On the Signs of Recurrence Sequences

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Abstract

We consider the following question: Which real sequences $(a(n))$ that satisfy a linear recurrence with constant coefficients are positive for sufficiently large n? We show that the answer is negative for both $(a(n))$ and $(-a(n))$, if the dominating characteristic roots in the representation of $a(n)$ as a generalized power sum comprise either two pairs of conjugate complex roots or one negative real root and one pair of conjugate complex roots. The proof uses results from Diophantine approximation and the geometry of numbers. Concerning the case of a real positive dominating root we show what the answer to the question of positivity is for almost all values of the other dominating roots, provided that all dominating roots are simple.

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1 Introduction and Main Result

Several classical special function inequalities, such as Fejér's Inequality [1] or the Askey-Gasper Inequality [1], assert the positivity of an object that can be defined by a linear recurrence with polynomial coefficients. Even for the special case of linear recurrences

$$a(n+d) = s_1 a(n+d-1) + \cdots + s_{d-1} a(n+1) + s_d a(n), \quad n \in \mathbb{N}, \qquad (1)$$

with constant coefficients $s_1, \ldots, s_d \in \mathbb{R}$ it is not always a simple matter to decide from the recurrence coefficients and the real initial values $a(0), \ldots, a(d-1)$ whether the solution is positive or not. We call sequences $(a(n))$ that satisfy a recurrence of the form (1) *recurrence sequences.* Zeilberger [2] gives them the more suggestive name *C-finite sequences*. Linear combinations (with constant coefficients) of recurrence sequences are recurrence sequences again, so positivity results are useful for comparing the magnitude of two sequences, too.

It is well known [3] that the sequence $(a(n))$ can be written in terms of the roots $\alpha_1, \ldots, \alpha_s$ of the *characteristic polynomial*

$$z^d - s_1 z^{d-1} - \cdots - s_{d-1} z - s_d$$

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of the recurrence as a generalized power sum

$$a(n) = C_1(n)\alpha_1^n + \dots C_d(n)\alpha_s^n, \tag{2}$$

where the $C_k(n)$ are polynomials in n with complex coefficients. We refer to the α_k that occur in (2) with nonzero coefficient as *characteristic roots of* $(a(n))$.

Example 1. *Consider the recurrence*

$$\begin{aligned} a(n+5) = 3a(n+4) - 2(\sqrt{5}+1)a(n+3) + 6(\sqrt{5}+1)a(n+2) \\ - 16a(n+1) + 48a(n). \end{aligned}$$

Its characteristic polynomial is

$$(z-3)(z-2\mathrm{e}^{7\mathrm{i}\pi/5})(z-2\mathrm{e}^{-7\mathrm{i}\pi/5})(z-2\mathrm{e}^{2\mathrm{i}\pi/5})(z-2\mathrm{e}^{-2\mathrm{i}\pi/5}),$$

so the solution is given by

$$\begin{aligned} a(n) &= c_0 3^n + 2^n \left(c_1 \mathrm{e}^{7n\mathrm{i}\pi/5} + \overline{c}_1 \mathrm{e}^{3n\mathrm{i}\pi/5} + c_2 \mathrm{e}^{2n\mathrm{i}\pi/5} + \overline{c}_2 \mathrm{e}^{8n\mathrm{i}\pi/5} \right) \\ &= c_0 3^n + O(2^n), \end{aligned} \tag{3}$$

where the coefficients $c_0 \in \mathbb{R}$ and $c_1, c_2 \in \mathbb{C}$ depend on the real initial values $a(0), \dots, a(3)$. We may ask ourselves whether $3^{-n}a(n)$ approaches c_0 from one side only. If c_1 and c_2 do not both vanish, it is natural to expect (and will be established in this paper) that this does not hold, because the $O(2^n)$ term seems to oscillate.

Conjecture 2. *Let $(a(n))$ be a recurrence sequence with no real positive dominating characteristic root. Then there are infinitely many n with $a(n) > 0$ and infinitely many n with $a(n) < 0$.*

The sequence $(a(n))$ might not oscillate if there is a real positive dominating characteristic root. See Section 7 for more on this.

So far Conjecture 2 has only been verified for one dominating characteristic root (trivial) and for one pair of conjugate complex roots [4]. We cannot follow an argument from Nagasaka and Shiue [5], viz. that this special case should immediately imply the truth of the conjecture in general. The main goal of this paper is to establish the following theorem by an extension of Burke and Webb's proof.

Theorem 3 (Main Theorem). *Let $(a(n))$ be a recurrence sequence with at most four dominating characteristic roots, none of which is real positive. Then there are infinitely many n with $a(n) > 0$ and infinitely many n with $a(n) < 0$.*

The rest of the paper is organized as follows. In Section 2 we reduce Theorem 3 from multiple roots to simple roots and subsequently to a geometric statement about the distribution modulo one of integer multiples of a real vector $\boldsymbol{\xi} = (\xi_1, \xi_2) = (\arg\alpha_1, \arg\alpha_2)/2\pi$, except for some special cases of Theorem 3 that are settled in Section 6. In Section 3 we deduce the desired result from Kronecker's Approximation Theorem, provided that one of ξ_1, ξ_2 is irrational. The proof in the case where both are rational is the subject of Sections 4 and 5. Section 7 presents a metric result that deals with the case of a positive real root. In the conclusion we comment, among other things, on extending our approach to Conjecture 2 to an arbitrary number of dominating characteristic roots.

2 Notation and Preliminaries

We write $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}_0^+$, $\mathbb{R}$ and $\mathbb{C}$ for the sets of natural numbers (including zero), integers, rational numbers, non-negative real numbers, real numbers and complex numbers, respectively. The conjugate of a complex number z is denoted by $\overline{z}$. Whenever $\boldsymbol{v}$ is a vector, we use the same letter with a subscript for its components, as in $\boldsymbol{v} = (v_1, \dots, v_m)$. For vectors $\boldsymbol{\xi} = (\xi_1, \dots, \xi_m)$, $\boldsymbol{\rho} = (\rho_1, \dots, \rho_m)$ of real numbers and a real number ρ, we write

$$\begin{aligned}\boldsymbol{\xi} \bmod \boldsymbol{\rho} &= (\xi_1 \bmod \rho_1, \dots, \xi_1 \bmod \rho_1) \quad \text{and} \\ \boldsymbol{\xi} \bmod \rho &= (\xi_1 \bmod \rho, \dots, \xi_1 \bmod \rho).\end{aligned}$$

We define the open rectangle parallel to the axes with side lengths $2\lambda_1, 2\lambda_2 \in \mathbb{R}$ centred at $\boldsymbol{c} = (c_1, c_2) \in \mathbb{R}^2$ as

$$\mathcal{R}_{\lambda_1,\lambda_2}(\boldsymbol{c}) := \left\{\boldsymbol{x} \in \mathbb{R}^2 : |x_1 - c_1| < \lambda_1, |x_2 - c_2| < \lambda_2\right\}.$$

For an open square parallel to the axes we write

$$\mathcal{S}_\lambda(\boldsymbol{c}) := \mathcal{R}_{\lambda,\lambda}(\boldsymbol{c}), \quad \lambda \in \mathbb{R}, \boldsymbol{c} \in \mathbb{R}^2.$$

A *lattice* [6] is a discrete subgroup $\Lambda \subset \mathbb{R}^m$. Its determinant is denoted by $\boldsymbol{d}(\Lambda)$. The lattice $L_g(a_1, a_2)$ is defined in Section 4.

Let $(a(n))$ be as in Theorem 3. We order the characteristic roots $\alpha_1, \dots, \alpha_s$ of $(a(n))$ such that $\alpha_1, \dots, \alpha_l$ contain all real dominating characteristic roots, precisely one element of every pair of conjugate non-real dominating characteristic roots and no other roots.

