Numerical-Symbolic Methods for Parameter-Dependent Geometric Differential Equations

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ABSTRACT

In this paper, we propose a numerical-symbolic approximation method for some parameter-dependent elliptic geometric equations. In order to perform a detailed analysis of the discretization methods we investigate a model problem related to meancurvature type equations.

The main idea of our approach is to construct suitable finite element discretizations of the non-linear elliptic equations leading to systems of algebraic equations, which are subsequently solved using symbolic elimination methods. The main advantage of this approach is the possibility of computing numerical approximations in a symbolic way with respect to some parameters, even if the discretization has more than one solutions.

Due to the high computational effort inherent in symbolic elimination method, the symbolic solutions can be computed on a very coarse grid only. For this sake we propose to use the symbolic elimination method as a coarse grid solver within a multigrid framework, and discuss some its issues.

Keywords

Finite element methods, symbolic elimination, nonlinear elliptic partial differential equations, systems of algebraic form, two-grid algorithms

1. INTRODUCTION

Many geometric optimization problems can be formulated via a representation of the geometric unknown as a graph or implicitely in level set form. The corresponding optimality system then yields a system of nonlinear partial differential equations with boundary conditions, like the famous Plateau problem (minimal surfaces) [8], other free boundary problems in variational form (cf. e.g. [2]), problems in mathematical imaging (cf. e.g. [1]) or, similarly, in level set approaches to shape optimization and reconstruction (cf. e.g [16, 4]).

Due to the strong nonlinearity of differential operators involved in geometric and similar partial differential equation, sometimes it is not easy to compute numerical solutions, e.g. from a generated finite element discretization. For minimizing the nonlinear discrete variational form, the standard Newton method applied to the parameter-free case will not converge globally in general, if the discrete operator is nonconvex. For parameter-dependent systems (like viscous solutions corresponding to first-order equations), finding a good initial guess is not trivial and very difficult by purely numerical methods. As we shall see in this paper, this problem can be resolved by using finite element discretization combined with a cascadic multigrid approach, where the coarse grid solution is computed directly by symbolic methods (cf. e.g. [18, 19]).

We shall investigate the numerical-symbolic approach for a specific model problem of the form

$$\sqrt{1+|\nabla u|^2}\ \mathrm{div}\ \left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \varepsilon \qquad \text{in } \Omega, \qquad (1)$$

subject to Dirichlet boundary conditions $u = f$ for a given sufficiently smooth function f on $\partial\Omega$, with Ω being a regular bounded domain in $\mathbb{R}^d$. Problem (1) is a geometric partial differential equation for the curve or surface determined by the graph of u, whose mean curvature is determined as

$$H = \ \mathrm{div}\ \left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right).$$

For a real parameter ε, the partial differential equation (1) is therefore related to the problem of determining curves of prescribed curvature (cf. [7]) and to a forward Euler time-discretization of the mean curvature flow of graphs (cf. [6])

$$\sqrt{1+|\nabla u|^2}\ \mathrm{div}\ \left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \varepsilon(u-u_0),$$

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where $\varepsilon = (\Delta t)^{-1}$ for the time step Δt and u_0 is the (given) initial value.

A major challenge in order to apply symbolic elimination methods to finite element discretization of nonlinear partial differential equations is to transfer the discrete nonlinear system into a system of algebraic equations. The obvious way of obtaining an algebraic system after discretization consists of rewriting the geometric partial differential equation as an equation with polynomial nonlinearity before discretization. This approach immediately yields algebraic equations after finite element discretization, but as we shall see below it may destroy a divergence structure in the equation so that the weak formulation involves also second derivatives. As a consequence, one might have to use very particular discrete subspaces and the convergence analysis of the discretization may require rather strong conditions. We shall detail these issues in spatial dimension two.

As an alternative to the finite element discretization based on the algebraic form, one can also perform a discretization using a weak form related to the divergence structure, i.e.,

$$\int_\Omega \frac{\nabla u \cdot \nabla v + \varepsilon}{\sqrt{1+|\nabla u|^2}}\, dx = 0, \qquad (2)$$

for all suitable test functions v (following the classical approach by Johnson and Thomeé [15] for the Plateau problem). Such a discretization does not yield an algebraic system, but as we shall show below, the discrete system can be approximated by an algebraic system after a simple perturbation. The perturbation is of higher order in terms of the discretization size and therefore will not destroy the convergence properties of the finite element method.

The paper is organized as follows: In the Section 2 we investigate the numerical-symbolic solution of the model problem for one dimension ($d = 1$), i.e., the graph of u representing a curve, which serves to present the basic idea in a simple way. Section 3 is devoted to a discussion of the two-dimensional case, and the difficulties arising in the direct discretization of the algebraic form, for which we provide a convergence results. In Section 4 we present results of computational experiments in the two-dimensional case and illustrate the properties of the parameter-dependent discrete solution(s) obtained by symbolic computation.

2. NUMERICAL-SYMBOLIC COMPUTATION OF CURVES

In this section, we initiate the idea by investigating (1) in the one-dimensional case $\Omega = [0,1]$, where

$$H = \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = \frac{d^2u}{dx^2}\left(1+|\frac{du}{dx}|^2\right)^{-3/2}$$

represents the curvature of the curve $\Gamma = \{(x,u(x)) \mid x \in \Omega\}$, and $Q = \sqrt{1+|\frac{du}{dx}|^2}$ is the infinitesimal curve length. Using the relation for H, (1) can be rewritten as the elliptic differential equation

$$-\frac{d^2u}{dx^2} + \varepsilon(1+(\frac{du}{dx})^2) = 0 \quad \text{in } \Omega, \qquad u(0) = u(1) = 0. \quad (3)$$

For simplicity we restrict our attention to the case $f \equiv 0$, but analogous reasoning is possible for arbitrary boundary values.

