MACMAHON'S PARTITION ANALYSIS XI: HEXAGONAL PLANE PARTITIONS

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Abstract. In this paper we continue the partition explorations made possible by Omega, the computer algebra implementation of MacMahon's Partition Analysis. The focus of our work has been partitions associated with directed graphs. The graphs considered here are made up of chains of hexagons, and the related generating functions are infinite products. Somewhat unexpectedly, while the generating functions are infinite products, they are most emphatically not modular forms.

1. Introduction

In his pioneering book "Combinatory Analysis" [11, Vol. II, Sect. VIII, pp. 91–170] MacMahon introduced Partition Analysis as a computational method for solving combinatorial problems in connection with systems of linear diophantine inequalities and equations. In particular, he devotes Chapter II of Section IX to the study of plane partitions as a natural application domain for his method.

In the course of a joint project devoted to Partition Analysis, the authors have turned MacMahon's method into an algorithm described in full detail in [4, 5]. As demonstrated in references [1]–[10], the resulting computer algebra package Omega[1] has been used as a powerful tool for combinatorial investigation. In particular, in [7, 9] we considered new variations of plane partitions, a study which will be extended in the present paper to plane partitions of "hexagonal shape".

The "most simple case" of classical plane partitions, treated by MacMahon in [11, Vol. II, p. 183], is the situation where the non-negative integer parts a_i of the partition are placed at the corners of a square such that the following order relations are satisfied:

$$a_1 \geq a_2,\ a_1 \geq a_3,\ a_2 \geq a_4,\ \text{and } a_3 \geq a_4. \tag{1}$$

It will be convenient to use arrows as an alternative description for $\geq$ relations; for instance, Fig. 1 represents the relations (1). Here and throughout the following it will be understood that an arrow pointing from a_i to a_j is interpreted as $a_i \geq a_j$.

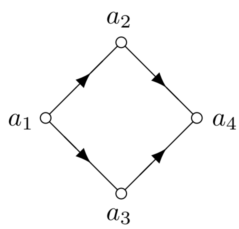

Figure 1. The inequalities (1)

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[1]available at *http://www.risc.uni-linz.ac.at/research/combinat/risc/software/Omega/*

By using Partition Analysis MacMahon derives that

$$\begin{aligned}\varphi &:= \sum x_1^{a_1} x_2^{a_2} x_3^{a_3} x_4^{a_4} \\ &= \frac{1 - x_1^2 x_2 x_3}{(1-x_1)(1-x_1x_2)(1-x_1x_3)(1-x_1x_2x_3)(1-x_1x_2x_3x_4)},\end{aligned} \tag{2}$$

where the sum is taken over all non-negative integers a_i satisfying (1). Furthermore, he observes that if $x_1 = x_2 = x_3 = x_4 = q$, the resulting generating function is

$$\frac{1}{(1-q)(1-q^2)^2(1-q^3)}.$$

In order to see how Partition Analysis works on (2) we need to recall the key ingredient of MacMahon's method, the Omega operator $\Omega_{\geqq}$.

Definition 1. The operator $\Omega_{\geqq}$ is given by

$$\underset{\geqq}{\Omega} \sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_r=-\infty}^{\infty} A_{s_1,\ldots,s_r} \lambda_1^{s_1} \cdots \lambda_r^{s_r} := \sum_{s_1=0}^{\infty} \cdots \sum_{s_r=0}^{\infty} A_{s_1,\ldots,s_r},$$

where the domain of the $A_{s_1,\ldots,s_r}$ is the field of rational functions over $\mathbb{C}$ in several complex variables and the λ_i are restricted to a neighborhood of the circle $|\lambda_i| = 1$. In addition, the $A_{s_1,\ldots,s_r}$ are required to be such that any of the series involved is absolute convergent within the domain of the definition of $A_{s_1,\ldots,s_r}$.

To avoid confusion we will always have $\Omega_{\geqq}$ operate on variables denoted by letters in the middle of the Greek alphabet (e.g. λ, μ, ν). The parameters unaffected by $\Omega_{\geqq}$ will be denoted by letters from the Latin alphabet.

We emphasize that it is essential to treat everything analytically rather than formally because the method relies on unique Laurent series representations of rational functions.

Another fundamental aspect of Partition Analysis is the use of elimination rules which describe the action of the Omega operator on certain base cases. MacMahon begins the discussion of his method by presenting a catalog [11, Vol. II, pp. 102–103] of twelve fundamental evaluations. Subsequently he extends this table by new rules whenever he is forced to do so. Once found, most of these fundamental rules are easy to prove. This is illustrated by the following examples which are taken from MacMahon's list.

