Coupled Boundary and Finite Element Tearing and Interconnecting Methods.

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Summary. We have recently introduced the Boundary Element Tearing and Interconnecting (BETI) methods as boundary element counterparts of the well-established Finite Element Tearing and Interconnecting (FETI) methods. Since Finite Element Methods (FEM) and Boundary Element Methods (BEM) have certain complementary properties, it is sometimes very useful to couple these discretization techniques and to benefit from both worlds. Combining our BETI techniques with the FETI methods gives new, quite attractive tearing and interconnecting parallel solvers for large scale coupled boundary and finite element equations. There is an unified framework for coupling, handling, and analyzing both methods. In particular, the FETI methods can benefit from preconditioning components constructed by boundary element techniques. This is especially true for sparse versions of the boundary element method such as the fast multipole method which avoid fully populated matrices arising in classical boundary element methods.

1 Introduction

In Langer and Steinbach [2003] we have recently introduced the Boundary Element Tearing and Interconnecting (BETI) methods as boundary element counterparts of the well-established Finite Element Tearing and Interconnecting (FETI) methods. The first FETI methods were introduced by Farhat and Roux [1991] (see also Farhat and Roux [1994] for a more detailed description by the same authors). Since then the FETI methods have successfully been applied to different engineering problems, new FETI versions have been proposed (see Farhat et al. [2000]), and the analysis has been developed as well (see Mandel and Tezaur [1996, 2001], Klawonn and Widlund [2001], Klawonn et al. [2002], Brenner [2003]). Nowadays, the FETI method is one of the most widely used domain decomposition (DD) methods in parallel codes including commercial codes. This success of the FETI methods is certainly related to the wide applicability of the FETI methods, the possibility of the use of standard components in the solution process, the moderate dependence of the iteration number on the complexity of the problem (see Mandel and Tezaur [1996], Klawonn and Widlund [2001], Brenner [2003]), the scalability (see, e.g., Stefanica [2001]) and, last but not least, the robustness (see Klawonn and Widlund [2001], Klawonn et al. [2002], Brenner [2003]). These facts are true for the BETI methods as well (see Langer and Steinbach [2003]).

In this paper we generalize the tearing and interconnecting technique to symmetric coupled boundary and finite element equations. Since Finite Element Methods (FEM) and Boundary Element Methods (BEM) have certain complementary properties, it is sometimes very useful to couple these discretization techniques and to benefit from the advantages of both worlds. This concerns not only the treatment of unbounded domains (BEM), but also the right handling of singularities (BEM), moving parts (BEM), air regions in electromagnetics (BEM), source terms (FEM), non-linearities (FEM) etc. The symmetric coupling of BEM and FEM goes back to Costabel [1987]. During the last decade iterative substructuring solvers for symmetric coupled boundary and finite element equations have been developed by Langer [1994], Haase et al. [1998], Hsiao et al. [2000], Steinbach [2003] for elliptic boundary value problems in bounded and unbounded, two and three-dimensional domains, and have been successfully applied to real-life problems. Parallel implementations showed high performance on several platforms (see Haase et al. [1998]). Especially in 3D, the preconditioning of the global (assembled) boundary and finite element Schur complement, living on the skeleton of the domain decomposition, is the crucial point for constructing an efficient iterative substructuring method. This weak point of iterative substructuring methods can be avoided by the dual approach. Indeed, combining our BETI techniques with the FETI methods gives new, quite attractive tearing and interconnecting parallel solvers for large scale coupled boundary and finite element equations. Moreover, there is an unified framework for coupling, handling, and analyzing both methods. In particular, the FETI methods can benefit from preconditioning components constructed by boundary element techniques. This is especially true for sparse versions of the boundary element method. Sparse approximation techniques such as the fast multipole method (see Greengard and Rokhlin [1987]) avoid fully populated matrices arising in classical boundary element methods. Our sparse hypersingular BETI/FETI-preconditioner that is based on symmetry and kernel preserving fast multipole techniques requires only $O((H/h)^{d-1}(\log(H/h))^2)$ arithmetical operations in a parallel regime, where d is the dimension of our computational domain ($d = 2$, or 3), and H and h denote the usual scaling of the subdomains and the elements, respectively. Similar to the FETI methods, the relative spectral condition number grows only like $O((1 + \log(H/h))^2)$ and is independent of the jumps in the coefficients of the partial differential equation.

The rest of the paper is organized as follows. In Section 2, we introduce the coupled BETI/FETI techniques for solving large scale coupled boundary and finite element DD equations, and discuss some algorithmical aspects. Section 3 is devoted to the preconditioning and analysis of the combined BETI/FETI solver. Finally, in Section 4, we draw some conclusions for using and developing the tearing and interconnecting technique in both the boundary and finite element worlds.

