde{g}|_{\Gamma_2} = g$. Now, by integration by parts we derive the variational formulation for problem (2.1)-(2.3):

Find $u \in H^1(\Omega)$ such that

$$u - \tilde{g} \in V$$

and

$$\int_\Omega q(u)\nabla u \cdot \nabla v \, dx = \int_\Omega f(x) v \, dx + \int_{\Gamma_1} hv \, d\Gamma_1 \quad \text{for all } \ v \in V \tag{2.4}$$

hold. Under the assumption

$$q \in H^1(\mathbb{R}) \ \text{ and } \ 0 < \alpha_1 \le q \le \alpha_2,$$

there exists a unique solution to the variational problem (2.4) in $H^1(\Omega)$. The proof - based on the continuous embedding $H^1(\mathbb{R}) \subseteq C(\mathbb{R})$ and the theory of quasi-monotone operators - can for example be found in [22] (see Proposition 5.1 and the subsequent relaxation to quasimonotone operators). Furthermore, there is an a-priori estimate for the solution, i.e. there is a constant $C > 0$ depending only on α_2, h, f and g, such that

$$\|u_q\|_{H^1(\Omega)} \le C, \tag{2.5}$$

where, in order to emphasize the fact that the solution u depends on the parameter q, we use the notation u_q, or $u_q(x)$.

Note that u_q can also be considered as the weak solution of the linear equation

$$\begin{aligned} -\nabla \cdot (\tilde{q}(x)\nabla u) &= f \ \text{ in } \ \Omega \\ \tilde{q}(x)\frac{\partial u}{\partial n} &= h \ \text{ on } \ \Gamma_1 \\ u(x) &= g \ \text{ on } \ \Gamma_2, \end{aligned}$$

with $\tilde{q}(x) = q(u_q(x))$. Already with respect to the inverse problem we cite [3] for the weak maximum principle: If we choose f and h such that

$$\int_\Omega f(x) v \, dx + \int_{\Gamma_1} hv \, d\Gamma_1 \le 0$$

holds for all essentially non-negative $v \in V$, we get

$$\text{ess.} \sup_{x \in \Omega} u_q \le \text{ess.} \sup_{x \in \Gamma_1} \max \{u_q, 0\} \tag{2.6}$$

(or u_q is a positive constant). Hence, the upper bound of the temperature range covered by u_q in Ω is given by the maximum temperature on Γ_1 (or by 0, if the latter is negative).

3 The Inverse Problem

Given a single boundary observation of the solution of the direct problem, the inverse problem is to recover the physical parameter q on the real interval covered by the temperature using the observation data. In order to overcome the ill-posedness of this identification problem, we choose Tikhonov regularization for its stabilization.

3.1 The Interval of Identifiability

Identifying the nonlinearity $q(u)$ is theoretically and numerically challenging, since the interval, on which the parameter can be recovered, is a-priori not known. Obviously, the parameter, as a function of one variable, cannot be reconstructed on the whole of $\mathbb{R}$, but at the most on the interval $[u_{\min}, u_{\max}]$, where $u_{\min}$ and $u_{\max}$ denote the extremal values of the temperature distributed over $\bar{\Omega}$. Outside this interval, no physical information is available, making the identification impossible in advance.

In the case that only boundary temperature measurements are given, the data need not necessarily cover all of $[u_{\min}, u_{\max}]$. If one still wants to recover the parameter on the whole interval, the following experimental setup for indirect measurements volunteers. Assume that the heat conductivity q is known up to a temperature value u_0 from maybe direct measurements, but inaccessible at temperatures above. Then, we set

$$g = u_0 \quad \text{(constant)} \tag{3.1}$$

in (2.3). By tuning f and h in (2.1) and (2.2), we drive the temperature on the boundary Γ_1 to values higher than u_0. Finally, we measure the temperature trace along Γ_1, whose maximum value we call u_1.

We know from the maximum principle (2.6) that

$$u_{\max} = u_1$$

holds (for f and h chosen appropriately). Unfortunately, we cannot guarantee $u_{\min} = u_0$ but only have $u_{\min} \leq u_0$. Nevertheless, since we assume to know q up to u_0, we can consider the identification of q on the interval

$$[u_{\min}, u_{\max}].$$

3.2 Output Least Squares Formulation

Denoting by $z(x)$ the measured temperature trace along Γ_1, we want to identify the true thermal conductivity $q^\dagger$ out of a set of admissible parameters, satisfying

$$\gamma u_{q^\dagger} = z, \tag{3.2}$$

where γ denotes the trace operator

$$\begin{aligned} \gamma : H^1(\Omega) &\to L^2(\Gamma_1) \\ u &\to u|_{\Gamma_1}, \end{aligned}$$

and $u_{q^\dagger}$ is the solution of the direct problem (2.1) - (2.3) with $q = q^\dagger$. We always assume the existence of a true parameter $q^\dagger$, i.e., that the exact data z are attainable. Of course, the measured (noisy) data need not be attainable.

We already mentioned in the previous section that $q^\dagger$ can at most be identified on the range of $u_{q^\dagger}$. Nevertheless, during the numerical solution of the inverse problem temperature values corresponding to other parameters than $q^\dagger$ may occur. Hence, the parameters have to be defined on an even larger range than that of $u_{q^\dagger}$. This crucial numerical point is discussed in [15] and [16] for the case of distributed temperature measurements.

For defining the set of admissible parameters, we choose positive constants α_1 and α_2 such that the temperatures u_{α_1} and u_{α_2} (obtained by solving the direct problem) contain (at least) the minimal and maximal values of the data, i.e., the measured temperature trace on Γ_1, respectively. Since α_1 and α_2 are constant, and hence regular parameters, regularity results, see for instance [17], yield that u_{α_1} and u_{α_2} are continuous on $\bar{\Omega}$, i.e., there are constants I_1 and I_2 such that

$$\begin{aligned} I_1 \leq u_{\alpha_1} \leq I_2, \\ I_1 \leq u_{\alpha_2} \leq I_2. \end{aligned}$$

Then, we can use the finite interval

$$I = [I_1, I_2], \quad I_1, I_2 \in \mathbb{R} \tag{3.3}$$

in order to define the set of admissible parameters as

$$K = \left\{ q \in H^1(\mathbb{R}) \mid \alpha_1 \leq q(\tau) \leq \alpha_2 \text{ for } \tau \in I \text{ and } q \text{ is fixed on } \mathbb{R}\backslash I \right\}. \tag{3.4}$$

Here, the attribute fixed has to be understood as

$$q_1(\tau) - q_2(\tau) \equiv 0 \ \text{ on } \mathbb{R}\backslash I \tag{3.5}$$

for any q_1, $q_2 \in K$. The only requirement for the behaviour of q on $\mathbb{R}\backslash I$ is that $q \in H^1(\mathbb{R})$ is not violated. Then, any $q \in K$ is continuous and bounded due to the continuous embedding $H^1(\mathbb{R}) \subseteq C_b(\mathbb{R})$. Of course, we can only identify $q^\dagger$ on a subdomain of I where we have information about the system from the data z. Outside this domain, we have no information, so that an identification is impossible in advance. In this sense we should look in (3.3) for an interval I of minimal length. Again, we refer to [16] for a possible numerical approach.

