MACMAHON'S PARTITION ANALYSIS X: PLANE PARTITIONS WITH DIAGONALS

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Abstract. We examine two-rowed plane partitions with a new diagonal constraint between the rows. The related generating function is an infinite product; surprisingly the numerator factors of the product are not cyclotomic polynomials.

1. Introduction

In his pioneering book "Combinatory Analysis" [10, Vol. II, Sect. VIII, pp. 91–170] MacMahon introduced Partition Analysis as a computational method for solving combinatorial problems in connection with systems of linear diophantine inequalities and equations. He devotes Chapter II of Section IX to the study of plane partitions as a natural application domain for his method. MacMahon starts out with the "most simple case" [10, Vol. II, p. 183], namely where non-negative integers a_i are placed at the corner of a square such that the order relations shown in Figure 1 are satisfied. It is understood that an arrow pointing from a_i to a_j is interpreted as $a_i \geq a_j$.

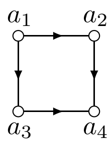

Figure 1.

By using Partition Analysis MacMahon then derives that

$$\begin{aligned} D_1 &:= \sum x_1^{a_1} x_2^{a_2} x_3^{a_3} x_4^{a_4} \\ &= \frac{1 - x_1^2 x_2 x_3}{(1-x_1)(1-x_1x_2)(1-x_1x_3)(1-x_1x_2x_3)(1-x_1x_2x_3x_4)}, \end{aligned}$$

where the summation ranges over all non-negative integers a_i satisfying the relations from Figure 1. Furthermore, he observes that if all x_i are set to q, the resulting generating function reduces to

$$\frac{1}{(1-q)(1-q^2)^2(1-q^3)}.$$

Subsequently MacMahon turns to a more general situation and tries to derive the full generating function for plane partitions of m rows, l columns and each part

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not exceeding n. However, his exploration culminates in the conclusion [10, Vol. II, p. 187]: "Our knowledge of the Ω operation is not sufficient to enable us to establish the final form of result. This will be accomplished by the aid of new ideas which will be brought forward in the following chapters."

Despite this negative statement, it turns out that Partition Analysis nevertheless is a powerful tool for investigating new variations of plane partitions, as it has been demonstrated for instance by the authors in [7]. Our object here is to study yet another variant of plane partitions, namely one with additional constraints on the diagonals, for which the full generating function may be computed.

Furthermore, we want to point out that MacMahon's method has been turned into an algorithm [4, 5]. Many of the results shown below have been verified, some even found, with the help of the Omega package[1].

2. Plane Partitions with Diagonals

First of all we need to recall the key ingredient of MacMahon's method, the Omega operator $\Omega_{\geqq}$.

Definition 1. The operator $\Omega_{\geqq}$ is given by

$$\underset{\geqq}{\Omega} \sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_r=-\infty}^{\infty} A_{s_1,\dots,s_r} \lambda_1^{s_1} \cdots \lambda_r^{s_r} := \sum_{s_1=0}^{\infty} \cdots \sum_{s_r=0}^{\infty} A_{s_1,\dots,s_r},$$

where the domain of the $A_{s_1,\dots,s_r}$ is the field of rational functions over $\mathbb{C}$ in several complex variables and the λ_i are restricted to a neighborhood of the circle $|\lambda_i| = 1$. In addition, the $A_{s_1,\dots,s_r}$ are required to be such that any of the series involved is absolute convergent within the domain of the definition of the $A_{s_1,\dots,s_r}$.

We emphasize that it is essential to treat everything analytically rather than formally because the method relies on unique Laurent series representations of rational functions; see also the discussion in [4].