Moreover, let $\alpha_1, \dots, \alpha_l$ be ordered such that

$$D := \deg C_1 = \cdots = \deg C_m > \deg C_{m+1} \geq \cdots \geq \deg C_l$$

for some $1 \leq m \leq l \leq 2$. Then we obtain [7]

$$n^{-D} a(n) = \sum_{k=1}^{m} (c_k \alpha_k^n + \overline{c}_k \overline{\alpha}_k^n) + O(n^{-1} |\alpha_1|^n),$$

where c_k is the leading coefficient of $C_k(n)$. This formula shows that Theorem 3 can be deduced from Burke and Webb's result ($m = 1$) and the following theorem ($m = 2$).

Theorem 4. *Let $\alpha_1, \alpha_2 \in \mathbb{C} \backslash \mathbb{R}_0^+$, $|\alpha_1| = |\alpha_2| = 1$, $\alpha_1 \neq \alpha_2 \neq \overline{\alpha}_1$. Let further c_1, c_2 be nonzero complex numbers and*

$$b(n) := c_1 \alpha_1^n + \overline{c}_1 \overline{\alpha}_1^n + c_2 \alpha_2^n + \overline{c}_2 \overline{\alpha}_2^n, \qquad n \geq 0. \tag{4}$$

Then there is $\delta > 0$ such that $b(n) > \delta$ for infinitely many n and $b(n) < -\delta$ for infinitely many n.

Note that if δ was replaced by zero, it might happen that, e.g., all negative values $b(n)$ are so small in absolute value that the remainder term of $a(n)$, which comes from the characteristic roots of smaller modulus, takes over and makes the corresponding values $a(n)$ positive. This uniformity condition was missed by Burke and Webb [4]. They only argue that $c_1\alpha_1^n + \overline{c}_1\overline{\alpha}_1^n$ has infinitely many positive and infinitely many negative values, which is not sufficient, but their proof can be easily repaired.

Now let α_1, α_2, c_1, c_2 be as in Theorem 4. Replacing (α_k, c_k) by $(\overline{\alpha}_k, \overline{c}_k)$ and vice versa if necessary, we may assume $\mathrm{Im}(c_k) \geq 0$. Putting $\theta_k := \arg \alpha_k$, we obtain by standard formulas

$$\begin{aligned} b(n) &= 2\sum_{k=1}^{2} \mathrm{Re}\,(c_k \exp\,(in\theta_k)) \\ &= 2\sum_{k=1}^{2} (\mathrm{Re}(c_k)\cos n\theta_k - \mathrm{Im}(c_k)\sin n\theta_k) \\ &= \sum_{k=1}^{2} w_k \sin(n\theta_k + \varphi_k), \end{aligned}$$

where the coefficients are nonzero real numbers

$$w_k := \begin{cases} -2|c_k|, & c_k \in \mathbb{C}\backslash\mathbb{R}; \\ 2c_k, & c_k \in \mathbb{R}, \end{cases}$$

and the φ_k are given by

$$\varphi_k := \begin{cases} -\arctan\frac{\mathrm{Re}(c_k)}{\mathrm{Im}(c_k)}, & c_k \in \mathbb{C}\backslash\mathbb{R}; \\ \frac{1}{2}\pi, & c_k \in \mathbb{R}. \end{cases}$$

We turn our attention to the signs of $\sin(n\theta_k + \varphi_k)$. If we can prove that for every pair (S_1, S_2) of $+1$'s and -1's there are infinitely many n such that the sign of $\sin(n\theta_k + \varphi_k)$ equals S_k for $k = 1, 2$, we will have shown that $(b(n))$ oscillates, whatever the values of the c_k (and thus the w_k) are. In other words, we are looking for n such that

$$(n\theta_k + \varphi_k) \bmod 2\pi \in \,]0, \pi[$$

or

$$(n\theta_k + \varphi_k) \bmod 2\pi \in \,]\pi, 2\pi[,$$

respectively. To get the δ in Theorem 4, we have to shrink the intervals to

$$]\epsilon, \pi - \epsilon[\quad \text{and} \quad]\pi + \epsilon, 2\pi - \epsilon[$$

for some small $\epsilon > 0$, of course independent from n. Now we rescale to the unit interval.

Theorem 5. *Let $\xi_1, \xi_2 \in \,]0,1[\backslash\{\frac{1}{2}\}$ such that $\xi_1 \not\equiv \pm\xi_2 \pmod 1$ and, if both ξ_1 and ξ_2 are rational, then the pair of their denominators is none of $(5,5)$, $(6,3)$, $(8,4)$. Then for all $\boldsymbol{c} \in \mathbb{R}^2$ there is $\epsilon > 0$ such that there are infinitely many n with*

$$n(\xi_1, \xi_2) \bmod 1 \;\in\; \mathcal{S}_{1/4-\epsilon}(\boldsymbol{c}) \bmod 1.$$

Applying this theorem with $(\xi_1, \xi_2) = (\theta_1/2\pi, \theta_2/2\pi)$ and $c_k = \frac{1}{4} - \varphi_k/2\pi$ to make $\sin(n\theta_k + \varphi_k)$ positive and $c_k = \frac{3}{4} - \varphi_k/2\pi$ for a negative sign proves Theorem 4, unless one of the α_k is a negative real number or $\theta_1/2\pi, \theta_2/2\pi$ are rational numbers with denominators in $\{(5,5),(6,3),(8,4)\}$. Section 6 deals with these special cases of Theorem 4.

In the proof of Theorem 5 we distinguish the following three cases:

(1) $\xi_1, \xi_2, 1$ are linearly independent over $\mathbb{Q}$

(2) ξ_1, ξ_2 are not both rational, but satisfy a linear relation $u_1\xi_1 + u_2\xi_2 = v$ with $u_1, u_2, v \in \mathbb{Z}$

(3) ξ_1 and ξ_2 are both rational

Section 3 settles the first two cases. The proof of Theorem 5 in Case 3 is the content of Sections 4 and 5.

We remark that in order to prove Conjecture 2 for one pair of conjugate complex dominating roots, it suffices to show that for every real number $\xi \neq \frac{1}{2}$ with $0 < \xi < 1$ and every real number c there is $\epsilon > 0$ such that for infinitely many n

$$n\xi \bmod 1 \;\in\; \left]c - \tfrac{1}{4} + \epsilon, c + \tfrac{1}{4} - \epsilon\right[\bmod 1.$$

This is essentially what was done (without ϵ, cf. the introduction) by Burke and Webb [4].

3 The Irrational Cases

The closure of the set of integer multiples of a vector $\boldsymbol{\xi} = (\xi_1, \xi_2)$ modulo one is described by a classical result from Diophantine approximation.

Theorem 6 (Kronecker's Theorem). *Let ξ_1, ξ_2 be real numbers.*

(i) If $\xi_1, \xi_2, 1$ are linearly independent over the rationals, then the points $n\boldsymbol{\xi} \bmod 1$, $n \in \mathbb{N}$, lie dense in the unit square.

(ii) If ξ_1, ξ_2 are not both rational, but satisfy a relation $u_1\xi_1 + u_2\xi_2 = v$ with $u_1, u_2, v \in \mathbb{Z}$ and $\gcd(u_1, u_2, v) = 1$, then the points $n\boldsymbol{\xi} \bmod 1$, $n \in \mathbb{N}$, lie dense on the portions of the lines

$$\ell_t := \left\{\boldsymbol{x} \in \mathbb{R}^2 : u_1x_1 + u_2x_2 = t\right\}, \quad t \in \mathbb{Z},$$

which lie within the unit square.