As we shall see below, this construction of K is mainly needed for technical reasons. Things would simplify, if one assumes the existence (but not the exact knowledge) of a finite interval I that covers all temperatures u_q for q belonging to

$$\tilde{K} = \left\{ q \in H^1(I) \mid \alpha_1 \leq q(\tau) \leq \alpha_2 \text{ for } \tau \in I \right\}$$

with α_1, α_2 appropriately chosen. This assumption may be supported by the finiteness of physical temperatures.

For later use, we introduce the set of the indefinite integrals of the parameters $q \in K$

$$S = \left\{ Q \in H^2(\mathbb{R}) \mid \frac{dQ}{d\tau} \in K \text{ and } Q(g) = 0 \right\}, \tag{3.6}$$

where $g \in I$ is the constant from (2.3), (3.1). Because of (3.4), we have a common Lipschitz constant, namely α_2, for the functions $Q \in S$:

$$|Q(\tau_1) - Q(\tau_2)| \leq \alpha_2 |\tau_1 - \tau_2|, \ \tau_1, \ \tau_2 \in \mathbb{R} \tag{3.7}$$

for all $Q \in S$.

In applications, the exact data $z(x)$ are not known precisely due to measurement errors. Hence, the actual data are available in the form

$$z^\delta(x) = z(x) + \text{noise},$$

where one needs some information

$$\|z - z^\delta\|_{L^2(\Gamma_1)} \leq \delta \tag{3.8}$$

about the noise level. Due to the data noise, the ill-posedness of the inverse problem requires some type of regularization in order to determine $q^\dagger$ in (3.2) in a stable way. Choosing Tikhonov regularization, we consider the following output-least-squares problem:

Let the set K of admissible parameters and noisy data z^δ be given as in (3.4) and (3.8). Assume that the exact data z is attainable from a parameter $q^\dagger \in K$. Then, for $\beta > 0$, find a parameter $q_\beta^\delta \in K$ that minimizes

$$J_\beta(q) = \int_{\Gamma_1} |\gamma u_q - z^\delta|^2 d\Gamma_1 + \beta \|q - q^*\|^2_{H^1(\mathbb{R})} \tag{3.9}$$

over K for an appropriate choice of β and $q^* \in K$. The selection of q^* is crucial for the results about the convergence rate in Section 4. Available a-priori information about the true parameter $q^\dagger$ should be used for the choice of q^*, i.e., q^* should be interpreted as some kind of a-priori guess for $q^\dagger$. Because of (3.5), the $H^1(\mathbb{R})$-norm can be replaced by the $H^1(I)$-norm, which in the following is denoted by $\|\cdot\|_I$ (other penalty terms are possible, see Section 6). Note that q^* also determines the "identified" parameter $q^\dagger$ outside the domain of information in the case of I chosen too large.

Before discussing aspects of stability and convergence of the regularized solutions q_β^δ towards $q^\dagger$, we make sure that the given boundary data are sufficient to identify the parameter uniquely.

3.3 Identifiability

Investigating the identifiability of q we are interested in the injectivity of the parameter-to-output map

$$q \to \gamma u_q.$$

We show that the temperature trace on Γ_1 determines the parameter uniquely on that range.

Theorem 3.1. *Let u_{q_1} and $u_{q_2} \in H^1(\Omega)$ be the solutions of the direct problem corresponding to parameters q_1 and $q_2 \in K$. Then $\gamma u_{q_1} = \gamma u_{q_2}$ implies $q_1 = q_2$ on the range of u_{q_1} on Γ_1.*

Proof. For $i = 1, 2$ we define $w_i = Q_i(u_{q_i})$ where $Q_i \in S$ is the anti-derivative to q_i. Then, w_i satisfies $w_i|_{\Gamma_2} = 0$ and the linear equation

$$\int_\Omega \nabla w_i \cdot \nabla v dx = \int_\Omega f(x) v dx + \int_{\Gamma_1} h v d\Gamma_1 \quad \text{for all } \; v \in V.$$

Hence, the difference $w = w_1 - w_2$ fulfills

$$\int_\Omega \nabla w \cdot \nabla v dx = 0 \quad \text{for all } \; v \in V,$$

which gives $w = 0$ according to the unique solvability of the homogeneous problem. Hence, we have $Q_1(u_{q_1}) = Q_2(u_{q_2})$ in Ω. From the trace theorem we get $\gamma Q_1(u_{q_1}) = \gamma Q_2(u_{q_2})$, the continuity of Q_i yields $Q_1(\gamma u_{q_1}) = Q_2(\gamma u_{q_2})$. From the assumption $\gamma u_{q_1} = \gamma u_{q_2}$ we then can conclude that $Q_1(\tau) = Q_2(\tau)$, and hence $q_1(\tau) = q_2(\tau)$ for τ out of the range of u_{q_1} on Γ_1. $\square$

3.4 Existence, Stability and Convergence of the Regularized Solutions

Returning to problem (3.9), we have to make sure that

- a minimizer q_β^δ exists for any data $z^\delta \in L^2(\Gamma_1)$ (existence)
- for a fixed regularization parameter β the minimizers of (3.9) depend continuously on the data z^δ (stability)
- the regularized solutions q_β^δ converge towards the true parameter $q^\dagger$ as both the noise level δ and the regularization parameter β (chosen by an apriori rule) tend to zero (convergence)

The proof of the desired properties is standard (see [4], [5], [16] or [14]), once the weak closedness of the mapping $q \to \gamma u_q$ is provided:

Proposition 3.1 (weak closedness). *For* $q_n \rightharpoonup q \in K$ *in* $H^1(\mathbb{R})$ *and* $\gamma u_{q_n} \rightharpoonup y$ *in* $L^2(\Gamma_1)$, *we have*

$$\gamma u_q = y.$$

Proof. From (2.5) we know that the sequence $\{u_{q_n}\}$ is bounded in $H^1(\Omega)$. Therefore, there exists a subsequence $\{u_{q_{n_k}}\}$ such that

$$u_{q_{n_k}} \rightharpoonup u^* \quad \text{in} \quad H^1(\Omega) \tag{3.10}$$

with $u^*|_{\Gamma_2} = g$. As the embedding of $H^1(\Omega)$ into $L^2(\Omega)$ is compact, we also have

$$u_{q_{n_k}} \rightarrow u^* \quad \text{in} \quad L^2(\Omega). \tag{3.11}$$

First, we prove that $u_{q_{n_k}} \rightharpoonup u_q$ in $H^1(\Omega)$, for which we have to show that $u^* = u_q$.