We start with the base case where

$$h_0 = h_0(x_1, x_2) = \sum_{\substack{a_1,a_2 \geq 0 \\ a_1 \geq a_2}} x_1^{a_1} x_2^{a_2}.$$

Then by geometric series summation,

$$h_0(x_1, x_2) = \frac{1}{(1-x_1)(1-x_1x_2)}. \tag{1}$$

Note that by using the $\Omega_{\geqq}$ operator, the h_0 series can be rewritten as

$$h_0 = \underset{\geqq}{\Omega} \sum_{a_1,a_2 \geq 0} x_1^{a_1} x_2^{a_2} \lambda_1^{a_1-a_2} = \underset{\geqq}{\Omega} \frac{1}{(1-x_1\lambda_1)\left(1-\frac{x_2}{\lambda_1}\right)}. \tag{2}$$

Expressions like the one on the right-hand side of (2) are called "crude generating functions" in [10]. Using this representation, relation (1) turns into

$$\underset{\geqq}{\Omega} \frac{1}{(1-x_1\lambda_1)\left(1-\frac{x_2}{\lambda_1}\right)} = \frac{1}{(1-x_1)(1-x_1x_2)}, \tag{3}$$

which can be interpreted as a rule to eliminate the variable λ_1 from the crude generating function. Informally, MacMahon's method can be described as follows. Let

[1]Available at *http://www.risc.uni-linz.ac.at/research/combinat/risc/software/Omega/*

$f(x_1,\dots,x_n)$ be defined by $\sum x_1^{a_1}\cdots x_n^{a_n}$, where the summation is over all nonnegative integers a_i satisfying a given system of r linear Diophantine inequalities. It is a well-known fact that such generating functions have a closed form representation as a rational function. In order to compute it via Partition Analysis one first transforms f into its crude generating function, i.e.,

$$f(x_1,\dots,x_n) = \underset{\geqq}{\Omega}\, g(x_1,\dots,x_n;\lambda_1,\dots,\lambda_r).$$

Then in the next step one successively applies elimination rules like (3) until one arrives at a rational function expression which is free of all λ_i. Elementary examples can be found in [4]; see also the articles [1, 2, 3, 5, 6, 7, 8, 9].

We generalize the h_0 as follows. Let

$$H_1 := \{(a_1,\dots,a_6)\in\mathbb{N}^6 : \text{the } a_i \text{ satisfy the order relations in Figure 2}\}.$$

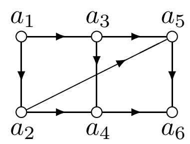

FIGURE 2.

Define

$$h_1 := h_1(x_1,\dots,x_6) := \sum_{(a_1,\dots,a_6)\in H_1} x_1^{a_1}\cdots x_6^{a_6}.$$

Obviously,

$$\begin{aligned} h_1 = \underset{\geqq}{\Omega}\, &\frac{1}{(1-x_1\lambda_1\lambda_2)\big(1-\frac{x_2\lambda_3\lambda_5}{\lambda_1}\big)\big(1-\frac{x_3\lambda_4\lambda_6}{\lambda_2}\big)} \\ &\cdot \frac{1}{\big(1-\frac{x_4\lambda_7}{\lambda_3\lambda_6}\big)\big(1-\frac{x_5\lambda_8}{\lambda_4\lambda_5}\big)\big(1-\frac{x_6}{\lambda_7\lambda_8}\big)}. \end{aligned} \tag{4}$$

The following elimination rules are special instances of the base cases (2.4) and (2.2) used in [5]. In fact, (6) is explicitly given by MacMahon [10, Vol. II, p. 102].

Lemma 1.

$$\begin{aligned} &\underset{\geqq}{\Omega}\, \frac{1}{(1-a\lambda)\big(1-\frac{b_1}{\lambda}\big)\big(1-\frac{b_2}{\lambda}\big)\big(1-\frac{b_3}{\lambda}\big)} \\ &\qquad = \frac{1}{(1-a)(1-ab_1)(1-ab_2)(1-ab_3)}; \end{aligned} \tag{5}$$

$$\begin{aligned} &\underset{\geqq}{\Omega}\, \frac{1}{\big(1-\frac{a}{\lambda}\big)(1-b_1\lambda)(1-b_2\lambda)} \\ &\qquad = \frac{1-ab_1b_2}{(1-b_1)(1-b_2)(1-ab_1)(1-ab_2)}. \end{aligned} \tag{6}$$

In order to keep expressions as simple as possible it will be convenient to introduce the following short-hand notation which will be used for the rest of the paper.