Proof. See, e.g., Niven [8, Theorems 3.4 and 3.6]. □

Part (i) of Theorem 6 settles Case 1 of Theorem 5. We proceed to Case 2. Let $\boldsymbol{c} \in \mathbb{R}^2$ be arbitrary but fixed and ℓ_t be as in part (ii) of Theorem 6. Since

$$\bigcup_{t\in\mathbb{Z}} \ell_t + \mathbb{Z}^2 = \bigcup_{t\in\mathbb{Z}} \ell_t,$$

it suffices to find infinitely many $n\boldsymbol{\xi} \bmod 1$ in the set

$$\mathcal{S}_{1/4-\epsilon}(\boldsymbol{c}) \cap \bigcup_{t\in\mathbb{Z}} \ell_t,$$

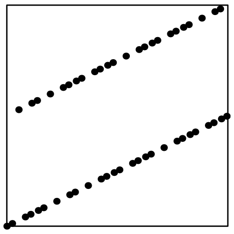

Figure 1: The unit square with $(n\xi_1, n\xi_2) \bmod 1$ for $\xi_1 = 2\sqrt{2}$, $\xi_1 - 2\xi_2 = 2$ and $n = 0, \ldots, 50$.

where $\epsilon > 0$ is yet to be chosen. First suppose that ξ_1 and ξ_2 are irrational. Then the parallel lines ℓ_t are neither horizontal nor vertical, since $u_1 u_2 \neq 0$. Two adjacent lines ℓ_t, ℓ_{t+1} have horizontal distance $1/|u_1|$ and vertical distance $1/|u_2|$. Since $\xi_1 \not\equiv \pm\xi_2 \pmod 1$, one of these quantities must be smaller than or equal to $\frac{1}{2}$. Thus

$$\mathcal{S}_{1/4}(\boldsymbol{c}) \cap \bigcup_{t\in\mathbb{Z}} \ell_t \neq \emptyset.$$

In fact this set is not only non-empty but contains a line segment. Clearly, we can find $\epsilon > 0$ such that the set $\mathcal{S}_{1/4-\epsilon}(\boldsymbol{c}) \cap \bigcup_{t\in\mathbb{Z}} \ell_t$ still contains a line segment of length greater than zero. Filling this line segment densely with points $n\boldsymbol{\xi} \bmod 1$ requires infinitely many n.

Now let ξ_1 be rational and ξ_2 be irrational. This implies $u_2 = 0$. Then the lines ℓ_t are vertical, and the horizontal distance between ℓ_t and ℓ_{t+1} is $1/b_1 \leq \frac{1}{3}$, since $b_1 > 2$ by the assumptions of Theorem 5. Case 2 of Theorem 5 is proved.

4 The Rational Case

Theorem 7. *Let $a_1, a_2, b_1, b_2 \in \mathbb{N}$, $2 \leq b_2 \leq b_1$, $1 \leq a_k < b_k$, $\gcd(a_k, b_k) = 1$ for $k = 1, 2$ and $\frac{a_1}{b_1} \not\equiv \pm\frac{a_2}{b_2} \pmod 1$. Then there is $\boldsymbol{c} \in [0,1]^2$ such that for all $n \in \mathbb{N}$*

$$n(\tfrac{a_1}{b_1}, \tfrac{a_1}{b_1}) \bmod 1 \notin \mathcal{S}_{1/4}(\boldsymbol{c}) \bmod 1$$

provided that

$$(b_1, b_2) \in \{(5,5), (6,3), (8,4)\} \cup \{(b_1, 2) : 2 \leq b_1 \in \mathbb{N}\}, \tag{5}$$

and there is no such $\boldsymbol{c}$ if (5) *does not hold.*

To see that Case 3 of Theorem 5 follows from Theorem 7, note that the purely periodic sequence

$$n(\tfrac{a_1}{b_1}, \tfrac{a_2}{b_2}) \bmod 1 = (\tfrac{na_1 \bmod b_1}{b_1}, \tfrac{na_2 \bmod b_2}{b_2}), \quad n \geq 0,$$

assumes each of its finitely many values infinitely often. The ϵ has disappeared because the set of all $n(\frac{a_1}{b_1}, \frac{a_2}{b_2}) \bmod 1$ is finite and $\mathcal{S}_{1/4}(\boldsymbol{c})$ is open.

Proof of the right to left implication of Theorem 7. If $b_2 = 2$, we necessarily have $a_2 = 1$ and we may take $c_2 = \frac{1}{4}$ and $c_1 \in \mathbb{R}$ arbitrary. (See Figure 2 for an example.) If $(b_1, b_2) = (5,5)$, it is easy to see that for all $\boldsymbol{a}$ in question the set of integer multiples modulo one is one of the two sets obtained from $\boldsymbol{a} = (1, 2)$ and $\boldsymbol{a} = (-1, 2)$. The same holds for $(b_1, b_2) = (6, 3)$. It is also true for $(b_1, b_2) = (8, 4)$, if we take $\boldsymbol{a} = (\pm 3, 1)$ instead of $(\pm 1, 2)$. The number of $\boldsymbol{a}$'s to check can be reduced further by taking advantage of some obvious symmetries. By the subsequent lemma, the alternative with negative first entry can be discarded in each of the three cases. Figure 2 shows that in the remaining cases we may take $\boldsymbol{c} = (\frac{1}{2}, \frac{1}{2})$, $(\frac{1}{12}, \frac{1}{3})$ and $(\frac{1}{2}, \frac{1}{2})$, respectively. $\square$

Lemma 8. *Define the maps s and τ on $\mathbb{R}^2$ by*

$$s(x_1, x_2) = ((1 - x_1) \bmod 1, x_2) \quad \text{and} \quad \tau(x_1, x_2) = (x_2, x_1).$$

Then for all real numbers ξ_1, ξ_2

$$s((\xi_1, \xi_2) \bmod 1) = s(\xi_1, \xi_2) \bmod 1 \quad \text{and} \quad \tau((\xi_1, \xi_2) \bmod 1) = \tau(\xi_1, \xi_2) \bmod 1.$$

Proof. Obvious. $\square$

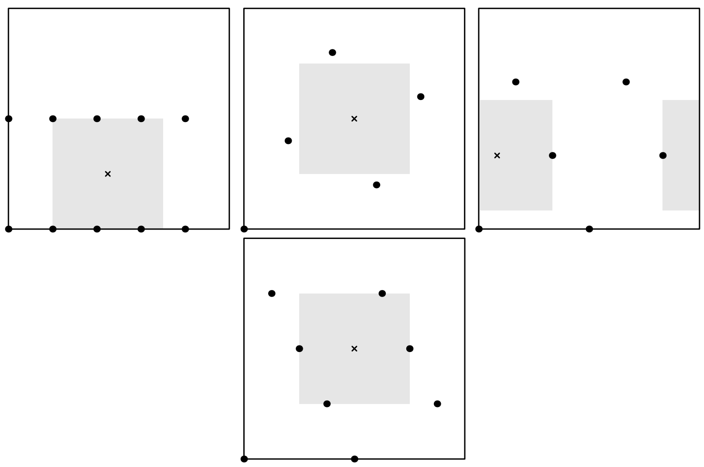

Figure 2: The unit square with the set $\{n(\frac{a_1}{b_1}, \frac{a_2}{b_2}) \bmod 1 : n \in \mathbb{N}\}$ for $(\frac{a_1}{b_1}, \frac{a_2}{b_2}) = (\frac{1}{5}, \frac{1}{2})$, $(\frac{1}{5}, \frac{2}{5})$, $(\frac{1}{6}, \frac{2}{3})$ and $(\frac{3}{8}, \frac{1}{4})$, respectively.

We have shown this implication just for the sake of completeness. The interesting part of Theorem 7 for our purpose is the converse implication. Its proof is the content of the remainder of this section and of the following section.