By the help of the the triangle inequality we get

$$\begin{aligned}
&\left|\int_\Omega q_{n_k}(u_{n_k})\nabla u_{n_k}\cdot\nabla v dx - \int_\Omega q(u^*)\nabla u^*\cdot\nabla v dx\right| \\
\leq\ &\left|\int_\Omega \{q_{n_k}(u_{n_k})\nabla u_{n_k} - q(u^*)\nabla u_{n_k}\}\cdot\nabla v dx\right| && (3.12)\\
&+\left|\int_\Omega \{q(u^*)\nabla u_{n_k} - q(u^*)\nabla u^*\}\cdot\nabla v dx\right| && (3.13)
\end{aligned}$$

Defining a linear functional l on $H^1(\Omega)$ by

$$l(u) = \int_\Omega q(u^*)\nabla v\cdot\nabla u dx,$$

we obtain from the weak convergence (3.10) that (3.13) vanishes for $k \to \infty$.

Applying once more the triangle inequality to (3.12) yields

$$\begin{aligned}
&\left|\int_\Omega \{q_{n_k}(u_{n_k})\nabla u_{n_k} - q(u^*)\nabla u_{n_k}\}\cdot\nabla v dx\right| \\
\leq\ &\left|\int_\Omega \{q_{n_k}(u_{n_k}) - q(u_{n_k})\}\nabla u_{n_k}\cdot\nabla v dx\right| && (3.14)\\
&+\left|\int_\Omega \{q(u_{n_k}) - q(u^*)\}\nabla u_{n_k}\cdot\nabla v dx\right| && (3.15)
\end{aligned}$$

Because of (3.5) and the Cauchy-Schwarz inequality we get

$$\begin{aligned}
&|\int_\Omega (q_{n_k}(u_{n_k}) - q(u_{n_k}))\nabla u_{n_k}\cdot\nabla v dx| \\
\leq\ &\|q_{n_k} - q\|_{C(I)}\|u_{n_k}\|_{H^1(\Omega)}\|v\|_{H^1(\Omega)} \\
\leq\ &\tilde{C}\|q_{n_k} - q\|_{C(I)} && (3.16)
\end{aligned}$$

for a constant $\tilde{C}$ not depending on n_k because of (2.5). Since the embedding of $H^1(I)$ into $C(I)$ is compact due to the finiteness of I, (3.14) tends to zero for $k \to \infty$.

Because of the boundedness of q, we also can apply the Cauchy Schwarz inequality to (3.15) and obtain

$$\begin{aligned} & \left| \int_\Omega \{q(u_{n_k}) - q(u^*)\} \nabla u_{n_k} \cdot \nabla v dx \right| \\ \leq \; & C \| q(u_{n_k}) \nabla v - q(u^*) \nabla v \|_{L^2(\Omega)} \end{aligned} \tag{3.17}$$

by means of (2.5). Furthermore, the continuity of q and (3.11) yield $\lim_{k\to\infty} q(u_{n_k}(x)) = q(u_*(x))$ for almost every $x \in \Omega$. Because of $\partial v / \partial x_i \in L^2(\Omega)$ and the boundedness of q, the dominated convergence theorem, see [9], shows that (3.17) vanishes for $k \to \infty$.

Summarizing these results we obtain for $k \to \infty$ that

$$\int_\Omega q(u^*) \nabla u^* \cdot \nabla v dx = \int_\Omega f(x) v dx + \int_{\Gamma_1} h v d\Gamma_1 \quad \text{for all } \; v \in V$$

with $u^* - \tilde{g} \in V$. Hence, u^* is the weak solution of (2.1) - (2.3) for the parameters q, f, g and h. As the weak solution is unique, we conclude that $u^* = u_q$. Finally, as $u_{q_{n_k}} \rightharpoonup u_q$ holds for any subsequence $u_{q_{n_k}}$, we get

$$u_{q_n} \rightharpoonup u_q \;\text{ in }\; H^1(\Omega). \tag{3.18}$$

According to our assumption, we have $\gamma u_{q_n} \rightharpoonup y$ in $L^2(\Gamma_1)$, because of (3.18) and the continuity of the trace operator γ we also know $\gamma u_{q_n} \rightharpoonup \gamma u_q$ in $L^2(\Gamma_1)$. The uniqueness of the weak limit yields $\gamma u_q = y$. □

Hence, existence, stability and convergence of the regularized solutions are guaranteed. The special construction of the set K was only needed in order to derive estimate (3.16). Furthermore, the proof shows that Proposition 3.1 also holds if one considers $\tilde{K}$ as set of admissible parameters.

4 Convergence Rates

Opposed to the general theory [4], we introduce a weak source condition for the convergence rate, which allows a full interpretation in Section 5. Though based on concepts from [16], both the formulation and the proof of the convergence rate theorem are different to [16], since we now have to deal with boundary terms.

Theorem 4.1. *Assume that there exists a function*

$$w \in V \tag{4.1}$$

such that

$$\left(q^\dagger - q^*, \psi\right)_I = \int_{\Gamma_1} \Psi(u_{q^\dagger}) \frac{\partial w}{\partial n} d\Gamma_1 \ \ \forall \psi \in H^1(I) \tag{4.2}$$

holds, where Ψ *is the antiderivative to* ψ*, fixed by*

$$\Psi(g) = 0. \tag{4.3}$$

Furthermore, assume that $\frac{\partial w}{\partial n} \in L^2(\Gamma_1)$ *with*

$$\Delta w = 0 \ \ in \ \ \Omega. \tag{4.4}$$

Then, with $\beta \sim \delta$*, we have*

$$\|\gamma u_{q_\beta^\delta} - z^\delta\|_{L^2(\Gamma_1)} = O(\delta)$$

and

$$\|q_\beta^\delta - q^\dagger\|_I = O(\sqrt{\delta}),$$

where q_β^δ *is the minimizer of* (3.9).