The solution of (3) only exists for $-\pi < \varepsilon < \pi$ (for larger values of $|\varepsilon|$ the curves with prescribed curvature are not graphs of a function) and can be computed analytically in dependence of ε as

$$u(x;\varepsilon) = \frac{1}{\varepsilon}\left[\log\left(\cos\frac{\varepsilon}{2}\right) - \log\left(\cos(\varepsilon x - \frac{\varepsilon}{2})\right)\right].$$

The graph of u as a function of x and ε is illustrated in Figure 1.

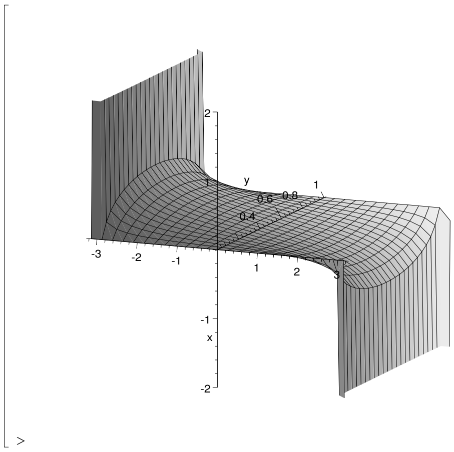

Figure 1: $u(x;\varepsilon)$

2.1 Discretization of the Algebraic Form

Let $0 = x_0 < x_1 < \ldots < x_N < x_{N+1} = 1$ and let $h = \sup_i |x_{j+1} - x_j|$. We discretize the problem into piecewise linear finite elements, i.e., we define the standard basis functions φ_j, $j = 1, 2, \ldots, N$, via

$$\varphi_j(x) = \begin{cases} \frac{x - x_{j-1}}{x_j - x_{j-1}} & \text{if } x_{j-1} \le x \le x_j, \\ \frac{x_{j+1} - x}{x_{j+1} - x_j} & \text{if } x_j \le x \le x_{j+1}, \\ 0 & \text{else,} \end{cases}$$

and $\mathcal{V}^h$ =span$\{\varphi_j\}$. For $\varepsilon > 0$, we call $u_h(.;\varepsilon) \in \mathcal{V}^h$ a discrete solution of (3) if

$$\int_0^1 \frac{du_h}{dx}(x;\varepsilon)\frac{dv}{dx}(x)dx + \varepsilon\int_0^1 (1+\left(\frac{du_h}{dx}(x;\varepsilon)\right)^2)v(x)dx = 0,$$

for all $v \in \mathcal{V}^h$.

For $\varepsilon = 0$ the equation has contains a unique solution and the finite element convergence theory is standard (also for non-homogeneous boundary values, cf. [5]). In the case of homogeneous boundary values, it is easy to see that the unique discrete solution satisfies $u_h(\cdot;0) \equiv 0$. The variation with respect to ε at $\varepsilon = 0$, i.e., $w_h := \frac{\partial u_h}{\partial \varepsilon}(\cdot;0)$ satisfies,

$$\int_0^1 \frac{dw_h}{dx}\frac{dv}{dx}\,dx + \varepsilon \int_0^1 v(x)\,dx = 0, \qquad \forall\, v \in \mathcal{V}^h,$$

and again by standard theory for the finite element discretization of linear elliptic equations, we conclude that there is a unique solution w_h. Hence, by the implicit function theorem we obtain the existence and local uniqueness of a discretized solution $u_h(x;\varepsilon)$ for ε sufficiently small, and the finite element convergence theory can be carried out in a straight-forward way. For large ε, there is no solution of (3) and hence, a convergence theory is not of interest anyway.

If we use the unique representation in the form

$$u_h(x;\varepsilon) = \sum_{j=1}^{N} c_j(\varepsilon)\ \varphi_j(x),$$

with a vector $\mathbf{c}(\varepsilon) = (c_j(\varepsilon))_{j=1}^N \in \mathbb{R}^N$, then the finite element discretization yields a system of quadratic equations for the vector $\mathbf{c}(\varepsilon)$. The solution of this algebraic system is then computed directly in dependence of ε via symbolic elimination approaches (cf. [18]).

For the numerical-symbolic solution we used a regular grid $x_j = \frac{j}{N+1}$. In all computational experiments we observed that indeed there exists no real solutions for ε large, in particular $|\varepsilon| > \pi$, and that the finite element solution is unique for $\varepsilon = 0$. For intermediate values of $|\varepsilon|$ however, we found two different real solutions of the discrete problem, with a lower branch approximating the real solution and an upper branch diverging as $\varepsilon \to 0$. The behavior is illustrated in Figures 2 and 3 for $h = \frac{1}{4}$ and $h = \frac{1}{11}$, respectively, by plotting $u(x;\varepsilon)$ versus ε at $x = \frac{1}{2}$, respectively. One observes in particular that the second branch includes oscillations for the larger value of ε, which indicates its instability.

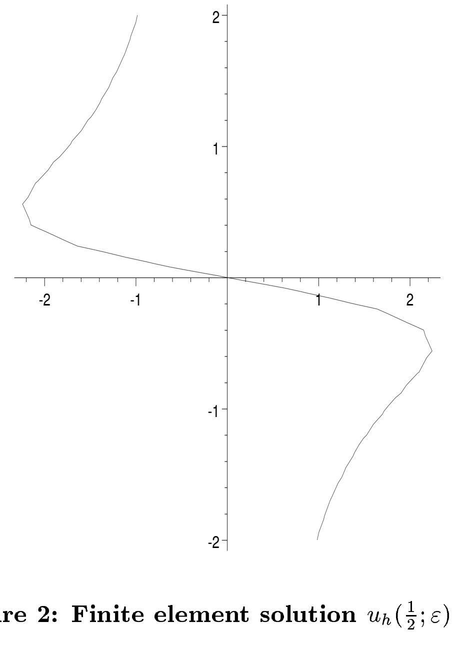

Figure 2: Finite element solution $u_h(\frac{1}{2};\varepsilon)$ for $h = \frac{1}{4}$.