Proposition 1. *For integer $s \geq 1$,*

$$\underset{\geqq}{\Omega} \frac{1}{(1-\lambda A)\left(1-\frac{B}{\lambda^s}\right)} = \frac{1}{(1-A)(1-A^s B)}; \tag{3}$$

$$\underset{\geqq}{\Omega} \frac{1}{(1-\lambda A)(1-\lambda B)\left(1-\frac{C}{\lambda}\right)} = \frac{1-ABC}{(1-A)(1-B)(1-AC)(1-BC)}. \tag{4}$$

We prove (3); the proof of (4) is analogous and is left to the reader.

Proof of (3). By geometric series expansion the left hand side equals

$$\underset{\geqq}{\Omega} \sum_{i,j\geq 0} \lambda^{i-sj} A^i B^j = \underset{\geqq}{\Omega} \sum_{j,k\geq 0} \lambda^k A^{sj+k} B^j,$$

where the summation parameter i has been replaced by $sj+k$. But now $\Omega_{\geqq}$ sets λ to 1 which completes the proof. □

Now we are ready for deriving the closed form expression for φ with Partition Analysis.

Proof of (2). First, in order to get rid of the diophantine constraints, one rewrites the sum expression in (2) into what MacMahon called the "crude form" of the generating function,

$$\begin{aligned}\varphi &= \Omega_{\geq} \sum_{a_1,a_2,a_3,a_4 \geq 0} \lambda_1^{a_1-a_2}\lambda_2^{a_1-a_3}\lambda_3^{a_2-a_4}\lambda_4^{a_3-a_4} x_1^{a_1}x_2^{a_2}x_3^{a_3}x_4^{a_4} \\ &= \Omega_{\geq} \frac{1}{\left(1-\lambda_1\lambda_2 x_1\right)\left(1-\frac{\lambda_3}{\lambda_1}x_2\right)\left(1-\frac{\lambda_4}{\lambda_2}x_3\right)\left(1-\frac{x_4}{\lambda_3\lambda_4}\right)}.\end{aligned}$$

Next by rule (3) we eliminate successively λ_1, λ_3, and λ_4,

$$\begin{aligned}\varphi &= \Omega_{\geq} \frac{1}{\left(1-\lambda_2 x_1\right)\left(1-\lambda_2\lambda_3 x_1x_2\right)\left(1-\frac{\lambda_4}{\lambda_2}x_3\right)\left(1-\frac{x_4}{\lambda_3\lambda_4}\right)} \\ &= \Omega_{\geq} \frac{1}{\left(1-\lambda_2 x_1\right)\left(1-\lambda_2 x_1x_2\right)\left(1-\frac{\lambda_4}{\lambda_2}x_3\right)\left(1-\frac{\lambda_2x_1x_2x_4}{\lambda_4}\right)} \\ &= \Omega_{\geq} \frac{1}{\left(1-\lambda_2 x_1\right)\left(1-\lambda_2 x_1x_2\right)\left(1-\frac{x_3}{\lambda_2}\right)\left(1-x_1x_2x_3x_4\right)}.\end{aligned}$$

Finally, applying rule (4) eliminates λ_2 and completes the proof of (2). □

Instead of glueing squares together as in the case of standard plane partitions, in [7] we considered configurations shown in Fig. 2. In the present paper we shall

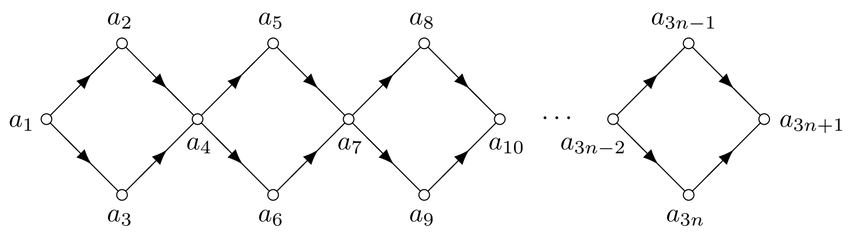

FIGURE 2. A plane partition diamond of length n

study the natural generalization depicted in Fig. 3 where we use hexagons instead of squares as building blocks of the chain.

