Numerical stability of surface implicitization ⋆

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Abstract

For a numerically given parametrization we cannot compute an exact implicit equation, just an approximate one. We introduce a condition number to measure the worst effect on the solution when the input data is perturbed by a small amount. Using this condition number the perturbation behavior of various implicitization methods can be analyzed.

Key words: Condition number, Implicitization

1 Introduction

The implicit formulation of curves and surfaces has several advantages: The implicit equation of curves and surfaces essentially unique, and the degree is apparent from their implicit equation. We can also easily determine whether a point lies on a curve or on a surface using the implicit form. Many curves and surfaces used in computer aided design are given in parametric form. In principle rationally given curves and surfaces can always be implicitized, e.g. they have an algebraic representation. The conversion from the parametric form to the implicit one is called implicitization. The reverse problem is called parametrization. Both representations has its own advantages and disadvantages. To avoid the weaknesses of these representations and to exploit the strength of both of them, the conversion problem is of fundamental importance.

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There are several symbolic implicitization techniques based on resultants (Busé, 2001; Marco and Martínez, 2002), Gröbner–basis (Alonso et al., 1995; Buchberger, 1988), moving surfaces (Zheng et al., 2003), or on residue calculus (Elkadi and Mourrain, 2004). For a given parametric representation of a surface $G_i(y_1, y_2, y_3)$, $i = 1 \ldots 4$, we can find the implicit form $F(x_1, x_2, x_3, x_4) = 0$ such that $F(G_1, G_2, G_3, G_4) = 0$.

In geometric applications the input data is often given in terms of floating point numbers. However for numerically given parametrization we can never compute the exact implicit equation, but an approximate one (Busé et al., 2003; Chen, 2003; Corless et al., 2001; Dokken, 2001; Dokken and Thomassen, 2003).

Given numeric input data the question is how precise we can say something about the output. Typically the output error can be upper estimated by the input error times a constant, called condition number, which measures the stability of the problem (or the algorithm).

Normally the condition number depends on the algorithm. The condition number for the best possible algorithm is the condition number of the problem itself: no matter which algorithm we use, we can not say anything more on the accuracy of the output than described by this condition number.

In this paper we introduce the condition number of the implicitization problem. It depends not only on the input, but also on the estimation of the degree of the implicit form. Such an estimation must always be used, see Corless et al. (2001), Dokken (2001).

The paper is organized as follows. In Section 2 we define the condition number of the implicitization problem and we give an algorithm for the computation. In Section 3 we show, how the condition number can be used to bound the difference of the computed, and the nominal implicit equation. This section contains two theorems, which provide an error analysis using the defined condition number. We also give an example to demonstrate the usage of these theorems. Section 4 contains observations and remarks on the condition number. Furthermore, we propose a way how to guess the implicit degree numerically.

2 Definition and computation of the condition number

In this section we define a constant, called condition number, for the surface implicitization problem, and we show how to compute it. Throughout this paper we work in the projective setting over the real numbers.

2.1 Definition of the condition number–special case

Let $n, m \in \mathbb{Z}$. Let P be the set of four–tuples of polynomials of degree n in the variables $y_1, \ldots, y_3$ over $\mathbb{R}$. Let I be the set of all homogenous polynomials of degree m in the variables $x_1, \ldots, x_4$ over $\mathbb{R}$; and we denote by R the set of homogenous polynomials of degree nm in the variables $y_1, \ldots, y_3$ over $\mathbb{R}$. (The letters P, I, R stand for parametrization, implicitization, and residuals respectively.) The sets P, I, R are real vector spaces and come with an inner product, which depends on the choice of a basis. We usually choose either the monomial or the Bernstein basis. We define an evaluation map $ev : I \times P \to R$ by $(H, G) \mapsto ev(H, G) = H(G)$. Note that the evaluation map is linear in the first entry, but not linear in the second.