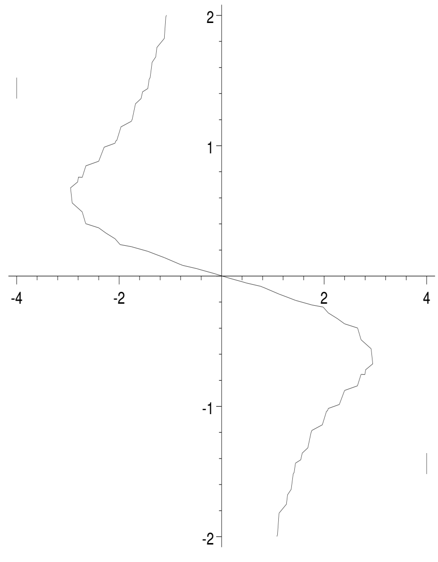

Figure 3: Finite element solution $u_h(\frac{1}{2};\varepsilon)$ for $h = \frac{1}{11}$.

From our numerical results, it becomes clear that a Newton-type method might converge to the wrong discrete solution. Even if additional continuation is used, further problem might appear for ε being close to π since the convergence radius will go to zero. However, those problems can be avoided if we can use the symbolic computation.

2.2 Discretization of the Divergence Form

As an alternative of a direct discretization of the algebraic form, we consider an approach via the weak form (2). We use the same finite element discretization and basis functions as introduced in the previous section, for the sake of simplicity with $x_j = \frac{j}{N+1} = jh$ (but analogous reasoning is possible for arbitrary meshes). We consequently consider the computation of a discrete solution $\tilde{u}_h(.,\varepsilon)$ satisfying

$$\int_\Omega \frac{\nabla \tilde{u}_h(.,\varepsilon)\cdot\nabla v + \varepsilon}{\sqrt{1+|\nabla \tilde{u}_h(.,\varepsilon)|^2}}\,dx = 0, \tag{4}$$

where in this case $\nabla = \frac{d}{dx}$. Using again the representation with respect to the basis functions in the form $\tilde{u}_h(x;\varepsilon) = \sum \tilde{c}_j(\varepsilon)\varphi_j(x)$, we obtain the discrete system (with $\tilde{c}_0 = \tilde{c}_{N+1} = 0$)

$$\frac{\tilde{c}_j - \tilde{c}_{j-1} + \frac{\varepsilon}{2}h^2}{\sqrt{h^2 + (\tilde{c}_j - \tilde{c}_{j-1})^2}} + \frac{\tilde{c}_j - \tilde{c}_{j+1} + \frac{\varepsilon}{2}h^2}{\sqrt{h^2 + (\tilde{c}_j - \tilde{c}_{j+1})^2}} = 0,$$

$j = 1,\ldots,N$. We now multiply the equations by $h^{-2}Q_j^3$ where

$$Q_j = \sqrt{h^2 + \frac{1}{2}(\tilde{c}_j - \tilde{c}_{j-1})^2 + \frac{1}{2}(\tilde{c}_j - \tilde{c}_{j+1})^2},$$

to obtain the nonlinear system

$$\begin{aligned} A_j(\tilde{\mathbf{c}}) &= h^{-2}Q_j^3\Big(\tfrac{\tilde{c}_j - \tilde{c}_{j-1} + \frac{\varepsilon}{2}h^2}{\sqrt{h^2+(\tilde{c}_j-\tilde{c}_{j-1})^2}} \\ &+ \tfrac{\tilde{c}_j - \tilde{c}_{j+1} + \frac{\varepsilon}{2}h^2}{\sqrt{h^2+(\tilde{c}_j-\tilde{c}_{j+1})^2}}\Big) = 0. \end{aligned}$$

In order to obtain an approximating algebraic system, we use a Taylor expansion. More precisely, let $t_j := \frac{1}{2}(\tilde{c}_j - \tilde{c}_{j-1})^2 - \frac{1}{2}(\tilde{c}_j - \tilde{c}_{j+1})^2$, then

$$\begin{aligned}\frac{h^{-2}Q_j^3}{\sqrt{h^2+(\tilde{c}_j-\tilde{c}_{j-1})^2}} &= \frac{h^{-2}Q_j^3}{\sqrt{Q_j^2+t_j}} \\ &= Q_j^2 - \frac{1}{2}h^{-2}t_j + \mathcal{O}(t_j^2), \\ \frac{h^{-2}Q_j^3}{\sqrt{h^2+(\tilde{c}_j-\tilde{c}_{j+1})^2}} &= \frac{h^{-2}Q_j^3}{\sqrt{Q_j^2-t_j}} \\ &= Q_j^2 + \frac{1}{2}h^{-2}t_j + \mathcal{O}(t_j^2).\end{aligned}$$