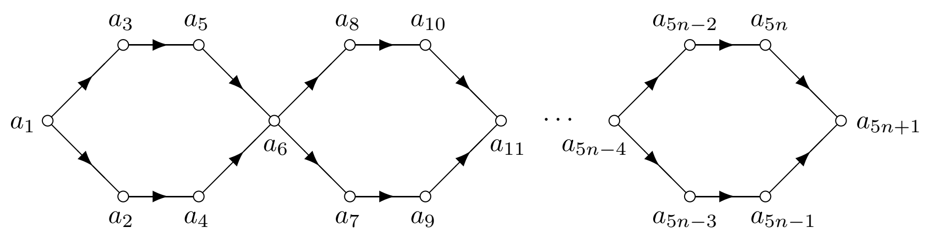

FIGURE 3

Definition 2. For $n \geq 1$ define

$$\begin{aligned}H_n &:= \{(a_1,\ldots,a_{5n+1}) \in \mathbb{N}^{5n+1} : \text{the } a_i \text{ satisfy the order relations in Fig. 3}\}, \\ h_n &:= h_n(x_1,\ldots,x_{5n+1}) := \sum_{(a_1,\ldots,a_{5n+1})\in H_n} x_1^{a_1}\cdots x_{5n+1}^{a_{5n+1}},\end{aligned}$$

and

$$h_n(q) := h_n(q,\ldots,q).$$

In Section 2 we shall derive a closed form (9) for the full generating function $h_n(x_1,\dots,x_{5n+1})$. For the specialization $x_1=\cdots=x_{5n+1}=q$ this will give the enumerative generating function of the following form.

Theorem 1. *For $n\geq 1$ we have*

$$h_n(q)=\sum_{(a_1,\dots,a_{5n+1})\in H_n} q^{a_1+\cdots+a_{5n+1}} = \frac{\prod_{j=0}^{n-1}(1+q^{5j+2}+2q^{5j+3}+q^{5j+4}+q^{10j+6})}{\prod_{j=1}^{5n+1}(1-q^j)}.$$

Note that the numerator does not factor into cyclotomic polynomials as in the case of classic plane partition generating functions. However, in heuristical studies using the Omega package we discovered a refinement H_n^* of H_n where the associated generating function indeed factors completely into cyclotomic factors. Namely, consider the order relations depicted in Fig. 4.

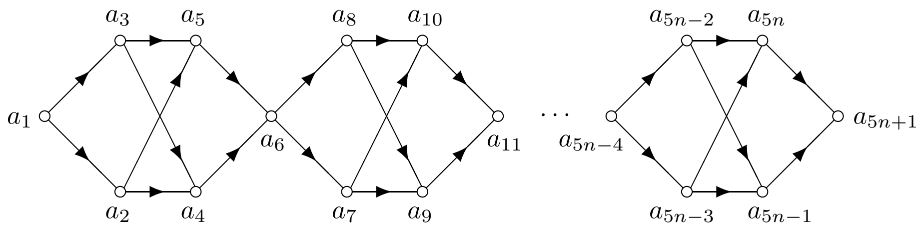

Figure 4

Definition 3. For $n\geq 1$ define

$$H_n^* := \{(a_1,\dots,a_{5n+1})\in\mathbb{N}^{5n+1} : \text{the } a_i \text{ satisfy the order relations in Fig. 4}\},$$

$$h_n^* := h_n^*(x_1,\dots,x_{5n+1}) := \sum_{(a_1,\dots,a_{5n+1})\in H_n^*} x_1^{a_1}\cdots x_{5n+1}^{a_{5n+1}},$$

and

$$h_n^*(q) := h_n^*(q,\dots,q).$$

In Section 3 we shall derive a closed form (13) for the full generating function $h_n^*(x_1,\dots,x_{5n+1})$. For the specialization $x_1=\cdots=x_{5n+1}=q$ this will give the enumerative generating function of the following form.

Theorem 2. *For $n\geq 1$ we have*

$$h_n^*(q)=\sum_{(a_1,\dots,a_{5n+1})\in H_n^*} q^{a_1+\cdots+a_{5n+1}} = \frac{\prod_{j=0}^{n-1}(1+q^{5j+2})(1+q^{5j+4})}{\prod_{j=1}^{5n+1}(1-q^j)}.$$

In Section 4 concluding remarks are made. In particular, we compare the combinatorics considered in Section 3 with previous work described in [9].

2. Vertex-Joined Hexagons

In this section we shall prove Theorem 1. To this end we shall derive a closed form representation for the full generating function $h_n=h_n(x_1,\dots,x_{5n+1})$.