Definition 2. For $k\ge 1$, we define

$$X_k := x_1x_2\cdots x_k.$$

Proposition 1.

$$h_1 = \frac{(1 - x_1X_3)(1 - X_5X_3)}{(1 - X_1)(1 - X_2)\bigl(1 - \frac{X_3}{x_2}\bigr)(1 - X_3)} \cdot \frac{1}{(1 - X_4)\bigl(1 - \frac{X_5}{x_4}\bigr)(1 - X_5)(1 - X_6)}. \tag{7}$$

Proof. We start with the crude generating function representation of h_1 as in (4) and apply rule (5) to it with respect to λ_8, λ_7, λ_6, λ_5, and λ_2 in exactly this order and arrive at

$$h_1 = \underset{\geqq}{\Omega} \frac{1}{(1 - X_6)(1 - x_1\lambda_1)\bigl(1 - \frac{x_2\lambda_3}{\lambda_1}\bigr)} \cdot \frac{1}{\bigl(1 - \frac{x_2x_5\lambda_3}{\lambda_1\lambda_4}\bigr)(1 - x_1x_3\lambda_1\lambda_4)\bigl(1 - \frac{x_1x_3x_4\lambda_1\lambda_4}{\lambda_3}\bigr)}.$$

Applying to this expression rule (6) with respect to λ_4, λ_3, and λ_1, in this order gives (7). $\square$

Definition 3. For $n \geq 1$ define

$$H_n := \{(a_1, \ldots, a_{4n+2}) \in \mathbb{N}^{4n+2} : \text{the } a_i \text{ satisfy the order relations in Figure 3}\}$$

and

$$h_n := h_n(x_1, \ldots, x_{4n+2}) := \sum_{(a_1,\ldots,a_{4n+2}) \in H_n} x_1^{a_1} \cdots x_{4n+2}^{a_{4n+2}}.$$

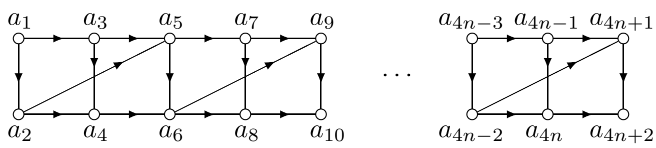

FIGURE 3.

The two following crude generating function representations are obvious:

Proposition 2.

$$h_2 = \underset{\geqq}{\Omega} \frac{1}{(1 - x_1\lambda_1\lambda_2)\bigl(1 - \frac{x_2\lambda_3\lambda_5}{\lambda_1}\bigr)\bigl(1 - \frac{x_3\lambda_4\lambda_6}{\lambda_2}\bigr)} \cdot \frac{1}{\bigl(1 - \frac{x_4\lambda_7}{\lambda_3\lambda_6}\bigr)\bigl(1 - \frac{x_5\lambda_8\lambda_9}{\lambda_4\lambda_5}\bigr)\bigl(1 - \frac{x_6\lambda_{10}\lambda_{11}}{\lambda_7\lambda_8}\bigr)} \cdot \frac{1}{\bigl(1 - \frac{x_7\lambda_{12}\lambda_{13}}{\lambda_9}\bigr)\bigl(1 - \frac{x_8\lambda_{14}}{\lambda_{10}\lambda_{13}}\bigr)\bigl(1 - \frac{x_9\lambda_{15}}{\lambda_{11}\lambda_{12}}\bigr)\bigl(1 - \frac{x_{10}}{\lambda_{14}\lambda_{15}}\bigr)}. \tag{8}$$