2 Formulation of BETI/FETI

For a bounded domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ with Lipschitz boundary $\Gamma = \partial\Omega$ we consider the Dirichlet boundary value problem

$$-\mathrm{div}\,[\alpha(x)\nabla u(x)] = f(x) \quad \text{for } x \in \Omega, \quad u(x) = g(x) \quad \text{for } x \in \Gamma. \tag{1}$$

We assume that there is given a non–overlapping quasi regular domain decomposition,

$$\overline{\Omega} = \bigcup_{i=1}^{p} \overline{\Omega}_i, \Omega_i \cap \Omega_j = \emptyset \text{ for } i \neq j, \Gamma_i = \partial\Omega_i, \Gamma_{ij} = \Gamma_i \cap \Gamma_j, \Gamma_S = \bigcup_{i=1}^{p} \Gamma_i.$$

Moreover, we assume that the coefficient function α is piecewise constant,

$$\alpha(x) = \alpha_i \quad \text{for } x \in \Omega_i, i = 1, \ldots, p.$$

Instead of the global boundary value problem (1) we now consider the local problems

$$-\alpha_i \Delta u_i(x) = f(x) \quad \text{for } x \in \Omega_i, \quad u_i(x) = g(x) \quad \text{for } x \in \Gamma_i \cap \Gamma \tag{2}$$

together with the transmission conditions

$$u_i(x) = u_j(x), \quad \alpha_i \frac{\partial}{\partial n_i} u_i(x) + \alpha_j \frac{\partial}{\partial n_j} u_j(x) = 0 \quad \text{for } x \in \Gamma_{ij}. \tag{3}$$

The solution of the local Dirichlet boundary value problems

$$-\alpha_i \Delta u_i(x) = f(x) \quad \text{for } x \in \Omega_i, \quad u_i(x) = g_i(x) \quad \text{for } x \in \Gamma_i \tag{4}$$

defines the Dirichlet–Neumann map

$$t_i(x) := \alpha_i \frac{\partial}{\partial n_i} u_i(x) = (S_i u_i)(x) - (N_i f)(x) \quad \text{for } x \in \Gamma_i \tag{5}$$

with the Steklov–Poincaré operator $S_i : H^{1/2}(\Gamma_i) \to H^{-1/2}(\Gamma_i)$ and with some Newton potential $N_i : \widetilde{H}^{-1}(\Omega_i) \to H^{-1/2}(\Gamma_i)$, see, e.g., Steinbach [2003]. The coupled boundary value problem (2)–(3) is therefore equivalent to find $(u_i, t_i) \in H^{1/2}(\Gamma_i) \times H^{-1/2}(\Gamma_i)$ for $i = 1, \ldots, p$ such that

$$\begin{aligned} t_i(x) &= (S_i u_i)(x) - (N_i f)(x) && \text{for } x \in \Gamma_i, \\ u_i(x) &= g(x) && \text{for } x \in \Gamma_i \cap \Gamma, \\ u_i(x) &= u_j(x) && \text{for } x \in \Gamma_{ij}, \\ 0 &= t_i(x) + t_j(x) && \text{for } x \in \Gamma_{ij}. \end{aligned}$$

Let $\widetilde{g} \in H^1(\Omega)$ be some arbitrary but fixed bounded extension of the given Dirichlet data $g \in H^{1/2}(\Gamma)$. Introducing the trace space $H^{1/2}(\Gamma_S)$ on the skeleton Γ_S and the subspace

$$H_0^{1/2}(\Gamma_S, \Gamma) := \left\{ v \in H^{1/2}(\Gamma_S) \,:\, v(x) = 0 \quad \text{for } x \in \Gamma \right\},$$

we arrive at the skeleton problem: find a function $u_0 \in H_0^{1/2}(\Gamma_S, \Gamma)$ such that

$$(S_i u_i)(x) + (S_j u_j)(x) = (N_i f)(x) + (N_j f)(x) \quad \text{for } x \in \Gamma_{ij}$$

is satisfied on all local coupling boundaries Γ_{ij} and where $u_i(x) := u_0(x) + \widetilde{g}(x)$ for $x \in \Gamma_i$. The resulting variational problem is to find $u_0 \in H_0^{1/2}(\Gamma_S, \Gamma)$ such that

$$\sum_{i=1}^{p} \int_{\Gamma_i} (S_i u_0)(x) v(x) ds_x = \sum_{i=1}^{p} \int_{\Gamma_i} [(N_i f)(x) - (S_i \widetilde{g})(x)] v(x) ds_x \tag{6}$$

is satisfied for all $v \in H_0^{1/2}(\Gamma_S, \Gamma)$.

Let

$$S_h^1(\Gamma_S) = \text{span}\{\varphi_k^1\}_{k=1}^{M} \subset H_0^{1/2}(\Gamma_S, \Gamma)$$

be a conformal finite dimensional trial space of piecewise linear continuous basis functions φ_k^1. By

$$S_h^1(\Gamma_i) := S_h^1(\Gamma_S)_{|\Gamma_i} = \text{span}\{\varphi_{k,i}^1\}_{k=1}^{M_i} \subset H_0^{1/2}(\Gamma_i, \Gamma)$$

we denote the restriction of the global trial space $S_h^1(\Gamma_S)$ onto the local subdomain boundaries Γ_i. For a global vector $\underline{v} \in \mathbb{R}^M$ and the corresponding finite element function $v_h \in S_h^1(\Gamma_S)$ ($\underline{v} \in \mathbb{R}^M \leftrightarrow v_h \in S_h^1(\Gamma_S)$), we consider the restriction $v_{h,i} := v_{h|\Gamma_i} \in S_h^1(\Gamma_i) \leftrightarrow \underline{v}_i \in \mathbb{R}^{M_i}$. Using connectivity matrices $A_i \in \mathbb{R}^{M_i \times M}$ this can be written as $\underline{v}_i = A_i \underline{v}$. Due to the implicit definition of the local Dirichlet–Neumann map (5) it is in general not possible to discretize the variational problem (6) in an exact manner. Hence we have to approximate the local Dirichlet problems which are involved in the definition of the local Dirichlet to Neumann map. This can be done either by finite or boundary elements, see Steinbach [2003].