Definition 9. *Let g be a positive integer and a_1, a_2 be integers relatively prime to g. Then we define the* lattice of multiples of $\boldsymbol{a} = (a_1, a_2)$ modulo g *as*

$$L_g(a_1, a_2) := \{\boldsymbol{u} \in \mathbb{Z}^2 : n\boldsymbol{a} \equiv \boldsymbol{u} \pmod{g} \quad \text{for some } n \in \mathbb{N}\}.$$

Alternatively [9], $L_g(a_1, a_2)$ can be defined as the lattice generated by the vectors $(0, g), (g, 0)$ and (a_1, a_2).

Lemma 10. *Let a_1, a_2 be integers and b_1, b_2 be positive integers with $\gcd(a_k, b_k) = 1$ for $k = 1, 2$ and $g := \gcd(b_1, b_2)$. Then*

(i) $\left\{n(\frac{a_1}{b_1}, \frac{a_2}{b_2}) \bmod 1 : n \in \mathbb{N}\right\} = \left\{(\frac{u_1}{b_1}, \frac{u_2}{b_2}) : \boldsymbol{u} \in L_g(a_1, a_2), 0 \le u_k < b_k\right\}$

(ii) $L_g(a_1, a_2) = \{\boldsymbol{u} \in \mathbb{Z}^2 : a_1 u_2 \equiv a_2 u_1 \pmod{g}\}$

Proof. We have

$$\begin{aligned}
&\left\{n\left(\tfrac{a_1}{b_1}, \tfrac{a_2}{b_2}\right) \bmod 1 : n \in \mathbb{N}\right\} \\
&= \left\{\left(\tfrac{na_1 \bmod b_1}{b_1}, \tfrac{na_2 \bmod b_2}{b_2}\right) : n \in \mathbb{N}\right\} \\
&= \left\{\left(\tfrac{u_1}{b_1}, \tfrac{u_2}{b_2}\right) : n\boldsymbol{a} \equiv \boldsymbol{u} \pmod{\boldsymbol{b}}, 0 \le u_k < b_k,\ k = 1, 2, \quad \text{for some } n \in \mathbb{N}\right\} \\
&= \left\{\left(\tfrac{u_1}{b_1}, \tfrac{u_2}{b_2}\right) : \boldsymbol{u} \in L_g(a_1, a_2), 0 \le u_k < b_k\right\}.
\end{aligned}$$

The latter equality and assertion (ii) follow from the Generalized Chinese Remainder Theorem [10, Exercise 4.3.2.3]. □

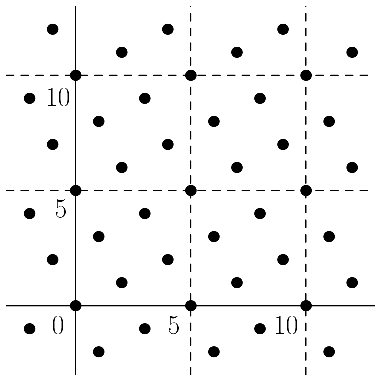

Figure 3: The lattice $L_5(2, 1)$.

Example 11. *In Example 1 the $O(2^n)$ term yields $\xi_1 = \theta_1/2\pi = \frac{7}{10}$ and $\xi_2 = \theta_2/2\pi = \frac{2}{5}$. The corresponding lattice $L_5(7, 1) = L_5(2, 1)$ is displayed in Figure 3.*

Let $\boldsymbol{a} = (a_1, a_2)$ and $\boldsymbol{b} = (b_1, b_2)$ be as in the assumptions of Theorem 7, but such that $\boldsymbol{b}$ is not in the set (5), and put $g := \gcd(b_1, b_2)$. In the light of Lemma 10, it is an immediate consequence of the periodicity property

$$L_g(a_1, a_2) = L_g(a_1, a_2) + g\mathbb{Z}^2$$

that searching a point $n(\frac{a_1}{b_1}, \frac{a_2}{b_2}) \bmod 1$ in a 'modded' square $\mathcal{S}_{1/4}(\boldsymbol{c}) \bmod 1$ amounts to looking for a point of the lattice $L_g(a_1, a_2)$ in the rectangle $\mathcal{R}_{b_1/4, b_2/4}(b_1c_1, b_2c_2)$ with side lengths $b_1/2$, $b_2/2$. We let c_k absorb b_k and write again $\boldsymbol{c} = (c_1, c_2)$ for the arbitrary centre (b_1c_1, b_2c_2).

Example 12. *If we want to show that the $O(2^n)$ term in (3) oscillates, we are lead to the problem of finding a point of $L_5(2, 1)$ in any rectangle $\mathcal{R}_{5/2, 5/4}(\boldsymbol{c})$, $\boldsymbol{c} \in \mathbb{R}^2$.*

If the numbers b_1/g and b_2/g are large, it is easy to find a point of $L_g(a_1, a_2)$ in the rectangle, whereas $b_1 = b_2 = g$ is the most difficult case. This is so because if we fix a_1, a_2 and g and enlarge b_1/g and b_2/g, the lattice $L_g(a_1, a_2)$ remains invariant, while the rectangle becomes bigger.

At first glance, the problem seems to be easily reducible to the case of equal denominators $b_1 = b_2 = g$. In Example 12, if we could show that any square $\mathcal{S}_{5/4}(\boldsymbol{c})$ contains a point of $L_5(2,1)$, then it would follow at once that every rectangle $\mathcal{R}_{5/2,5/4}(\boldsymbol{c})$ contains a point of $L_5(2,1)$. But we have already seen that there are squares $\mathcal{S}_{5/4}(\boldsymbol{c})$ without points of $L_5(2,1)$. In general, the catch is that even if (a_1, a_2, b_1, b_2) satisfy the requirements of Theorem 7 and (b_1, b_2) is not in the set (5), it may happen that $(a_1 \bmod g, a_2 \bmod g, g, g)$ violate the requirements of Theorem 7 or that (g, g) is in (5). Therefore we choose a different approach for the case $b_1 \neq b_2$.

For relatively prime b_1 and b_2 the lattice $L_g(a_1, a_2)$ equals $\mathbb{Z}^2$. All rectangles $\mathcal{R}_{b_1/4,b_2/4}(\boldsymbol{c})$ with $\boldsymbol{c} \in \mathbb{R}^2$ have side lengths greater than one and therefore contain a point of $\mathbb{Z}^2$. If $g = 2$, then a_1 and a_2 must be odd, hence

$$L_g(a_1, a_2) = \left\{\boldsymbol{u} \in \mathbb{Z}^2 : u_1 \equiv u_1 \pmod 2\right\}.$$

Since $b_1 > 4$ in this case, it is easy to see that this lattice contains a point of any rectangle $\mathcal{R}_{b_1/4,b_2/4}(\boldsymbol{c})$.

From now on we assume $g \geq 3$. The following proposition deals with the case $(b_1, b_2) = (2g, g)$. Recall that $(b_1, b_2) = (4, 2)$, $(6, 3)$ and $(8, 4)$ need not be considered, because they are in the set (5).

Proposition 13. *Let a_1, a_2, b_1, b_2 be as in Theorem 7. Suppose $g \geq 5$, $b_1 = 2g$ and $b_1 = g$. Then for all $\boldsymbol{c} \in \mathbb{R}^2$*

$$L_g(a_1, a_2) \cap \mathcal{R}_{b_1/4,b_2/4}(\boldsymbol{c}) \neq \emptyset.$$

Proof. Observe that by the periodicity of $L_g(a_1, a_2)$ it suffices to find a point of the lattice in the set

$$\mathcal{R}_{b_1/4,b_2/4}(\boldsymbol{c}) + g\mathbb{Z}^2. \tag{6}$$

Let $\boldsymbol{p}$ be the lower left corner of $\mathcal{R}_{b_1/4,b_2/4}(\boldsymbol{c})$. We assume w.l.o.g. $0 \leq p_1, p_2 < g$ and define $I := \,]p_2, p_2 + \frac{1}{2}g[$. Then (6) contains the set

$$([0, g[\setminus\{p_1\}) \times I = ([0, g[\times I) \setminus (\{p_1\} \times I). \tag{7}$$

The interval I contains at least two integers, since its length is $\frac{1}{2}g > 2$. Thus there are at least two points of $L_g(a_1, a_2)$ in $[0, g[\times I$ by part (ii) of Lemma 10, and at least one of them lies in (7). ☐

Now we consider values of b_1 that are at least $3g$, which completes the case $b_1 \neq b_2$ of Theorem 7.

