Convergence Rate Analysis of a Derivative Free Landweber Iteration for Parameter Identification in Certain Elliptic PDEs

Philipp Kügler [1]

Abstract

We consider the nonlinear inverse problem of identifying a parameter from knowledge of the physical state in an elliptic partial differential equation. For a derivative free Landweber method, convergence rates are proven under a weak source condition not involving the standard Fréchet derivative of the nonlinear parameter-to-output map. This source condition is discussed both for the estimation of state- and space-dependent parameters in higher dimensions. Finally, numerical results are presented.

1 Introduction

The problem of identifying a parameter $q \in Q \subset X$ from knowledge of the physical state $z \in Y$ in an elliptic partial differential equation can be formulated as an operator equation

$$F(q) = z. \tag{1.1}$$

Here, the nonlinear *parameter-to-output map* F maps the parameter $q \in Q \subset X$ onto the (unique) solution $u_q \in Y$ of a possibly nonlinear elliptic state equation. Examples are the estimation of q in

$$-\nabla \cdot (q(x)\nabla z) + b(z) = f \quad \text{in} \;\; \Omega \subset \mathbb{R}^d,$$

($d = 1, 2$ or 3 throughout this paper), or

$$-\nabla(q(z)\nabla z) = f \quad \text{in} \;\; \Omega \subset \mathbb{R}^d \tag{1.2}$$

(both with appropriate boundary conditions on z) for given $z \in Y$, i.e., the goal is to find a parameter q_* such that $u_{q_*} = z$ holds. Considering the direct problem, i.e., the partial differential equation, in its variational formulation, allows to define F as an operator acting between two appropriately chosen Hilbert spaces X and Y.

The inverse problem (1.1) is typically ill-posed, i.e., its solution q_* (assumed to exist,

[1] Institut für Industriemathematik, Johannes Kepler Universität, A–4040 Linz, Austria. E-Mail: kuegler@indmath.uni-linz.ac.at. This work was supported by the Austrian National Science Foundation FWF under grant SFB F013 project F1308.

but not necessarily unique) does not depend continuously on the data z. This is especially crucial if the data z are not known exactly and only a rough approximation z^δ with

$$\|z - z^\delta\| \leq \delta \tag{1.3}$$

is given, where δ denotes the noise level. Then, a numerically stable and reliable approximation of q_* can merely be obtained by the use of regularization techniques, see [4], [18], [2] for a general overview. They can be roughly divided into methods related to Tikhonov regularization, where the approximate solution is sought as the minimizer of

$$\|z^\delta - F(q)\|^2 + \beta\|q - q_0\|^2, \tag{1.4}$$

with $\beta > 0$, see [5], and iterative methods based on solving the normal equation

$$F'(q)^*(z^\delta - F(q)) = 0$$

via successive iteration starting from an initial guess q_0, see [7], such as the Landweber method

$$q_{k+1}^\delta = q_k^\delta + \lambda F'(q_k^\delta)^*(z^\delta - F(q_k^\delta)) - \beta_k(q_k^\delta - q_0), \tag{1.5}$$

see [10], [21]. Here, $F'(\cdot)^*$ denotes the Hilbert space adjoint operator to the Fréchet derivative $F'(\cdot)$, β_k is a given sequence of non-negative parameters and λ is scaling parameter. The stabilizing effect in (1.4) is due to the penalty term multiplied by a positive regularization parameter β, in iterative methods stability is obtained by stopping the iteration "at the right time". According to the popular discrepancy principle the stopping index $k_*(\delta)$ is determined by

$$\|z^\delta - F(q_{k_*}^\delta)\| \leq \tau\delta < \|z^\delta - F(q_k^\delta)\|, \quad 0 \leq k < k_*, \tag{1.6}$$

for some sufficiently large $\tau > 0$. In any case, regularization always means to balance between stability and accuracy, for further rules for the choice of the regularization parameter or the stopping index we refer to [18], [7].

The label *regularization method* is only awarded to a solution technique for an ill-posed problem if besides of stability also convergence of the approximations towards q_* can be guaranteed as the noise level δ and the regularization parameter, i.e., $1/k_*(\delta)$ in case of an iterative method, tend to zero. Since the convergence can in general be arbitrarily slow, see [24], rate estimates are of special interest in the analysis of a regularization method. Such estimates could so far only be obtained under source-conditions of the form

$$\exists w \in Y, \quad q_* - q_0 = \lambda F'(q_*)^* w, \tag{1.7}$$

see [4], [2], [7], i.e., with the Fréchet derivative of the nonlinear operator F playing a central role. In the context of parameter identification, the meaning of source conditions has - if at all - only been discussed for one-dimensional linear direct problems (see [4],

[12], [21]), then usually requiring some additional smoothness and prescribed boundary behaviour for $q_* - q_0$.

However, in order to obtain convergence rates, these source conditions are not sufficient. Beneath smallness assumptions on the source function w, the theories require either range conditions on the Fréchet derivative of the parameter-to-output map F that are hard to verify, as in [10], [3], or even the Lipschitz continuity of F' with a sufficiently small Lipschitz constant, as in [21], [5].

In [13], we introduced a *derivative free Landweber method*

$$q_{k+1}^\delta = q_k^\delta + \lambda L(q_k^\delta)^*(z^\delta - F(q_k^\delta)) \tag{1.8}$$

for solving the parameter identification problem (1.1), where the iteration operator $L(\cdot)$ is directly coupled to the operator describing the underlying PDE, see Section 3. The regularizing properties of (1.8) could simply be derived from assumptions already needed for the unique solvability of the underlying partial differential equation. That way, the Fréchet differentiability of the parameter-to-output map as well as conditions restricting its nonlinearity were no longer required. Using an additionally stabilizing term in (1.8), we continue the convergence analysis in this paper and provide rate estimates under the weak source condition

$$\exists w \in Y, \quad q_* - q_0 = \lambda L(q_*)^* w, \tag{1.9}$$

motivated by the work presented in [8] and [16]. Neither for (1.9) nor at any other part of our theory the Fréchet derivative of F is needed, especially, standard Lipschitz and range invariance conditions on F' become redundant.

The paper is organized as follows. Section 2 contains the abstract formulation of the underlying partial differential equation based on which the derivative free Landweber method is defined in Section 3. The convergence rates results are given in Section 4 followed by a discussion of the weak source condition in Section 5. Finally, Section 6 contains numerical results obtained by application of our method for identifying a nonlinearity of the type $q = q(|\nabla z|)$.

2 The Direct Problem

Let Y_0 be a closed (not necessarily strict) subspace of Y. Both for the Hilbert spaces X and Y, the inner products and norms are denoted by $(\cdot,\cdot)$ and $\|\cdot\|$, their meaning can always be identified from the context in which they appear. Furthermore, let Y_0^* be the dual space to Y_0, equipped with the duality product $\langle\cdot,\cdot\rangle$ and the duality map $J : Y_0^* \to Y_0$.