We start to consider a finite element approximation of the local Steklov–Poincaré operators S_i to realize the Dirichlet to Neumann map in the subdomains Ω_i, $i = 1, \ldots, q \leq p$. Let

$$S_h^1(\Omega_i) := \text{span}\{\phi_{\kappa,i}^1\}_{\kappa=1}^{\bar{M}_i} \subset H_0^1(\Omega_i)$$

be the local finite element spaces of piecewise linear and continuous basis functions $\phi_{\kappa,i}^1$ which vanish on the subdomain boundary Γ_i. The finite element discretization of the local Dirichlet boundary value problems (4) for given Dirichlet data $\underline{u}_{C,i}$ then leads to the linear systems

$$\begin{pmatrix} K_{II,i} & K_{CI,i} \\ K_{CI,i}^\top & K_{CC,i} \end{pmatrix} \begin{pmatrix} \underline{u}_{I,i} \\ \underline{u}_{C,i} \end{pmatrix} = \begin{pmatrix} \underline{f}_{I,i} \\ \underline{f}_{C,i} \end{pmatrix} \tag{7}$$

with block matrices defined by

$$K_{CC,i}[\ell,k] := \int_{\Omega_i} \alpha_i \nabla\varphi^1_{k,i}(x)\nabla\varphi^1_{\ell,1}(x)dx,$$

$$K_{II,i}[\lambda,\kappa] := \int_{\Omega_i} \alpha_i \nabla\phi^1_{\kappa,i}(x)\nabla\phi^1_{\lambda,i}(x)dx,$$

$$K_{CI}[\lambda,k] := \int_{\Omega_i} \alpha_i \nabla\varphi^1_{k,i}(x)\nabla\phi^1_{\lambda,i}(x)dx$$

and the right-hand side

$$f_{I,i,\lambda} := \int_{\Omega_i} f(x)\phi^1_{\lambda,i}(x)dx - \int_{\Omega_i} \alpha_i\nabla\widetilde{g}(x)\nabla\phi^1_{\lambda,i}(x)dx,$$

$$f_{C,i,\ell} := \int_{\Omega_i} f(x)\varphi^1_{\ell,i}(x)dx - \int_{\Omega_i} \alpha_i\nabla\widetilde{g}(x)\nabla\varphi^1_{\ell,i}(x)dx$$

for all $k,\ell = 1,\ldots,M_i$ and $\kappa,\lambda = 1,\ldots,\bar{M}_i$. Eliminating $\underline{u}_{I,i}$, we now obtain the finite element approximation

$$S^{\mathrm{FEM}}_{i,h}\underline{u}_{C,i} = \left[K_{CC,i} - K^\top_{CI,i}K^{-1}_{II,i}K_{CI,i}\right]\underline{u}_{C,i} = \underline{f}_{C,i} - K^\top_{CI,i}K^{-1}_{II,i}\underline{f}_{I,i} = \underline{f}^{\mathrm{FEM}}_i$$

of the local Dirichlet to Neumann map (5). In the finite element subdomains the coefficients α_i can, of course, depend on x, but should not vary too much on Ω_i for $i = 1,\ldots,q$.
In the remaining subdomains Ω_i, $i = q+1,\ldots,p$ we assume $f(x) = 0$ for $x \in \Omega_i$. Hence we may use a symmetric boundary element method to approximate the local Steklov–Poincaré operators S_i. The fundamental solution of the Laplace operator is given by

$$U^*(x,y) = \begin{cases} -\dfrac{1}{2\pi}\log|x-y| & \text{for } d = 2, \\ \dfrac{1}{4\pi}\dfrac{1}{|x-y|} & \text{for } d = 3. \end{cases}$$

The relation between the local Cauchy data $[t_i, u_i]$ can then be described by the system of boundary integral equations (Calderón projection),

$$\begin{pmatrix} u_i \\ t_i \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I - K_i & V_i \\ D_i & \frac{1}{2}I + K_i' \end{pmatrix} \begin{pmatrix} u_i \\ t_i \end{pmatrix},$$

where we used the standard notations for the single layer potential operator V_i, for the double layer potential operator K_i and its adjoint K_i' and for the hypersingular integral operator D_i defined by

$$
\begin{aligned}
(V_i t_i)(x) &:= \alpha_i \int_{\Gamma_i} U^*(x,y) t_i(y) ds_y \quad \text{for } x \in \Gamma_i, \\
(K_i u_i)(x) &:= \alpha_i \int_{\Gamma_i} \frac{\partial}{\partial n_y} U^*(x,y) u_i(y) ds_y \quad \text{for } x \in \Gamma_i, \\
(K_i' t_i)(x) &:= \alpha_i \int_{\Gamma_i} \frac{\partial}{\partial n_x} U^*(x,y) t_i(y) ds_y \quad \text{for } x \in \Gamma_i, \\
(D_i u_i)(x) &:= -\alpha_i \frac{\partial}{\partial n_x} \int_{\Gamma_i} \frac{\partial}{\partial n_y} U^*(x,y) u_i(y) ds_y \quad \text{for } x \in \Gamma_i,
\end{aligned}
$$

respectively. The mapping properties of all of these boundary integral operators are well known (see Costabel [1988]), in particular, the local single layer potential $V_i : H^{-1/2}(\Gamma_i) \to H^{1/2}(\Gamma_i)$ is $H^{-1/2}(\Gamma_i)$–elliptic and therefore invertible (Hsiao and Wendland [1977]); for $d = 2$ we assume $\operatorname{diam} \Omega_i < 1$ that can be always obtained by scaling the computational domain. Then we obtain a symmetric boundary integral operator representation of the Steklov–Poincaré operator $S_i : H^{1/2}(\Gamma_i) \to H^{-1/2}(\Gamma_i)$,

$$
(S_i u_i)(x) \;=\; \left[D_i + (\frac{1}{2} I + K_i') V_i^{-1} (\frac{1}{2} I + K_i) \right] u_i(x) \quad \text{for } x \in \Gamma_i.
$$