Proposition 3. *For $n \geq 3$,*

$$
\begin{aligned}
h_n = \underset{\geqq}{\Omega} & \frac{1}{(1-x_1\lambda_1\lambda_2)\big(1-\frac{x_2\lambda_3\lambda_5}{\lambda_1}\big)\big(1-\frac{x_3\lambda_4\lambda_6}{\lambda_2}\big)} \\
& \cdot \frac{1}{\big(1-\frac{x_4\lambda_7}{\lambda_3\lambda_6}\big)\big(1-\frac{x_5\lambda_8\lambda_9}{\lambda_4\lambda_5}\big)\big(1-\frac{x_6\lambda_{10}\lambda_{11}}{\lambda_7\lambda_8}\big)} \\
& \vdots \\
& \cdot \frac{1}{\big(1-\frac{x_{4n-5}\lambda_{7n-9}\lambda_{7n-8}}{\lambda_{7n-12}}\big)\big(1-\frac{x_{4n-4}\lambda_{7n-7}}{\lambda_{7n-11}\lambda_{7n-8}}\big)\big(1-\frac{x_{4n-3}\lambda_{7n-6}\lambda_{7n-5}}{\lambda_{7n-10}\lambda_{7n-9}}\big)} \\
& \cdot \frac{1}{\big(1-\frac{x_{4n-2}\lambda_{7n-4}\lambda_{7n-3}}{\lambda_{7n-7}\lambda_{7n-6}}\big)\big(1-\frac{x_{4n-1}\lambda_{7n-2}\lambda_{7n-1}}{\lambda_{7n-5}}\big)\big(1-\frac{x_{4n}\lambda_{7n}}{\lambda_{7n-4}\lambda_{7n-1}}\big)} \\
& \cdot \frac{1}{\big(1-\frac{x_{4n+1}\lambda_{7n+1}}{\lambda_{7n-3}\lambda_{7n-2}}\big)\big(1-\frac{x_{4n+2}}{\lambda_{7n}\lambda_{7n+1}}\big)}.
\end{aligned}
\tag{9}
$$

The next proposition is immediately implied by the previous representations (8) and (9).

Proposition 4. *For $n \geq 1$,*

$$
\begin{aligned}
h_{n+1} = \underset{\geqq}{\Omega} & \, h_n(x_1,\dots,x_{4n},x_{4n+1}\lambda_{7n+2},x_{4n+2}\lambda_{7n+3}\lambda_{7n+4}) \\
& \cdot \frac{1}{\big(1-\frac{x_{4n+3}\lambda_{7n+5}\lambda_{7n+6}}{\lambda_{7n+2}}\big)\big(1-\frac{x_{4n+4}\lambda_{7n+7}}{\lambda_{7n+3}\lambda_{7n+6}}\big)\big(1-\frac{x_{4n+5}\lambda_{7n+8}}{\lambda_{7n+4}\lambda_{7n+5}}\big)\big(1-\frac{x_{4n+6}}{\lambda_{7n+7}\lambda_{7n+8}}\big)}.
\end{aligned}
$$

In the next section we shall use this proposition to prove the following theorem.

Theorem 1. *For $n \geq 0$, $X_{-1} := 1$, and $x_0 := 1$,*

$$
\begin{aligned}
\frac{h_{n+1}}{h_n} = & \frac{1-X_{4n+5}X_{4n+3}}{1-\frac{X_{4n+5}}{x_{4n+4}}} \, \frac{P_n(x_1,\dots,x_{4n+3})}{\big(1-\frac{X_{4n+3}}{x_{4n}x_{4n+2}}\big)\big(1-\frac{X_{4n+3}}{x_{4n+2}}\big)} \\
& \cdot \frac{1}{(1-X_{4n+3})(1-X_{4n+4})(1-X_{4n+5})(1-X_{4n+6})(1-X_{4n+1}X_{4n-1})},
\end{aligned}
\tag{10}
$$