Assume that $G \in P$, $\|G\| = 1$ is a parametrization and $F \in I$ is the implicit equation of the same surface. Then F is the unique solution H of $ev(H, G) = 0$. Let

$$F^{\perp} := \{J \in I | \langle F, J \rangle_I = 0\}.$$

(Recall, that $F^{\perp}$ depends on the chosen basis.) Then $ev(J, G)$ is a nonzero vector for all $J \in F^{\perp}$. The following amount:

$$\kappa := \min_{J \in F^{\perp}, \|J\|_I = 1} \|ev(J, G)\|_V$$

is a numerical measurement of the uniqueness of the implicitization problem. If $\kappa = 0$, then there are several linearly independent equations H with $ev(H, G) = 0$. If κ is small, we are close to such a case. The condition number is defined as:

$$K := 1/\kappa,$$

in the case where we have a parametrization G and the implicit equation F of the same surface.

If $\|G\| \neq 1$, then the condition number is always the condition number of the normed equation .

2.2 General definition and computation of the condition number

For any $F \in I$, $G \in P$ we can write

$$ev(F, G) = M_G \cdot F$$

where M_G is a matrix depending on G of size $\bar{m} \times \bar{n}$, where $\bar{m} = \frac{(mn+1)(mn+2)}{2}$, $\bar{n} = \frac{(m+1)(m+2)(m+3)}{6}$. We can write it as $U \cdot \Sigma \cdot V^t$, where $\Sigma \in \mathbb{R}^{\bar{m} \times \bar{n}}$ is diagonal, $U \in \mathbb{R}^{\bar{m} \times \bar{m}}$, $V \in \mathbb{R}^{\bar{n} \times \bar{n}}$ are orthogonal matrices, by singular value decomposition.

Proposition 1 *If $G \in P$, $\|G\| = 1$ is a parametrization and $F \in I$, $\|F\| = 1$ is the implicit equation of the same surface, then the following are true:*

- *The smallest singular value is zero.*
- *The right singular vector belonging to the smallest singular value is F.*
- *$F^{\perp}$ is spanned by the first $\bar{n} - 1$ right singular vector.*
- *The second smallest singular value is κ.*
- *The right singular vector belonging to the second smallest singular value minimizes the function $H \mapsto ev(H, G)$ in the unit sphere of $F^{\perp}$.*

For an arbitrary nonzero vector $G \in P$, $\|G\| = 1$ we define the condition number K as the reciprocal of the *formally second smallest* singular value of M_G. With "formally second smallest singular value", we mean that we take multiplicities into account. For instance, if 0 is a multiple singular value, then the condition number is infinity. Note that the condition number K depends not just on G, but also on the integer m.

Remark 1 To compute the condition number of an implicitization problem, the implicit equation of the parametrically given surface does not need to be computed. Computation of the formally second smallest singular value is easier than the computation of the implicit equation, at least numerically, because the singular value is numerically stable, whereas the implicitization problem can be very bad conditioned. The last step in the condition number computation, taking inverse, can also be very bad conditioned, when the singular value is small.

Here is an algorithm to compute the condition number.

Algorithm "Condition Number"

Input: A quadruple of polynomials $G = (G_1, \ldots, G_4)$ of total degree n in the parameters y_1, y_2, y_3, such that $\|G\| = 1$, and an $m \in \mathbb{Z}$.
Output: Condition number of the implicitization problem.

(1) Initialize M_G by an empty matrix.
For each b_i in the basis B_I of I, $i = 1, \ldots, \bar{n}$
  (a) substitute G into b_i,
  (b) expand the result in the basis B_R of R
  (c) append the column to M_G
  (Now we constructed the matrix M_G)
(2) Compute the singular value decomposition of the matrix M_G.
(3) $1/\sigma_{\bar{n}-1}$ is the condition number, where $\bar{n} = \frac{(m+1)(m+2)(m+3)}{6}$

2.3 An example

As we originally started this research with the error analysis of a particular implicitization method for cubic surfaces, and we already had experimented with some implicitization tools (Berry and Patterson, 2001; Dokken et al., 2001; Zheng et al., 2003) for this class of surfaces, we decided to take cubic surfaces as test examples. This class of algebraic surfaces admit both parametric and implicit representation (with the exception of a cone over an elliptic cubic curve). In the following example we compute the implicit condition number of a cubic surface given in parametric form. The computation is done using monomial basis.