First we consider the case $n=1$. Obviously, we have

$$h_1=\underset{\geqq}{\Omega}\frac{1}{(1-x_1\lambda_1\lambda_2)\big(1-\frac{x_2\lambda_3}{\lambda_1}\big)\big(1-\frac{x_3\lambda_4}{\lambda_2}\big)\big(1-\frac{x_4\lambda_5}{\lambda_3}\big)\big(1-\frac{x_5\lambda_6}{\lambda_4}\big)\big(1-\frac{x_6}{\lambda_5\lambda_6}\big)}.$$

In order to eliminate the λ-variables we apply rule (3) with $s = 1$ to the variables λ_2, λ_3, λ_4, λ_5, and λ_6, in this order, which results in

$$h_1 = \frac{1}{1 - X_6} \underset{\geqq}{\Omega} \frac{1}{\left(1 - \frac{x_2}{\lambda_1}\right)\left(1 - \frac{x_2 x_4}{\lambda_1}\right)(1 - x_1\lambda_1)(1 - x_1x_3\lambda_1)(1 - x_1x_3x_5\lambda_1)}, \tag{5}$$

where $X_6 = x_1 x_2 \cdots x_6$. Subsequently we will often use an extended version of this short-hand notation.

Definition 4. For $n \geq 1$ define

$$X_n := \prod_{k=1}^{n} x_k.$$

For the following it will be convenient to introduce two further abbreviations.

Definition 5. We define

$$f(a; b_1, b_2, b_3) := \frac{1 + (b_1b_2b_3 - b_1b_2 - b_1b_3 - b_2b_3)a + b_1b_2b_3a^2}{(1 - ab_1)(1 - ab_2)(1 - ab_3)}$$

and

$$g(a_1, a_2; b_1, b_2, b_3) := f(a_1; b_1, b_2, b_3) - a_2 f(a_1a_2; b_1, b_2, b_3).$$

In order to eliminate λ_1 from the right hand side of (5) we apply the following rule which is an extension of rule (4) and which can be found in MacMahon's table [11, Vol. II, Art. 348], namely,

$$\underset{\geqq}{\Omega} \frac{1}{\left(1 - \frac{a}{\lambda}\right)(1 - b_1\lambda)(1 - b_2\lambda)(1 - b_3\lambda)} = \frac{f(a; b_1, b_2, b_3)}{(1 - b_1)(1 - b_2)(1 - b_3)} \tag{6}$$

whenever the variables a, b_1, b_2, b_3 are free of λ. Rule (6) can be applied to the right hand side of (5) after carrying out the partial fraction decomposition

$$\frac{1}{\left(1 - \frac{x_2}{\lambda_1}\right)\left(1 - \frac{x_2x_4}{\lambda_1}\right)} = \frac{1}{1 - x_4}\left(\frac{1}{1 - \frac{x_2}{\lambda_1}} - \frac{x_4}{1 - \frac{x_2x_4}{\lambda_1}}\right);$$

these steps then give

$$h_1 = h_1(x_1, \ldots, x_6) = \frac{1}{(1 - X_6)(1 - x_4)} \frac{g\left(x_2, x_4; X_1, \frac{X_3}{x_2}, \frac{X_5}{x_2x_4}\right)}{(1 - X_1)\left(1 - \frac{X_3}{x_2}\right)\left(1 - \frac{X_5}{x_2x_4}\right)}. \tag{7}$$

It is straight-forward to verify that the crude generating functions for h_2 and for h_n with $n \geq 3$ are as follows:

$$h_2 = \underset{\geqq}{\Omega} \frac{1}{(1 - x_1\lambda_1\lambda_2)\left(1 - x_2\frac{\lambda_3}{\lambda_1}\right)\left(1 - x_3\frac{\lambda_4}{\lambda_2}\right)\left(1 - x_4\frac{\lambda_5}{\lambda_3}\right)\left(1 - x_5\frac{\lambda_6}{\lambda_4}\right)\left(1 - x_6\frac{\lambda_7\lambda_8}{\lambda_5\lambda_6}\right)}$$
$$\cdot \frac{1}{\left(1 - x_7\frac{\lambda_9}{\lambda_7}\right)\left(1 - x_8\frac{\lambda_{10}}{\lambda_8}\right)\left(1 - x_9\frac{\lambda_{11}}{\lambda_9}\right)\left(1 - x_{10}\frac{\lambda_{12}}{\lambda_{10}}\right)\left(1 - \frac{x_{11}}{\lambda_{11}\lambda_{12}}\right)}$$

and

(8)