where

$$
\begin{aligned}
P_n(x_1,\dots,x_{4n+3}) = {} & 1 - X_{4n-1}X_{4n+1} - \frac{X_{4n-1}X_{4n+3}}{x_{4n+2}} - \frac{X_{4n+1}X_{4n+3}}{x_{4n}x_{4n+2}} \\
& - \frac{X_{4n+1}X_{4n+3}}{x_{4n}} - X_{4n+1}X_{4n+3} + \frac{X_{4n-1}X_{4n+1}X_{4n+3}}{x_{4n}x_{4n+2}} \\
& + \frac{X_{4n-1}X_{4n+1}X_{4n+3}}{x_{4n+2}} + X_{4n-1}X_{4n+1}X_{4n+3} + \frac{X_{4n+1}^2X_{4n+3}}{x_{4n}} \\
& + \frac{X_{4n+1}X_{4n+3}^2}{x_{4n}x_{4n+2}} - \frac{X_{4n-1}X_{4n+1}^2X_{4n+3}^2}{x_{4n}x_{4n+2}}.
\end{aligned}
\tag{11}
$$

Setting $x_i = q$ for all $i \geq 1$ and $x_0 = 1$, $X_{-1} = 0$, we observe that

$$
\begin{aligned}
&\frac{1 - X_{4n+5}X_{4n+3}}{\left(1 - \frac{X_{4n+5}}{x_{4n+4}}\right)\left(1 - X_{4n+1}X_{4n-1}\right)} \; \frac{P_n(x_1, \ldots, x_{4n+3})}{\left(1 - \frac{X_{4n+3}}{x_{4n}x_{4n+2}}\right)\left(1 - \frac{X_{4n+3}}{x_{4n+2}}\right)} \\
&\quad = \frac{1 + q^{4n+4}}{1 - q^{8n}} \; \frac{(1 - q^{4n+3})(1 - q^{8n}) - q(1 - q^{4n})(1 - q^{8n+1})}{1 - q} \\
&\quad = \frac{1 + q^{4n+4}}{1 + q^{4n}} \; \frac{(1 - q^{4n+3})(1 + q^{4n}) - q(1 - q^{8n+1})}{1 - q} \\
&\quad = \frac{1 + q^{4n+4}}{1 + q^{4n}} \; (1 + q^{4n} + q^{4n+1} + q^{4n+2} + q^{8n+2}).
\end{aligned} \tag{12}
$$

If we define for $n \geq 0$,

$$Q_n := 1 + q^{4n} + q^{4n+1} + q^{4n+2} + q^{8n+2},$$

then due to (12), equation (10) reduces to

$$\frac{h_{n+1}(q)}{h_n(q)} = \frac{1 + q^{4n+4}}{1 + q^{4n}} \; \frac{Q_n}{(1 - q^{4n+3})(1 - q^{4n+4})(1 - q^{4n+5})(1 - q^{4n+6})}$$

for all $n \geq 1$, where

$$h_n(q) = h_n(q, q, \ldots, q).$$

We summarize in the form of a corollary.

Corollary 1. *For $n \geq 1$,*

$$h_n(q) = (1 + q^2)(1 + q^{4n}) \frac{Q_1 Q_2 \cdots Q_{n-1}}{(q; q)_{4n+2}}.$$

For $n \to \infty$ we obtain the following.

Corollary 2.

$$h_\infty(q) = (1 + q^2) \prod_{n=1}^{\infty} \frac{1 + q^{4n} + q^{4n+1} + q^{4n+2} + q^{8n+2}}{1 - q^n}.$$

3. Proof of Theorem 1

We shall prove Theorem 1 in the following equivalent form.