Under sufficient smoothness assumptions we have at least $t_j = \mathcal{O}(h^2)$, and hence the perturbation is of order h^4. Ignoring higher-order terms with respect to t_j, we obtain the approximate equation operator

$$\begin{aligned}B_j(\mathbf{c}) &= (c_j - c_{j-1} + \frac{\varepsilon}{2}h^2)(1 + \frac{3}{4}h^{-2}(c_j - c_{j+1})^2 \\ &\quad + \frac{1}{2}h^{-2}(c_j - c_{j-1})^2) \\ &\quad + (c_j - c_{j+1} + \frac{\varepsilon}{2}h^2)(1 + \frac{3}{4}h^{-2}(c_j - c_{j-1})^2 \\ &\quad + \frac{1}{2}h^{-2}(c_j - c_{j+1})^2) \\ &= (2c_j - c_{j-1} - c_{j+1}) + \varepsilon(h^2 + \frac{1}{2}(c_j - c_{j+1})^2 \\ &\quad + \frac{1}{2}(c_j - c_{j-1})^2) \\ &\quad + \frac{1}{4}h^{-2}(2c_j - c_{j-1} - c_{j+1})^3.\end{aligned}$$

We subsequently compute the discrete solution as

$$u_h(x;\varepsilon) = \sum c_j(\varepsilon)\varphi_j(x)$$

with $\mathbf{c}(\varepsilon)$ solving

$$B_j(\mathbf{c}(\varepsilon)) = 0, \qquad j = 1, \ldots, N.$$

Note that the discrete operator $\tilde{B}$ corresponding to the discretization of the algebraic form in the previous section is given by

$$\tilde{B}_j(\mathbf{c}(\varepsilon)) = (2c_j - c_{j-1} - c_{j+1}) + \varepsilon(h^2 + \frac{1}{2}(c_j - c_{j+1})^2 + \frac{1}{2}(c_j - c_{j-1})^2)$$

so that the discrete equations only differ by the higher-order term $\frac{1}{4}h^{-2}(2c_j - c_{j-1} - c_{j+1})^3$. Consequently, we may expect similar behaviour of the discrete solutions, which is indeed confirmed by the numerical experiments. In particular, we obtain again two branches of solutions, as illustrated for $h = \frac{1}{4}$ in Figure 4.

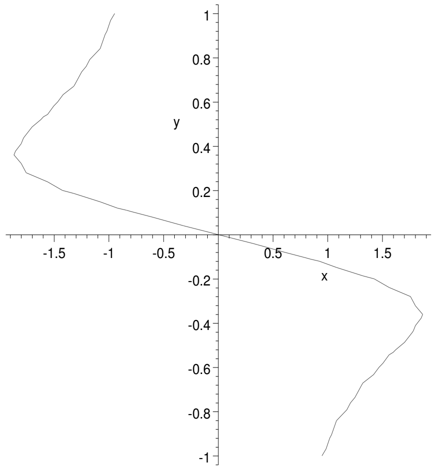

Figure 4: Finite element solution $u_h(\frac{1}{2};\varepsilon)$ of the divergence form.

3. NUMERICAL-SYMBOLIC COMPUTATION OF SURFACES

We now consider problem (1) for $\Omega \subset \mathbb{R}^2$. With the notation $\frac{\partial u}{\partial x} = u_x$, $\frac{\partial^2 u}{\partial xy} = u_{xy}$, we can rewrite (1) in algebraic form as

$$\begin{aligned}&\text{div } ((1+u_y^2)u_x, (1+u_x^2)u_y) - 6u_{xy}u_xu_y \\ &+\varepsilon(1+u_x^2+u_y^2) = 0\end{aligned} \tag{5}$$

with the boundary value $u = f$ on $\partial\Omega$, where f is smooth function and Ω is a given regular domain.

Multiplying with a smooth test function v and integrating the divergence term by part, we obtain a weak formulation, namely to find u with $u = f$ on $\partial\Omega$ such that

$$\begin{aligned}&\int_\Omega [(1+u_y^2)u_xv_x + (1+u_x^2)u_yv_y + 6u_{xy}u_xu_yv \\ &-\varepsilon(1+u_x^2+u_y^2)v] = 0,\end{aligned} \tag{6}$$

for all sufficiently smooth functions v vanishing on $\partial\Omega$. Using a suitable finite element space $S_h(\Omega)$, the discretization consists in finding $u_h \in S_h(\Omega)$ with $u_h|_{\partial\Omega} = f_h$ satisfying

$$\begin{aligned}&\int_\Omega [(1+u_{h,y}^2)u_{h,x}v_x + (1+u_{h,x}^2)u_{h,y}v_y \\ &+6u_{h,xy}u_{h,x}u_{h,y}v - \varepsilon(1+u_{h,x}^2+u_{h,y}^2)v] \\ &= 0, \qquad \forall\, v \in S_h^0(\Omega),\end{aligned} \tag{7}$$

where $S_h^0(\Omega)$ is the subspace of functions in $S_h(\Omega)$ vanishing on the boundary. The discretization then corresponds again to an algebraic system for the coefficients with respect to a suitable set of basis functions, and this algebraic system is then solved by symbolic methods, which are introduced for the case $\varepsilon = 0$ in [11, 10].

A major difficulty in this approach is that the weak form still includes a mixed second derivative, which cannot be eliminated. In particular, in order to obtain a well-defined discrete form, the finite element space $S_h(\Omega)$ must admit mixed second derivatives, i.e.,

$$S_h(\Omega) \subset \{u \in W^{1,\infty}(\Omega) \mid u_{xy} \in L^2(\Omega)\}.$$

This property is satisfied e.g. for $S_h(\Omega)$ being a subspace of piecewise bilinear functions on a rectangular mesh (for a rectangular domain Ω), where the edges are parallel to the coordinate axis, or for standard C^2 elements on arbitrary meshes and domains. In particular the finite element convergence analysis becomes rather complicated in such cases, as we shall work out in detail in the case of bilinear elements on a rectangular mesh.