$$h_n = \underset{\geqq}{\Omega} \frac{1}{(1-x_1\lambda_1\lambda_2)(1-x_2\frac{\lambda_3}{\lambda_1})(1-x_3\frac{\lambda_4}{\lambda_2})(1-x_4\frac{\lambda_5}{\lambda_3})(1-x_5\frac{\lambda_6}{\lambda_4})(1-x_6\frac{\lambda_7\lambda_8}{\lambda_5\lambda_6})}$$
$$\cdot \frac{1}{(1-x_7\frac{\lambda_9}{\lambda_7})(1-x_8\frac{\lambda_{10}}{\lambda_8})(1-x_9\frac{\lambda_{11}}{\lambda_9})(1-x_{10}\frac{\lambda_{12}}{\lambda_{10}})(1-x_{11}\frac{\lambda_{13}\lambda_{14}}{\lambda_{11}\lambda_{12}})}$$
$$\vdots$$
$$\cdot \frac{1}{(1-x_{5n-8}\frac{\lambda_{6n-9}}{\lambda_{6n-11}})(1-x_{5n-7}\frac{\lambda_{6n-8}}{\lambda_{6n-10}})(1-x_{5n-6}\frac{\lambda_{6n-7}}{\lambda_{6n-9}})(1-x_{5n-5}\frac{\lambda_{6n-6}}{\lambda_{6n-8}})}$$
$$\cdot \frac{1}{(1-x_{5n-4}\frac{\lambda_{6n-5}\lambda_{6n-4}}{\lambda_{6n-7}\lambda_{6n-6}})(1-x_{5n-3}\frac{\lambda_{6n-3}}{\lambda_{6n-5}})(1-x_{5n-2}\frac{\lambda_{6n-2}}{\lambda_{6n-4}})}$$
$$\cdot \frac{1}{(1-x_{5n-1}\frac{\lambda_{6n-1}}{\lambda_{6n-3}})(1-x_{5n}\frac{\lambda_{6n}}{\lambda_{6n-2}})(1-\frac{x_{5n+1}}{\lambda_{6n-1}\lambda_{6n}})}.$$

Proposition 2. *For $n \geq 1$ we have*

$$h_{n+1} = \underset{\geqq}{\Omega} h_n(x_1,\ldots,x_{5n},x_{5n+1}\lambda_{6n+1}\lambda_{6n+2})$$
$$\cdot \frac{1}{(1-x_{5n+2}\frac{\lambda_{6n+3}}{\lambda_{6n+1}})(1-x_{5n+3}\frac{\lambda_{6n+4}}{\lambda_{6n+2}})(1-x_{5n+4}\frac{\lambda_{6n+5}}{\lambda_{6n+3}})}$$
$$\cdot \frac{1}{(1-x_{5n+5}\frac{\lambda_{6n+6}}{\lambda_{6n+4}})(1-\frac{x_{5n+6}}{\lambda_{6n+5}\lambda_{6n+6}})}.$$

Proof. The result follows immediately from (8). □

Theorem 3. *For $n \geq 1$ we have*

(9)

$$h_n = \frac{1}{1-X_{5n+1}} \prod_{j=0}^{n-1} \frac{1}{1-x_{5j+4}} \prod_{j=0}^{n-1} \frac{g\left(x_{5j+2}, x_{5j+4}; X_{5j+1}, \frac{X_{5j+3}}{x_{5j+2}}, \frac{X_{5j+5}}{x_{5j+2}x_{5j+4}}\right)}{(1-X_{5j+1})(1-\frac{X_{5j+3}}{x_{5j+2}})(1-\frac{X_{5j+5}}{x_{5j+2}x_{5j+4}})}.$$

Proof. We proceed by induction on n. The case $n = 1$ is immediate by (7). For the induction step observe that from Proposition 2 and the induction hypothesis for n we obtain that

$$h_{n+1} = \prod_{j=0}^{n-1} \frac{1}{1-x_{5j+4}} \prod_{j=0}^{n-1} \frac{g\left(x_{5j+2}, x_{5j+4}; X_{5j+1}, \frac{X_{5j+3}}{x_{5j+2}}, \frac{X_{5j+5}}{x_{5j+2}x_{5j+4}}\right)}{(1-X_{5j+1})(1-\frac{X_{5j+3}}{x_{5j+2}})(1-\frac{X_{5j+5}}{x_{5j+2}x_{5j+4}})}$$
$$\cdot \underset{\geqq}{\Omega} \frac{1}{1-X_{5n+1}\lambda_{6n+1}\lambda_{6n+2}} \frac{1}{(1-x_{5n+2}\frac{\lambda_{6n+3}}{\lambda_{6n+1}})(1-x_{5n+3}\frac{\lambda_{6n+4}}{\lambda_{6n+2}})}$$
$$\cdot \frac{1}{(1-x_{5n+4}\frac{\lambda_{6n+5}}{\lambda_{6n+3}})(1-x_{5n+5}\frac{\lambda_{6n+6}}{\lambda_{6n+4}})(1-\frac{x_{5n+6}}{\lambda_{6n+5}\lambda_{6n+6}})}.$$