Theorem 2. *Let $g_n = g_n(x_1, \ldots, x_{4n+6})$ denote the right-hand side of equality* (10). *Then for $n \geq 0$,*

$$h_{n+1} = g_n g_{n-1} \cdots g_0 h_0.$$

Proof. We proceed by induction on n. The case $n = 0$ is immediate from (1) and (7). For the step from n to $n + 1$ we invoke Proposition 4, namely

$$h_{n+1} = \underset{\geqq}{\Omega}\, h_n(x_1, \ldots, x_{4n}, x_{4n+1}\lambda_{7n+2}, x_{4n+2}\lambda_{7n+3}\lambda_{7n+4})\, T_n, \tag{13}$$

where

$$T_n = \frac{1}{\left(1 - \frac{x_{4n+3}\lambda_{7n+5}\lambda_{7n+6}}{\lambda_{7n+2}}\right)\left(1 - \frac{x_{4n+4}\lambda_{7n+7}}{\lambda_{7n+3}\lambda_{7n+6}}\right)\left(1 - \frac{x_{4n+5}\lambda_{7n+8}}{\lambda_{7n+4}\lambda_{7n+5}}\right)\left(1 - \frac{x_{4n+6}}{\lambda_{7n+7}\lambda_{7n+8}}\right)}.$$

According to the induction hypothesis,

$$
\begin{aligned}
h_n(x_1, \ldots, x_{4n+2}) = {} & g_{n-1}(x_1, \ldots, x_{4n+2}) g_{n-2}(x_1, \ldots, x_{4n-2}) \cdots g_0(x_1, \ldots, x_6) \\
& \cdot h_0(x_1, x_2).
\end{aligned} \tag{14}
$$

Consequently, (13) and (14) give

$$
\begin{aligned}
h_{n+1} &= \overbrace{g_{n-2}\cdots g_0 h_0}^{h_{n-1}}\\
&\quad\cdot \underset{\mu_1,\dots,\mu_7}{\Omega_{\geqq}}\; g_{n-1}(x_1,\dots,x_{4n},x_{4n+1}\mu_1,x_{4n+2}\mu_2\mu_3)\\
&\quad\cdot \underbrace{\frac{1}{\left(1-\frac{x_{4n+3}\mu_4\mu_5}{\mu_1}\right)\left(1-\frac{x_{4n+4}\mu_6}{\mu_2\mu_5}\right)\left(1-\frac{x_{4n+5}\mu_7}{\mu_3\mu_4}\right)\left(1-\frac{x_{4n+6}}{\mu_6\mu_7}\right)}}_{T(\mu_1,\dots,\mu_7)}.
\end{aligned}
\tag{15}
$$

So our task is to show that the $\Omega_{\geqq}$ expression in (15) is indeed $g_n g_{n-1}$, which corresponds to showing that

$$
\begin{aligned}
g_n g_{n-1} = \underset{\geqq}{\Omega}\, &\frac{1-X_{4n+1}X_{4n-1}\mu_1}{1-X_{4n-3}X_{4n-5}}\;\frac{P_{n-1}(x_1,\dots,x_{4n-1})}{\left(1-\frac{X_{4n-1}}{x_{4n-4}x_{4n-2}}\right)\left(1-\frac{X_{4n-1}}{x_{4n-2}}\right)}\\
&\cdot\frac{1}{(1-X_{4n-1})(1-X_{4n})(1-X_{4n+1}\mu_1)(1-X_{4n+2}\mu_1\mu_2\mu_3)}\\
&\cdot\frac{1}{1-\frac{X_{4n+1}\mu_1}{x_{4n}}}\;T(\mu_1,\dots,\mu_7).
\end{aligned}
$$

