SECOND-ORDER NECESSARY CONDITIONS FOR NONLINEAR OPTIMIZATION PROBLEMS: THE DEGENERATE CASE

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Abstract. In this paper we derive second-order necessary conditions for optimality for an optimization problem with abstract constraints in Banach spaces. Results for the non-degenerate case derived earlier [13] are extended to the degenerate case. For the mathematical programming problem, where the constraints are given by equality and finitely many inequality constraints, our approach applies to the abnormal case, when the equality constraints are not regular and our results appear to be new even in this special case. Our second-order necessary conditions are contained in the gap between the standard necessary and sufficient conditions, where the only difference is the change from a non-strict to a strict inequality. Our results are formulated in such a way to be applicable also to vector optimization problems.

Key words. Second-order optimality conditions, abstract constraints, vector optimization.

AMS subject classifications. 49K27, 90C29, 90C30

1. Introduction. In this paper we study necessary second-order optimality conditions for minimization problems with abstract constraints of the form

$$g(x) \in K, \tag{1.1}$$

where the constraint mapping $g : X \to V$ carries a Banach space X into another Banach space V and K is a closed convex subset of V.

The feasible sets of various optimization problems can be formulated in the form (1.1) in a natural way. For instance, the possibly infinite dimensional mathematical programming problem with finitely many inequality constraints $g_i(x) \le 0$, $i = 1, \dots, m$ and an equality constraint $G(x) = 0$ fits into the scheme with $V = \mathbb{R}^m \times \hat{V}$, $K = \mathbb{R}^m_- \times \{0\}$ and $g = (g_1, \dots, g_m, G)$. Other examples are provided by semi-infinite programming problems, semi-definite programming problems and optimal control problems.

When studying necessary optimality conditions at a point $\bar{x}$ satisfying (1.1), usually a condition on the constraints is needed, since otherwise the necessary conditions trivially hold and therefore their utility for describing optimality is very limited. One classical condition in this setting is Robinson's condition [28]:

$$0 \in \operatorname{int}\,(g(\bar{x}) + g'(\bar{x})X - K) \tag{1.2}$$

Note, that in the case of the finite dimensional mathematical programming problem condition (1.2) reduces to the classical Mangasarian-Fromovitz constraint qualification. Under Robinson's condition, the structure of the set of tangent directions of the constraints is well established (see, e.g. [9]) and second-order conditions have been formulated by several authors, see [8], [12], [15], [21], [26], [27].

In a recent paper [13], we presented second-order necessary optimality conditions, which are shown to be best possible in a certain sense, under the non-degeneracy assumption

$$\operatorname{int}\,(g'(\bar{x})X - K) \neq \emptyset. \tag{1.3}$$

This condition is clearly weaker than Robinson's condition (1.2). For instance, for the mathematical programming problem condition (1.3) reduces to a condition on the equality constraint only, namely the well-known Lyusternik-condition $G'(\bar{x})X = \hat{V}$, which has been used by several authors (see, e.g. [6], [7], [25], [14]). However, it is well-known that there exist second-order necessary conditions for the mathematical programming problem which do not require the Lyusternik-condition

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$G'(\bar{x})X = \hat{V}$ to be satisfied. Avakov [3],[4] presented several generalizations of the Lyusternik theorem to the abnormal case of non-surjective operators $G'(\bar{x})$. These results have been extended by several authors, see e.g. [1], [2], [5], [10], [17], [23], [24].

In this paper we will derive second-order necessary optimality conditions in the degenerate case, i.e. condition (1.3) is dropped. The obtained results appear to be new even in the special case of the mathematical programming problem. Our approach is essentially based on the ideas presented in [13]. We will use both an observation made by Robinson [30], that a certain multifunction built by the objective and the constraints obtains a singular behavior at a local minimizer, and an accurate characterization of metric regularity of this multifunction by means of a certain signed distance function.

With this approach, when dealing with general constraints of the form (1.1), it does not require great effort to consider also the case of general objective functions. Thus the problem we consider in this paper is given by

$$(P) \qquad \text{L-minimize}\, f(x) \quad \text{subject to} \quad g(x) \in K,$$

where $f : X \to U$ is a mapping from the Banach space X to another Banach space U and where $L \subset U$ is a closed convex cone with nonempty interior, $\operatorname{int} L \neq \emptyset$. We define different kinds of local L-minimizers as follows.