The $\Omega_{\geqq}$-expression is nothing but

$$h_1(X_{5n+1}, x_{5n+2}, x_{5n+3}, x_{5n+4}, x_{5n+5}, x_{5n+6}),$$

hence applying (7) completes the proof of the induction step. □

Setting all x_i to q, a straight-forward simplification gives the generating function for $h_n(q) = h_n(q,\ldots,q)$ as in Theorem 1. For $n \to \infty$ in Theorem 1, i.e., for a chain of infinitely many hexagons with order relations as in Fig. 3 one obtains

$$h_\infty(q) = \prod_{j=1}^{\infty} \frac{1+q^{5j-3}+2q^{5j-2}+q^{5j-1}+q^{10j-4}}{1-q^j}.$$

3. Vertex-Joined Enriched Hexagons

In this section we shall prove Theorem 2. To this end we shall derive a closed form representation for the full generating function $h_n^* = h_n^*(x_1,\dots,x_{5n+1})$.

Again we consider the case $n=1$ first. For this we need two lemmas.

Lemma 1.

$$\Omega_{\geqq} \frac{f\big(a\lambda; b_1, b_2, \frac{b_3}{\lambda}\big)}{1-\frac{b_3}{\lambda}} = \frac{1-ab_1b_2}{(1-ab_1)(1-ab_2)(1-ab_3)}.$$

Proof. First observe that

$$\begin{aligned}\frac{f\big(a\lambda; b_1, b_2, \frac{b_3}{\lambda}\big)}{1-\frac{b_3}{\lambda}} &= \frac{1-ab_1b_3-ab_2b_3+ab_1b_2b_3}{(1-ab_1\lambda)(1-ab_2\lambda)(1-ab_3)\big(1-\frac{b_3}{\lambda}\big)} \\ &\quad - \frac{\lambda ab_1b_2}{(1-ab_1\lambda)(1-ab_2\lambda)\big(1-\frac{b_3}{\lambda}\big)}.\end{aligned} \tag{10}$$

Because of (4) and the similar rule ([5, (2.2)])

$$\Omega_{\geqq} \frac{\lambda}{(1-\lambda A)(1-\lambda B)\big(1-\frac{C}{\lambda}\big)} = \frac{1+C-AC-BC}{(1-A)(1-B)(1-AC)(1-BC)},$$

the result of applying the $\Omega_{\geqq}$-operator to the right hand side of (10) equals

$$\begin{aligned}&\frac{(1-ab_1b_3-ab_2b_3+ab_1b_2b_3)(1-a^2b_1b_2b_3)}{(1-ab_1)(1-ab_2)(1-ab_3)(1-ab_1b_3)(1-ab_2b_3)} \\ &\quad - \frac{ab_1b_2(1+b_3-ab_1b_3-ab_2b_3)}{(1-ab_1)(1-ab_2)(1-ab_1b_3)(1-ab_2b_3)},\end{aligned}$$

which simplifies to

$$\frac{1-ab_1b_2}{(1-ab_1)(1-ab_2)(1-ab_3)}.$$

□

Lemma 2.

$$\begin{aligned}&\Omega_{\geqq} \frac{1}{1-\frac{a_2}{\lambda_2}} \frac{g\big(a_1\lambda_1, \frac{a_2}{\lambda_2}; b_1, b_2\lambda_2, b_3\frac{\lambda_2}{\lambda_1}\big)}{(1-b_2\lambda_2)\big(1-b_3\frac{\lambda_2}{\lambda_1}\big)} \\ &\quad = \frac{(1-a_1b_1b_2)(1-a_1^2a_2b_2b_3)}{(1-b_2)(1-a_1b_1)(1-a_1b_2)(1-a_1b_3)(1-a_1a_2b_2)(1-a_1a_2b_3)}.\end{aligned}$$

Proof. According to Definition 5, we have

$$\begin{aligned}&\frac{1}{1-\frac{a_2}{\lambda_2}} \frac{g\big(a_1\lambda_1, \frac{a_2}{\lambda_2}; b_1, b_2\lambda_2, b_3\frac{\lambda_2}{\lambda_1}\big)}{(1-b_2\lambda_2)\big(1-b_3\frac{\lambda_2}{\lambda_1}\big)} \\ &\quad = \frac{1}{1-\frac{a_2}{\lambda_2}} \frac{f\big(a_1\lambda_1; b_1, b_2\lambda_2, b_3\frac{\lambda_2}{\lambda_1}\big)}{(1-b_2\lambda_2)\big(1-b_3\frac{\lambda_2}{\lambda_1}\big)} - \frac{a_2}{\lambda_2}\frac{1}{1-\frac{a_2}{\lambda_2}} \frac{f\big(a_1\lambda_1\frac{a_2}{\lambda_2}; b_1, b_2\lambda_2, b_3\frac{\lambda_2}{\lambda_1}\big)}{(1-b_2\lambda_2)\big(1-b_3\frac{\lambda_2}{\lambda_1}\big)}.\end{aligned}$$