Let

$$
S_h^0(\Gamma_i) = \operatorname{span}\{\psi_{\kappa,i}^0\}_{\kappa=1}^{N_i} \subset H^{-1/2}(\Gamma_i) \tag{8}
$$

be the trial space of piecewise constant basis functions $\psi_{\kappa,i}^0$ to approximate the local Neumann data $t_i \in H^{-1/2}(\Gamma_i)$. This Galerkin approximation of the local Steklov–Poincaré operator S_i gives the matrix representation

$$
S_{i,h}^{\text{BEM}} \;:=\; D_{i,h} + (\frac{1}{2} M_{i,h}^\top + K_{i,h}^\top) V_{i,h}^{-1} (\frac{1}{2} M_{i,h} + K_{i,h})
$$

with

$$
\begin{aligned}
D_{i,h}[\ell, k] &= \langle D_i \varphi_{k,i}^1, \varphi_{\ell,i}^1 \rangle_{L_2(\Gamma_i)}, \\
V_{i,h}[\lambda, \kappa] &= \langle V_i \psi_{\kappa,i}^0, \psi_{\lambda,i}^0 \rangle_{L_2(\Gamma_i)}, \\
K_{i,h}[\lambda, k] &= \langle K_i \varphi_{k,i}^1, \psi_{\lambda,i}^0 \rangle_{L_2(\Gamma_i)}, \\
M_{i,h}[\lambda, k] &= \langle \varphi_{k,i}^1, \psi_{\lambda,i}^0 \rangle_{L_2(\Gamma_i)}
\end{aligned}
$$

for all $k, \ell = 1, \ldots, M_i$ and $\kappa, \lambda = 1, \ldots, N_i$.

Note that both finite and boundary element approximations $S_{i,h}^{\text{BEM/FEM}}$ of the local Steklov–Poincaré operators S_i are symmetric and spectrally equivalent to the exact Galerkin matrices $S_{i,h}$. This holds true for an almost arbitrary choice of the local trial spaces $S_h^1(\Omega_i)$ and $S_h^0(\Gamma_i)$, respectively, see Steinbach [2003]. Moreover, the error $S_{i,h} - S_{i,h}^{\text{BEM/FEM}}$ of the approximate Steklov–Poincaré operators can be controlled by the approximation properties of the local trial spaces $S_h^1(\Omega_i)$ and $S_h^0(\Gamma_i)$, respectively.

The Galerkin discretization of the variational problem (6) with the boundary and finite element approximations of the local Dirichlet problems discussed above leads now to the linear system

$$\sum_{i=1}^{q} A_i^\top S_{i,h}^{\text{FEM}} A_i \underline{u} + \sum_{i=q+1}^{p} A_i^\top S_{i,h}^{\text{BEM}} A_i \underline{u} = \sum_{i=1}^{q} A_i^\top \underline{f}_i^{\text{FEM}} - \sum_{i=q+1}^{p} A_i^\top S_{i,h}^{\text{BEM}} A_i \underline{g} \quad (9)$$

which is uniquely solvable due to the positive definitness of the assembled stiffness matrix. Discretization error estimates are given in Steinbach [2003]. The aim of tearing and interconnecting domain decomposition methods is to design efficient solution strategies to solve the global linear system (9). When introducing local vectors $\underline{u}_i = A_i \underline{u}$ the continuity of the primal variables across the interfaces can be written by the constraint

$$\sum_{i=1}^{p} B_i \underline{u}_i = \underline{0}$$

where $B_i \in \mathbb{R}^{M \times M_i}$. Each row of the matrix $B = (B_1, \ldots, B_p)$ is connected with a pair of matching nodes across the interface. The entries of such a row are 1 and -1 for the indices corresponding to the matching nodes and 0 otherwise. By introducing the Lagrange multiplier $\underline{\lambda} \in \mathbb{R}^M$ we have to solve the linear system

$$\begin{pmatrix} S_{1,h}^{\text{FEM}} & & & & & & B_1^\top \\ & \ddots & & & & & \vdots \\ & & S_{q,h}^{\text{FEM}} & & & & B_q^\top \\ & & & S_{q+1,h}^{\text{BEM}} & & & B_{q+1}^\top \\ & & & & \ddots & & \vdots \\ & & & & & S_{p,h}^{\text{BEM}} & B_p^\top \\ B_1 & \cdots & B_q & B_{q+1} & \cdots & B_p & 0 \end{pmatrix} \begin{pmatrix} \underline{u}_1 \\ \vdots \\ \underline{u}_q \\ \underline{u}_{q+1} \\ \vdots \\ \underline{u}_p \\ \underline{\lambda} \end{pmatrix} = \begin{pmatrix} \underline{f}_1^{\text{FEM}} \\ \vdots \\ \underline{f}_q^{\text{FEM}} \\ \underline{f}_{q+1}^{\text{BEM}} \\ \vdots \\ \underline{f}_p^{\text{BEM}} \\ \underline{0} \end{pmatrix} \quad (10)$$

with $\underline{f}_i^{\text{BEM}} := -S_{i,h}^{\text{BEM}} A_i \underline{g}$ for $i = q+1, \ldots, p$. For $i = 1, \ldots, p$ we now consider the solvability of the local systems

$$S_{i,h}^{\text{FEM/BEM}} \underline{u}_i = \underline{f}_i^{\text{FEM/BEM}} - B_i^\top \underline{\lambda}. \quad (11)$$