Given a parameter q out of an admissible set $Q \subset X$, the direct problem consists in solving the abstract elliptic state equation

$$C(q)u = f \quad \text{in } Y_0^*, \tag{2.1}$$

for which we shall assume

Assumption 1. *Let $Q \subset X$ be a set of admissible parameters. For $q \in Q$ the operator $C(q)$ maps from Y_0 into the dual space Y_0^*, i.e.*

$$C(q) : Y_0 \to Y_0^*.$$

Furthermore, there exist positive constants α_1 and α_2 such that

$$\alpha_1 \|v - w\|^2 \leq \langle C(q)v - C(q)w, v - w\rangle \quad v, w \in Y_0 \tag{2.2}$$

and

$$\langle C(q)v - C(q)w, y\rangle \leq \alpha_2 \|v - w\| \|y\| \quad v, w, y \in Y \tag{2.3}$$

hold for all $q \in Q$.

Under Assumption 1, which states the strict monotonicity and the Lipschitz continuity of $C(q)$, the direct problem (2.1) is uniquely solvable for $f \in Y_0^*$ and any $q \in Q$, see, e.g., [25]. In order to emphasis the dependence on the parameter, the solution is denoted by $u_q \in Y_0$.

Already with respect to the parameter identification problem, we also assume

Assumption 2. *For all $p \in X$ and $u \in Y_0$ the operator $C(p)$ satisfies*

$$C(p) = B + A(p) \tag{2.4}$$

with

$$A(\cdot)u \in \mathcal{L}(X, Y_0^*) \tag{2.5}$$

and a parameter independent, possibly nonlinear operator B acting from Y_0 to Y_0^*.

Hence, on the one hand the parameter q shall appear linearly in the direct problem (2.1). On the other hand, we also require $A(p)u \in Y_0^*$ not only for $p \in Q$ - which would already be given by Assumption 1 - but also for $p \in X$. Note that $C(p) : Y_0 \to Y_0^*$ still only has to be invertible if $p \in Q$. Furthermore, we emphasize that despite of (2.5), the nonlinearity of (2.1) may be due to the unknown parameter q itself.

Simple examples of partial differential equations that can be treated in this abstract framework with $Y = H^1(\Omega)$ and $Y_0 = H_0^1(\Omega)$ are (see [13] for the verification of Assumptions 1 and 2)

Example 1.

$$\begin{aligned}-\Delta u + q(u) &= f \quad \textit{in } \Omega,\\ u &= 0 \quad \textit{on } \partial\Omega,\end{aligned}$$

with

$$\begin{aligned}\langle A(q)u, v\rangle &= \int_\Omega q(u)v\, dx, \\ \langle Bu, v\rangle &= \int_\Omega \nabla u \nabla v\, dx,\end{aligned} \tag{2.6}$$

$X = H^1(I)$ *for an appropriate real interval* I*, and*

$$Q = \left\{q \in X \mid \underline{\gamma} \le q' \le \bar{\gamma}\right\},$$

or

Example 2.

$$\begin{aligned}-\nabla(q(x)\nabla u) + b(u) &= f \quad \textit{in } \Omega,\\ u &= 0 \quad \textit{on } \partial\Omega,\end{aligned}$$

with

$$\begin{aligned}\langle A(q)u, v\rangle &= \int_\Omega q(x)\nabla u \nabla v\, dx, \\ \langle Bu, v\rangle &= \int_\Omega b(u)v\, dx,\end{aligned}$$

$X \subset L^\infty(\Omega)$ *and*

$$Q = \left\{q \in X \mid \underline{\gamma} \le q \le \bar{\gamma}\right\}.$$

A priori, there are no restrictions on the type of nonlinearity, i.e., functions in (2.1) may depend on x, and u, ∇u, $|\nabla u|$, ... Furthermore, our assumptions are not limited to second order pdes, see [15], and the unknown parameter may also appear in higher order terms of the pde-operator. Besides, neither the Dirichlet-type nor the homogeneity of the boundary condition are essential for our theory.

In case that Assumption 1 cannot be verified we may replace it by

Assumption 3. *Let* $C(q)$ *be defined as in Assumption 1 and let* $\tilde{Y}$ *be a Hilbert space with* $Y \subseteq \tilde{Y}$*. There exist positive constants* α_1*,* α_2 *and a linear operator* $S : \tilde{Y} \to Y_0$ *such that*

$$\alpha_1 \|v - w\|_{\tilde{Y}}^2 \le \langle C(q)v - C(q)w, S(v - w)\rangle \quad v, w \in Y_0,$$

and

$$\langle C(q)v - C(q)w, Sy\rangle \le \alpha_2 \|v - w\|_{\tilde{Y}} \|y\|_{\tilde{Y}} \quad v, w, y \in Y$$

for all $q \in Q$*.*

instead. Assumption 3 is satisfied by Example (1.2), $u|_{\partial\Omega} = 0$, with

$$\langle A(q)u, v\rangle = \int_\Omega q(u)\nabla u \nabla v \, dx,$$

$B = 0$, $S = -\Delta^{-1}$, $\tilde{Y} = L^2(\Omega)$, $Y = H^1(\Omega)$, $Y_0 = H^1_0(\Omega)$ and

$$Q = \left\{ q \in H^1(I) \mid \underline{\gamma} \le q \le \bar{\gamma} \right\},$$

see [13].

3 The Derivative Free Landweber Method

Assumption 2 allows to construct the linear operator $L(q) : X \to Y_0$ for $q \in Q \subset X$ by

$$L(q)p = -JA(p)u_q \tag{3.1}$$

and to introduce the derivative free Landweber method

$$q^\delta_{k+1} = q^\delta_k + \lambda L(q^\delta_k)^*(z^\delta - F(q^\delta_k)) - \beta_k(q^\delta_k - q_0). \tag{3.2}$$

The iteration operator (3.1) is in fact motivated by ideas presented in [8] and [16], where convergence rates for Tikhonov regularization have been proven under a derivative free source condition by means of certain parameter identification problems. A generalization of these results led to the abstract formulation (1.9), finally suggesting the iteration operator (3.1) used in (1.8). In this paper, the latter is enhanced by the additionally stabilizing term involving β_k for mainly proof techniqual reasons, also used in [21], [1], [3].

For Example 1, (3.2) translates into

$$(q^\delta_{k+1}, p) = (q^\delta_k, p) - \lambda \int_\Omega p(u_{q^\delta_k}) \cdot (z^\delta - u_{q^\delta_k}) \, dx - \beta_k(q^\delta_k - q_0, p),$$

for Example 2, one obtains

$$(q^\delta_{k+1}, p) = (q^\delta_k, p) - \lambda \int_\Omega p(x)\nabla u_{q^\delta_k} \cdot \nabla(z^\delta - u_{q^\delta_k}) \, dx - \beta_k(q^\delta_k - q_0, p),$$

where $p \in X$ is a test function. In general, derivatives of the parameter or of known functions in B with respect to the solution u are no longer required by our method since $F(\cdot)'$ is not involved. With numerical differentiation being an ill-posed problem, see [4], this positively affects the stabilizing properties of (3.2).