Proof. For the sake of simplicity, we now omit the explicit notation of γ. Then, as q_β^δ is a minimizer of (3.9), we get $J_\beta(q_\beta^\delta) \leq J_\beta(q^\dagger)$. This implies

$$\|u_{q_\beta^\delta} - z^\delta\|^2_{L^2(\Gamma_1)} + \beta\|q_\beta^\delta - q^*\|_I^2 \leq \delta^2 + \beta\|q^\dagger - q^*\|_I^2,$$

from which we obtain

$$\begin{aligned}
& \|u_{q_\beta^\delta} - z^\delta\|^2_{L^2(\Gamma_1)} + \beta\|q^\dagger - q_\beta^\delta\|_I^2 \\
\leq \ & \delta^2 + \beta\|q^\dagger - q^*\|_I^2 + \beta\left\{\|q^\dagger - q_\beta^\delta\|_I^2 - \|q_\beta^\delta - q^*\|_I^2\right\} \\
= \ & \delta^2 + 2\beta\left(q^\dagger - q^*, q^\dagger - q_\beta^\delta\right)_I.
\end{aligned} \tag{4.5}$$

As integration by parts yields

$$\int_{\partial\Omega} \Psi(u_{q^\dagger}) \frac{\partial w}{\partial n} dS - \int_\Omega \Psi(u_{q^\dagger}) \Delta w dx = \int_\Omega \psi(u_{q^\dagger}) \nabla u_{q^\dagger} \nabla w dx,$$

(4.3) and (4.4) give

$$\int_{\Gamma_1} \Psi(u_{q^\dagger}) \frac{\partial w}{\partial n} d\Gamma_1 = \int_\Omega \psi(u_{q^\dagger}) \nabla u_{q^\dagger} \nabla w dx$$

for the right-hand side of the source condition (4.2). Hence, choosing $\psi = q^\dagger - q_\beta^\delta$ in the source condition (4.2) leads to

$$\left(q^\dagger - q^*, q^\dagger - q_\beta^\delta\right)_I = \int_\Omega \left(q^\dagger(u_{q^\dagger}) - q_\beta^\delta(u_{q^\dagger})\right) \nabla u_{q^\dagger} \nabla w dx. \tag{4.6}$$

Using the direct problem formulation (see (2.4)) for $u_{q_\beta^\delta}$ and $u_{q^\dagger}$, we see by taking the difference that

$$\int_\Omega \left(q_\beta^\delta(u_{q_\beta^\delta}) \nabla u_{q_\beta^\delta} - q^\dagger(u_{q^\dagger}) \nabla u_{q^\dagger} \right) \cdot \nabla w dx = 0 \tag{4.7}$$

holds. Multiplying (4.6) by β and adding zero in the form of (4.7), it follows that

$$\begin{aligned}
\beta \left(q^\dagger - q^*, q^\dagger - q_\beta^\delta \right)_I = \quad & \beta \int_\Omega \left(q^\dagger(u_{q^\dagger}) - q_\beta^\delta(u_{q^\dagger}) \right) \nabla u_{q^\dagger} \nabla w dx \\
+ \quad & \beta \int_\Omega q_\beta^\delta(u_{q_\beta^\delta}) \nabla u_{q_\beta^\delta} \cdot \nabla w dx \\
- \quad & \beta \int_\Omega q^\dagger(u_{q^\dagger}) \nabla u_{q^\dagger} \cdot \nabla w dx.
\end{aligned} \tag{4.8}$$

We simplify the right hand side of (4.8) to

$$I_1 = \beta \int_\Omega \left(q_\beta^\delta(u_{q_\beta^\delta}) \nabla u_{q_\beta^\delta} - q_\beta^\delta(u_{q^\dagger}) \nabla u_{q^\dagger} \right) \cdot \nabla w dx.$$

Using the antiderivative $Q_\beta^\delta \in S$ (see (3.6)) of q_β^δ, we obtain

$$I_1 = \beta \int_\Omega \left(\nabla Q_\beta^\delta(u_{q_\beta^\delta}) - \nabla Q_\beta^\delta(u_{q^\dagger}) \right) \cdot \nabla w dx.$$

Integration by parts leads to

$$\begin{aligned}
I_1 \quad = \quad & \beta \int_{\partial\Omega} \left(Q_\beta^\delta(u_{q_\beta^\delta}) - Q_\beta^\delta(u_{q^\dagger}) \right) \frac{\partial w}{\partial n} \, dS \\
& -\beta \int_\Omega \left(Q_\beta^\delta(u_{q_\beta^\delta}) - Q_\beta^\delta(u_{q^\dagger}) \right) \Delta w \; dx.
\end{aligned}$$

Because of $u_{q_\beta^\delta}|_{\Gamma_2} = u_{q^\dagger}|_{\Gamma_2} = g$ and (4.4), we finally obtain

$$I_1 = \beta \int_{\Gamma_1} \left(Q_\beta^\delta(u_{q_\beta^\delta}) - Q_\beta^\delta(u_{q^\dagger}) \right) \frac{\partial w}{\partial n} \, d\Gamma_1$$

Next, we estimate I_1 by (3.7) and the Cauchy-Schwarz inequality in order to get

$$|I_1| \leq \beta\alpha_2 \|u_{q^\dagger} - u_{q_\beta^\delta}\|_{L^2(\Gamma_1)} \|\frac{\partial w}{\partial n}\|_{L^2(\Gamma_1)}.$$

Applying the triangle inequality and Young's inequality

$$a \cdot b \leq \varepsilon a^2 + \frac{b^2}{4\varepsilon},$$

for any $\varepsilon > 0$, we obtain

$$\begin{aligned}
|I_1| &\leq \beta\alpha_2 \|u_{q^\delta_\beta} - z^\delta\|_{L^2(\Gamma_1)} \|\frac{\partial w}{\partial n}\|_{L^2(\Gamma_1)} \\
&\quad + \beta\alpha_2 \|z^\delta - u_{q^\dagger}\|_{L^2(\Gamma_1)} \|\frac{\partial w}{\partial n}\|_{L^2(\Gamma_1)} \\
&\leq \varepsilon\alpha_2^2\delta^2 + \frac{\beta^2}{2\varepsilon}\|\frac{\partial w}{\partial n}\|^2_{L^2(\Gamma_1)} \\
&\quad + \varepsilon\alpha_2^2 \|u_{q^\delta_\beta} - z^\delta\|^2_{L^2(\Gamma_1)}.
\end{aligned}$$