The right-hand side equals

$$
\begin{aligned}
&\frac{P_{n-1}(x_1,\dots,x_{4n-1})}{\left(1-\frac{X_{4n-1}}{x_{4n-4}x_{4n-2}}\right)\left(1-\frac{X_{4n-1}}{x_{4n-2}}\right)}\;\frac{1}{(1-X_{4n-3}X_{4n-5})(1-X_{4n-1})(1-X_{4n})}\\
&\quad\cdot\underset{\geqq}{\Omega}\,\frac{(1-X_{4n+1}X_{4n-1}\mu_1)T(\mu_1,\dots,\mu_7)}{(1-X_{4n+1}\mu_1)(1-X_{4n+2}\mu_1\mu_2\mu_3)\left(1-\frac{X_{4n+1}\mu_1}{x_{4n}}\right)},
\end{aligned}
$$

so in other words, by $\Omega_{\geqq}$ elimination we have to derive that

$$
\begin{aligned}
L &:= \underset{\geqq}{\Omega}\,\frac{(1-X_{4n+1}X_{4n-1}\mu_1)T(\mu_1,\dots,\mu_7)}{(1-X_{4n+1}\mu_1)(1-X_{4n+2}\mu_1\mu_2\mu_3)\left(1-\frac{X_{4n+1}\mu_1}{x_{4n}}\right)}\\
&= g_n\,\frac{1-X_{4n+1}X_{4n-1}}{(1-X_{4n+1})(1-X_{4n+2})\left(1-\frac{X_{4n+1}}{x_{4n}}\right)}.
\end{aligned}
\tag{16}
$$

To this end we apply rule (5) to eliminate from the left-hand side of (16) the variables μ_7, μ_6, μ_5, and μ_3, in this order, and arrive at

$$
\begin{aligned}
L = &\frac{1}{1-X_{4n+2}x_{4n+3}x_{4n+4}x_{4n+5}x_{4n+6}}\\
&\cdot\underset{\geqq}{\Omega}\,\frac{1-X_{4n-1}X_{4n+1}\mu_1}{(1-X_{4n+1}\mu_1)(1-X_{4n+2}\mu_1\mu_2)\left(1-\frac{X_{4n+1}\mu_1}{x_{4n}}\right)}\\
&\cdot\frac{1}{\left(1-\frac{x_{4n+3}\mu_4}{\mu_1}\right)\left(1-\frac{x_{4n+3}x_{4n+4}\mu_4}{\mu_1\mu_2}\right)\left(1-\frac{X_{4n+2}x_{4n+5}\mu_1\mu_2}{\mu_4}\right)}.
\end{aligned}
$$

In the next step we apply rule (6) to eliminate μ_2 and μ_4, in this order, which gives

$$
L = C\cdot\underset{\geqq}{\Omega}\,\frac{1-\mu_1X_{4n-1}X_{4n+1}}{(1-X_{4n+1}\mu_1)(1-X_{4n+2}\mu_1)\left(1-\frac{X_{4n+1}\mu_1}{x_{4n}}\right)\left(1-\frac{x_{4n+3}}{\mu_1}\right)}
\tag{17}
$$

with

$$C = \frac{1 - X_{4n+3}X_{4n+5}}{(1 - X_{4n+4})\big(1 - \frac{X_{4n+5}}{x_{4n+4}}\big)(1 - X_{4n+5})(1 - X_{4n+6})}.$$

For the final elimination step let us denote the $\Omega_{\geqq}$ expression in (17) by L'. In order to eliminate μ_1 from it, we need to extend rule (6) by one more term and with 1 or λ in the numerator.

Lemma 2. *We have*

$$\begin{aligned}(18)\qquad &\underset{\geqq}{\Omega} \frac{1}{\big(1 - \frac{a}{\lambda}\big)(1 - b_1\lambda)(1 - b_2\lambda)(1 - b_3\lambda)} \\ &\quad = \frac{1 + (b_1b_2b_3 - b_1b_2 - b_1b_3 - b_2b_3)a + b_1b_2b_3a^2}{(1 - b_1)(1 - b_2)(1 - b_3)(1 - ab_1)(1 - ab_2)(1 - ab_3)}\end{aligned}$$

and

$$\begin{aligned}(19)\qquad &\underset{\geqq}{\Omega} \frac{\lambda}{\big(1 - \frac{a}{\lambda}\big)(1 - b_1\lambda)(1 - b_2\lambda)(1 - b_3\lambda)} \\ &\quad = \frac{1 + (1 - b_1 - b_2 - b_3)a + b_1b_2b_3a^2}{(1 - b_1)(1 - b_2)(1 - b_3)(1 - ab_1)(1 - ab_2)(1 - ab_3)}.\end{aligned}$$