The idea of using symbolic computation on a coarse grid has been used for the Plateau problem without parameter-dependence (cf. [9, 11]), and integrated into a two-grid algorithms with purely numerical computations on the fine grid (cf. [20, 21]). We shall discuss the solution by multigrid methods

3.1 Convergence of the Discretization

To prove the convergence theory associated to the finite element approximation (7), we first need the following lemma:

LEMMA 3.1. *Let $u \in W^{2,\infty}(\Omega)$ satisfy (5), and let for $v, w \in S_h(\Omega)$,*

$$R(u,w,v) := A(w,v) - A(u,v) - A'(u; w-u, v),$$

where

$$A'(u;w,v) = \int_\Omega (\sum_{i,j=1}^{2} a_{ij} w_{(i)} v_{(j)} + \sum_{i=1}^{2} b_i w_{(i)} v)$$

is the Jacobian of (5), with the coefficients $a_{ij}, b_i, c,\ i,j = 1,2$ given by

$$a_{11}(u) = 1 + u_y^2,\ a_{22}(u) = 1 + u_x^2,$$

$$a_{12}(u) = a_{21}(u) = -u_y u_x,$$

$$b_1(u) = 3u_y u_{xy} - 3u_{yy} u_x - 2\varepsilon u_x,$$

and

$$b_2(u) = 3u_x u_{xy} - 3u_{xx} u_y - 2\varepsilon u_y.$$

Then $u_h \in S_h(\Omega)$ solves (5) if and only if

$$A'(u; u_h, v) = R(u, u_h, v), \qquad \forall v \in S_h(\Omega).$$

Moreover, if

$$\|\partial_{xy} u_h\|_{0,\infty} + \|u_h\|_{1,\infty} \le K$$

and $S_h\Omega) = S_h^1\Omega)$ is constructed by bilinear rectangular mesh interpolation, then the remainder R satisfies

$$R(u, u_h, v) \le C(K) \|u - u_h\|_{1,\infty}^2 \|v\|_{1,1}$$

with $C(K)$ independent of ε.

PROOF. Set $G(t) = A(u + t(u_h - u), v)$. Then we have identity

$$G(1) = G(0) + G_t(0) + \int_0^1 G_{tt}(t)(1-t)dt.$$

By applying identity

$$\begin{aligned}
&\int_\Omega (w_{xy} - w_{h,xy}) u_x u_y v \\
&= -\int_\Omega (w_x - w_{h,x}) u_{xy} u_y v + (w_x - w_{h,x}) u_x u_{yy} v \\
&\quad + (w_x - w_{h,x}) u_x u_y v_y \\
&= -\int_\Omega (w_y - w_{h,y}) u_{xx} u_y v + (w_y - w_{h,y}) u_x u_{xy} v \\
&\quad + (w_y - w_{h,y}) u_x u_y v_x,
\end{aligned}$$

then

$$\begin{aligned}
&G_t(0) = \partial_t|_{t=0} A(u + t(u_h - u), v) \\
&= \int_\Omega [(1 + u_y^2)(u_{h,x} - u_x) v_x + (1 + u_x^2)(u_{h,y} - u_y) v_y \\
&\quad + 2u_x u_y (u_{h,x} - u_x) v_y + 2u_x u_y (u_{h,y} - u_y) v_x \\
&\quad + 6u_{xy} u_y (u_{h,x} - u_x) v + 6u_{xy} u_x (u_{h,y} - u_y) v \\
&\quad + 6u_x u_y (u_{h,xy} - u_{yx}) v - 2\varepsilon u_x (u_{h,x} - u_x) v \\
&\quad - 2\varepsilon u_y (u_{h,y} - u_y) v] \\
&= A'(u; w_h - w, v),
\end{aligned}$$

where the coefficients of A' satisfy the given conditions of the lemma.

And, by taking

$$R(u, u_h, v) := \int_0^1 G_{tt}(t)(1-t)dt,$$

a standard estimate using the Hölder inequality yields

$$\begin{aligned}
&|R(u, u_h, v)| \le \max|G_{tt}(t)| \\
&\le C(K) \|u - u_h\|_{1,\infty}^2 \|v\|_{1,1}.
\end{aligned}$$

Finally, if u_h solves (5), then $G(1) = 0$ and this completes the proof. □

We mention that the above estimate can be derived in an analogous way for any finite element discretizations $S_h(\Omega)$ if $\|u_h\|_{2,\infty} \le K$. The convergence theory can now be derived using the fact that $A'(u; \cdot, \cdot)$ is elliptic for $\varepsilon \ne 0$.

THEOREM 3.1. *Let $\Omega = [0,1]^2$, $u \in W^{2,\infty}(\Omega)$ be a solution of (5) with $u = f$ on $\partial\Omega$. Moreover, let $S_h(\Omega)$ be a finite element subspace consisting of piecewise bilinear continuous functions on a rectangular grid. Moreover, let f_h be a linear interpolation f on the boundary segments of the grid. Then there exists a constant $c > 0$, such that for h sufficiently small, there exists a solution $u_h \in S_h(\Omega)$ of (7) with $u_h = f_h$ satisfying*

$$\|u_h - u\|_{1,\infty} \le c\ h. \tag{8}$$

PROOF. For any solution u, we define a nonlinear operator $\Phi : S_h(\Omega) \to S_h(\Omega)$ via

$$A'(u; \Phi(v) - u, \phi) = R(u, v, \phi), \qquad \forall \phi \in S_h(\Omega). \tag{9}$$