Using this estimate for $|I_1|$, we get from (4.5) that

$$\begin{aligned}
\|u_{q^\delta_\beta} - z^\delta\|^2_{L^2(\Gamma_1)} + \beta\|q^\delta_\beta - q^\dagger\|^2_I \leq \\
\leq \delta^2 + 2\alpha_2^2\varepsilon\|u_{q^\delta_\beta} - z^\delta\|^2_{L^2(\Gamma_1)} + \\
+ 2\alpha_2^2\varepsilon\delta^2 + \frac{\beta^2}{\varepsilon}\|\frac{\partial w}{\partial n}\|^2_{L^2(\Gamma_1)},
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
\left\{1 - 2\alpha_2^2\varepsilon\right\} \|u_{q^\delta_\beta} - z^\delta\|^2_{L^2(\Gamma_1)} + \\
+ \beta\|q^\delta_\beta - q^\dagger\|^2_I \leq \delta^2 + 2\varepsilon\alpha_2^2\delta^2 + \\
+ \frac{\beta^2}{\varepsilon}\|\frac{\partial w}{\partial n}\|^2_{L^2(\Gamma_1)}.
\end{aligned}$$

With $\varepsilon < \frac{1}{2\alpha_2^2}$ we finally obtain that

$$\|u_{q^\delta_\beta} - z^\delta\|_{L^2(\Gamma_1)} = O(\delta) \tag{4.9}$$

(hence, by (3.8), also $\|u_{q^\delta_\beta} - z\|_{L^2(\Gamma_1)} = O(\delta)$) and

$$\|q^\delta_\beta - q^\dagger\|_I = O(\sqrt{\delta}) \tag{4.10}$$

hold. The constants in the O-terms in (4.9) and (4.10) can be derived from the proof. $\square$

Now, the theoretical analysis of our identification problem is complete. We have shown stability, convergence and given conditions on the rate of convergence. In the next section, we give sufficient conditions for the existence of a source function w that satisfies (4.1)-(4.4), which allows the interpretation of the source condition (4.2). Again, we modify the approach of [16] appropriately.

5 Discussion of the Source Condition

Opposed to the general theory (compare with (1.1)), where Fréchet-differentiability of the forward operator F and Lipschitz continuity of $F'(q)$ already require a more regular parameter, the formulation of (4.2) itself does not impose any more regularity on q^* and on $q^\dagger$ than they being in $H^1(I)$. Furthermore, in the general theory the adjoint of $F'(q^\dagger)$ is needed which makes the source condition usually very difficult to interpret. The new approach only uses the parameter-to-solution map $u_{q^\dagger}$ itself, which has a direct physical meaning, not its linearization.

Usually, source conditions as (1.1) mean severe restrictions on the parameter, and are readily interpretable only in the one-dimensional case. We next construct a source function w for (4.2)- even for the higher dimensional case - under quite natural conditions. The interpretation is based on our work in [16], but now for single boundary measurements.

If we denote the range of the true temperature $u_{q^\dagger}$ on Γ_1 (and hence on Ω, see Section 3.1) by the interval $I^\dagger = \left[I^\dagger_{\min}, I^\dagger_{\max}\right]$, i.e.

$$I^\dagger_{\min} = \min_{x\in\Gamma_1} u_{q^\dagger} \text{ and } I^\dagger_{\max} = \max_{x\in\Gamma_1} u_{q^\dagger},$$

we have

$$I^\dagger \subseteq I.$$

If we can assume to know the true parameter $q^\dagger$ outside the range $I^\dagger$ already from q^*, i.e., for $\rho := q^\dagger - q^*$ we have

$$\rho = 0 \text{ for } \tau \in I\backslash I^\dagger, \tag{5.1}$$

the source condition (4.2) turns to

$$(\rho, \psi)_{I^\dagger} = \int_{\Gamma_1} \Psi(u_{q^\dagger}) \frac{\partial w}{\partial n} d\Gamma_1 \;\; \forall \psi \in H^1(I).$$

This is a very natural assumption as outside of $I^\dagger$ the parameter is of no use for the physical system. Also, from the inverse point of view we only can expect to identify the parameter on the range of the true temperature as outside of $I^\dagger$ no information is available. Hence, on $I\backslash I^\dagger$ $q^\dagger$ is already determined by the choice of q^*.

We now assume that

$$q^\dagger - q^* \in H^4(I) \tag{5.2}$$

and require the trace of the true temperature $u_{q^\dagger}$ to satisfy

$$\gamma u_{q^\dagger} : \Gamma_1 \to I^\dagger \text{ is Lipschitz.} \tag{5.3}$$

Then - because of the compact embedding $H^4(I^\dagger) \subset C^3(I^\dagger)$ - our assumptions on the a priori knowledge about $\rho = q^\dagger - q^*$ (see (5.1)) result in

$$\rho^{(j)}(I^\dagger_{\min}) = \rho^{(j)}(I^\dagger_{\max}) = 0 \;\text{ for }\; j = 0, 1, 2, 3. \tag{5.4}$$

Because of assumption (5.3) the change of variables formula (see [9], [16]) can be applied with the transformation $t = \gamma u_{q^\dagger}$, whose level-sets are isotherms on Γ_1. This gives

$$\int_{\Gamma_1} s(x) J\gamma u_{q^\dagger}(x) d\Gamma_1 = \int_{I^\dagger} \left[\int_{\gamma u_{q^\dagger}^{-1}\{\tau\}} s dH^{n-2} \right] d\tau \tag{5.5}$$

for any L^{n-1}-summable function $s : \mathbb{R}^{n-1} \to \mathbb{R}$, where $J\gamma u_{q^\dagger}$ denotes the Jacobian of $\gamma u_{q^\dagger}$ and H^{n-2} is the $(n-2)$-dimensional Hausdorff measure. Next, we have to find a suitable function s in (5.5) for our purpose. First, we define m to be the (n-2)-dimensional Hausdorff-measure of the level sets of $\gamma u_{q^\dagger}$, i.e.

$$m(\tau) = \int_{\gamma u_{q^\dagger}{}^{-1}\{\tau\}} dH^{n-2} \;\text{ for }\; \tau \in I^\dagger,$$

and assume the trace of $u_{q^\dagger}$ on Γ_1 to behave such that

$$(\rho'''(\gamma u_{q^\dagger}) - \rho'(\gamma u_{q^\dagger})) \cdot \frac{1}{m(\gamma u_{q^\dagger})} \cdot J\gamma u_{q^\dagger} \in L^2(\Gamma_1). \tag{5.6}$$