For a unique framework we define the modified matrices

$$\widetilde{S}_{i,h}^{\rm BEM/FEM} := S_{i,h}^{\rm BEM/FEM} + \beta_i \underline{e}_i \underline{e}_i^\top$$

with $\beta_i = 0$ for non–floating subdomains Ω_i, i.e., the subdomain boundary $\Gamma_i = \partial\Omega_i$ contains some part of the Dirichlet boundary $\Gamma = \partial\Omega$, and some suitable chosen $\beta_i > 0$ for floating subdomains Ω_i with $\Gamma_i \cap \Gamma = \emptyset$. When requiring the solvability condition

$$\underline{e}_i^\top \left[\underline{f}_i^{\rm BEM/FEM} - B_i^\top \underline{\lambda} \right] = 0, \tag{12}$$

the local linear systems (11) are equivalent to the modified systems

$$\widetilde{S}_{i,h}^{\rm FEM/BEM} \underline{u}_i = \underline{f}_i^{\rm FEM/BEM} - B_i^\top \underline{\lambda} \tag{13}$$

which are now unique solvable. However, for floating subdomains we have to incorporate the rigid body motions. Hence the general solutions of the modified linear systems (13) are given by

$$\underline{u}_i = \left[\widetilde{S}_{i,h}^{\rm BEM/FEM} \right]^{-1} \left[\underline{f}_i^{\rm BEM/FEM} - B_i^\top \underline{\lambda} \right] + \gamma_i \underline{e}_i \tag{14}$$

with $\gamma_i = 0$ for all non–floating subdomains. Inserting these local solutions into the last equation of (10) we obtain the Schur complement system

$$\sum_{i=1}^p B_i \left[\widetilde{S}_{i,h}^{\rm BEM/FEM} \right]^{-1} B_i^\top \underline{\lambda} - \sum_{i=1}^p \gamma_i B_i \underline{e}_i = \sum_{i=1}^p B_i \left[\widetilde{S}_{i,h}^{\rm BEM/FEM} \right]^{-1} \underline{f}_i^{\rm BEM/FEM}$$

where the compatibility condition (12) is to be assumed for all floating subdomains. Hence we have to solve the linear system

$$\begin{pmatrix} F & -G \\ G^\top & \end{pmatrix} \begin{pmatrix} \underline{\lambda} \\ \underline{\gamma} \end{pmatrix} = \begin{pmatrix} \underline{d} \\ \underline{e} \end{pmatrix} \tag{15}$$

with

$$F := \sum_{i=1}^p B_i \left[\widetilde{S}_{i,h}^{\rm BEM/FEM} \right]^{-1} B_i^\top, \quad G := (B_i \underline{e}_i)_{i:\Gamma_i \cap \Gamma = \emptyset}$$

and

$$\underline{d} := \sum_{i=1}^p B_i \left[\widetilde{S}_{i,h}^{\rm BEM/FEM} \right]^{-1} \underline{f}_i^{\rm BEM/FEM}, \quad \underline{e} := \left(\underline{e}_i^\top \underline{f}_i^{\rm FEM/BEM} \right)_{i:\Gamma_i \cap \Gamma = \emptyset}.$$

Defining now the orthogonal projection

$$P := I - G(G^\top G)^{-1} G^\top : \Lambda := \mathbb{R}^M \to \Lambda_0 := \ker G^\top = (\mathrm{range}\, G)^\perp$$

with respect to the Euclidian scalar product, we can split the computation of λ from the definition of $\underline{\gamma}$. Indeed, applying P to the first equation in (15) gives the equation

$$PF\underline{\lambda} = \underline{P}\,\underline{d} \tag{16}$$

since $PG\underline{\gamma} = \underline{0}$. Once λ is determined by solving (16), we obtain

$$\gamma := (G^\top G)^{-1} G^\top (F\underline{\lambda} - \underline{d}).$$

Finally, we get the vectors $\underline{u}_i$ from (14). Let us mention that in the case of jumping coefficients the scalar product in Λ has to be changed according to the proposal made by Klawonn and Widlund [2001] on pages 63 and 75 (see also Brenner [2003]).

The dual problem (16) is solved by a preconditioned conjugate gradient subspace iteration. The matrix-by-vector multiplication with the stiffness matrix F involves the application of the inverse modified discrete Steklov–Poincaré operators $[\widetilde{S}_{i,h}^{\text{BEM/FEM}}]^{-1}$ to some vector $B_i^\top \underline{\lambda}$ resulting in some $\underline{w}_i = [\widetilde{S}_{i,h}^{\text{BEM/FEM}}]^{-1} B_i^\top \underline{\lambda}$. This can be done by solving directly extended systems for the local boundary and finite element Neumann problems. This is the standard technique in the FETI methods (see Langer and Steinbach [2003] for BETI). Khoromskij et al. [2004] propose the application of the $\mathcal{H}$-matrix technique for an approximate inversion of the boundary and finite element Schur complements resulting in a sparse representation of the approximate inverse Schur complements in $\mathcal{H}$-matrix formate. This representation allows us to perform this matrix-by-vector multiplication with almost optimal complexity. Other approaches are discussed by Langer and Steinbach [2003].