Definition 1.1. *An element $\bar{x} \in X$ is called a* local weak minimizer *for (P), if $g(\bar{x}) \in K$ and if there exists a neighborhood N of $\bar{x}$ such that for each $x \in N$ with $g(x) \in K$, one has $f(x) - f(\bar{x}) \notin -\operatorname{int} L$. A local weak minimizer $\bar{x}$ is called a* strict local minimizer *for (P), if for each $x \in N \setminus \{\bar{x}\}$ with $g(x) \in K$, one has $f(x) - f(\bar{x}) \notin -L$. Finally, a weak local minimizer $\bar{x}$ is is called an* essential local minimizer of second order *for problem (P), if there exists some real $\beta > 0$ such that*

$$\max\{d(f(x) - f(\bar{x}), -L), d(g(x), K)\} \geq \beta \|x - \bar{x}\|^2, \quad \textit{for all } x \in N. \tag{1.4}$$

Of course, each essential local minimizer of second order is also a strict local minimizer.

Note that (P) includes the very common problem of constrained scalar minimization, for which $U = \mathbb{R}$ and $L = \mathbb{R}_+$. Local weak minimizers for (P) then amount to usual local minimizers and for essential local minimizers the so-called *quadratic growth condtion* is satisfied:

$$f(x) - f(\bar{x}) \geq \beta \|x - \bar{x}\|^2, \quad \forall x \in N : \; g(x) \in K$$

Another important particular case, for instance, is the Pareto maximization optimization problem with $U = \mathbb{R}^p$, $L = \mathbb{R}^p_-$.

For second-order optimality conditions for problem (P) in the multicriteria case we refer to [7], [11], [13], [18]-[20].

Given a feasible point $\bar{x} \in g^{-1}(K)$, fixed throughout this paper, we will now define a certain multifunction associated with (P) and $\bar{x}$. Let $h : X \to U \times V$ defined by

$$h(x) := (f(x) - f(\bar{x}),\ g(x)), \qquad C := (-L) \times K, \qquad Y := U \times V. \tag{1.5}$$

Then the multifunction $\Gamma : X \rightrightarrows Y$, given by

$$\Gamma(x) := h(x) - C, \tag{1.6}$$

will form the basis of our investigations. Throughout this paper we will use the following smoothness assumption on h:

Assumption 1. *h is Fréchet differentiable at $\bar{x}$ and for some radius $\bar{r} > 0$ and some scalar $\eta \geq 0$ we have*

$$\|h(x_1) - h(x_2) - h'(\bar{x})(x_1 - x_2)\| \leq \eta \max\{\|x_1 - \bar{x}\|, \|x_2 - \bar{x}\|\} \|x_1 - x_2\|$$

for all $x_1, x_2 \in \bar{x} + \bar{r}\mathcal{B}_X$.

In what follows we will use the norm $\|(u,v)\| := \max\{\|u\|, \|v\|\}$ on the product space $Y = U \times V$.

Our notation is fairly standard. In a normed space Z, $\mathcal{B}_Z := \{z \in Z : \|z\| \le 1\}$ denotes the closed unit ball and $\mathcal{S}_Z := \{z \in Z : \|z\| = 1\}$ denotes the unit sphere. The topological dual space is denoted by Z^*. $\langle z^*, z\rangle$ is the value $z^*(z)$ of the linear functional $z^* \in Z^*$ at $z \in Z$. For a set $D \subset Z$ we denote by $\sigma_D(\cdot)$ its support function, i.e. $\sigma_D(z^*) := \sup_{z\in D}\langle z^*, z\rangle$ and by $d(\cdot, D)$ the distance function, i.e. $d(z, D) = \inf_{y\in D} \|z - y\|$. For a convex set $D \subset Z$ we denote by $T_D(z)$ (respectively, $N_D(z)$) the common tangent cone (respectively, normal cone) of convex analysis at a point $z \in D$, i.e. we have $N_D(z) := \{z^* \in Z^* : \langle z^*, \zeta - z\rangle \le 0,\ \forall \zeta \in D\}$ and

$$T_D(x) = \{s : \liminf_{t\to 0_+} \frac{d(z + ts, D)}{t} = 0\} = \{s : \limsup_{t\to 0_+} \frac{d(z + ts, D)}{t} = 0\}.$$