The only term that might cause a violation of (5.6) is $\frac{1}{m(\gamma u_{q^\dagger})}$. In two dimensions ($n = 2$), the H^{n-2}-measure is the counting measure. Then, we have $m(\tau) \neq 0$ on $I^\dagger$ as every $\tau \in I^\dagger$ is at least attained once by $u_{q^\dagger}$ for $x \in \Gamma_1$ (by definition). This is distinct from [16] where even in two dimensions condition (5.6) cannot be guaranteed a-priori. In three dimensions (5.6) is certainly fulfilled, if $m(\gamma u_{q^\dagger})$ is bounded away from 0, i.e., if all temperatures in $I^\dagger$ are assumed on sets of non-vanishing H^1-measure, and if these measures depend in a reasonable way on the temperatures. This is a (weak) regularity condition on the measures of the isotherms, and it is reasonable, since one cannot expect identifiability of $q^\dagger$ for temperatures which are assumed only on a "small" set (of H^1-measure zero). But even such "small isotherms" are not excluded by (5.6): The only temperature values possibly attained by $\gamma u_{q^\dagger}$ at a set of H^1-measure zero are $I^\dagger_{\min}$ or $I^\dagger_{\max}$. This is a consequence of the continuity of the trace of $u_{q^\dagger}$ on Γ_1 and the intermediate value theorem. But both the first term in the product (see (5.4)) and $J\gamma u_{q^\dagger}$ (necessary condition for an extremum) vanish in the respective critical situation. Now if the product of these two expressions tends faster to zero than $m(\gamma u_{q^\dagger})$, the L^2-boundedness in (5.6) is maintained, even then.

Now we again omit the explicit notation of γ and look at the Poisson equation

$$\Delta w = 0 \text{ in } \Omega \tag{5.7}$$

$$\frac{\partial w}{\partial n} = (\rho'''(u_{q^\dagger}) - \rho'(u_{q^\dagger})) \cdot \frac{1}{m(u_{q^\dagger})} \cdot J u_{q^\dagger} \;\text{ on }\; \Gamma_1 \tag{5.8}$$

$$w = 0 \text{ on } \Gamma_2. \tag{5.9}$$

for which the existence of a unique (weak) solution $w \in V$ is guaranteed because of (5.6). Similar as in [16], one can show that the solution w of (5.7)-(5.9) satisfies the source condition (4.2). Note that the conditions (4.1) and (4.4) of the convergence rate theorem are automatically fulfilled by this approach. The essential assumptions needed for the proof are:

- sufficient smoothness of $q^\dagger - q^*$ as required in (5.2)
- sufficient smoothness of the trace of $u_{q^\dagger}$ on Γ_1: (5.3)
- sufficient knowledge about $q^\dagger$ on the boundary of the temperature interval where measurements are available: (5.4)
- condition (5.6) (only needed for $n = 3$), which essentially says that the isotherms of $u_{q^\dagger}$ on Γ_1 depend in a sufficiently regular way on the temperature level, where this regularity is rather weak.

Since under these conditions the source condition can be verified, the convergence rates from Theorem 4.1 are valid.

In any dimension, the heat-dependent conductivity is identified as a function of one variable. Hence, it is remarkable that the interpretation edges down more in two dimensions, where the boundary temperature represents only a one dimensional data manifold, than in three, where a two dimensional data manifold is available.

6 Variants

In Section 2 we introduced a nonlinear mixed boundary problem for which we considered in Section 4 a constant boundary temperature u_0 on Γ_2 for technical reasons. From the practical (with respect to industry) point of view, the pure Neumann type problem

$$-\nabla \cdot (q(u)\nabla u) = f(x) \text{ in } \Omega \tag{6.1}$$

$$q(u)\frac{\partial u}{\partial n} = h \text{ on } \partial\Omega \tag{6.2}$$

could be more realistic, where (6.2) then describes, e.g., the cooling of a steel strand (see [11]). If we choose the space of test functions V as

$$V = \left\{ v \in H^1(\Omega) \mid \int_\Omega v dx = 0 \right\}$$

and require

$$\int_\Omega f dx + \int_{\partial\Omega} h dS = 0,$$

the existence of a unique weak solution u_q in V to (6.1) - (6.2) can be guaranteed by [22]. Then the theory developed in Sections 3 and 4 remains valid if we replace Γ_1 by $\partial\Omega$. This means that the measurements are now done on all of $\partial\Omega$ and the Tikhonov functional is

$$J_\beta(q) = \int_{\partial\Omega} |u_q - z^\delta|^2 dS + \beta\|q - q^*\|_I^2.$$

The source condition

$$\left(q^\dagger - q^*, \psi\right)_I = \int_{\partial\Omega} \Psi(u_{q^\dagger})\frac{\partial w}{\partial n} dS \quad \forall \psi \in H^1(I), \tag{6.3}$$

which yields the rates

$$\|u_{q_\beta^\delta} - z^\delta\|_{L^2(\partial\Omega)} = O(\delta)$$

and

$$\|q_\beta^\delta - q^\dagger\|_I = O(\sqrt{\delta}),$$

can be interpreted as in Section 5. Note that in order to show the existence of a unique (weak) solution to problem (5.7) - (5.9) in V (with $\Gamma_1 = \partial\Omega$, $\Gamma_2 = \emptyset$), we now in addition have to check

$$\int_{\partial\Omega} \frac{\partial w}{\partial n} dS = 0. \tag{6.4}$$

But this follows from applying the coarea formula (5.5) with

$$s(x) = (\rho'''(u_{q^\dagger}(x)) - \rho'(u_{q^\dagger}(x)))\frac{1}{m(u_{q^\dagger}(x))},$$

which gives

$$\begin{aligned} \int_{\partial\Omega} \frac{\partial w}{\partial n} dS &= \int_{I^\dagger} (\rho'''(\tau) - \rho'(\tau))\frac{1}{m(\tau)} m(\tau) d\tau \\ &= \int_{I^\dagger} (\rho''' - \rho') d\tau \\ &= 0, \end{aligned} \tag{6.5}$$

where the last equality holds because of (5.4).