As in the case of the classical Landweber iteration (and all the variants discussed in the literature), the method (3.2) can only converge if the iteration operator $L(\cdot)^*$ is (locally) uniformly bounded and the scaling parameter λ is properly chosen. Hence, we assume for our analysis (see [13] for a verification for our examples) that

$$\|L(q)\| \leq \hat{L}, \quad q \in \mathcal{B}_\rho(q_0) \tag{3.3}$$

for a ball $\mathcal{B}_\rho(q_0)$ of radius ρ around q_0 satisfying

$$\mathcal{B}_\rho(q_0) \subset \mathcal{D}(F).$$

Assumptions 1 and 2 yield that

$$\begin{aligned} \alpha_1 \|u_q - u_{\tilde{q}}\|^2 &\leq \langle C(\tilde{q})u_q - C(\tilde{q})u_{\tilde{q}}, u_q - u_{\tilde{q}}\rangle \\ &= \langle A(\tilde{q})u_q - A(q)u_q, u_q - u_{\tilde{q}}\rangle \\ &= (L(q)(q-\tilde{q}), u_q - u_{\tilde{q}}) \end{aligned}$$

for q, $\tilde{q}$ in $\mathcal{B}_\rho(q_0)$. Hence, (3.3) can be understood as sufficient condition for the Lipschitz continuity of the parameter-to-output map F with Lipschitz constant $\hat{L}/\alpha_1$. However, condition (3.3) does not imply the Fréchet differentiability of F since the operator B is not involved.

Under Assumption 3 instead of Assumption 1, the linear operator S has to be incorporated into (3.2) leading to

$$q_{k+1}^\delta = q_k^\delta + \lambda L(q_k^\delta)^* S(z^\delta - F(q_k^\delta)) - \beta_k(q_k^\delta - q_0). \tag{3.4}$$

For Example (1.2), the iteration (3.4) reads as

$$(q_{k+1}^\delta, p) = (q_k^\delta, p) + \int_\Omega p(u_{q_k^\delta}) \nabla u_{q_k^\delta} \nabla[\Delta^{-1}(z^\delta - u_{q_k^\delta})]\, dx - \beta_k(q_k^\delta - q_0, p).$$

Then, assumption (3.3) turns into

$$\|S^* L(q)\| \leq \hat{L}, \quad q \in \mathcal{B}_\rho(q_0).$$

The convergence properties of the derivative free Landweber method have been analyzed in [13], [14]. Since convergence of any iterative regularization method for inverse and ill-posed problems, i.e., $q_k \to q_*$ for $k \to \infty$ (in case of exact data) or $q_{k_*}^\delta \to q_*$ for $\delta \to 0$ may be arbitrarily slow, see [24], convergence rates can only be obtained under additional assumptions. In all the related theories, the Fréchet differentiability of the parameter-to-output map F has so far been indispensable.

4 Convergence Rates Result

In [21], convergence rates for (1.5) were proven based on the classical source condition (1.7) as well as on the Lipschitz continuity of the Fréchet derivative with a small Lipschitz constant $\hat{c}$. For a certain decay behaviour of the strictly positive regularization parameters β_k, the rate

$$\|q^\delta_{k_*(\delta)} - q_*\| = \mathcal{O}(\sqrt{\beta_{k_*}}) \tag{4.1}$$

was obtained when stopping the iteration according to the discrepancy principle (1.6). In addition, the rate

$$\|q^\delta_{N_0(\delta)+1} - q_*\| = \mathcal{O}(\sqrt{\delta}) \tag{4.2}$$

could be guaranteed if (1.5) is stopped when

$$\frac{\delta}{\beta_k} \le \tilde{C}$$

for a positive constant $\tilde{C}$ is violated for the first time.

In this section, we prove the convergence rates (4.1) and (4.2) for the derivative free Landweber method (3.2) only using the weak source condition (1.9) and Assumptions 1 and 2 on the direct problem, i.e., $F'(\cdot)$ is not involved at all. Though we adopt the approach of [21], all estimates concerning the iteration operator now have to be done differently.

Theorem 4.1 (Convergence Rates). *Let q_* be a solution of* (1.1) *in $\mathcal{B}_{\rho/2}(q_0)$ and suppose that Assumptions 1, 2 and* (3.3) *are satisfied. Choose a sequence of decaying regularization parameters β_k such that*

$$\beta_0 \le 1/8 \tag{4.3}$$

and

$$a_k := 2 - \beta_k - \frac{1}{\beta_k} + \frac{\beta_{k+1}}{\beta_k^2} \ge \eta \tag{4.4}$$

holds for all $k \in \mathbb{N}_0$, where η is a fixed positive constant.

- *A posteriori stopping criterion: Choose the parameters λ and τ such that*

$$(\alpha_1 + 2\lambda\hat{L}^2 - \frac{7}{4}(\alpha_1 - \frac{\alpha_2}{\tau}) + \frac{\alpha_1}{\tau^2}) \le E < 0 \tag{4.5}$$

holds for a fixed negative constant E, and let the source condition (1.9) *be fulfilled. If* (3.2) *is stopped according to* (1.6) *with τ satisfying* (4.5)*, and if*

$$2\|q_* - q_0\|^2 + 2\lambda\frac{\alpha_2^2}{\alpha_1}\|w\|^2 \le \eta\frac{\rho^2}{4\beta_0} \tag{4.6}$$

holds, then

$$\|q^\delta_{k_*} - q_*\| = \mathcal{O}(\sqrt{\beta_{k_*}}). \tag{4.7}$$

- *In the case of exact data, i.e., $\delta = 0$, we set $\tau = \infty$ in (4.5) and replace (4.6) by*
$$2\|q_* - q_0\|^2 + \lambda \frac{\alpha_2^2}{\alpha_1}\|w\|^2 \le \eta \frac{\rho^2}{4\beta_0}. \tag{4.8}$$
Then, we obtain
$$\|q_k - q_*\| = \mathcal{O}(\sqrt{\beta_k})$$
for all $k \in \mathbb{N}_0$.