Proof. Rule (18) is an entry in MacMahon's list [10, Vol. 2, Art. 348]. Rule (19) is proved in [5]. □

Applying rules (18) and (19), to the first, resp. second, part of L' results in

$$\begin{aligned}(20)\qquad &\underset{\geqq}{\Omega} \frac{1}{(1 - X_{4n+1}\mu_1)(1 - X_{4n+2}\mu_1)\big(1 - \frac{X_{4n+1}\mu_1}{x_{4n}}\big)\big(1 - \frac{x_{4n+3}}{\mu_1}\big)} \\ &= D \cdot \Big(1 - X_{4n-1}X_{4n+1} - \frac{X_{4n+1}X_{4n+3}}{x_{4n}x_{4n+2}} - \frac{X_{4n+1}X_{4n+3}}{x_{4n}} + \frac{X_{4n+1}^2X_{4n+3}}{x_{4n}} \\ &\qquad + \frac{X_{4n+1}X_{4n+2}^2}{x_{4n}x_{4n+2}}\Big)\end{aligned}$$

and

(21)

$$\begin{aligned}&-\underset{\geqq}{\Omega} \frac{\mu_1 X_{4n-1}X_{4n+1}}{(1 - X_{4n+1}\mu_1)(1 - X_{4n+2}\mu_1)\big(1 - \frac{X_{4n+1}\mu_1}{x_{4n}}\big)\big(1 - \frac{x_{4n+3}}{\mu_1}\big)} \\ &= D \cdot \Big(-X_{4n-1}X_{4n+1} - \frac{X_{4n-1}X_{4n+3}}{x_{4n+2}} + \frac{X_{4n-1}X_{4n+1}X_{4n+3}}{x_{4n+2}} \\ &\qquad + X_{4n-1}X_{4n+1}X_{4n+3} + \frac{X_{4n-1}X_{4n+1}X_{4n+3}}{x_{4n+2}x_{4n}} - \frac{X_{4n-1}X_{4n+1}^2X_{4n+2}^2}{x_{4n}x_{4n+2}}\Big),\end{aligned}$$

where

$$\begin{aligned}D = &\frac{1}{(1 - X_{4n+1})(1 - X_{4n+2})\big(1 - \frac{X_{4n+1}}{x_{4n}}\big)\big(1 - \frac{X_{4n+3}}{x_{4n+2}}\big)} \\ &\cdot \frac{1}{(1 - X_{4n+3})\big(1 - \frac{X_{4n+3}}{x_{4n+2}x_{4n}}\big)}.\end{aligned}$$

Finally, combining (20) and (21) with (17), and recalling the definition of P_n from (11), gives

$$\begin{aligned} L &= C \cdot D \cdot P_n \\ &= g_n \, \frac{1 - X_{4n+1}X_{4n-1}}{(1 - X_{4n+1})(1 - X_{4n+2})\big(1 - \frac{X_{4n+1}}{x_{4n}}\big)}, \end{aligned}$$

which proves (16). This completes the proof of Theorem 2, and hence the proof of Theorem 1. □

4. Conclusion

Once again we find that MacMahon's Partition Analysis implemented in the Omega package is a powerful exploratory tool. The generating function in Corollary 2 is totally unexpected from past experience. All of the previous infinite product generating functions for ordinary and plane partitions consisted of products of cyclotomic polynomials. Now for plane partitions with diagonals we discover the lovely product in Corollary 2.

Naturally, we are led to plane partitions with other diagonals and with more rows. We have found further partitions that are related to those considered here and that have striking infinite product generating functions. These will be the subject of a subsequent paper.

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