The example we chose consist of a quadruple of cubics through the points as in the table below. These *base points* determine the cubics up to a linear change of coordinates. (We do not write out the cubic polynomials here because of space reason.) It is well–known that four cubics through six base points parameterize in general a cubic surface, see Sederberg (1990). Hence we have $n = m = 3$.

base points:	$[1, 0, 0], [0, 1, 0], [0, 0, 1], [1, 1, 1], [1, 2, 3], [2, -1, 1]$
$\kappa = \sigma_{19}$:	$0.29202 \cdot 10^{-1}$
condition number:	34.24411

Let $b_1, \ldots, b_{\bar{n}}$ denote the basis of I, and $\bar{b}_1, \ldots, \bar{b}_{\bar{m}}$ the basis of R, where $\bar{n} = 20, \bar{m} = 55$. Furthermore let $G_1, \ldots, G_4$ denote the cubics through the base points mentioned above. To compute the i -th element of the j -th column of M_G, substitute $G_1, \ldots, G_4$ into b_j, and take the coefficient with respect to $\bar{b}_i$. In the singular value decomposition of M_G we get the following singular values:

$$0.34594, 0.32343, 0.26387, 0.23315, 0.20552, 0.18776, 0.18361,$$
$$0.17378, 0.17310, 0.12436, 0.11745, 0.10767, 0.082498, 0.078262,$$
$$0.056372, 0.053370, 0.042189, 0.032242, 0.029202, 0.18272 \cdot 10^{-10}$$

The smallest but one singular value gives κ, its inverse 34.24411 provides the condition number, which is small for the class of cubic surfaces. (The last singular value is zero, the result above is due to numerical errors in the singular value decomposition.)

3 Error analysis based on the condition number

The condition number can be used to give an upper bound for the error in the computed implicit equation. We show, if there are two parametrizations close to each other, then the difference between the computed implicit equations can be estimated by the condition number.

Theorem 2 *Let G_1 be a quadruple of homogenous polynomials of parametric degree n in y_1, y_2, y_3, with $\|G_1\| = 1$. Let $F_1 \in I$ be a homogenous polynomial of degree m in the variables $x_1, \ldots, x_4$ with $\|F_1\| = 1$, such that $\|ev(F_1, G_1)\| \leq \epsilon_1$. Then for any parametrization G_2 and implicitization F_2, where $\|G_2\| = 1$, $\|F_2\| = 1$ and $\|ev(F_2, G_2)\| \leq \epsilon_1$, with $\|G_1 - G_2\| \leq \epsilon_2$, we have one of the following*

$$\|F_1 - F_2\| \leq K \cdot c_{m,n} \cdot \max\{\epsilon_1, \epsilon_2\}$$

$$\|F_1 + F_2\| \leq K \cdot c_{m,n} \cdot \max\{\epsilon_1, \epsilon_2\}$$

where K is the condition number of G_1 and $c_{m,n}$ is a constant.

Proof:

Let F_3 be such that $\|F_3\| = 1$ and $\|ev(F_3, G_1)\|$ is minimal. It follows that $\|ev(F_3, G_1)\| \leq \|ev(F_1, G_1)\| \leq \epsilon_1$

Let

$$R_1 := F_1 - \lambda_1 F_3,$$

$$R_2 := \lambda_2 F_3 - F_2,$$

where $\lambda_1 := \langle F_3, F_1 \rangle, \lambda_2 := \langle F_3, F_2 \rangle$. Then $R_1, R_2 \in F_3^{\perp}$.

From the definition of κ we have

$$\|ev(R_i, G_1)\| \geq \kappa \cdot \|R_i\|$$

$$\|R_i\| \leq \|ev(R_i, G_1)\| / \kappa \tag{1}$$

for $i = 1, 2$.