3 Preconditioners and Analysis

In this section we will describe and analyze an efficient solution of the linear system (16) by some projected preconditioned conjugate gradient method. The preconditioning matrix C to be used in the PCG algorithm should be spectrally equivalent to the matrix F on the subspace $\Lambda_0 = \ker G^\top$, i.e.

$$c_1\,(C\underline{\lambda}, \underline{\lambda}) \;\le\; (F\underline{\lambda}, \underline{\lambda}) \;\le\; c_2\,(C\underline{\lambda}, \underline{\lambda}) \quad \text{for all } \underline{\lambda} \in \Lambda_0 \tag{17}$$

with positive spectral equivalence constants c_1 and c_2 such that the relative spectral condition number $\kappa(PC^{-1}P^\top P^\top FP)$ respectively its bound c_2/c_1 is as small as possible and the application of the preconditioner is as cheap as possible.

Following the FETI approach a first preconditioner is built from the local Schur complements $S_{i,h}^{\text{FEM/BEM}}$,

$$C_{\text{FETI}}^{-1} := (BC_\alpha^{-1}B^\top)^{-1} BC_\alpha^{-1} \left[\sum_{i=1}^{p} B_i S_{i,h}^{\text{FEM/BEM}} B_i^\top \right] C_\alpha^{-1} B^\top (BC_\alpha^{-1}B^\top)^{-1}$$

where $C_\alpha = \operatorname{diag}(C_{\alpha,i})_{i=1:p}$ and $C_{\alpha,i} = \operatorname{diag}(c_\ell^i)_{\ell=1:M_i}$ are diagonal matrices with appropriately chosen weights c_ℓ^i, e.g. as proposed in Klawonn and Widlund [2001], see also Brenner [2003].

The proof of the spectral equivalence inequalities (17) is essentially based on the spectral equivalence inequalities of the approximated Steklov–Poincaré operators $S_{i,h}^{\text{FEM/BEM}}$ with the exakt Galerkin approximation $S_{i,h}$. The application C_{FETI}^{-1} of the preconditioning matrix C_{FETI} mainly consists in the application of the local approximate Steklov–Poincaré operators $S_{i,h}^{\text{FEM/BEM}}$, i.e. the solution of local Dirichlet boundary value problems by either finite or boundary element methods. Here we will propose a more efficient preconditioning strategy when replacing the approximate Steklov–Poincaré operators $S_{i,h}^{\text{FEM/BEM}}$ by discrete hypersingular integral operators $D_{i,h}$ which are defined with respect to all subdomain boundaries Γ_i and $i = 1, \ldots, p$.

Lemma 1. *The local boundary element Schur complement matrix $S_{i,h}^{\text{BEM}}$ and the local finite element Schur complement matrix $S_{i,h}^{\text{FEM}}$ are spectrally equivalent to the exact Galerkin matrix $S_{i,h}$ of the local Steklov–Poincaré operator S_i and to the boundary element matrix $D_{i,h}$ of the local hypersingular boundary integral operator D_i, i.e.*

$$S_{i,h}^{\text{BEM}} \simeq S_{i,h}^{\text{FEM}} \simeq S_{i,h} \simeq D_{i,h}$$

for all $i = 1, \ldots, p$, where $A \simeq B$ means that the matrices A and B are spectrally equivalent.

Proof. It is well known (see, e.g., theorem 3.5, p. 64 in Steinbach [2003]), that the finite element Schur complement is spectrally equivalent to the $H^{-1/2}(\Gamma_i)$–semi–norm squared, i.e., there exist universal positive constants c_1 and c_2 such that

$$c_1\, |v_{i,h}|^2_{H^{1/2}(\Gamma_i)} \le (S_{i,h}^{\text{FEM}} \underline{v}_i, \underline{v}_i) \le c_2\, |v_{i,h}|^2_{H^{1/2}(\Gamma_i)}$$

is satisfied for all $v_{i,h} \in S_h^1(\Gamma_i) \leftrightarrow \underline{v}_i \in \mathbb{R}^{M_i}$. On the other hand, the boundary element Schur complement $S_{i,h}^{\text{BEM}}$ is spectrally equivalent to the Galerkin matrix $D_{i,h}$ of the local hypersingular boundary integral operator $D_{i,h}$ see lemma 3.1 in Langer and Steinbach [2003]. Since the energy of the local hypersingular integral operator D_i is also equivalent to the $H^{1/2}(\Gamma_i)$–semi–norm squared, the proof is completed. □

The resulting scaled hypersingular BETI preconditioner is now given by

$$C_{\text{BETI}}^{-1} := (BC_\alpha^{-1}B^\top)^{-1}BC_\alpha^{-1}\left[\sum_{i=1}^{p} B_i D_{i,h} B_i^\top\right] C_\alpha^{-1}B^\top(BC_\alpha^{-1}B^\top)^{-1}. \quad (18)$$

Theorem 1 (Theorem 3.1 in Langer and Steinbach [2003]). *For the scaled hypersingular BETI preconditioner (18), the condition estimate*

$$\kappa(PC_{\text{BETI}}^{-1}P^\top P^\top FP) \le c\left(1 + \log\frac{H}{h}\right)^2$$

holds, where the positive constant c is independent of the local mesh size h, the average subdomain size H, the number p of subdomains and of the coefficients

α_i (coefficient jumps). The matrix by vector operation $D_{i,h}\underline{v}_i$ that is the most expensive operation in the preconditioning step costs ops$(D_{i,h}\underline{v}_i) = \mathcal{O}((H/h)^2)$ and ops$(D_{i,h}\underline{v}_i) = \mathcal{O}((H/h)^4)$ arithmetical operations for $d = 2$ and for $d = 3$, respectively.