Finally, we change the settings of the direct problem to the pure Dirichlet case:

$$-\nabla \cdot (q(u)\nabla u) = f(x) \text{ in } \Omega \tag{6.6}$$
$$u = g \text{ on } \partial\Omega. \tag{6.7}$$

The existence of a unique solution in $H^1(\Omega)$ is given by the theory quoted in Section 2. As now the temperature flux $q(u)\frac{\partial u}{\partial n}$ is measured on (all of) the boundary $\partial\Omega$, we need a higher regularity of the solution u_q than in our previous discussions, if we still want to measure our data in the L^2-setting, i.e., consider that the measurements are in L^2. If we choose the set of admissible parameters

$$\hat{K} = \left\{ q \in H^2(I) \mid \alpha_1 \leq q(\tau) \leq \alpha_2 \text{ for } \tau \in I \quad \wedge \quad \|\frac{dq}{d\tau}\|_{L^\infty(I)} \leq \infty \right\}$$

and require

$$g \in H^{3/2}(\partial\Omega),$$

then, for $q \in \hat{K}$, the existence of a unique solution u_q even in $H^2(\Omega)$ can be shown (see [2]). Now, the continuous embedding $H^2(\mathbb{R}) \subseteq C^1(\mathbb{R})$ and the trace theorem yield

$$q(u_q)\frac{\partial u_q}{\partial n} \in H^{1/2}(\partial\Omega).$$

Hence, the Tikhonov functional

$$J_\beta(q) = \int_{\partial\Omega} |\gamma_q u_q - z^\delta|^2 dS + \beta\|q - q^*\|_I^2$$

with

$$\begin{array}{rcl} \gamma_q : H^2(\Omega) & \to & L^2(\partial\Omega) \\ u & \to & q(u)\dfrac{\partial u}{\partial n} \end{array}$$

is meaningful. Under the source condition

$$\left(q^\dagger - q^*, \psi\right)_I = \int_{\partial\Omega} \Psi(g)\frac{\partial w}{\partial n} dS \quad \forall \psi \in H^1(I) \tag{6.8}$$

(g now is no longer needed to be constant) the rates

$$\|q_\beta^\delta - q^\dagger\|_I = O(\sqrt{\delta})$$

and

$$\|\gamma_{q_\beta^\delta} u_{q_\beta^\delta} - z^\delta\|_{L^2(\partial\Omega)} = O(\delta)$$

can be shown. Note, that (6.8) does now not even depend explicitly on the unknown temperature $u_{q^\dagger}$, which is a major difference to the theory of convergence rate developed so far. Once more, the source function $w \in H^1(\Omega)$ can be found as the solution of

$$\begin{array}{rcl} \Delta w & = & 0 \text{ in } \Omega \\ \dfrac{\partial w}{\partial n} & = & (\rho'''(u_{q^\dagger}) - \rho'(u_{q^\dagger})) \cdot \dfrac{1}{m(u_{q^\dagger})} \cdot J u_{q^\dagger} \text{ on } \partial\Omega, \end{array}$$

for the proof of the convergence rate the variational formulation

$$\int_\Omega q(u)\nabla u \cdot \nabla v dx = \int_\Omega h(x) v dx + \int_{\partial\Omega} q(u)\frac{\partial u}{\partial n} v dS$$

with test functions $v \in H^1(\Omega)$ is needed. The advantage of considering pure Dirichlet data for the direct problem is that the condition

$$\gamma u_{q^\dagger} : \partial\Omega \to I^\dagger \text{ is Lipschitz}$$

can now be automatically satisfied by choosing the problem input g regular enough because of

$$\gamma u_{q^\dagger} = g. \tag{6.9}$$

Furthermore, we now can drive the interval $I^\dagger$ on which we want to identify the parameter in a straightforward way by the choice of g.

Finally we mention, that in all our optimization problems, the H^1-regularization term can be replaced by a L^2-term, i.e. we can consider

$$\min J(q) := \{L^2\text{-norm of residual}\}^2 + \beta \int_I (q - q^*)^2 d\tau. \tag{6.10}$$

Results for stability, convergence and rate of convergence can be proven in a completely analogous way, where the H^1-scalar product $(q^\dagger - q^*, \psi)_I$ in (4.2) is replaced by the L^2-scalar product $\int_I (q^\dagger - q^*)\psi d\tau$. A solution to the source condition can be constructed as in Section 5 under even weaker regularity assumptions on $q^\dagger$.

The existence of minimizers of (6.10) cannot be guaranteed by Theorem 3.1, but this difficulty can be resolved by incorporating a tolerance η into the minimization, i.e., replacing minimizers of (6.10) by elements $q^\delta_{\beta,\eta}$ such that

$$J(q^\delta_{\beta,\eta}) \le \inf J(q) + \eta.$$

As long as $\eta = O(\delta^2)$, all proofs carry over (see [5]). This can of course also be done for (3.9).

7 Numerical Experiments

In order to test the identification of the heat conductivity by Tikhonov regularization, we carry out numerical simulations using the temperature trace $u|_{\Gamma_1}$ as data. Considering a rectangular domain $\Omega = [0, 0.5] \times [0, 2]$ with boundaries $\Gamma_1 = \{0\} \times [0, 2] \cup [0, 0.5] \times \{2\} \cup \{0.5\} \times [0, 2]$ and $\Gamma_2 = [0, 0.5] \times \{0\}$, and a temperature field

$$u_{q^\dagger}(x, y) = \frac{y}{2}$$

we want to recover the nonlinearity in

$$\begin{aligned} -\nabla \cdot ((2 + \cos(2\pi u))\nabla u) &= \pi \cdot \sin(\pi y) \text{ in } \Omega \\ (2 + \cos(2\pi u))\frac{\partial u}{\partial n} &= 0 \text{ on } \Gamma_1 \\ u &= 0 \text{ on } \Gamma_2 \end{aligned}$$

from "observations" of

$$u|_{\Gamma_1}.$$

While we already know the true data $u_{q^\dagger}|_{\Gamma_1}$ by construction, a sequence of noisy data z^{δ_i} is generated by artificially perturbing $u_{q^\dagger}|_{\Gamma_1}$ with high frequency noise. Then, the regularized solutions $q_\beta^{\delta_i}$ are defined as the minimizers of

$$J_\beta(q) = \int_{\Gamma_1} |\gamma u_q - z^{\delta_i}|^2 d\Gamma_1 + \beta \|q - q^*\|_I^2. \tag{7.11}$$

Though the true data $u_{q^\dagger}|_{\Gamma_1}$ (and hence the temperature distribution $u_{q^\dagger}$) only cover the range $I^\dagger = [0, 1]$, we choose a larger interval

$$I = [-0.2, 1.2]$$

in (7.11), as both noisy data and computed forward solutions during the minimization procedure may exceed $I^\dagger$. Of course, we only can expect to recover the heat conductivity on $I^\dagger$, outside it will be determined by the initial guess q^*. With

$$q^* = 3,$$

we then have (see Figure 1)

$$q^\dagger(\tau) = \begin{cases} 2 + \cos(2\pi\tau) & \text{for} \quad \tau \in I^\dagger \\ 3 & \text{for} \quad \tau \in I \backslash I^\dagger. \end{cases}$$

Since we know the exact parameter $q^\dagger$ we can compute the error $\|q^\dagger - q_\beta^{\delta_i}\|_{H^1(I)}$, allowing

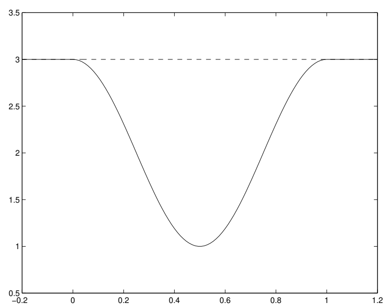

Figure 1: $q^\dagger$ and q^*

us to investigate the behaviour of Tikhonov regularization with respect to stability and rate of convergence. Note that we at least satisfy condition (5.4) for $j = 1, 2$ by the choice of q^* and in addition conditions (5.3) and (5.6) by the construction of our example.