Using ellipticity of A', it can be shown by standard arguments that Φ is well-defined and a continuous operator. Let P_h be the standard Galerkin projection operator associated to the bilinear form A', i.e., $P_h(u) = f_h$ on $\partial\Omega$ and

$$A'(u; P_h(u) - u, \phi) = 0, \quad \forall \phi \in S_h^0(\Omega).$$

For $u \in W^{2,\infty}(\Omega)$, the assumptions of Theorem 8.1.11 and Corollary 8.1.12 in [3] are satisfied, which implies the existence of a positive constant C such that

$$\|u - P_h(u)\|_{1,\infty} \le C\ h\ \|u\|_{2,\infty}.$$

associated

Define the set

$$B = \{v \in S_h(\Omega) : \|v - P_h(u)\|_{1,\infty} \le Ch\ \},$$

then by inverse estimates, there exist $C_1, C_2 > 0$ depending on u only, such that

$$\begin{aligned}\|\partial_{xy}(v-u)\|_{0,\infty} &\le \|\partial_{xy}(v - P_h(u))\|_{0,\infty} \\ &\quad + \|\partial_{xy}(P_h(u) - u)\|_{0,\infty} \\ &\le C_1 h^{-1}\|v - P_h(u)\|_{1,\infty} + C_2 \\ &\le 2C_1 C + C_2.\end{aligned}$$

Hence, $\|\partial_{xy} v\|_{0,\infty}$ is uniformly bounded with respect to h for v in the subspace $S_h(\Omega)$ of bilinear elements.

We now prove that $\Phi(B) \subset B$. In fact, when We substitute ϕ in (9) by using discrete Green functions $\phi = g^z_{h,x}$ and $\phi = g^z_{h,y}$, where

$$A'(u; v, g^z_{h,x}) = v_x(z),$$

$$A'(u; v, g^z_{h,y}) = v_y(z),$$

apply the definition of Galerkin projection operator P_h, the last inequality of Lemma 3.1 and the properties of discrete Green functions (cf. [17]), then we obtain for all $v \in B$,

$$\begin{aligned}\|\Phi(v) - P_h(u)\|_{1,\infty} &\le C_0|\log h|\|u - v\|^2_{1,\infty} \\ &\le 2C_0|\log h|(\|P_h(u) - v\|^2_{1,\infty} \\ &\quad + \|u - P_h(u)\|^2_{1,\infty}) \\ &\le 2C_0|\log h|(C^2h^2 + C^2h^2) \\ &= 4C_0C^2|\log h|h^2 (\le Ch),\end{aligned}$$

for h sufficiently small.

By Brouwer's fixed point theorem, there exist a solution $u_h \in B$, such that $\Phi(u_h) = u_h$. And according to Lemma 3.1, u_h solves (7) and satisfies

$$\|u_h - u\|_{1,\infty} \le \|u_h - P_h(u)\|_{1,\infty} + \|u - P_h(u)\|_{1,\infty} \le 2Ch.$$

□

3.2 Discretization of the Divergence Form

In the following we discuss a two-dimensional version of the approximation of the divergence form (2), which can be performed on arbitrary triangular grids. For this sake we choose a standard finite element subspace $\mathcal{V}^h$ consisting of continuous piecewise linear functions on a triangular grid of size h. A finite element discretization consists in finding $\tilde{u}_h \in \mathcal{V}_h$ satisfying.

$$\int_\Omega \frac{\nabla\tilde{u}_h \cdot \nabla v + \varepsilon v}{\sqrt{1 + |\nabla\tilde{u}_h|^2}}\, dx = 0, \qquad \forall\, v \in \mathcal{V}_h, \tag{10}$$

With the standard set of nodal basis functions φ_j satisfying $\varphi_j(x_i) = \delta_{ij}$ for all grid points x_i we can represent the discrete solution as

$$\tilde{u}(x) = \sum_{j=1}^N \tilde{c}_j \varphi_j(x).$$

Using the local support of the basis functions and the fact that $\nabla\tilde{u}_j$ is constant on each triangle, the finite element approximation (10) can equivalently be written as

$$Q_j^3 \sum_{T\in\mathcal{T}(x_j)} \frac{1}{\sqrt{1 + |\nabla\tilde{u}_h|_T|^2}} \int_T (\nabla\tilde{u}_h \cdot \nabla\varphi_j + \varepsilon\varphi_j)\, dx = 0,$$

$j = 1, \dots, N$ with an arbitrary positive factor Q_j^3, where $\mathcal{T}(x_i)$ is the set of triangles whose nodes include x_j. Now let $A_j = \sum_{T\in\mathcal{T}(x_j)} |T|$ be the area of the triangles surrounding x_j, then we define

$$Q_j := \sqrt{1 + \sum_{T\in\mathcal{T}(x_j)} \frac{|T|}{A_j} |\nabla\tilde{u}_h|_T|^2}.$$

Then we can derive a similar first-order Taylor expansion of

$$\frac{Q_j^3}{\sqrt{1 + |\nabla\tilde{u}_h|_T|^2}} = \frac{Q_j^3}{\sqrt{Q_j^2 + t_j(T)}}$$

with respect to

$$\begin{aligned}t_j(T) &= |\nabla\tilde{u}_h|_T|^2 - \textstyle\sum_{T'\in\mathcal{T}(x_j)} \frac{|T'|}{A_j} |\nabla\tilde{u}_h|_{T'}|^2 \\ &= \textstyle\sum_{T'\in\mathcal{T}(x_j)} \frac{|T'|}{A_j} \left(|\nabla\tilde{u}_h|_T|^2 - |\nabla\tilde{u}_h|_{T'}|^2\right).\end{aligned}$$

By similar reasoning as in the one-dimensional case we can derive an approximating weak form this way, which is polynomial (of order three) in the coefficients c_j. The arising system of algebraic system for the coefficients can subsequently be solved by symbolic elimination techniques.