Proposition 14. *Let a_1, a_2, b_1, b_2 be as in Theorem 7. Suppose $g \geq 3$ and $b_1 \geq 3g$. Then for all $\boldsymbol{c} \in \mathbb{R}^2$*

$$L_g(a_1, a_2) \cap \mathcal{R}_{b_1/4,b_2/4}(\boldsymbol{c}) \neq \emptyset.$$

Proof. It suffices to consider $b_1 = 3g$ and $b_2 = g$. Proceeding analogously to the proof of Proposition 13, we arrive at the set $[0, g[\times I$ instead of (7). The result follows from part (ii) of Lemma 10 and $\frac{1}{2}g > 1$. ☐

5 The Rational Case With Equal Denominators

Proposition 15. *Let a_1, a_2, b_1, b_2 be as in Theorem 7. Suppose $b_1 = b_2 = g \neq 5$. Then for all $\boldsymbol{c} \in \mathbb{R}^2$*

$$L_g(a_1, a_2) \cap \mathcal{S}_{g/4}(\boldsymbol{c}) \neq \emptyset.$$

If $L_g(a_1, a_2)$ contains one or two sufficiently short vectors, its points are dense enough so that the square $\mathcal{S}_{g/4}(\boldsymbol{c})$ is populated by at least one lattice point. This is the basic idea of our proof of Proposition 15. Although there are algorithms [11, 9] tailored to $L_g(a_1, a_2)$ for computing a reduced lattice basis, we do not know of any specialized a priori bounds for the norm of the basis elements. Therefore, we appeal to the standard bound [12].

Definition 16. *Let $\mathcal{K}$ be a subset of $\mathbb{R}^m$ and $\Lambda \subset \mathbb{R}^m$ be a lattice. Then the* successive minima *of $\mathcal{K}$ w.r.t. Λ are defined for $1 \leq k \leq m$ by*

$$\lambda_k(\mathcal{K}, \Lambda) := \inf \{\lambda > 0 : \lambda\mathcal{K} \text{ contains } k \text{ linearly independent points of } \Lambda\}.$$

Theorem 17 (Minkowski's Second Theorem). *If Λ is an m-dimensional lattice in $\mathbb{R}^m$ and $\mathcal{K} \subset \mathbb{R}^m$ is a bounded zero-symmetric convex set with volume $V(\mathcal{K})$, then*

$$\lambda_1(\mathcal{K}, \Lambda) \cdots \lambda_m(\mathcal{K}, \Lambda) V(\mathcal{K}) \leq 2^m \boldsymbol{d}(\Lambda).$$

From this theorem we can deduce that $L_g(a_1, a_2)$ must contain either two 'short' linearly independent vectors or one 'very short' nonzero vector. For the second case we could not find an applicable result in the literature that would ensure a lattice point in the square, so we provide one now.

Lemma 18. *Let $\Lambda \subset \mathbb{R}^2$ be a lattice and $\boldsymbol{r} = (r_1, r_2)$ be a primitive point of Λ with $0 < r_2 \leq r_1$. Let further $\mathcal{Q}$ be an open square with sides parallel to the axes and side length $A > 0$. If $\mathcal{Q}$ contains no point of Λ, then*

$$A \leq \max\left(r_1, \tfrac{\boldsymbol{d}(\Lambda) + 2r_1 r_2}{r_1 + r_2}\right).$$

Proof. There is a family $\mathfrak{L}$ of parallel equidistant lines with slope $s := r_2/r_1$ such that $\Lambda \subset \bigcup \mathfrak{L}$ and the perpendicular distance between two adjacent lines of $\mathfrak{L}$ is $\boldsymbol{d}(\Lambda)/\|\boldsymbol{r}\|_2$ [6, Lemma III.5]. Then the vertical distance between two adjacent lines is $D := \boldsymbol{d}(\Lambda)/r_1$. We claim

$$\min_{\boldsymbol{c} \in \mathbb{R}^2} \max_{\ell \in \mathfrak{L}} \big(\text{horizontal length of } \ell \cap \mathcal{S}_{A/2}(\boldsymbol{c})\big)$$

$$= \begin{cases} A, & D \leq A(1-s); \\ \frac{A(1+s) - D}{2s}, & A(1-s) \leq D \leq A(1+s); \\ 0, & D \geq A(1+s). \end{cases} \quad (8)$$

If $D \leq A(1-s)$, then for each square $\mathcal{S} = \mathcal{S}_{A/2}(\boldsymbol{c})$ there is a line in $\mathfrak{L}$ that goes through the left and the right edge of the square (see Figure 5). If D is larger than $A(1+s)$, there is a square that is not intersected by any line from $\mathfrak{L}$. We are left with the intermediate case $A(1-s) \leq D \leq A(1+s)$. To achieve the minimum in (8), we must certainly place $\mathcal{S}$ such that there is no line from $\mathfrak{L}$ in the parallelogram $\mathcal{P}(\mathcal{S})$ of Figure 5. But then there is always a line $\ell \in \mathfrak{L}$ that intersects $\mathcal{S} \backslash \mathcal{P}(\mathcal{S})$, say in the upper triangle of $\mathcal{S} \backslash \mathcal{P}(\mathcal{S})$. If no line intersects the lower triangle of $\mathcal{S}\backslash\mathcal{P}(\mathcal{S})$, we can make the maximum in (8) smaller by pushing $\mathcal{S}$ downwards. The smallest possible value of the maximum is achieved as soon as the intersections of $\mathcal{S}$ with ℓ and the line from $\mathfrak{L}$ just below ℓ have equal length. It is easy to see that these intersections both have horizontal length $(A(1+s)-D)/2s$.

Now that (8) is established, let $\mathcal{Q}$ be an open square with sides parallel to the axes and side length

$$A > \max\left(r_1, \tfrac{\boldsymbol{d}(\Lambda)+2r_1r_2}{r_1+r_2}\right).$$

Our goal is to show $\mathcal{Q} \cap \Lambda \neq \emptyset$. If the first case in the right hand side of (8) occurs, we are well off: Since $A > r_1$, the line segment in $\mathcal{Q} \cap \bigcup \mathfrak{L}$ of horizontal length A must contain a point of Λ. The third case in (8) cannot happen, since it would imply $\boldsymbol{d}(\Lambda) \geq A(r_1+r_2)$. As for the second case, $A > \frac{\boldsymbol{d}(\Lambda)+2r_1r_2}{r_1+r_2}$ implies

$$r_1 < \frac{A(r_1+r_2)-\boldsymbol{d}(\Lambda)}{2r_2} = \frac{A(1+s)-D}{2s},$$

hence $\mathcal{Q} \cap \Lambda \neq \emptyset$. $\square$

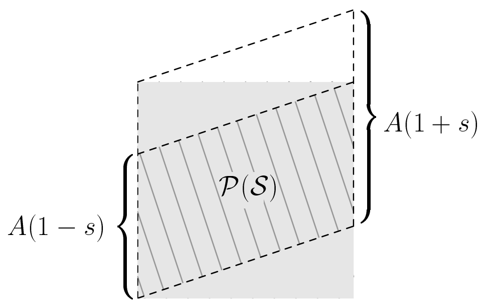

Figure 4: The square $\mathcal{S}$ (shaded) and the parallelogram $\mathcal{P}(\mathcal{S})$ (hatched), which lies between two lines of slope s that go through the upper right and the lower left corner of $\mathcal{S}$, respectively.