If W is another normed space, we denote by $L(Z, W)$ the space of all continuous linear operators from Z into W. If $A \in L(Z, W)$, then $A^* : W^* \to Z^*$ denotes the adjoint operator of A. Finally, we denote by $\mathbb{T}$ the set of all sequences $(t_n) \to 0_+$.

Fritz-John-type optimality conditions for problem (P) can be written in the form

$$f'(\bar{x})^*u^* + g'(\bar{x})^*v^* = 0, \quad 0 \ne (u^*, v^*) \in L^* \times N_K(g(\bar{x})) \subset U^* \times V^*, \tag{1.7}$$

where $L^* := \{u^* \in U^* : \langle u^*, u\rangle \ge 0,\ \forall u \in L\}$ is the dual cone of the cone L. Setting $y^* := (u^*, v^*)$ and using the notation of h and C, the condition (1.7) can also be written more shortly as

$$h'(\bar{x})^*y^* = 0, \quad y^* \in N_C(h(\bar{x})), \quad y^* \ne 0. \tag{1.8}$$

In the sequel we will denote the set of multipliers y^* satisfying the Fritz-John conditions (1.8) by Λ_{FJ}. It should be noted that in general Λ_{FJ} may be empty. An additional condition has to be imposed to ensure the existence of a nontrivial multiplier y^* at a local weak minimizer for (P).

2. Preliminaries. In this section we will recapitulate partially the basic theory on second-order optimality conditions as presented in [13].

In a very general form, these conditions can be formulated by means of a function $\hat{d}_C(y, A, \kappa) : Y \times L(X, Y) \times \mathbb{R} \to \mathbb{R}$ given by

$$\hat{d}_C(y, A, \kappa) := \sup_{y^*\in\mathcal{S}_{Y^*}} \{\langle y^*, y\rangle - \sigma_C(y^*) - \kappa\|A^*y^*\|\}.$$

THEOREM 2.1. *([13, Theorem 3.2]) Suppose that Assumption 1 is satisfied at $\bar{x}$. If $\bar{x}$ is a local weak minimizer for (P) then*

$$\liminf_{\substack{x\to\bar{x}\\ \tau\to 0_+}} \frac{\hat{d}_C(h(x), h'(\bar{x}), \tau\|x - \bar{x}\|)}{\|x - \bar{x}\|^2} \ge 0. \tag{2.1}$$

Moreover, a feasible point $\bar{x}$ is an essential local minimizer of second order for (P) if and only if

$$\liminf_{\substack{x\to\bar{x}\\ \tau\to 0_+}} \frac{\hat{d}_C(h(x), h'(\bar{x}), \tau\|x - \bar{x}\|)}{\|x - \bar{x}\|^2} > 0. \tag{2.2}$$

The following Theorem give further details on the optimality conditions of Theorem 2.1 (see [13, Theorems 3.5, 3.6]

THEOREM 2.2.

1. *Suppose that a feasible point $\bar{x}$ is not an essential local minimizer of second order for (P). Then there exists a twice continuously differentiable mapping $\delta h := (\delta f, \delta g)$ satisfying $\delta h(\bar{x}) = 0$, $\delta h'(\bar{x}) = 0$ and $\delta h''(\bar{x}) = 0$, such that $\bar{x}$ is not a local weak minimizer for (P) with f and g replaced by $f + \delta f$ and $g + \delta g$, respectively.*

2. *Assume that at a feasible point $\bar x$ condition (2.1) holds and assume that*

$$\operatorname{int}(h'(\bar x)X - C) \neq \emptyset. \tag{2.3}$$

Then there exists a mapping $\delta h = (\delta f, \delta g) : X \to Y$ with $\delta h(x) = \psi(\|x - \bar x\|)y$, where $y \in Y$ and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a twice continuously differentiable function satisfying $\psi(0) = \psi'(0) = \psi''(0) = 0$, such that $\bar x$ is a strict local minimizer for (P) with f and g replaced by $f + \delta f$ and $g + \delta g$, respectively.