To obtain a more efficient preconditioning strategy we may use some fast boundary element method such as the fast multipole method to realize the local matrix by vector multiplication with $D_{i,h}$. The resulting sparse version of the scaled hypersingular BETI preconditioner then reads

$$C_{\mathrm{sBETI}}^{-1} := (BC_\alpha^{-1}B^\top)^{-1}BC_\alpha^{-1}\left[\sum_{i=1}^{p} B_i\widetilde{D}_{i,h}B_i^\top\right]C_\alpha^{-1}B^\top(BC_\alpha^{-1}B^\top)^{-1}. \quad (19)$$

We start the analysis of the sparse hypersingular BETI preconditioner (19) with some considerations of the local sparse approximations $\widetilde{D}_{i,h}$. Using integration by parts (see Nédélec [1982]) the bilinear form of the local hypersingular boundary integral operator D_i can be rewritten as

$$\langle D_i u_i, v_i\rangle_{L_2(\Gamma_i)} = -\frac{\alpha_i}{2\pi}\int\limits_{\Gamma_i}\dot{v}(x)\int\limits_{\Gamma_i}\log|x-y|\dot{u}(y)ds_y ds_x$$

for $d = 2$ where $\dot{u}$ means the derivative with respect to the arc length. Similarly, for $d = 3$ we have

$$\langle D_i u_i, v_i\rangle_{L_2(\Gamma_i)} = \frac{\alpha_i}{4\pi}\int\limits_{\Gamma_i}\int\limits_{\Gamma_i}\frac{\mathrm{curl}_{\Gamma_i}u_i(y)\cdot\mathrm{curl}_{\Gamma_i}v_i(x)}{|x-y|}ds_y ds_x,$$

where

$$\mathrm{curl}_{\Gamma_i}u_i(x) := n_i(x)\times\nabla_x u_i^*(x) \quad \text{for } x\in\Gamma_i$$

is the surface curl and u_i^* is an extension of u_i into a neighborhood of Γ_i. When using an interface triangulation of plane triangles and piecewise linear continuous basis functions φ_k^1, $\mathrm{curl}_{\Gamma_i}\varphi_k^1 \in \mathbb{R}^3$ is piecewise constant. Then the local Galerkin matrix $D_{i,h}$ can be represented in the form ($d = 3$)

$$D_{i,h} = C_{i,h}^\top\begin{pmatrix} V_{i,h} & & \\ & V_{i,h} & \\ & & V_{i,h}\end{pmatrix}C_{i,h},$$

where $V_{i,h}$ is the local Galerkin matrix of the related single layer potential with piecewise constant basis functions. Moreover, $C_{i,h}$ is an appropriate $3N_i \times M_i$ matrix which describes the transformation of the coefficient vector $\underline{v}_i \in \mathbb{R}^{M_i}$ of $v_{h,i} \in S_h^1(\Gamma_i)$ to the piecewise constant vector–valued result in $\mathbb{R}^{3N_i}$ of $\mathrm{curl}_{\Gamma_i}v_{h,i}$. A fast realization $\widetilde{D}_{i,h}$ of the discrete hypersingular integral operator is now reduced to three fast applications $\widetilde{V}_{i,h}$ of the discrete single layer potential,

$$\widetilde{D}_{i,h} \;=\; C_{i,h}^{\top} \begin{pmatrix} \widetilde{V}_{i,h} & & \\ & \widetilde{V}_{i,h} & \\ & & \widetilde{V}_{i,h} \end{pmatrix} C_{i,h}. \tag{20}$$

Since the curl of a constant function vanishes we conclude $C_{i,h}\underline{e}_i = \underline{0}$ and therefore $\ker \widetilde{D}_{i,h} = \ker D_{i,h}$, i.e., this approach is kernel–preserving for any possible fast application $\widetilde{V}_{i,h}$ of the approximate discrete single layer potential. Let $V_{i,h}$ be the Galerkin matrix of the local single layer potential operator V_i when using piecewise constant basis functions $\psi^0_{\kappa,i} \in S^0_h(\Gamma_i)$. The matrix by vector product $\underline{v}_i = V_{i,h}\underline{w}_i$ then reads ($d = 3$)

$$v_{i,\lambda} \;=\; \sum_{\kappa=1}^{N_i} V_{i,h}[\lambda,\kappa] w_{i,\kappa} \;=\; \frac{\alpha_i}{4\pi} \sum_{\kappa=1}^{N_i} w_{i,\kappa} \int_{\tau_\lambda} \int_{\tau_\kappa} \frac{1}{|x-y|} ds_y ds_x.$$