Proof of Proposition 15. We begin this proof, which is the core of the proof of Theorem 3, by settling the cases where g is at most 9. The only numbers to consider are $g = 7, 8, 9$, since for smaller $g \neq 5$ there are no a_1, a_2 that satisfy the requirements of Theorem 7. First let $g = 7$. If we have proved the desired result for a pair (a_1, a_2), we need not consider the five pairs

$$(a_2, a_1), (g-a_1, a_2), (a_1, g-a_2), (g-a_2, a_1) \quad \text{and} \quad (a_2, g-a_1)$$

any more by Lemma 8. It is readily seen that under our restrictions on a_1, a_2 all lattices $L_7(a_1, a_2)$ are equal to $L_7(1,3)$ modulo these symmetries. Similarly, for $g = 8$ and $g = 9$ it suffices to consider $L_8(3,1)$ and $L_9(2,1)$, respectively. In all three cases it is easy to verify the desired result.

From now on we assume $g \geq 10$. Put $\Lambda := L_g(a_1, a_2)$ and let

$$\mathcal{B} := \left\{\boldsymbol{x} \in \mathbb{R}^2 : \|\boldsymbol{x}\|_2 \leq 1\right\}$$

be the unit circle. It is not difficult to see [11] that the determinant of Λ is $\boldsymbol{d}(\Lambda) = g$. Then Theorem 17 shows

$$\lambda_1(\mathcal{B},\Lambda)\lambda_2(\mathcal{B},\Lambda)\pi \leq 4g.$$

First suppose $\lambda_2(\mathcal{B},\Lambda) < g/4$. Bender [13] has shown that if the quotient of area and perimeter of a bounded convex set $\mathcal{C} \subset \mathbb{R}^2$ is greater than

$$\tfrac{1}{2}\max\left(\|\boldsymbol{w}_1\|_2, \|\boldsymbol{w}_2\|_2 \sin\vartheta\right),$$

where $\{\boldsymbol{w}_1, \boldsymbol{w}_2\}$ is a basis of a lattice $\Xi \subset \mathbb{R}^2$ and $0 < \vartheta < \pi$ is the angle between $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$, then $\mathcal{C}$ contains a point of Ξ. In our situation, the quotient is $\frac{g^2}{4}/2g = g/8$, hence we can apply Bender's result.

If, on the other hand, $\lambda_2(\mathcal{B},\Lambda) \geq g/4$, then we have $\lambda_1(\mathcal{B},\Lambda) \leq \frac{16}{\pi}$, which provides us with a nonzero point $\boldsymbol{r} \in \Lambda$ with $\|\boldsymbol{r}\|_2 \leq \frac{16}{\pi}$. W.l.o.g. assume that $\boldsymbol{r}$ is primitive with $0 < r_2 \leq r_1$. According to Lemma 18, it suffices to show

$$\frac{g}{2} > \frac{g + 2r_1r_2}{r_1 + r_2},$$

i.e.,

$$4r_1r_2 < g(r_1 + r_2 - 2). \tag{9}$$

This inequality is satisfied for $g \geq 10$ and

$$\boldsymbol{r} \in \{(2,1),(3,1),(4,1),(2,2),(3,2),(4,2),(3,3),(4,3)\},$$

which are all values of $\boldsymbol{r}$ in question. Observe that $a_1 \not\equiv a_2 \pmod g$ implies $(1,1) \notin \Lambda$. $\square$

This completes the proof of Theorems 7 and 5. We remark that the successive minima approach from the preceding proof can be applied to the case of distinct denominators b_1, b_2, too. However, the number of special cases that have to be checked separately is much larger than for equal denominators.

6 Completion of the Proof of the Main Theorem

If ξ_2 from Theorem 5 equals $\frac{1}{2}$, which corresponds to $\arg\alpha_2 = \theta_2 = \pi$ and thus to a real negative dominating characteristic root α_2 in Theorem 3, then the squares centred at $\boldsymbol{c} = (c_1, \frac{1}{4})$ or $\boldsymbol{c} = (c_1, \frac{3}{4})$, $c_1 \in \mathbb{R}$, do not contain any point $n\boldsymbol{\xi}$ mod 1. But in this case we need not consider all squares:

$$w_2 \sin(n\pi + \varphi_2) = (-1)^n w_2 \sin\varphi_2,$$

hence φ_2 can be absorbed in w_2, and we retain full generality if we assign a convenient value to φ_2.

Proposition 19. *Let $\xi_1 \in]0,1[\setminus\{\frac{1}{2}\}$. Then for all $c_1 \in \mathbb{R}$ there are $\epsilon > 0$ and $c_2 \in \mathbb{R}$ such that for infinitely many $n \in \mathbb{N}$*

$$n(\xi_1, \tfrac{1}{2}) \bmod 1 \ \in\ \mathcal{S}_{1/4-\epsilon}(\boldsymbol{c}) \bmod 1.$$

Proof. Let c_1 be an arbitrary real number with $0 \le c_1 < 1$. It suffices to show that there is $n \in \mathbb{N}$ with

$$n\xi_1 \bmod 1 \ \in \ \left]c_1 - \tfrac{1}{4}, c_1 + \tfrac{1}{4}\right[\bmod 1.$$

If ξ_1 is irrational, this follows immediately from Theorem 6. Now let ξ_1 be a rational number with denominator $b_1 > 0$. If b_1 is odd, then Lemma 10 shows

$$\left\{n(\xi_1, \tfrac{1}{2}) \bmod 1 : n \in \mathbb{N}\right\} = \left\{0, \tfrac{1}{b_1}, \ldots, \tfrac{b_1-1}{b_1}\right\} \times \left\{0, \tfrac{1}{2}\right\}.$$

For even b_1 we obtain

$$\begin{aligned}\left\{n(\xi_1, \tfrac{1}{2}) \bmod 1 : n \in \mathbb{N}\right\} &= \left\{0, \tfrac{2}{b_1}, \ldots, \tfrac{b_1-2}{b_1}\right\} \times \{0\} \\ &\quad \cup \left\{\tfrac{1}{b_1}, \tfrac{3}{b_1}, \ldots, \tfrac{b_1-1}{b_1}\right\} \times \left\{\tfrac{1}{2}\right\}.\end{aligned}$$

In both cases the desired result follows from $1/b_1 < \frac{1}{2}$. □

Proposition 19 settles the case of Theorem 4 where α_1 or α_2 is a negative real number.

Finally let us see what happens if the pair of denominators of $(\theta_1/2\pi, \theta_2/2\pi)$ in Theorem 4 is $(5,5)$, $(6,3)$ or $(8,4)$, where $\theta_k = \arg \alpha_k$ as usual. According to Theorem 7, our argument with lattice points in squares fails for these values. Fortunately, it is straightforward to show directly that the purely periodic sequences $(b(n))$ arising from these values oscillate.

Once again we can appeal to the symmetries noted in Lemma 8. Indeed, swapping θ_1 and θ_2 does no harm, and the sign of θ_k can be absorbed in w_1 and φ_1. It turns out that for all three pairs of denominators it suffices to consider the numerators $(a_1, a_2) = (2, 2)$.

Proposition 20. *Let w_1, w_2 be nonzero real numbers and φ_1, φ_2 be real numbers. Then the sequence defined by*

$$b(n) := w_1 \sin(n\theta_1 + \varphi_1) + w_2 \sin(n\theta_2 + \varphi_2), \quad n \ge 0,$$

has a positive and a negative entry for each of the following values of (θ_1, θ_2).