Remark: It follows from the proof of [13, Theorem 3.6] that the assertion of the second part of Theorem 2.2 does not depend on the special form of h and C, respectively. Indeed, for any closed convex set $C \subset Y$ and any mapping $h : X \to Y$, differentiable at some point $\bar x$ satisfying $h(\bar x) \in C$ and conditions (2.1) and (2.3), there exist a neighborhood N of $\bar x$ and a mapping $\delta h(x) = \psi(\|x - \bar x\|)y$ of the same kind as in the second part of Theorem 2.2, such that $d\big((h + \delta h)(x), C\big) > 0,\ \forall x \in N \setminus \{\bar x\}$.

Definition 2.3. *We call $\bar x$* non-degenerate *for the problem (P), if* $\operatorname{int}(h'(\bar x)X - C) \neq \emptyset$. *Conversely, if* $\operatorname{int}(h'(\bar x)X - C) = \emptyset$, *the element $\bar x$ is said to be* degenerate *for (P).*

The following theorem states some geometrical properties of the function $\hat d_C$: It can be treated as a signed distance function for certain sets.

Theorem 2.4. *([13, Theorem 2.6]) For each $A \in L(X, Y)$, each $y \in Y$ and each $\kappa \geq 0$ let $D_C(y, A, \kappa)$ be given by $D_C(y, A, \kappa) := y + \kappa A\mathcal{B}_X - C$. Then one has*

$$\hat d_C(y, A, \kappa) = \begin{cases} d(0, D_C(y, A, \kappa)) & \text{if } 0 \notin \operatorname{cl} D_C(y, A, \kappa) \\ 0 & \text{if } 0 \in \operatorname{bd} D_C(y, A, \kappa) \\ -\sup\{\rho \,:\, \rho\mathcal{B}_Y \subset D_C(y, A, \kappa)\} & \text{if } 0 \in \operatorname{int} D_C(y, A, \kappa) \end{cases}$$

It follows easily from the definition that $\hat d_C(\cdot, A, \kappa)$ is Lipschitz continuous with constant 1. Moreover we have the following property

Lemma 2.5. *([13, Lemma 2.8])Let $A \in L(X, Y)$, $y \in Y$ and $\kappa \geq 0$ be such that $\hat d_C(y, A, \kappa) < 0$. Then*

$$d(0, A^{-1}(C - y)) \leq \frac{\kappa}{d(y, C) - \hat d_C(y, A, \kappa)} d(y, C).$$

For the sake of completeness we mention some material from ([13]) also used in this paper. First, let us recall the notion of (local) metric regularity :

Definition 2.6. *Let $\Psi : X \rightrightarrows Y$ be a set-valued map, $\bar y \in \Psi(\bar x)$. The multifunction Ψ is called* metrically regular near $(\bar x, \bar y)$, *if there are neighborhoods $N_{\bar x}$, $N_{\bar y}$ of $\bar x$, $\bar y$, respectively and some $k > 0$ such that $d(x, \Psi^{-1}(y)) \leq k\, d(y, \Psi(x))$ for each $(x, y) \in N_{\bar x} \times N_{\bar y}$.*

The following two theorems are the base for the necessary optimality conditions of Theorem 2.1 and are also of substantial significance for this paper.

Theorem 2.7. *([13, Theorem 2.4]) Let (x_n) be a sequence converging to $\bar x$ such that for each n, Γ is metrically regular near $(x_n, 0)$. Then $\bar x$ is not a local weak minimizer for (P).*