To estimate $\|R_1\|$ we write:

$$\begin{aligned} \|R_1\| &= K \cdot \|ev(R_1, G_1)\| = K \cdot \|ev(F_1 - \lambda_1 F_3, G_1)\| \\ &\leq K \cdot (\|ev(F_1, G_1)\| + \|ev(\lambda_1 F_3, G_1)\|) \\ &\leq K \cdot (1 + \lambda_1)\epsilon_1, \end{aligned} \tag{2}$$

Let $\mu : \mathbb{R}^{4\frac{(n+1)(n+2)}{2}} \to \mathbb{R}^{\bar{n}\times\bar{m}}$, $G \mapsto M_G$. The map μ is differentiable.

$$\begin{aligned}\|ev(F_2, G_1) - ev(F_2, G_2)\| &\leq \|M_{G_1} - M_{G_2}\| \cdot \|F_2\| \\ &= \|M_{G_1} - M_{G_2}\| \\ &= \|\mu(G_1) - \mu(G_2)\| \\ &\leq \|Jac(\mu)(G)\| \cdot \|G_1 - G_2\| \\ &\leq \bar{c}_{m,n} \cdot \|G_1 - G_2\|,\end{aligned}$$

Here $\bar{c}_{m,n} = \max_{\|G\|=1} \|Jac(\mu)(G)\|$

To estimate $\|R_2\|$ we write:

$$\begin{aligned}\|R_2\| &= K \cdot \|ev(R_2, G_1)\| = K \cdot \|ev(\lambda_2 F_3 - F_2, G_1)\| \\ &\leq K \cdot (\|ev(\lambda_2 F_3, G_1)\| + \|ev(F_2, G_1)\|) \\ &\leq K \cdot (\lambda_2 \epsilon_1 + \|ev(F_2, G_1) - ev(F_2, G_2)\| + \|ev(F_2, G_2)\|) \\ &\leq K \cdot (\lambda_2 \epsilon_1 + \bar{c}_{m,n} \cdot \epsilon_2 + \epsilon_1),\end{aligned} \tag{3}$$

Let $c_{m,n} = 4(4 + \bar{c}_{m,n})$. We distinguish two cases.

Case1: $\|\lambda_1\|, \|\lambda_2\| \geq 1/2$

Combining (1), (2), (3) we get the following:

$$\begin{aligned}\|F_1 - F_2\| &\leq \|F_1 - F_3\| + \|F_3 - F_2\| \\ &\leq \|R_1\|/\lambda_1 + \|R_2\|/\lambda_2 \\ &\leq \|ev(R_1, G_1)\|/\lambda_1\kappa + \|ev(R_2, G_1)\|/\lambda_2\kappa \\ &\leq (\lambda_1 + 1)\epsilon_1/\lambda_1\kappa + ((1 + \lambda_2)\epsilon_1 + \bar{c}_{m,n} \cdot \epsilon_2)/\lambda_2\kappa \\ &\leq 2K \cdot ((\lambda_1 + 1)\epsilon_1 + (1 + \lambda_2)\epsilon_1 + \bar{c}_{m,n} \cdot \epsilon_2) \\ &\leq 2K \cdot ((2 + \lambda_1 + \lambda_2) \cdot \max\{\epsilon_1, \epsilon_2\} + \bar{c}_{m,n} \cdot \max\{\epsilon_1, \epsilon_2\}) \\ &= 2K \cdot (2 + \lambda_1 + \lambda_2 + \bar{c}_{m,n}) \cdot \max\{\epsilon_1, \epsilon_2\} \\ &\leq K \cdot 2(4 + \bar{c}_{m,n}) \cdot \max\{\epsilon_1, \epsilon_2\} \\ &\leq K \cdot c_{m,n} \cdot \max\{\epsilon_1, \epsilon_2\},\end{aligned}$$

Case2: One of the λ_i is less than $1/2$.

Case2.1: Assume $\lambda_1 < 1/2$.

Then $R_1 > \sqrt{3}/2$. Combining it with (1), (2) we have:

$\sqrt{3} < \|R_1\| \leq 2K(1+\lambda_1)\epsilon_1 \leq 4K\epsilon_1$

$$\begin{aligned}\|F_1 - F_2\| &\leq 2 \\ &< 2\sqrt{3} \\ &< 4\|R_1\| \\ &< 4K(1+\lambda_1)\epsilon_1 \\ &< 8K\epsilon_1 \\ &< 2K \cdot (4+\bar{c}_{m,n}) \cdot \epsilon_1 \\ &< 2K \cdot (4+\bar{c}_{m,n}) \cdot \max\{\epsilon_1, \epsilon_2\} \\ &< K \cdot c_{m,n} \cdot \max\{\epsilon_1, \epsilon_2\}\end{aligned}$$

Case2.2: Assume $\lambda_2 < 1/2$.