- *A priori stopping criterion: Choose λ such that*
$$2\lambda \hat{L}^2 - \frac{11}{16}\alpha_1 \le 0 \tag{4.9}$$
is satisfied and let the source condition (1.9) be fulfilled. If (3.2) is stopped according to
$$\frac{\delta}{\beta_k} \le \tilde{C}, \quad 0 \le k \le N_0, \quad \frac{\delta}{\beta_{N_0+1}} > \tilde{C} \tag{4.10}$$
for some positive constant $\tilde{C}$, and if
$$2\|q_* - q_0\|^2 + \frac{\lambda}{\alpha_1}\alpha_2^2(16\tilde{C}^2 + \|w\|^2) + 2\lambda\tilde{C}\alpha_2\|w\| \le \eta \frac{\rho^2}{4\beta_0} \tag{4.11}$$
holds, then
$$\|q^\delta_{N_0+1} - q_*\| = \mathcal{O}(\sqrt{\delta}). \tag{4.12}$$

Proof. We first consider the a posteriori strategy. Because of $q_* \in \mathcal{B}_{\rho/2}(q_0)$ we have
$$\frac{\|q_* - q_0\|^2}{\beta_0} \le \frac{\rho^2}{4\beta_0}$$
Arguing by induction based on an idea from [1] also used in [21], we assume that
$$\frac{\|q^\delta_k - q_*\|^2}{\beta_k} \le \frac{\rho^2}{4\beta_0}$$
holds for an index $k < k_*(\delta)$, where $k_*(\delta)$ denotes the stopping index according to (1.6). Then, the iteration step (3.2) is well-defined, yielding
$$\begin{aligned}\|q^\delta_{k+1} - q_*\|^2 \;=\; & (1-\beta_k)^2\|q^\delta_k - q_*\|^2 + \beta_k^2\|q_* - q_0\|^2 + \lambda^2\|L(q^\delta_k)^*(z^\delta - u_k)\|^2 \\ & -2\beta_k(1-\beta_k)(q^\delta_k - q_*, q_* - q_0) \\ & -2(1-\beta_k)\lambda(z^\delta - u_k, L(q^\delta_k)(q_* - q^\delta_k)) \\ & +2\beta_k\lambda(q_0 - q_*, L(q^\delta_k)^*(z^\delta - u_k))\end{aligned}$$

With (1.9) it follows that

$$\begin{aligned}\|q_{k+1}^\delta - q_*\|^2 &\le (1-\beta_k)^2\|q_k^\delta - q_*\|^2 + 2\beta_k^2\|q_* - q_0\|^2 + 2\lambda^2\|L(q_k^\delta)^*(z^\delta - u_k)\|^2 \\ &\quad -2\beta_k(1-\beta_k)\lambda(L(q_*)(q_k^\delta - q_*), w) \\ &\quad -2(1-\beta_k)\lambda(z^\delta - u_k, L(q_k^\delta)(q_* - q_k^\delta)). \qquad (4.13)\end{aligned}$$

The following lines contain the major difference to [21] and are only possible for the special iteration operator (3.1). Because of (2.5) and

$$A(q_*)z + Bz = A(q_k^\delta)u_k + Bu_k \ \text{ in } \ Y_0^*,$$

one gets

$$\begin{aligned}&-(z^\delta - u_k, L(q_k^\delta)(q_* - q_k^\delta)) \\ = \ &\langle z^\delta - u_k, A(q_* - q_k^\delta)u_k\rangle \\ = \ &\langle z^\delta - u_k, A(q_*)u_k - A(q_*)z\rangle + \langle z^\delta - u_k, Bu_k - Bz\rangle \\ = \ &\langle z^\delta - u_k, C(q_*)u_k - C(q_*)z\rangle \\ = \ &-\langle z^\delta - u_k, C(q_*)z^\delta - C(q_*)u_k\rangle + \langle z^\delta - u_k, C(q_*)z^\delta - C(q_*)z\rangle \\ \le \ &-\alpha_1\|z^\delta - u_k\|^2 + \alpha_2\|z^\delta - u_k\|\|z^\delta - z\|,\end{aligned}$$

where the inequality holds because of (2.2) and (2.3). Furthermore, (2.3) yields

$$\begin{aligned}|(L(q_*)(q_k^\delta - q_*), w)| &= |\langle A(q_k^\delta - q_*)z, w\rangle| \\ &= |\langle C(q_k^\delta)z - C(q_k^\delta)u_k, w\rangle| \\ &\le \alpha_2\|z - u_k\|\|w\| \\ &\le \alpha_2(\|z^\delta - u_k\| + \delta)\|w\|. \qquad (4.14)\end{aligned}$$

As a consequence, (4.13) yields

$$\begin{aligned}\|q_{k+1}^\delta - q_*\|^2 &\le (1-\beta_k)^2\|q_k^\delta - q_*\|^2 + 2\beta_k^2\|q_* - q_0\|^2 + 2\lambda^2\|L(q_k^\delta)^*(z^\delta - u_k)\|^2 \\ &\quad +2\beta_k(1-\beta_k)\alpha_2\lambda\|z^\delta - u_k\|\|w\| - 2\alpha_1(1-\beta_k)\lambda\|z^\delta - u_k\|^2 \\ &\quad +2(1-\beta_k)\alpha_2\delta\lambda\left(\|z^\delta - u_k\| + \beta_k\|w\|\right). \qquad (4.15)\end{aligned}$$

From (1.6) it now follows that

$$\begin{aligned}&2(1-\beta_k)\alpha_2\delta\|z^\delta - u_k\| \le \frac{2\alpha_2}{\tau}(1-\beta_k)\|z^\delta - u_k\|^2, \\ &2(1-\beta_k)\alpha_2\delta\beta_k\|w\| \le \frac{\alpha_1}{\tau^2}\|z^\delta - u_k\|^2 + (1-\beta_k)^2\beta_k^2\frac{\alpha_2^2}{\alpha_1}\|w\|^2.\end{aligned}$$

Using these estimates and

$$2\beta_k(1-\beta_k)\alpha_2\|z^\delta - u_k\|\|w\| \le \alpha_1\|z^\delta - u_k\|^2 + \beta_k^2(1-\beta_k)^2\frac{\alpha_2^2}{\alpha_1}\|w\|^2$$

in (4.15) yields

$$\begin{aligned}\|q^\delta_{k+1}-q_*\|^2 &\le (1-\beta_k)^2\|q^\delta_k-q_*\|^2+2\beta_k^2\|q_*-q_0\|^2\\ &+\lambda(\alpha_1+2\lambda\hat{L}^2-2(\alpha_1-\frac{\alpha_2}{\tau})(1-\beta_k)+\frac{\alpha_1}{\tau^2})\|(z^\delta-u_k)\|^2\\ &+2\beta_k^2(1-\beta_k)^2\frac{\alpha_2^2}{\alpha_1}\lambda\|w\|^2. \qquad (4.16)\end{aligned}$$

From (4.5) we obtain

$$\lambda(\alpha_1+2\lambda\hat{L}^2-2(\alpha_1-\frac{\alpha_2}{\tau})(1-\beta_k)+\frac{\alpha_1}{\tau^2})\le\lambda E<0.$$

Hence, using the abbreviations

$$\gamma_k:=\frac{\|q^\delta_k-q_*\|^2}{\beta_k}\ \text{ and }\ A:=2\|q_*-q_0\|^2+2\lambda\frac{\alpha_2^2}{\alpha_1}\|w\|^2,$$

we obtain from (4.16)

$$\gamma_{k+1}\le(1-\beta_k)^2\frac{\beta_k}{\beta_{k+1}}\gamma_k+\frac{\beta_k^2}{\beta_{k+1}}A:=I(\gamma_k). \qquad (4.17)$$

Since (4.6) and (4.4) yield

$$A\le\eta\frac{\rho^2}{4\beta_0}\le a_k\frac{\rho^2}{4\beta_0},$$

we obtain

$$I(\frac{\rho^2}{4\beta_0})\le\frac{\rho^2}{4\beta_0}\left((1-\beta_k)^2\frac{\beta_k}{\beta_{k+1}}+\frac{\beta_k^2}{\beta_{k+1}}a_k\right).$$

Because $I(\gamma_k)$ is monotonically increasing as a function of γ_k and because of the definition of a_k in (4.4), we finally have

$$\gamma_{k+1}\le I(\gamma_k)\le I(\frac{\rho^2}{4\beta_0})\le\frac{\rho^2}{4\beta_0}.$$

Hence, the induction is complete, which especially shows that $q^\delta_{k+1}\in\mathcal{B}_\rho(q_0)$ and

$$\|q^\delta_{k+1}-q_*\|\le\sqrt{\frac{\rho^2}{4\beta_0}\beta_{k+1}}$$

hold for $0\le k<k_*$.