3.3 Multigrid Versions

At the current speed of symbolic elimination methods, the symbolic solution of the discretized problem can be carried out with reasonable efficiency on coarse grids only. Therefore it seems natural to couple the symbolic solution technique with a multigrid approach (cf. [13] for an overview of multigrid methods), where symbolic solutions are computed on the coarse grid, and purely numerical techniques are used on finer grids.

Since the step between finer grids with numerical techniques is standard, we only discuss a two grid version, with a coarse grid (of size H) and a fine grid (of size h). We assume to know suitable prolongation and interpolation operators $P_h^H : S_H(\Omega) \to S_h(\Omega)$ and $I_H^h : S_h(\Omega) \to S_H(\Omega)$, respectively. For given w, let $A'(w; \cdot, \cdot)$ be the bilinear map from Lemma 3.1. Then, we can immediately derive a cascadic multigrid method (similar to [9] for the parameter-free case) with exact symbolic solution on the coarse grid and a Newton-type correction at the fine grid.

Cascadic Two-Grid Algorithm.

1. Compute $u_H \in S_H(\Omega)$ such that
$$A(u_H, v) = 0, \quad \forall v \in S_H(\Omega).$$
2. Set $u_h^0 = P_h^H u_H$, and for $k = 1, \dots, k_*$ compute $u_h^k \in S_h(\Omega)$ from the linear equation
$$A'(u_h^{k-1}; u_h^k - u_h^{k-1}, v) = -A(u_h^{k-1}, v), \quad \forall v \in S_h(\Omega).$$
3. Set $u_h = u_h^{k_*}$.

We finally mention that in an analogous way, standard V-cycle multigrid methods can be constructed, using the symbolic solver as a coarse grid correction, taking advantage of the fact that the exact coarse grid can also be computed in dependence of multiple parameters. Hence, if these parameters represent the values of the restriction of the fine grid solution, the symbolic coarse-grid solution can be computed in a preprocessing step, and during the V-cycle one only has to evaluate the coarse grid solution without extra computational effort.

4. NUMERICAL EXPERIMENTS RELATED TO SURFACES

In this section, we illustrate the solutions of (7) obtained by the symbolic elimination. All results were computed using the computer algebra package "CASA" [14] based on Maple software.

4.1 Discrete Solutions for Fixed Parameters

Set the boundary condition $f = 0$ and let the finite element space consists of piecewise-bilinear functions on a regular rectangular grid in the domain $\Omega = [0, 1] \times [0, 1]$. We start with a grid consisting of 5×4 nodes, which produces (after elimination of the boundary nodes) 6 unknowns.

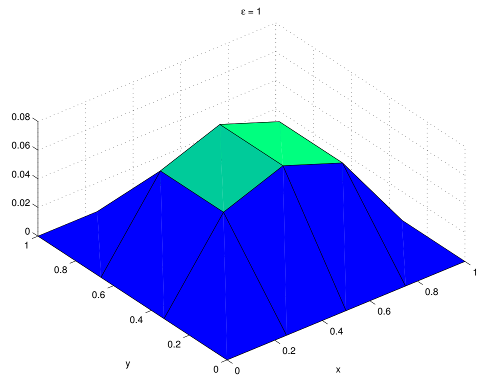

Figure 5: Finite element solution $u_h(\cdot;\varepsilon = 1)$.

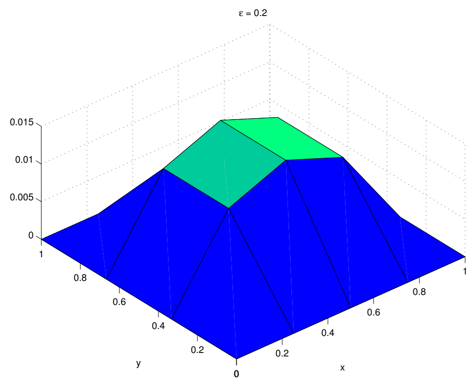

Figure 6: Finite element solution $u_h(\cdot;\varepsilon = 1/5)$.

We start by computing discrete solutions for fixed parameter ϵ on this mesh. The discrete solution turned out to be unique in all numerical experiments, which is a significant difference to the one-dimensional example, where the discrete solution was not unique or non-existent for a large range of the parameter values ϵ.. We illustrate the results for $\epsilon = 1$ and $\epsilon = 1/5$ in Figures 5 and 6.

If we change the average mesh to 5×5 nodes, we obtain a discretization fineness $h = 0.2$ and, after elimination of boundary nodes, 9 unknowns. The corresponding discrete solution for $\varepsilon = 1/5$ is plotted in Figure 7.

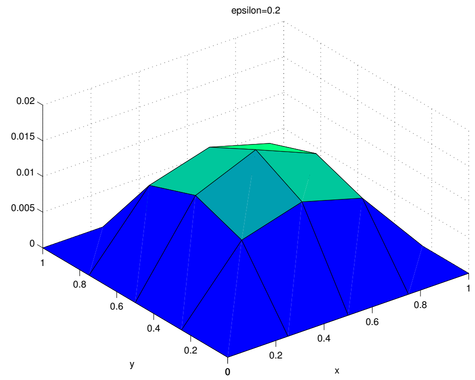

Figure 7: Finite element solution $u_h(\cdot;\varepsilon = 1/5)$, $h = 0.2$.

4.2 Parameter-Dependent Discrete Solutions

As mentioned above, the symbolic elimination methods directly compute the discrete solutions as a function of the parameter ε, and one may also consider u_h as a function of the parameter ε, a relation that can be illustrated by implicit function graphs at any point $(x, y) \in \Omega$. As an example Figure 8 shows the function graph of parameter-dependent solution $u_h(x = 0.5, y = 2/3; \varepsilon)$ computed at a rectangular grid with 5×4 nodes on $\Omega = [0, 1]^2$.