Now we use Lemma 1 twice to eliminate λ_1 from the right hand side and obtain

$$\begin{aligned}&\frac{1-a_1b_1b_2\lambda_2}{(1-a_1b_1)\big(1-\frac{a_2}{\lambda_2}\big)(1-b_2\lambda_2)(1-a_1b_2\lambda_2)(1-a_1b_3\lambda_2)} \\ &\quad - \frac{a_2(1-a_1a_2b_1b_2)}{\lambda_2(1-a_1a_2b_2)(1-a_1a_2b_3)\big(1-\frac{a_2}{\lambda_2}\big)\big(1-\frac{a_1a_2b_1}{\lambda_2}\big)(1-b_2\lambda_2)}.\end{aligned}$$

Finally, we eliminate λ_2 by using [5, (2.2)] and [5, (2.4)] from this expression to get the desired result. $\square$

Clearly, we have

$$h_1^* = \underset{\geqq}{\Omega} \frac{1}{(1-x_1\lambda_1\lambda_2)\left(1-\frac{x_2\lambda_3\lambda_5}{\lambda_1}\right)\left(1-\frac{x_3\lambda_4\lambda_6}{\lambda_2}\right)\left(1-\frac{x_4\lambda_7}{\lambda_3\lambda_6}\right)\left(1-\frac{x_6}{\lambda_7\lambda_8}\right)\left(1-\frac{x_5\lambda_8}{\lambda_4\lambda_5}\right)}.$$

By applying the rewrite rules

$$\lambda_5 \to \mu_1,\ \lambda_6 \to \mu_2,\ \lambda_7 \to \lambda_5,\ \lambda_8 \to \lambda_6,$$

we transform h_1^* into

$$h_1^* = \underset{\geqq}{\Omega} \frac{1}{(1-x_1\lambda_1\lambda_2)\left(1-\frac{x_2\lambda_3\mu_1}{\lambda_1}\right)\left(1-\frac{x_3\lambda_4\mu_2}{\lambda_2}\right)\left(1-\frac{x_4\lambda_5}{\lambda_3\mu_2}\right)\left(1-\frac{x_6}{\lambda_5\lambda_6}\right)\left(1-\frac{x_5\lambda_6}{\lambda_4\mu_1}\right)}.$$

Consequently we see that

$$h_1^* = \underset{\geqq}{\Omega}\, h_1\Big(x_1, x_2\mu_1, x_3\mu_2, \frac{x_4}{\mu_2}, \frac{x_5}{\mu_1}, x_6\Big),$$

and thus by (7) we have

$$h_1^* = \underset{\geqq}{\Omega} \frac{1}{(1-X_6)\left(1-\frac{x_4}{\mu_2}\right)} \frac{g\left(x_2\mu_1, \frac{x_4}{\mu_2}; X_1, \frac{X_3}{x_2}\mu_2, \frac{X_5}{x_2x_4}\frac{\mu_2}{\mu_1}\right)}{(1-X_1)\left(1-\frac{X_3}{x_2}\mu_2\right)\left(1-\frac{X_5}{x_2x_4}\frac{\mu_2}{\mu_1}\right)}.$$

Because of Lemma 2 we eventually obtain

$$h_1^* = \frac{1}{(1-X_1)\cdots(1-X_6)} \frac{(1-X_1X_3)(1-X_3X_5)}{\left(1-\frac{X_3}{x_2}\right)\left(1-\frac{X_5}{x_4}\right)}. \tag{11}$$

Similarly, the crude generating functions for h_2^* and h_n^* for $n \geq 3$ are found to be

$$h_2^* = \underset{\geqq}{\Omega}\, h_2\Big(x_1, x_2\mu_1, x_3\mu_2, \frac{x_4}{\mu_2}, \frac{x_5}{\mu_1}, x_6, x_7\mu_3, x_8\mu_4, \frac{x_9}{\mu_4}, \frac{x_{10}}{\mu_3}, x_{11}\Big)$$

and

$$\begin{aligned} h_n^* = \underset{\geqq}{\Omega}\, h_n\Big(&x_1, x_2\mu_1, x_3\mu_2, \frac{x_4}{\mu_2}, \frac{x_5}{\mu_1}, x_6, \ldots, \\ &x_{5n-3}\mu_{2n-1}, x_{5n-2}\mu_{2n}, \frac{x_{5n-1}}{\mu_{2n}}, \frac{x_{5n}}{\mu_{2n-1}}, x_{5n+1}\Big). \end{aligned} \tag{12}$$