Then $R_2 > \sqrt{3}/2$. Combining it with (1), (3) we have:

$\sqrt{3} < 2K((1+\lambda_2)\epsilon_1 + \bar{c}_{m,n}\epsilon_2) < 2K(2\epsilon_1 + \bar{c}_{m,n}\epsilon_2)$

$$\begin{aligned}\|F_1 - F_2\| &\leq 2 \\ &< 2\sqrt{3} \\ &< 4K(2\epsilon_1 + \bar{c}_{m,n}\epsilon_2) \\ &< 4K(2\epsilon_1 + \bar{c}_{m,n}\epsilon_2) + 8K\epsilon_1 \\ &< 4K \cdot (4+\bar{c}_{m,n}) \cdot \max\{\epsilon_1, \epsilon_2\} \\ &< K \cdot c_{m,n} \cdot \max\{\epsilon_1, \epsilon_2\}\end{aligned}$$

□

Remark 2 The computation of the constant $c_{n,m}$ is cumbersome and very technical, but it can be computed for each parametric degree n and implicit degree m. An upper estimate gives $c_{n,m} \leq n^2 \cdot (m!)^3$. If $\max\{\epsilon_1, \epsilon_2\}$ is small enough we can use first order approximation and the constant $c_{m,n}$ becomes smaller by a factor of 4.

Example We continue our example from the previous section. The theorem above allows to give a stability test of various implicitization techniques. The output error can be computed by applying the technique to a slightly perturbed input. If it is bigger than the upper bound in Theorem 2, then the use of the technique is responsible for the output error, and therefore the stability test rejects the technique for the given input. If it is smaller, than we cannot say anything (hence this test is only able to reject unstable techniques, but it cannot prove that a certain technique is stable).

For $n = m = 3$ we have $c_{3,3} = 1.8$. We compare the numerical stability of the following methods:

M1: Implicitization technique due to Berry and Patterson (2001)
M2: Moving planes method using Gauss elimination (Zheng et al., 2003)
M3: Dokken's method using SVD (Dokken et al., 2001)

input error:	$0.80000 \cdot 10^{-9}$
error bound:	$0.10255 \cdot 10^{-6}$
output error using different algorithms:	
M1:	$0.11371 \cdot 10^{-8}$
M2:	$0.42242 \cdot 10^{-8}$
M3:	$0.14239 \cdot 10^{-8}$

After introducing some error in the coefficients of the parametric form, the output error using all three methods is smaller than the worst case bound in Theorem 2. Thus in this example all three methods are accepted. We should point out that the test does not allow to rank the three methods, because the example is not statistically significant.

In the introduction we claimed that the condition number K is not a condition number of a particular algorithm, but a condition number of the implicitization problem. To justify this statement, we need to show that big output errors do arise when the condition number is big. Here is the precise statement.

Theorem 3 *Let G_1 be as in the previous theorem, and $F_1 \in I$ with $\|F_1\| = 1$, such that $\|ev(F_1, G_1)\| \leq \epsilon_1$. Then there exists a parametrization G_2, $\|G_1 - G_2\| \leq \epsilon_2$, and $F_2 \in I$ with $\|F_2\| = 1$, such that $\|ev(F_2, G_2)\| \leq \epsilon_1$, and*

$$\|F_1 - F_2\| \geq \frac{1}{2} \cdot \epsilon_1 \cdot K$$

Proof:

Let M_{G_1} the matrix belonging to G_1 in the matrix- vector decomposition. We choose F_1 as the right singular vector belonging to the smallest singular value of the matrix M_{G_1}. Then $\|F_1\| = 1$, and $\|ev(F_1, G_1)\| = \sigma_{\bar{m}}$, where $\sigma_{\bar{m}}$ is the smallest singular value of M_{G_1}.