(*i*) $\theta_1 = \frac{1}{3}\pi, \theta_2 = \frac{2}{3}\pi$

(*ii*) $\theta_1 = \frac{1}{4}\pi, \theta_2 = \frac{1}{2}\pi$

(*iii*) $\theta_1 = \frac{4}{5}\pi, \theta_2 = \frac{2}{5}\pi$

Proof. It suffices to consider $w_1 = 1$. We set $s_i := \sin\varphi_i$ and $c_i := \cos\varphi_i$ for $i = 1, 2$, suppose $b(n) \ge 0$ for all $n \ge 0$ and derive a contradiction.

(i) From

$$\begin{aligned} b(0) + b(1) + b(5) &= 2s_1 \\ b(2) + b(3) + b(4) &= -2s_1 \\ b(0) + b(3) &= 2s_2 w_2 \\ b(1) + b(2) + b(4) + b(5) &= -2s_2 w_2 \end{aligned}$$

we deduce $s_1 = s_2 w_2 = 0$, so $s_2 = 0$ or $w_2 = 0$. If $w_2 = 0$, then

$$\begin{aligned} b(1) &= \tfrac{1}{2}\sqrt{3}c_1 \\ b(4) &= -\tfrac{1}{2}\sqrt{3}c_1 \end{aligned}$$

implies $c_1 = 0$, which contradicts $c_1^2 + s_1^2 = 1$. If, on the other hand, $s_2 = 0$, then $c_1 = 0$ follows from

$$\begin{aligned} b(1) + b(2) &= \sqrt{3}c_1 \\ b(4) + b(5) &= -\sqrt{3}c_1. \end{aligned}$$

(ii) Now

$$\begin{aligned} b(1) + b(3) &= \sqrt{2}c_1 \\ b(5) + b(7) &= -\sqrt{2}c_1 \\ b(1) + b(7) &= \sqrt{2}s_1 \\ b(3) + b(5) &= -\sqrt{2}s_1 \end{aligned}$$

shows $c_1 = s_1 = 0$, again contradicting $c_1^2 + s_1^2 = 1$.

(iii) Since

$$\begin{aligned} b(0) &= s_1 + s_2 w_2 \\ b(1) + b(2) + b(3) + b(4) &= -s_1 - s_2 w_2, \end{aligned}$$

we have $s_2 w_2 = -s_1$. Then we obtain

$$\begin{aligned} b(1) + b(4) &= -\sqrt{5}s_1 \\ b(2) + b(3) &= \sqrt{5}s_1, \end{aligned}$$

hence $s_1 = 0$ and $c_1 = \pm 1$. Therefore $w_2 = 0$ or $s_2 = 0$. If $w_2 = 0$, then the values

$$b(1) = \tfrac{1}{2}\sqrt{\tfrac{1}{2}(5 - \sqrt{5})}c_1$$

and

$$b(2) = -\tfrac{1}{2}\sqrt{\tfrac{1}{2}(5 + \sqrt{5})}c_1$$

have opposite signs. It remains to consider the case $s_2 = 0$. For each of the four possible values of $(c_1, c_2) = (\pm 1, \pm 1)$ the inequalities $b(0) \geq 0, \ldots, b(4) \geq 0$ form a linear system of inequalities in w_2. It is easy to check that none of these four systems are solvable. □

The proof of Theorem 4 is complete, hence Theorem 3 is established.

7 A Positive Real Root

If one of the dominating characteristic roots $\alpha_1, \ldots, \alpha_m$ is real positive, Conjecture 2 is not applicable. Consider the sequence defined by

$$a(n) := \sum_{k=1}^{m} w_k \sin(n\theta_k + \varphi_k) + 1 + o(1), \quad n \geq 0, \tag{10}$$

where $\theta_1, \ldots, \theta_m$, $w_1, \ldots, w_m$ are nonzero real numbers and $\varphi_1, \ldots, \varphi_m$ are real numbers. Here and throughout this section we assume that the coefficient of the real positive root is positive (and thus w.l.o.g. equals one). Analogous considerations apply for a negative coefficient. The behaviour of $(a(n))$ depends on how 1 compares to

$$S := -\inf_{n \geq 0} \sum_{k=1}^{m} w_k \sin(n\theta_k + \varphi_k) \in \,]-W, W], \tag{11}$$

where $W = \sum_{k=1}^{m} |w_k|$. The sequence $(a(n))$ is positive for large n if $S < 1$ (in particular, if $W < 1$), and it oscillates if $S > 1$. If $S = 1$, the behaviour of $(a(n))$ depends on how well $\sum_{k=1}^{m} w_k \sin(n\theta_k + \varphi_k)$ approximates -1 and possibly on the $o(1)$ term.

The preceding discussion gives a handy criterion only for $W < 1$, which was already obtained by Burke and Webb [4]. For $W \geq 1$ we confine ourselves to showing how $(a(n))$ behaves for almost all values of the parameters θ_k, φ_k and w_k.

Lemma 21. *Let $\boldsymbol{\alpha} \in \mathbb{R}^m$ and let $(\psi(n))$ be a sequence of positive real numbers such that $\sum_{n \geq 0} \psi(n)^m$ converges. Then the set of inequalities*

$$(n\xi_k - \alpha_k) \bmod 1 < \psi(n), \quad 1 \leq k \leq m$$

has infinitely many solutions $n \in \mathbb{N}$ for almost no $\boldsymbol{\xi} \in \mathbb{R}^m$.

Proof. See Cassels [14, Lemma VII.2.1]. □

In the following theorem we require the dominating characteristic roots to be simple. This assumption makes the remainder term $r(n)$ go to zero exponentially. Parts (i) and (iii) hold for multiple roots as well, since they only require $r(n) = o(1)$. Our proof of part (ii), however, breaks down for $m = 1$ in case of a multiple root, because then we can ensure only $r(n) = O(n^{-1})$ and this leads to a divergent series in Lemma 21.

Theorem 22. *Let $w_1, \ldots, w_m$ be nonzero real numbers with $W := \sum_{k=1}^{m} |w_k|$, $\varphi_1, \ldots, \varphi_m$ be real numbers and $(r(n))$ be a real sequence with $r(n) = O(\omega^n)$ for some $0 < \omega < 1$.*

(i) If $W < 1$, then for all $\boldsymbol{\theta} \in \mathbb{R}^m$ the sequence $(a(n))$ defined by

$$a(n) := \sum_{k=1}^{m} w_k \sin(n\theta_k + \varphi_k) + 1 + r(n)$$

is positive for large n.

(ii) If $W = 1$, then for almost all $\boldsymbol{\theta} \in \mathbb{R}^m$ the sequence $(a(n))$ is positive for large n.

(iii) If $W > 1$, then $(a(n))$ oscillates for almost all $\boldsymbol{\theta} \in \mathbb{R}^m$.