Theorem 4. *For $n \geq 1$ we have*

$$h_n^* = \prod_{j=1}^{5n+1} \frac{1}{1-X_j} \prod_{j=0}^{n-1} \frac{(1-X_{5j+1}X_{5j+3})(1-X_{5j+3}X_{5j+5})}{\left(1-\frac{X_{5j+3}}{x_{5j+2}}\right)\left(1-\frac{X_{5j+5}}{x_{5j+4}}\right)}. \tag{13}$$

Proof. From (9) we get that

$$h_{n+1} = h_n \frac{1-X_{5n+1}}{1-X_{5n+6}} \frac{1}{1-x_{5n+4}} \frac{g\left(x_{5n+2}, x_{5n+4}; X_{5n+1}, \frac{X_{5n+3}}{x_{5n+2}}, \frac{X_{5n+5}}{x_{5n+2}x_{5n+4}}\right)}{(1-X_{5n+1})\left(1-\frac{X_{5n+3}}{x_{5n+2}}\right)\left(1-\frac{X_{5n+5}}{x_{5n+2}x_{5n+4}}\right)}.$$

Hence, using (12) with $\nu_1 := \mu_{2n+1}$ and $\nu_2 := \mu_{2n+2}$, we obtain

$$\begin{aligned} h_{n+1}^* &= h_n^* \frac{1-X_{5n+1}}{1-X_{5n+6}} \\ &\cdot \underset{\geqq}{\Omega} \frac{1}{1-\frac{x_{5n+4}}{\nu_2}} \frac{g\left(x_{5n+2}\nu_1, \frac{x_{5n+4}}{\nu_2}; X_{5n+1}, \frac{X_{5n+3}}{x_{5n+2}}\nu_2, \frac{X_{5n+5}}{x_{5n+2}x_{5n+4}}\frac{\nu_2}{\nu_1}\right)}{(1-X_{5n+1})\left(1-\frac{X_{5n+3}}{x_{5n+2}}\nu_2\right)\left(1-\frac{X_{5n+5}}{x_{5n+2}x_{5n+4}}\frac{\nu_2}{\nu_1}\right)}. \end{aligned}$$

Applying Lemma 2 we find that

$$h_{n+1}^* = h_n^* \frac{1}{(1-X_{5n+2})\cdots(1-X_{5n+6})} \frac{(1-X_{5n+1}X_{5n+3})(1-X_{5n+3}X_{5n+5})}{\left(1-\frac{X_{5n+3}}{x_{5n+2}}\right)\left(1-\frac{X_{5n+5}}{x_{5n+4}}\right)}.$$

Together with (11) this proves the assertion. □

Setting all x_i to q, immediately gives the generating function for $h_n^*(q) = h_n^*(q,\ldots,q)$ as in Theorem 2. For $n \to \infty$ in Theorem 2, i.e., for a chain of infinitely many hexagons with order relations as in Fig. 4 one obtains

$$h_\infty(q) = \prod_{j=1}^{\infty} \frac{(1+q^{5j-3})(1+q^{5j-1})}{1-q^j}.$$

4. Conclusion

In [9] we considered plane partitions with diagonals, i.e., the generating function $\sum x_1^{a_1} \cdots x_{4n+1}^{a_{4n+1}}$, where the a_i satisfy the order relations depicted in Fig. 5. As stated in [9, Thm. 1] its rational function representation involves complicated

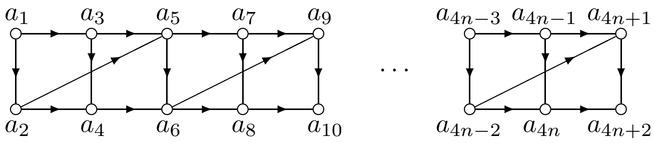

Figure 5

irreducible numerator polynomials of total degree 2. We want to note that despite the nice structure of the rational function representation of h_n^* in Theorem 4 above, the poset H_n^* can be viewed as a variation of the poset described by Fig. 5 if drawn in an equivalent alternative to Fig. 4. For instance, for $n = 3$ the poset H_3^* can be depicted as in Fig. 6.

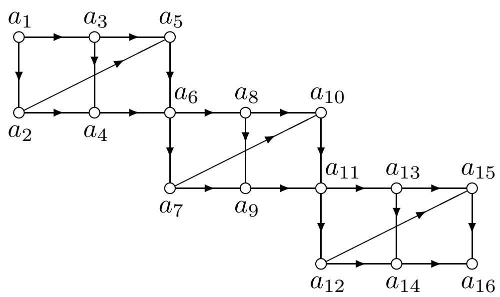

Figure 6

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