Let $G_2 := G_1$. Then we have $\|G_1 - G_2\| \leq \epsilon_2$ for any $\epsilon_2 > 0$. (This is the reason why ϵ_2 does not appear in the bound for the output error.) Let $F_2 := F_1 + \delta \cdot R$, where R is the right singular vector corresponding to the smallest but one singular value of the matrix M_{G_1}. We assume that $\epsilon_1 \geq 2 \cdot \sigma_r$, and choose $\delta = \frac{1}{2} \cdot \epsilon_1 \cdot K$.

Then

$$\begin{aligned}\|ev(F_2,G_2)\| = \|ev(F_2,G_1)\| &= \sqrt{\|ev(F_1,G_1)\|^2 + \delta^2 \cdot \|ev(R,G_1)\|^2} \\ &\le \sqrt{\frac{\epsilon_1^2}{4} + \frac{1}{4}\cdot\epsilon_1^2\cdot K^2\cdot\kappa^2} \\ &= \sqrt{\epsilon_1^2(\frac{1}{4}+\frac{1}{4})} \\ &= \epsilon_1\cdot\frac{\sqrt{2}}{2}\end{aligned}$$

The $\|ev(F_2,G_2)\| \le \epsilon_1$ requirement is fulfilled.

By the choice of δ above we have $F_2 = F_1 + \frac{1}{2}\cdot\epsilon_1\cdot K\cdot R$. From this it follows that

$$\|F_1 - F_2\| \ge \frac{1}{2}\cdot\epsilon_1\cdot K$$

□

4 Observations and remarks on the condition number

In most of our testing examples the condition number was between 1 and 100, in less examples between 100 and 500, and in some cases over 500. The best conditioned example we have found had condition number 18.137085.

In our test examples the cubic surfaces were given by six base points. Choosing different basis for the linear system passing through the given base points we got different condition numbers. If two basis differed only by an orthogonal linear transformation, the condition numbers only slightly changed. Changing the basis by a nonorthogonal transformation resulted in a noticeable change in the condition number. More precisely it seems that our condition number gets multiplied by a factor which is proportional to the square of the condition number of the nonorthogonal transformation.

The question how the geometry of the base points effect the condition number seems more difficult. This is a topic of future research. Maybe the methods in Castro et al. (2002) are useful for this investigation.

Due to our observations singularities do not effect the stability of the implicitization, i.e. surfaces with singularities do not have big condition number. To illustrate this behavior, we show an example.

Example In Table 1 we show two surfaces. The first one is a singular one as three of the base points are on a line. In Table 1 (b) we can see a modified example. One of the base points is slightly moved so that no three points are on a line. As the condition number shows, this problem has nearly the same stability as the previous one. The singular point does not destroy the stability of the problem.

(a) singular case

base points	$(1,1,1),(2,2,1),(3,3,1),(-1,1,1),(-1,-2,1),(2,-2,1)$
$\kappa=\sigma_{19}$:	$0.28113 \cdot 10^{-1}$
K:	35.57099

(b) nonsingular case

base points	$(1,1,1),(2,2.01,1),(3,3,1),(-1,1,1),(-1,-2,1),(2,-2,1)$
$\kappa=\sigma_{19}$:	$0.30226 \cdot 10^{-1}$
K:	33.08406

Table 1
Comparing singular and nonsingular examples

The following example shows that being nonsingular is not a sufficient criterion to have a well-conditioned problem.

Example

As the base points are generic, i.e. no three are on a line, not all of them lie on a conic,

$$[1,0,0],[5,4,1],[9,-1,1],[12,2,1],[-4,5,1],[-8,-4,1],$$

the surface is nonsingular. One would expect numerical stability, however, we get quite big condition number, $K = 9638.33951$, i.e. the implicitization problem is not stable in this case. There are small perturbations of the input which lead to big changes in the output. (The authors have not found any explanation yet.)

4.1 Numerical degree guessing

The condition number K depends on the integer m, which is the estimate of the implicit degree of the surface. In this section we describe a way of obtaining information that allows more accurate guessing of the degree.

Let G be an element of P. We compute the singular values of the matrix $M_{G,m}$ for $m = 1, 2, \ldots$ until the last singular value is sufficiently small. Then we know that for this value of m there is an equation F such that $ev(F, G)$ is small (namely the right singular vector belonging to the smallest singular value). We distinguish two cases.