Theorem 2.8. *([13, Theorem 2.10]) Let $\hat x \in X$ be given and suppose that there exist a continuous linear mapping $A \in L(X, Y)$, a vector $x^0 \in X$ and scalars $R > 0$, $\hat\kappa \geq 0$ and $\gamma > 0$, such that the following conditions are satisfied:*

$$\|h(x') - h(x) - A(x' - x)\| \leq \gamma\|x' - x\|,\ \forall x, x' \in \hat x + R\mathcal{B}_X, \tag{2.4}$$

$$h(\hat x) + A(x^0 - \hat x) \in C,\ r := \|x^0 - \hat x\| < R/2, \tag{2.5}$$

$$2\gamma(\hat\kappa + 3r) + \hat d_C(h(\hat x), A, \hat\kappa) < 0. \tag{2.6}$$

Then there exists some $\tilde x \in x^0 + r\mathcal{B}_X$, such that $h(\tilde x) \in C$ and Γ is metrically regular near $(\tilde x, 0)$. Moreover, $\hat d_C(h(\tilde x), A, \hat\kappa + \|\tilde x - \hat x\|) \leq \hat d_C(h(\tilde x), A, \hat\kappa) + \gamma\|\tilde x - \hat x\|$.

3. A general necessary condition. The following theorem states an abstract necessary optimality condition for the problem (P). In some sense it is contained in the gap between the necessary and sufficient optimality conditions of Theorem 2.1.

THEOREM 3.1. *Assume that Assumption 1 holds. Further assume that $\bar{x}$ is a local weak minimizer, but not an essential local minimizer of second order and let $(z_n) \subset \mathcal{S}_X$, $(t_n) \in \mathbb{T}$ and $(\tau_n) \in \mathbb{T}$ be sequences such that*

$$\limsup_{n\to\infty} t_n^{-2} \hat{d}_C(h(\bar{x} + t_n z_n), h'(\bar{x}), \tau_n t_n) \le 0. \tag{3.1}$$

Further assume that there is a sequence $(A_n) \subset L(X, Y)$ of continuous linear operators mapping X into Y such that, together with some positive scalars $\gamma', R' > 0$ and a sequence $(\varphi_n') \in \mathbb{T}$ one has

$$\begin{aligned} &\|h(x') - h(x) - A_n(x' - x)\| \\ &\qquad \le \big(\varphi_n' t_n + \gamma' \max\{\|\bar{x} + t_n z_n - x'\|, \|\bar{x} + t_n z_n - x\|\}\big)\|x' - x\| \end{aligned} \tag{3.2}$$

for all $x', x \in \bar{x} + t_n(z_n + R'\mathcal{B}_X)$ and for each n. Then for each $T > 0$ one has

$$\liminf_{n\to\infty} t_n^{-2} \hat{d}_C(h(\bar{x} + t_n z_n), A_n, T t_n) \ge 0. \tag{3.3}$$

Proof. By Theorem 2.4, for each n we can find some element $\delta z_n \in \mathcal{B}_X$ such that

$$\begin{aligned} \delta_n :&= t_n^{-2} d(h(\bar{x} + t_n z_n) + t_n \tau_n h'(\bar{x})\delta z_n, C) \\ &\le t_n^{-2} \max\{\hat{d}_C(h(\bar{x} + t_n z_n), h'(\bar{x}), \tau_n t_n), 0\} + \frac{1}{n} \end{aligned}$$

and from (3.1) it follows $\delta_n \to 0$. Let $z_n' := z_n + \tau_n \delta z_n$. Then $\|z_n'\| \le 1 + \tau_n$ and using Assumption 1 we obtain

$$\begin{aligned} \delta_n' &:= t_n^{-2} d(h(\bar{x} + t_n z_n'), C) \\ &\le t_n^{-2}\big(d(h(\bar{x} + t_n z_n) + t_n \tau_n h'(\bar{x})\delta z_n, C) \\ &\qquad\qquad + \|h(\bar{x} + t_n z_n') - h(\bar{x} + t_n z_n) - t_n \tau_n h'(\bar{x})\delta z_n\|\big) \\ &\le \delta_n + t_n^{-2} \eta \max\{\|t_n z_n'\|, \|t_n z_n\|\}\|t_n(z_n - z_n')\| \le \delta_n + \eta(1 + \tau_n)\tau_n \end{aligned}$$

yielding $\delta_n' \to 0$. We will now prove by contraposition that (3.3) holds for arbitrarily fixed $T > 0$. Assume on the contrary that

$$\liminf_{n\to\infty} t_n^{-2} \hat{d}_C(h(\bar{x} + t_n z_n), A_n, T t_n) \le -2\epsilon < 0$$