In the case of exact data, i.e. $\delta=0$, it follows from (4.15) that

$$\begin{aligned}\|q_{k+1}-q_*\|^2 &\le (1-\beta_k)^2\|q_k-q_*\|^2+2\beta_k^2\|q_*-q_0\|^2+2\lambda^2\|L(q_k)^*(z-u_k)\|^2\\ &+2\beta_k(1-\beta_k)\alpha_2\lambda\|z-u_k\|\|w\|-2(1-\beta_k)\alpha_1\lambda\|z-u_k\|^2\\ &\le (1-\beta_k)^2\|q_k-q_*\|^2+\beta_k^2((2\|q_*-q_0\|^2+(1-\beta_k)^2\lambda\frac{\alpha_2^2}{\alpha_1}\|w\|^2)\\ &+\|z-u_k\|^2\lambda(\alpha_1+2\lambda\hat{L}^2-2\alpha_1(1-\beta_k)).\end{aligned}$$

Using (4.8), the further proof is analogous to that for noisy data.

Next, we turn to the a priori strategy. Starting from (4.15), we again argue by induction. Let $k \le N_0$ and assume that $\|q_k^\delta - q_*\|^2/\beta_k \le \frac{\rho^2}{4\beta_0}$. From (4.10) it follows that

$$\begin{aligned}
2(1-\beta_k)\alpha_2\delta\|z^\delta - u_k\| &\le 2\tilde{C}\alpha_2\beta_k(1-\beta_k)\|z^\delta - u_k\| \\
&\le 16\tilde{C}^2\frac{\alpha_2^2}{\alpha_1}\beta_k^2(1-\beta_k)^2 + \alpha_1\frac{\|z^\delta - u_k\|^2}{16}, \\
2(1-\beta_k)\alpha_2\delta\beta_k\|w\| &\le 2\tilde{C}(1-\beta_k)\beta_k^2\alpha_2\|w\|.
\end{aligned}$$

Using these estimates and

$$2\beta_k(1-\beta_k)\alpha_2\|z^\delta - u_k\|\|w\| \le \alpha_1\|z^\delta - u_k\|^2 + \beta_k^2(1-\beta_k)^2\frac{\alpha_2^2}{\alpha_1}\|w\|^2$$

in (4.15) yields

$$\begin{aligned}
\|q_{k+1}^\delta - q_*\|^2 \le\; & (1-\beta_k)^2\|q_k^\delta - q_*\|^2 + 2\lambda\tilde{C}(1-\beta_k)\beta_k^2\alpha_2\|w\| \\
& +\beta_k^2\left(2\|q_* - q_0\|^2 + (1-\beta_k)^2\lambda\frac{\alpha_2^2}{\alpha_1}(16\tilde{C}^2 + \|w\|^2)\right) \\
& +\lambda(\alpha_1 + 2\lambda\hat{L}^2 - 2\alpha_1(1-\beta_k) + \frac{\alpha_1}{16})\|z^\delta - u_k\|^2.
\end{aligned}$$

From (4.9) and (4.3) it follows that

$$(\alpha_1 + 2\lambda\hat{L}^2 - 2\alpha_1(1-\beta_k) + \frac{\alpha_1}{16}) \le 0.$$

Hence, using the abbreviations

$$\gamma_k := \frac{\|q_k^\delta - q_*\|^2}{\beta_k} \quad \text{and} \quad A := 2\|q_* - q_0\|^2 + \lambda\frac{\alpha_2^2}{\alpha_1}(16\tilde{C}^2 + \|w\|^2) + 2\lambda\tilde{C}\alpha_2\|w\|,$$

we can continue as in the proof for the a-posteriori strategy. $\square$

Under Assumption 3 instead of Assumption 1, convergence rates for (3.4) corresponding to Theorem 4.1 can be obtained using the weak source condition

$$\exists\, w \in \tilde{Y} \quad q_* - q_0 = \lambda L(q_*)^* S w. \tag{4.18}$$

This follows easily from

$$\begin{aligned}
|(L(q_*)(q_k^\delta - q_*), Sw)| &= |\langle A(q_k^\delta - q_*)z, Sw\rangle| \\
&= |\langle C(q_k^\delta)z - C(q_k^\delta)u_k, Sw\rangle| \\
&\le \alpha_2\|z - u_k\|_{\tilde{Y}}\|w\|_{\tilde{Y}} \\
&\le \alpha_2(\|z^\delta - u_k\|_{\tilde{Y}} + \delta)\|w\|_{\tilde{Y}},
\end{aligned}$$

compare to (4.14).

Though the assumptions of Theorem 4.1, which do not explicitly involve the abstract parameter-to-output map F or its derivative, may act as a deterrent at the first sight, they are readable and satisfiable. The proof shows that the rate estimates are mainly determined by the decay of the regularization parameters β_k, a technique originating from [1]. In [21], the choice

$$\beta_k = (k + l_0)^{-\psi} \tag{4.19}$$

is suggested with $0 < \psi < 1$ and $l_0 \in \mathbb{N}$ taken large enough. This sequence is also suited for our theory since we can show that (4.4) is satisfied by (4.19) with $\eta = a_0$. The two lines at the top in Figure 1 serve as an illustration of the fact that a_k based on (4.19) is monotonically increasing with k and bounded by 2. If η in (4.4) is bounded - it certainly

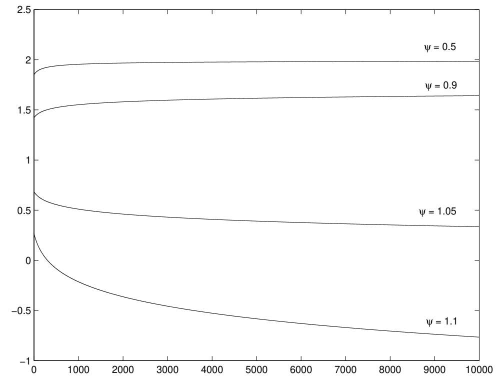

Figure 1: a_k for several choices of ψ in (4.19)

is bounded by $a_0 < 2$ for the choice (4.19) with $0 < \psi < 1$ -, the conditions (4.6), (4.8) or (4.11) represent smallness assumptions on the source function w. Since - for fixed λ - $\|q_* - q_0\|$ is bounded by $\|w\|$ via (1.9) and (3.3), these conditions are satisfied if $\|w\|$ is sufficiently small. That kind of assumption is inevitable for all available iterative methods, then often only formulated as “let $\|w\|$ be sufficiently small”.