For a fixed boundary element τ_λ we consider the collection of all boundary elements τ_κ, which are in farfield of τ_λ satisfying the admissibility condition

$$\operatorname{dist}(\tau_\kappa, \tau_\lambda) \;\geq\; \eta \, \max\{\operatorname{diam}\tau_\kappa, \operatorname{diam}\tau_\lambda\}$$

with some appropriately chosen parameter $\eta > 1$. The remaining boundary elements τ_κ are called to be in the nearfield of τ_λ. Using some numerical integration scheme in the farfield, the matrix by vector product can be rewritten as

$$v_{i,\lambda} \;=\; \sum_{\text{nearfield}} V_{i,h}[\lambda,\kappa] w_{i,\kappa} + \sum_{\text{farfield}} w_{i,\kappa} \sum_{m=1}^{N_{G,\kappa}} \sum_{n=1}^{N_{G,\lambda}} \frac{\omega_{\kappa,m}\omega_{\lambda,n}}{|x_{\kappa,m} - x_{\lambda,n}|}$$

where $x_{\kappa,m}$ are suitable chosen integration nodes and $\omega_{\kappa,m}$ are related integration weights, respectively. The evaluation of

$$v_{i,\lambda,n} \;=\; \sum_{\text{farfield}} w_{i,\kappa} \sum_{m=1}^{N_{G,\kappa}} \frac{\omega_{\kappa,m}}{|x_{\kappa,m} - x_{\lambda,n}|}$$

corresponds exactly to the fast multiple particle simulation algorithm as described in Greengard and Rokhlin [1987] and can be implemented efficiently (Of [2001]). This defines a fast multipole approximation $\widetilde{V}_{i,h}$ of the discrete local single layer potential $V_{i,h}$. In fact, the matrix by vector multiplication with the discrete single layer potential by means of the fast multipole method costs $\operatorname{ops}(\widetilde{V}_{i,h}\underline{w}_i) \;=\; \mathcal{O}(N_i \log^2 N_i)$ arithmetical operations ($d = 3$). Choosing both the numerical integration scheme in the farfield and the multipole parameters in an appropriate way, we obtain corresponding error estimates for the perturbed single layer potential $\widetilde{V}_{i,h}$. In fact, the approximated single layer potential $\widetilde{V}_{i,h}$ turns out to be $H^{-1/2}(\Gamma_i)$–elliptic, i.e.

$$(\widetilde{V}_{i,h}\underline{w}_i, \underline{w}_i) \;\geq\; \frac{1}{2} c_1^V \, \|w_{i,h}\|^2_{H^{-1/2}(\Gamma_i)} \quad \text{for all } w_{i,h} \in S^0_h(\Gamma_i) \leftrightarrow \underline{w}_i \in \mathbb{R}^{N_i}$$

where c_1^V is the ellipticity constant of the local single layer potential operator V_i. Combining the discrete ellipticity of the approximated single layer potential $\widetilde{V}_{i,h}$ with the representation (20) of the discrete hypersingular integral operator we get the following result.

Lemma 2. *The sparse representation $\widetilde{D}_{i,h}$ as given in (20) is symmetric and spectrally equivalent to the Galerkin matrix $D_{i,h}$ of the local hypersingular integral operator, i.e., there hold the spectral equivalence inequalities*

$$c_1\,(D_{i,h}\underline{v}_i,\underline{v}_i) \;\le\; (\widetilde{D}_{i,h}\underline{v}_i,\underline{v}_i) \;\le\; c_2\,(D_{i,h}\underline{v}_i,\underline{v}_i) \quad \textit{for all}\; \underline{v}_i \in \mathbb{R}^{M_i}.$$

From Lemma 1 and Lemma 2 we now conclude the spectral equivalence of the finite and boundary element Schur complements $S_{i,h}^{\text{FEM/BEM}}$ with the sparse representation $\widetilde{D}_{i,h}$ of the local hypersingular integral operator D_i. Hence we can reformulate Theorem 1 for the sparse version of the scaled hypersingular BETI preconditioners as defined in (19).

Theorem 2. *For the sparse version of the scaled hypersingular BETI preconditioner (19), the condition estimate*

$$\kappa(PC_{sBETI}^{-1}P^\top P^\top FP) \;\le\; c\left(1+\log\frac{H}{h}\right)^2$$

holds, where the positive constant c is independent of the local mesh size h, the average subdomain size H, the number p of subdomains and of the coefficients α_i (coefficient jumps). The matrix by vector operation $\widetilde{D}_{i,h}\underline{v}_i$ costs $ops(D_{i,h}\underline{v}_i) = \mathcal{O}((H/h)^2\log^2(H/h))$ arithmetical operations ($d=3$).

4 Concluding Remarks

In this paper we presented the BETI/FETI technique for solving large scale coupled boundary and finite element equations arising from the non-overlapping domain decomposition. Our BETI/FETI preconditioner was constructed from the discrete hypersingular operator that is especially efficient in its sparse version. In the latter case the complexity of the preconditioning operation is almost proportional to the number of unknowns living on the skeleton of our domain decomposition. Our analysis showed that the BETI/FETI method has the same nice numerical and practical properties as they are known from the well-established FETI methods. In Langer and Steinbach [2003] we report on the first numerical experiments with our BETI solver that shows the same numerical behaviour as it is typical for the FETI methods.

Klawonn and Widlund [2000] proposed a FETI version with inexact solvers. This technique avoids the exact solution of the local Neumann and

Dirichlet problems in the FETI methods and works only with the corresponding preconditioners. In a forthcoming paper we will develop sparse inexact BETI versions the total complexity of which is basically proportional to the number of the subdomain boundary unknowns. The coupling of both inexact techniques will lead to sparse inexact BETI/FETI methods of almost optimal total complexity.

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