Proof. (i) is clear. (iii) follows from Theorem 6, because $\theta_1/2\pi, \ldots, \theta_m/2\pi, 1$ are linearly inedependent over the rationals for almost all $\boldsymbol{\theta}$. We proceed to prove (ii). Suppose $a(n) \leq 0$ for all n in an infinite set $I \subseteq \mathbb{N}$. To make $a(n)$

non-positive, $\sin(n\theta_k + \varphi_k)$ has to be very close to -1 if $w_k > 0$ and very close to 1 if $w_k < 0$. To be precise, we must have

$$\lim_{\substack{n\to\infty \\ n\in I}} f(n) = 0$$

for

$$f(n) := (f_1(n), \dots, f_m(n))$$

with

$$f_k(n) := \begin{cases} (n\theta_k + \varphi_k - \frac{1}{2}\pi) \bmod 2\pi, & w_k < 0; \\ (n\theta_k + \varphi_k - \frac{3}{2}\pi) \bmod 2\pi, & w_k > 0. \end{cases}$$

By Taylor expansion, we obtain

$$\begin{aligned} \sum_{k=1}^{m} w_k \sin(n\theta_k + \varphi_k) + 1 &= -\sum_{k=1}^{m} |w_k| + \frac{1}{2}\sum_{k=1}^{m} |w_k| f_k(n)^2 + O(\sum_{k=1}^{m} f_k(n)^4) + 1 \\ &= \frac{1}{2}\sum_{k=1}^{m} |w_k| f_k(n)^2 + O(\sum_{k=1}^{m} f_k(n)^4) \quad \text{as } n \to \infty \text{ in } I. \end{aligned}$$

Removing finitely many elements from I if necessary, we thus have

$$\sum_{k=1}^{m} w_k \sin(n\theta_k + \varphi_k) + 1 > \frac{w}{3}\sum_{k=1}^{m} f_k(n)^2, \quad n \in I,$$

where $w := \min_{0\le k\le m} |w_k| > 0$. Since $a(n) \le 0$ for $n \in I$, this implies

$$\sum_{k=1}^{m} f_k(n)^2 < -\tfrac{3}{w} r(n) = O(\omega^n), \quad n \in I,$$

hence for $1 \le k \le m$

$$f_k(n) = O(\omega^{n/2}) \quad \text{as } n \to \infty \text{ in } I.$$

According to Lemma 21 this holds for almost no $\boldsymbol{\theta}$. $\square$

Finer questions may be asked about the sets of measure zero alluded to in Theorem 22. As for part (ii) of the theorem, we note that there are $\varphi_1, \dots, \varphi_m$, $r(n)$ and infinitely many $\boldsymbol{\theta}$ such that $(a(n))$ oscillates for all nonzero $w_1, \dots, w_m$ with $W = 1$. To see this, define

$$\varphi_k := \begin{cases} \frac{1}{2}\pi, & w_k < 0; \\ \frac{3}{2}\pi, & w_k > 0, \end{cases}$$

let $\boldsymbol{\theta} \in \mathbb{Q}^m$ be arbitrary and $r(n) := (-\omega)^{n+1}$ for some $0 < \omega < 1$. Then $a(n) \ge (-\omega)^{n+1} = \omega^{n+1} > 0$ for odd n, and $a(n) = -\omega < 0$ if n is two times a common multiple of the denominators of $\theta_1, \dots, \theta_m$.

The preceding example is a special case of the following proposition, wich completely describes the behaviour of $(a(n))$ under the assumptions of part (ii) of Theorem 22 and the additional constraint $\theta_k/\pi \in \mathbb{Q}$ for $1 \le k \le m$.

Proposition 23. *Let $\theta_1, \dots, \theta_m$ be real numbers such that $\theta_k/2\pi$ is a rational number a_k/b_k for $1 \le k \le m$, let $\varphi_1, \dots, \varphi_m$ be real numbers, let $w_1, \dots, w_m$ be nonzero real numbers with $\sum_{k=1}^m |w_k| = 1$ and define*

$$a(n) := \sum_{k=1}^{m} w_k \sin(n\theta_k + \varphi_k) + 1 + o(1), \quad n \ge 0.$$

(i) If there is a k such that φ_k/π is irrational, then $(a(n))$ is positive for large n.

(ii) Suppose that $\varphi_k/2\pi$ is a rational number c_k/d_k for $1 \le k \le m$. If for all $1 \le k, l \le m$

$$b_k(A_k d_k - 4c_k) \equiv b_l(A_l d_l - 4c_l) \pmod{4 \gcd(d_k b_k, d_l b_l)} \tag{12}$$

with

$$A_k := \begin{cases} 1, & w_k < 0; \\ 3, & w_k > 0, \end{cases}$$

then there are infinitely many n with $b(n) = 0$, where

$$b(n) := \sum_{k=1}^{m} w_k \sin(n\theta_k + \varphi_k) + 1 \ge 0,$$

and the behaviour of $(a(n))$ depends in an obvious way on the sign of the $o(1)$ term for these n. If there are k, l such that (12) does not hold, then $(a(n))$ is positive for large n.

Proof. The purely periodic sequence $(b(n))$ satisfies $b(n) \ge 0$ for all $n \ge 0$. If none of its finitely many values are zero, then $(a(n))$ is positive for large n. We have $b(n) = 0$ if and only if $\sin(n\theta_k + \varphi_k)$ equals 1 for the k's with $w_k < 0$ and -1 for the k's with $w_k > 0$, i.e.,

$$n\theta_k + \varphi_k \equiv \tfrac{1}{2} A_k \pi \pmod{2\pi}, \quad 1 \le k \le m,$$

which is equivalent to

$$n\tfrac{a_k}{b_k} + \tfrac{\varphi_k}{2\pi} \equiv \tfrac{1}{4} A_k \pmod{1}, \quad 1 \le k \le m.$$

Clearly, this cannot hold if one of the φ_k/π is irrational. Under the assumption of part (ii), we are lead to the system of congruences

$$4d_k a_k n \equiv b_k(A_k d_k - 4c_k) \pmod{4d_k b_k}, \quad 1 \le k \le m.$$

Now the result follows from the Generalized Chinese Remainder Theorem. □

8 Conclusion and Future Directions

In order to extend our approach to Conjecture 2 to m dominating characteristic roots, we need to show that infinitely many $n(\xi_1, \dots, \xi_m)$ mod 1 lie in any given m-dimensional hypercube (modulo one) with side length $\frac{1}{2} - \epsilon$. Theorem 6 generalizes in the following way [14, Theorem III.5.IV]: The points $n\boldsymbol{\xi}$ mod 1 lie dense in the set of all $\boldsymbol{x} \in \mathbb{R}^m$ that satisfy $\langle \boldsymbol{u}, \boldsymbol{x} \rangle \in \mathbb{Z}$ for all integer vectors $\boldsymbol{u}$ with $\langle \boldsymbol{u}, \boldsymbol{\xi} \rangle \in \mathbb{Z}$. Again the case of rational $\xi_1, \ldots, \xi_m$ with equal denominators will be the crux of the proof. This case seems to become more and more difficult for fixed denominator g as m increases, since the set

$$\left\{ n(\tfrac{a_1}{g}, \ldots, \tfrac{a_m}{g}) \bmod 1 : n \in \mathbb{N} \right\} \tag{13}$$

has g elements for all m, whereas the volume of the hypercube is $(\frac{1}{2} - \epsilon)^m$. Theorem 17 is certainly a valuable tool. Hadwiger [15] has extended Bender's two-dimensional result that we used in the proof of Proposition 15 to arbitrary dimension m. A significant extension of Lemma 18 is still needed. Anyway it is conceivable that the exceptional rational values $a_1/b_1, \ldots, a_m/b_m$ for which the hypercube might contain no point of (13) become unmanageable as m increases (Cf. Section 6).

Our results on a positive real characteristic root leave ample room for refinement. For instance, one could try to extend part (ii) of Theorem 22 to the case of a multiple root or to continue the discussion begun in Proposition 23 by relaxing the requirement that all θ_k/π be rational.

The Skolem–Mahler–Lech Theorem [3] describes the structure of the zero set $\{n : a(n) = 0\}$ of a recurrence sequence. It is the union of a finite set and finitely many arithmetic progressions. There might be an analogue of this result for the set $\{n : a(n) > 0\}$.

Finally, we have excluded algorithmics so far. We do not know whether the positivity of recurrence sequences is a decidable problem. Proving Conjecture 2 and giving an effective criterion instead of the metric Theorem 22 would lead to a decision procedure.

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