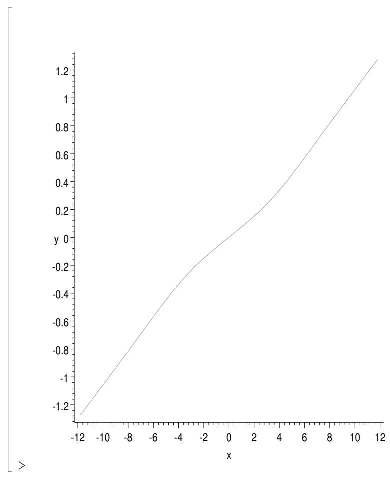

Figure 8: function curve of $u_h(x = 0.5, y = 2/3; \varepsilon)$

From a algebraic geometry point of view, the implicit function curve dependent on ε (represented by the x-axis) is isomorphic, and one observes that the elliptic problem in the two-dimensional domain contains a unique solution which does not depend on parameter ε. In other cases (such as in one dimension), where the existence or uniqueness of solutions might be changed with respect to the value of ε, hysteresis would be clearly reflected by singular points in those curves.

5. CONCLUSION

This paper discusses a combined way of solving parameter dependent elliptic equations by finite element methods and symbolic computation, highlighting difficulties that are obtained when discretizing geometric partial differential equations into an algebraic form. A related method has also been considered for the regularization of certain ill-posed problems in [12] (where the parameter is a regularization parameter), and can be generalized to various classes of problems, whose nonlinearity can be rewritten into (or approximated by) a polynomial form.

The size of the coarse grid problem is still strongly limited, since the symbolic elimination methods due not take advantage of the sparsity structure in the discrete problem. It might be a very important future task in symbolic computation to develop methods that can take sparsity into account.

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6. REFERENCES

[1] L. Alvarez, F. Guichard, P. Lions, and J. Morel. Axioms and fundamental equations of image processing. *Arch. Rat. Mech. Anal.*, 123:199–257, 1993.

[2] E. Bonnetier, R. Falk, and M. Grinfeld. Analysis of a one-dimensional variational model of the equilibrium shape of a deformable crystal. *Math. Model. Anal. Numer.*, 33:573–591, 1999.

[3] S. Brenner and R. Scott. *The Mathematical Theory of Finite Element Methods.* Springer-Verlag, 1994.

[4] M. Burger and S. Osher. A survey on level set methods for inverse problems and optimal design. *European J. Appl. Math.*, 2004. To appear.

[5] P. Ciarlet. *The Finite Element Method for Elliptic Problems.* North-Holland, 1978.

[6] K. Deckelnick and G. Dziuk. Error estimates for a semi-implicit fully discrete finite element scheme for the mean curvature flow of graphs. *Interfaces Free Bound.*, 2:341–359, 2000.

[7] G. Dziuk and J. Hutchinson. Finite element approximations and the dirichlet problem for surfaces of prescribed mean curvature. In *Mathematical visualization. Algorithms, applications, and numerics*, pages 73–87. Springer, Berlin, in H.C. Hege et al., eds., 1998.

[8] J. Gray. *The Hilbert Challenge.* Oxford University Press, 2000.

[9] H. Gu. The finite element approximation to the minimal surfaces subject to the plateau problems. In *Proceedings of The Sixth International Conference on Computational Structure Technology, Prague, Czech Rep.* Civil-comp Press, 2002.

[10] H. Gu. Generating minimal surfaces subject to the plateau problems by finite element method. In *Springer Lecture Notes in Computer Science 2542, Proceeding of the 5th International Conference of Numerical Methods and Applications*, pages 471–478. Bulgaria Acad. of Sci., 2002.

[11] H. Gu. Numerical methods and symbolic computations for generating minimal surfaces. In *RISC Tech. Report 03-10.* RISC-Linz, J.Kepler University, 2003.

[12] H. Gu and S. Kindermann. Solution analysis of denoising equations related to the mumford-shah functional. In *Proceedings of The 7th International Conference of Computational Structure Technology.* Lisboa, Portugal, Civil-comp Press, 2004.

[13] W. Hackbusch. *Multi-Grid Methods and Applications.* Springer, Berlin, 1985.

[14] R. Hemmecke, E. Hillgarter, and F. Winkler. Casa. In *Handbook of Computer Algebra: Foundations, Applications, systems*, pages 356–359. J.Grabmeier, E.Kaltofen, V.Weispfenning (eds.), Springer-Verlag, 2003.

[15] C. Johnson and V. Thomeé. Error estimates for a finite element approximation of a minimal surface. *Math. of Comp.*, 29(130):343–349, 1975.

[16] S. Osher and R. Fedkiw. *Level Set Methods and Dynamic Implicit Surfaces.* Springer-Verlag, New York, 2002.

[17] V. Thomeé, J. Xu, and N. Zhang. Superconvergence of gradient in piecewise linear finite element approximation to a parabolic problem. *SIAM J. of Numer. Anal.*, 26:389–414, 1989.

[18] D. Wang. *Elimination Methods.* Springer-Verlag, Vienna, 2001.

[19] F. Winkler. *Polynomial Algorithms in Computer Algebra.* Series "Texts and Monographs in Symbolic Computation", Springer-Verlag, Wien, New York, 1996.

[20] J. Xu. Two-grid discretization techniques for linear and nonlinear partial differential equation. *SIAM J. Numer. Anal.*, 33:1759–1777, 1996.

[21] J. Xu and A. Zhou. Local and parallel finite element algorithms based on two-grid discretizations for nonlinear problems. *Adv. Comput. Math.*, 14:393–327, 2001.