For the minimization of (7.11) we use a quasi-Newton method, approximating the Hessian matrix of J_β by a BFGS-update formula in each iteration step k. Given a search direction p_k by that rule, the parameter q_k is updated by

$$q_{k+1} = q_k + \alpha_k p_k$$

until a minimum is reached. In order to raise the convergence speed of the optimization procedure we also use a line search algorithm for the determination of the stepsize α_k. A more detailed discussion can be found in [15].

The first computations were done for $\beta = 0$, i.e. the approach to identify the parameter by simply minimizing the output least squares term

$$\int_{\Gamma_1} |\gamma u_q - z^{(\delta)}|^2 d\Gamma_1.$$

For the case of exact data $z = u_{q^\dagger}|_{\Gamma_1}$, the result $q_{nonoise}$ is shown in Figure 2. As predicted

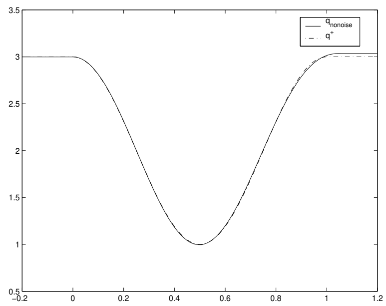

Figure 2: $q_{nonoise}$ identified from exact data

by our theory, the parameter is identifiable from observations of the temperature trace on the boundary, but of course only on the interval $I^\dagger$, where the data is available. Outside, the solution is given by the initial guess q^*. Figures 3 and 4 illustrate the ill-posedness of the identification problem. Perturbed data z^{δ_7} with 4.61% noise already have a dramatic impact on the recovery process. On the left hand side the relative error $\frac{\|q^\dagger - q_k\|_I}{\|q^\dagger\|_I}$ is plotted vs. the iteration index k in the optimization routine, the right hand sides records the result q_{noreg} after 80 steps. While the error in the residual $\|\gamma u_{q_k} - z^\delta\|_{L^2(\Gamma_1)}$ (not shown) is monotonically decreasing with k, the error in the parameter starts to increase after some 20 steps, leading to a solution that differs from $q^\dagger$ by more than 60% measured in the $H^1(I)$-norm. Only by introducing the penalty term

$$\beta \|q - q^*\|_I^2$$

(or alternatively stopping the iteration at "the right time", see [4] for an introduction to iterative regularization methods) these high numerical instabilities can be overcome.

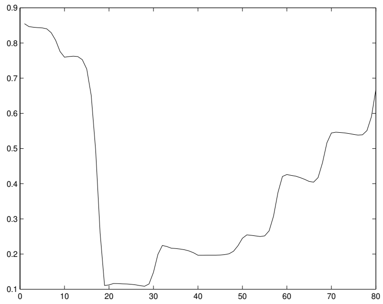

Figure 3: $\frac{\|q^\dagger - q_k\|_I}{\|q^\dagger\|_I}$ vs. k

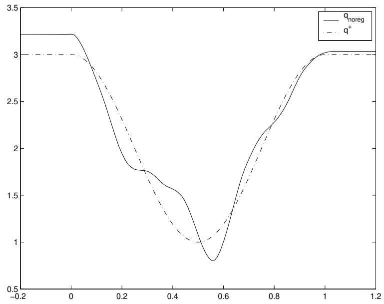

Figure 4: solution q_{noreg} after 80 steps

Though there are sophisticated methods for choosing the regularization parameter β from the knowledge of the noise level δ and the data z^δ (see [20]) itself, we content ourselves with the a-priori choice

$$\beta_i = 4 \cdot 10^{-4} \cdot \delta_i$$

for the sequence of perturbed data z^{δ_i}. This relation was found by trial and error, which is sufficient for our purposes. In order to test the rate of convergence behaviour of Tikhonov regularization predicted by Theorem 4.1, we only need to meet the requirement $\beta \sim \delta$. Figure 5 shows the error $\|q^\dagger - q_{\beta_i}^{\delta^i}\|_I$ plotted vs. the noise level $\delta_i = \|\gamma u_{q^\dagger} - z^{\delta_i}\|_{L^2(\Gamma_1)}$.

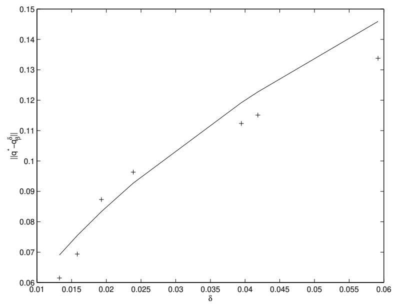

Figure 5: convergence rate $\|q^\dagger - q_{\beta_i}^{\delta_i}\|_I = O(\sqrt{\delta_i})$

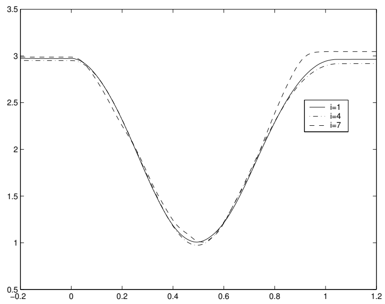

Figure 6: regularized solutions $q_{\beta_i}^{\delta_i}$

The solid line indicates that the convergence speed $\|q^\dagger - q_{\beta_i}^{\delta_i}\|_I = O(\sqrt{\delta_i})$ from Theorem 4.1 is obeyed, even though not all conditions of Section 5 are satisfied by our example. This gives hope that a source function w in Theorem 4.1 can be found under even weaker assumptions than in made in Section 5. Finally, Figure 6 shows the regularized solutions $q_{\beta_i}^{\delta_i}$ for $\delta_1 = 0.0132, \delta_4 = 0.0239$ and $\delta_7 = 0.0529$.

Acknowledgement: The author thanks Prof. Masahiro Yamamoto (University of Tokyo) and Prof. Jun Zou (Chinese University of Hong Kong) for fruitful discussions and the

Sohmen Foundation (Linz) for funding the stay in Asia. The numerical part of this work was done while the author was visiting OCIAM (University of Oxford).

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