The conditions (4.5) and (4.9) can always be satisfied by choosing λ sufficiently small and τ sufficiently large. Note that the use of a ”large” τ in the discrepancy principle (1.6) might cause a too early termination of the iteration. However, this problem is not specific to our iteration (3.2) but also appears in all the classical iteration methods [10], [21], [22]. Furthermore, our stopping rule no longer requires constants related to $F'(\cdot)$ but mainly depends on quantities associated to the direct problem.

In the next section we focus on the remaining assumption in Theorem 4.1, namely the weak source condition (1.9).

5 On the Weak Source Condition

As already mentioned, the concept of a weak source condition - the adjective weak relates to the fact that as opposed to classical source conditions the derivative of $F(q_*)$ is neglected - has been started in [8] and [16]. In [14], those ideas for Tikhonov regularization have been extended to a convergence rate theorem for (1.4) uniformly applicable to the class of inverse problems described by (1.1) and (2.1). While - in the context of parameter identification - the classical source conditions has only been discussed for some linear and one-dimensional direct problems, see [4], [12], [23], their weak counterparts have also been studied for higher dimensional and nonlinear direct problems, see [16], [14], [6], [15]. In the following we give a short summary.

Though Theorem 4.1 holds for both linear and nonlinear direct problems, a uniform interpretation of the source condition on an abstract level seems to be unrealistic such that (1.9) has to be investigated as the case arises. Nevertheless the course of action one should follow for the identification of space dependent parameters is distinguishable from that for state dependent parameters.

In the first case, (1.9) can be related to the so-called direct approach for solving (1.1): Given the solution z of the state equation (2.1), one might consider the latter in its classical formulation as an equation for the unknown parameter q. For instance, the hyperbolic PDE

$$-\nabla q(x) \cdot \nabla z - q(x)\Delta z + b(z) = f \quad \text{in} \ \ \Omega, \tag{5.1}$$

is obtained that way for Example 2. It has been shown in [14] that the weak source condition (1.9) for space dependent parameters can be understood as a solvability requirement on a PDE for the source function w whose principal part is exactly of the same type as that obtained by the direct approach. In case of Example 2, a sufficient condition for (1.9) to be satisfied - in addition to smoothness of q_* - is

$$\exists\, w \in Y \quad -\nabla w \cdot \nabla u_{q_*} = \frac{1}{\lambda}(I - \Delta)(q_* - q_0) \ \ \text{in} \ \ \Omega$$

with the same principal part as in (5.1) ($z = u_{q_*}$). Hence, for space dependent parameters the interpretation of the source condition can be linked to the analysis of a pde for the source funtion w. A full determination of w is not required since (1.9) only requires its existence.

We emphasize that even in case of exact data the direct approach is not necessarily applicable for solving the inverse problem. In [15], a parameter identification problem is studied that leads to a second order PDE for q changing its type between elliptic and hyperbolic in dependence on z. Due to the lack of a direct numerical routine, the application of a regularization technique is the only option. Another example for the failure of the direct approach is (1.2) containing a state dependent parameter. There, one obtains

$$-q(z)\Delta z - q'(z)|\nabla z|^2 = f \quad \text{in} \;\; \Omega \subset \mathbb{R}^d,$$

where it is for higher dimensions not clear how to proceed.

Regarding the identification of a state dependent parameter, an apparent link to the direct approach is no longer given since the parameter is sought as a function of a one-dimensional variable while the source function w still depends on the space variable. Hence, in analyzing the source condition one has to proceed differently. It turned out in [16], [14] that the the *co-area formula*, an integral transfomation rule, see [9], is a practical tool for that purpose. Base of operations is the variational formulation of (1.9), i.e.,

$$\exists w \in Y \; \forall p \in X \qquad (q_* - q_0, p) = -\lambda \langle A(p) u_{q_*}, w \rangle, \tag{5.2}$$

compare to (3.1). Using the co-area formula, the goal is to transform the expression on the right-hand side in (5.2), which is an integral over the domain $\Omega \subset \mathbb{R}^d$, see for instance (2.6), into an integral over the interval I on which the parameter q is defined. Then, the source condition (1.9) is satisfied if a function w can be found such that this integral transformation yields equality in (5.2). For Example 1, it is shown in [14] that

$$w = \frac{1}{\lambda}(q_*''(u_{q_*}) - q_0''(u_{q_*}) - q_*(u_{q_*}) + q_0(u_{q_*})) \cdot \frac{1}{m(u_{q_*})} \cdot Du_{q_*}$$

is an appropriate candidate, where m is the $(d-1)$-dimensional Hausdorff-measure of the level sets of u_{q_*}, i.e.,

$$m(\tau) = \int_{u_{q_*}^{-1}\{\tau\}} d\mathcal{H}^{d-1} \;\; \text{for} \;\; \tau \in I,$$

and Du_{q_*} denotes the Jacobian. Thinking of u_{q_*} as a temperature distribution, whose level sets then are isotherms in Ω, the essential conditions for (5.2) to eventually hold are sufficient smoothness of $q_* - q_0$, sufficient knowledge about q_* on the boundary of the temperature interval given by the data and sufficiently regular dependency of the isotherms of u_{q_*} on the temperature level, see [14] for details. Similar results have been obtained in [16] for Example 1.2 and (4.18).

Our discussion hints that even for the derivative free Landweber method rate estimates require smoothness assumptions on the unknown parameter in any case. However, we emphasize that this does not automatically yield the properties on $F'(\cdot)$ required by the classical theories for obtaining rates since the operator B from (2.4) is not involved in (1.9). For instance, if the nonlinear source term b in Example 2 only belongs to $H^s(I)$ with $s < 5/2$, then the corresponding Nemyckii operator - and hence the full parameter-to-output map F - is not Lipschitz continuously differentiable, see, e.g., [17].

Recalling that the Fréchet differentiability of the parameter-to-output map F plays a central role in the convergence analysis of the classical iteration methods, the advantage of the approach presented in this paper is that convergence rates can be obtained without this regularity assumption on F. Theorem 4.1 applies to a wide class of underlying elliptic direct problems, also allowing nonlinearities that are due to the unknown parameter and / or other parts of the pde-operator. For a special choice of the parameters β_k, convergence rates for (3.2) can be proven under the weak source condition (1.9).