First case: The second smallest singular value is big. In this case the implicitization problem is well conditioned, and we can say that the equation F above is the solution of the implicitization problem.

Second case: The second smallest singular value is small. The implicitization problem is ill conditioned. There are at least two equations F_1, F_2 for which the residuum is small. There are two possible explanations. Either G is numerically close to a parametrization of a curve (namely the intersection of F_1 and F_2). Or some small perturbations of F_1 and F_2 have a common factor. This common factor F_3 has degree smaller than m, and $ev(F_3, G)$ is small, therefore this case should have been noticed before.

(a) first case

degree	$\sigma_{\bar{n}}$	$\sigma_{\bar{n}-1}$
1	big	big
2	big	big
$\vdots$	$\vdots$	$\vdots$
$m-1$	big	big
m	small	big
$\vdots$	$\vdots$	$\vdots$

(b) second case

degree	$\sigma_{\bar{n}}$	$\sigma_{\bar{n}-1}$
1	big	big
2	big	big
$\vdots$	$\vdots$	$\vdots$
$m-1$	big	big
m	small	small
$\vdots$	$\vdots$	$\vdots$

Example Let G be the following quadruple:

$$\begin{aligned}
p_1 &= 0.33014\, y_1^2 y_2 + 0.11889\, y_1^2 y_3 - 0.62851\, y_1 y_2 y_3 + 0.23483\, y_1 y_2^2 \\
&\quad - 0.53898\, y_3^2 y_1 + 0.33994\, y_2^2 y_3 + 0.14368\, y_2 y_3^2 \\
p_2 &= 0.0091976\, y_1^2 y_2 + 0.17598\, y_1^2 y_3 + 0.17918\, y_1 y_2 y_3 - 0.58353\, y_1 y_2^2 \\
&\quad - 0.090543\, y_3^2 y_1 + 0.67456\, y_2^2 y_3 - 0.36484\, y_2 y_3^2 \\
p_3 &= -0.071139\, y_1^2 y_2 + 0.61051\, y_1^2 y_3 + 0.22714\, y_1 y_2 y_3 - 0.32573\, y_1 y_2^2 \\
&\quad - 0.44603\, y_3^2 y_1 - 0.36177\, y_2^2 y_3 + 0.36702\, y_2 y_3^2 \\
p_4 &= 0.39218\, y_1^2 y_2 + 0.18953\, y_1^2 y_3 - 0.49534\, y_1 y_2 y_3 - 0.47488\, y_1 y_2^2 \\
&\quad + 0.53353\, y_3^2 y_1 - 0.22121\, y_2^2 y_3 + 0.076182\, y_2 y_3^2
\end{aligned}$$

The table below shows the last two singular values computed for the corresponding implicit degree m.

m	σ_r		σ_{r-1}	
1	0.5	(big)	0.5	(big)
2	$0.87627 \cdot 10^{-1}$	(big)	0.12182	(big)
3	$0.33475 \cdot 10^{-6}$	(small)	$0.29202 \cdot 10^{-1}$	(big)

Hence $m = 3$ is the correct degree.

The right singular vector belonging to the smallest singular value gives the implicit surface.

$$
\begin{aligned}
F = {} & 0.12037\, x_1^3 + 0.234111\, x_1^2 x_2 - 0.28728\, x_1^2 x_3 + 0.21993\, x_1^2 x_4 - 0.33179\, x_1 x_2^2 \\
& - 0.287875\, x_1 x_2 x_3 + 0.26653\, x_1 x_2 x_4 + 0.16530\, x_1 x_3^2 - 0.04950\, x_1 x_3 x_4 \\
& - 0.12789\, x_1 x_4^2 - 0.028536\, x_2^3 - 0.37570\, x_2^2 x_3 + 0.12459\, x_2^2 x_4 \\
& - 0.20386\, x_2 x_3^2 - 0.36427\, x_2 x_3 x_4 + 0.32183\, x_2 x_4^2 + 0.086380\, x_3^3 \\
& + 0.036834\, x_3^2 x_4 + 0.13637\, x_3 x_4^2 - 0.16410\, x_4^3
\end{aligned}
$$

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