6 Numerical Results

For our numerical tests we consider the equation

$$
\begin{aligned}
-\nabla \cdot (q_*(|\nabla u|)\nabla u) &= 0 \text{ in } \Omega = \left\{(x,y)\,|\,x^2+y^2 \leq 1\right\} \qquad (6.1)\\
u &= x^2 \text{ on } \partial\Omega \qquad (6.2)
\end{aligned}
$$

with

$$
q_*(\tau) = \frac{1}{\sqrt{1+\tau^2}}. \qquad (6.3)
$$

It is well known that the solution u_{q_*} to this nonlinear PDE is the function of minimal surface among those defined on the unit circle and required to satisfy (6.2). As inverse minimal surface problem we study the identification of the true parameter (6.3) from knowledge of u_{q_*} or its noisy versions assumed to be obtained by measurements. The exact data $z = u_{q_*}$ and $|\nabla z|$, the argument of (6.3), are shown in Figures 2 and 3.

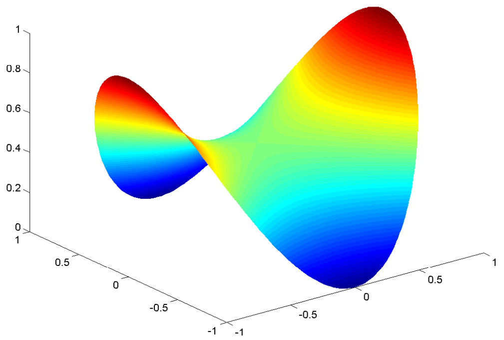

Figure 2: the exact data z

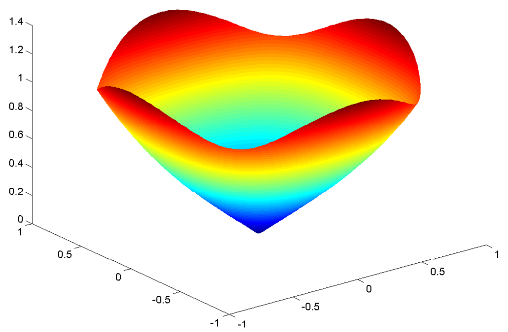

Figure 3: $|\nabla z|$, argument of q_*

Choosing the interval $I = [0, 1.5]$, which covers the range of $|\nabla z|$ and also gives some scope with respect to perturbed data, we set $X = H^1(I)$ as parameter Hilbert space.

As subset Q of admissible parameters, q_* among them, we consider those functions that satisfy

$$\alpha_1(s-t)^2 \le (q(s)s - q(t)t)(s-t), \quad s,t \ge 0,$$

and

$$|q(s)s - q(t)t| \le \alpha_2 |s-t|, \quad s,t \ge 0$$

with common constants α_1 and α_2. Then, with $Y = H^1(\Omega)$, $Y_0 = \{u \in Y \mid u|_{\partial\Omega} = x^2\}$, $B = 0$ and

$$\langle A(q)u, v\rangle = \int_\Omega q(|\nabla u|)\nabla u \nabla v \, dx \quad \forall u, v \in Y_0,$$

Assumptions 1 and 2 are satisfied, see [20], such that the iteration operator for the derivative free Landweber method (3.2) is well-defined. The sequence of stabilization parameters is chosen as

$$\beta_k = \frac{1}{(k+100)^{0.99}},$$

compare to (4.19), the scaling parameter is set as $\lambda = 5$.

The identification of nonlinearities that do not depend on the physical state in a straightforward way, i.e., $q_* = q_*(u)$, but are of more complicated types, e.g., $q_* = q_*(|\nabla u|)$, is so far hardly discussed in the literature. In [11], the magnetic reluctivity appearing in nonlinear Maxwell's equations is estimated by means of an inexact Newton type method based on the Fréchet derivative of the associated forward operator F. However, with non standard terms in the linearized direct problem due to the $|\cdot|$-expression and the ∇-operator "inside of" q_*, the classical source condition (1.7) has neither theoretically nor numerically been approached in [11]. Even though such non standard terms do not appear in the derivative free source condition (1.9) for our inverse minimal surface problem, its theoretical interpretation is an open challenge. In order to still accompany Theorem 4.1 with numerical tests, we simply choose an element $w \in Y$ and numerically construct an initial guess q_0 such that (1.9) is satisfied. The result is displayed in Figure 4.

For performing the derivative free Landweber method (3.2) the FEM-representation

$$q(\tau) = \sum_{i=1}^{m} c_i \varphi_i(\tau), \tau \in I$$

of the iterates q_k is based on cubic splines with $m = 30$. The corresponding solutions u_k of the direct problem are represented via Lagrange-quadratic elements over a regular mesh of 1289 nodes, all computations were done in MATLAB / FEMLAB.

The first two columns in Table 1 provide absolute and relative information about the quality of the sequence of data z^δ obtained by perturbation of the exact data z. The latter was generated on a different mesh than used in (3.2) in order to avoid inverse

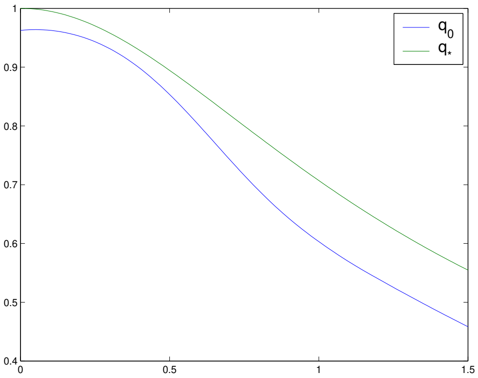

Figure 4: initial guess q_0 and solution q_*

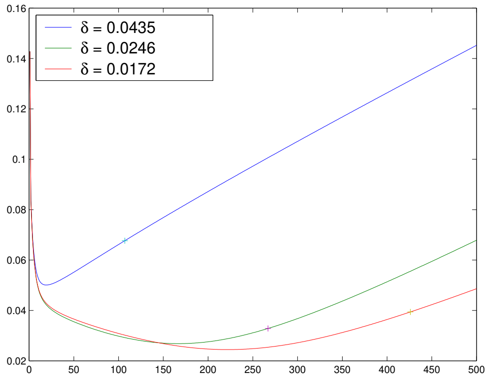

Figure 5: $\|q_k^\delta - q_*\|$ vs. k

$\delta = \|z^\delta - z\|$	$\frac{\|z^\delta - z\|}{\|z\|}$	N_0	$\|z^\delta - F(q_{N_0}^\delta)\|$	$\|q_{N_0}^\delta - q_*\|$
0.0876	0.0555	2	0.0891	0.08379
0.0435	0.0276	107	0.0435	0.06768
0.0347	0.0220	159	0.0347	0.03666
0.0246	0.0156	267	0.0246	0.03283
0.0172	0.0109	426	0.0172	0.03939
0.0144	0.0091	532	0.0144	0.02350

Table 1: results

crimes. Note that the data error is measured with respect to $Y = H^1(\Omega)$. For instance, 5.55% error in $H^1(\Omega)$ correspond to only 0.052% error in $L^2(\Omega)$, in other words the perturbed data z^δ can optically hardly be distinguished from the exact data shown Figure 2. Nevertheless, the data pertubation must not be neglected. Figure 5 shows the typical course of the iteration by means of three examples. Initially, the error in the parameter decreases but then starts to increase due to the amplification of the data error. Only if the iteration is stopped in dependence on the noise level, a reliable approximation of q_* can be expected. Table 1 contains the stopping index N_0 obtained by application of the a-priori stopping rule (4.10) with $\tilde{C} = 6.5$ as well as the corresponding errors in the output and the parameter. The latter numerically indicates the convergence rate $\mathcal{O}(\sqrt{\delta})$ given in Theorem 4.1, see also Figure 6. Finally, Figure 7 shows the regularized solutions $q_{N_0(\delta)}^\delta$ according to the stopping marks given in Figure 5. We recall that as opposed to classical regluarization methods, (3.2) does not require derivatives of the parameter q_k with respect to its argument, also contributing to a reduction of numerical costs.

A goal for future research is to extend the derivative free iteration technique to param-

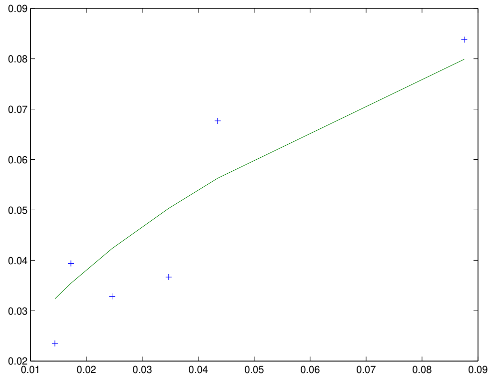

Figure 6: $\|q^\delta_{N_0(\delta)} - q_*\|$ vs. δ

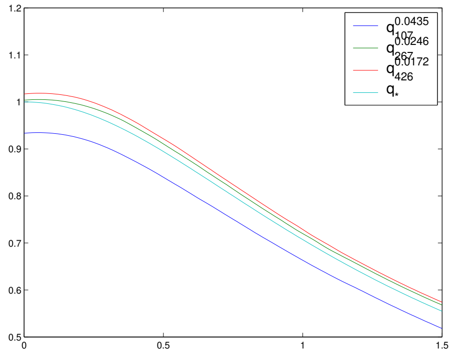

Figure 7: q_* and $q^\delta_{N_0(\delta)}$

eter identification problems, where only partial information about the physical state is available, e.g., observations in subdomains of Ω or boundary measurements. We do not know a single example where the standard assumptions of the classical theory based on F' could be verified for such problems. Another idea is to embed (3.2) into a Hilbert scale setting in order to accelerate the method, see [19] for such an approach again based on F'.

References

[1] A.B. Bakushinskii, The problem of the convergence of the iteratively regularized Gauss-Newton method, Comput. Math. Math. Phys. 32 (1992), 1353-1359.

[2] A. Bakushinsky, A. Goncharsky, Ill-Posed Problems: Theory and Applications, Kluwer Academic Publishers, 1994.

[3] B. Blaschke, A. Neubauer, O. Scherzer, On convergence rates for the iteratively regularized Gauss-Newton method, IMA Journal of Numerical Analysis 17 (1997), 421–436.

[4] H.W. Engl, M. Hanke, A. Neubauer, Regularization of Inverse Problems, Kluwer Academic Publishers, 1996.

[5] H.W. Engl, K. Kunisch, A. Neubauer, Convergence rates for Tikhonov regularisation of non-linear ill-posed problems, Inverse Problems 5 (1989), 523–540.

[6] H.W. Engl, P. Kügler, The influence of the equation type on iterative parameter identification problems which are elliptic or hyperbolic in the parameter, European Journal of Applied Mathematics 14 (2003), 129–163.

[7] H.W. Engl, O. Scherzer, Convergence rate results for iterative methods for solving nonlinear ill-posed problems, in: D. Colton, H.W. Engl, A.K. Louis, J. McLaughlin, W.F. Rundell (eds.), Survey on Solution Methods for Inverse Problems, Springer, Vienna/New York, 2000, 7-34.

[8] H.W. Engl, J. Zou, A new approach to convergence rate analysis of Tiknonov regularization for parameter identification in heat conduction, Inverse Problems 16 (2000), 1907–1923.

[9] L.C. Evans and R. Gariepy, Measure Theory and Fine Properties of Functions, CRC Press, 1992.

[10] M. Hanke, A. Neubauer, O. Scherzer, a convergence analysis of the Landweber iteration for nonlinear ill-posed problems, Numerische Mathematik 72 (1995), 21 – 37.

[11] B. Kaltenbacher, M. Kaltenbacher, S. Reitzinger, Identification of nonlinear B-H curves based on magnetic field computations and multigrid methods for ill-posed problems, SFB-Report 17 (2001), SFB F013, Austria.

[12] K. Kunisch and W. Ring, Regularization of nonlinear illposed problems with closed operators, Numer. Funct. Anal. and Optimiz. 14 (1993), 389-404.

[13] P. Kügler, A derivative-free Landweber iteration for parameter identification in certain elliptic PDEs, Inverse Problems 19 (2003), 1407–1426.

[14] P. Kügler, A Derivative Free Landweber Method for Parameter Identification in Elliptic Partial Differential Equations with Application to the Manufacture of Car Windshields, PhD Thesis, Johannes Kepler University, Linz, Austria, 2003.

[15] P. Kügler, A parameter identification problem of mixed type related to the manufacture of car windshields, SIAM J. Appl. Math. 64 (2004), 858–877.

[16] P. Kügler, H.W. Engl, Identification of a temperature dependent heat conductivity by Tikhonov regularization, Journal of Inverse and Ill-posed Problems 10 (2002), 67–90.

[17] J.L. Lions, E. Magenes, Non-Homogenuous Boundary Value Problems and Applications, Vol. I, Springer, Berlin, Heidelberg, New York, 1972.

[18] A. Morozov, Regularization Methods for Ill-Posed Problems, CRC Press, 1993.

[19] A. Neubauer, On Landweber iteration for nonlinear ill-posed problems in Hilbert scales, Numerische Mathematik 85 (2000), 309–328.

[20] C. Pechstein, Multigrid-Newton-Methods For Nonlinear Magnetostatic Problems, Diploma Thesis, Johannes Kepler University, Linz, Austria, 2004.

[21] O. Scherzer, A modified Landweber iteration for solving parameter estimation problems, Applied Mathematics and Optimization 38 (1998), 45–68.

[22] O. Scherzer, Convergence criteria of iterative methods based on Landweber iteration for solving nonlinear problems, Journal of Mathematical Analysis and Applications 194 (1995), 911–933.

[23] O. Scherzer, H.W. Engl, R.S. Anderssen, Parameter identification from boundary measurements in a parabolic equation arising from geophysics, Nonlinear Anal. 20 (1993), 127–156.

[24] E. Schock, Approximate solution of ill-posed equations: arbitrarily slow convergence vs. superconvergence, in: G. Hämmerlin and K.H. Hoffmann, eds., Construcitve Methods for the Practical Treatment of Integral Equations, Birkhäuser, Basel, 1985, 234–243.

[25] E. Zeidler, Nonlinear Functional Analysis and its Applications II/b, Springer-Verlag, New York, Berlin, Heidelberg, 1980.