for some $T > 0$. Using the Lipschitz continuity of $\hat{d}_C(\cdot, A_n, T t_n)$, Theorem 2.4 and condition (3.2) we obtain

$$\begin{aligned} \hat{d}_C(h(\bar{x} + t_n z_n'), A_n, T t_n) &\le \hat{d}_C(h(\bar{x} + t_n z_n) + \tau_n t_n A_n \delta z_n, A_n, T t_n) \\ &\qquad + \|h(\bar{x} + t_n z_n') - h(\bar{x} + t_n z_n) - \tau_n t_n A_n \delta z_n\| \\ &\le \hat{d}_C(h(\bar{x} + t_n z_n), A_n, (T + \tau_n) t_n) \\ &\qquad + (\varphi_n' t_n + \gamma' \tau_n t_n \|\delta z_n\|)\tau_n t_n \|\delta z_n\| \\ &= \hat{d}_C(h(\bar{x} + t_n z_n), A_n, (T + \tau_n) t_n) + o(t_n^2) \end{aligned}$$

Next define $T_n := T + \tau_n$ and $\mu_n := \max\{\tau_n, \varphi_n', \frac{T_n \delta_n'}{\epsilon}\}$ for each n. Since $\mu_n \to 0$, by passing to a subsequence if necessary, we may assume that

$$\begin{aligned} t_n^{-2} \hat{d}_C(h(\bar{x} + t_n z_n'), A_n, T_n t_n) &\le -\epsilon, \\ (2 + 8\gamma')(\mu_n T_n + 3\mu_n^2) - \epsilon &< 0, \\ \tau_n + 3\mu_n &\le R' \end{aligned}$$

for each n. Now let n be arbitrarily fixed. We will now show that the assumptions of Theorem 2.4 hold with data $\hat{x} = \bar{x} + t_n z_n'$, $A = A_n$, $R = 3\mu_n t_n$, $\hat{\kappa} = T_n t_n$ and $\gamma = (1+4\gamma')\mu_n t_n$. Since $\|t_n(z_n' - z_n)\| + R \le t_n(\tau_n + 3\mu_n) \le t_n R'$ it follows from (3.2) for all $x, x' \in \bar{x} + t_n z_n' + R\mathcal{B}_X \subset \bar{x} + t_n(z_n + R'\mathcal{B}_X)$ that

$$\begin{aligned}
&\|h(x') - h(x) - A_n(x' - x)\| \\
&\qquad \le (t_n\varphi_n' + \gamma' \max\{\|\bar{x} + t_n z_n - x'\|, \|\bar{x} + t_n z_n - x\|\})\|x - x'\| \\
&\qquad \le (t_n\varphi_n' + \gamma'(R + t_n\|z_n - z_n'\|))\|x - x'\| \\
&\qquad \le (\varphi_n' + \gamma'(3\mu_n + \tau_n))t_n\|x - x'\| \le (1+4\gamma')\mu_n t_n\|x - x'\|
\end{aligned}$$

and hence (2.4) holds. Using Lemma 2.5 we have

$$\begin{aligned}
d(0, A_n^{-1}(h(\bar{x} + t_n z_n') - C)) &\le \frac{T_n t_n d(h(\bar{x} + t_n z_n'), C)}{d(h(\bar{x} + t_n z_n'), C) - \hat{d}_C(h(\bar{x} + t_n z_n'), A_n, T_n t_n)} \\
&\le \frac{T_n t_n \delta_n' t_n^2}{\delta_n' t_n^2 + \epsilon t_n{}^2} < \frac{T_n \delta_n'}{\epsilon} t_n \le \mu_n t_n.
\end{aligned}$$

Thus there exists some x^0 such that

$$h(\bar{x} + t_n z_n') + A_n(x^0 - (\bar{x} + t_n z_n')) \in C, \quad r := \|x^0 - (\bar{x} + t_n z_n')\| \le \mu_n t_n = \frac{R}{3}$$

showing the validity of (2.5). Finally, we have

$$\begin{aligned}
2\gamma(\hat{\kappa} + 3r) + \hat{d}_C(h(\bar{x} + t_n z_n'), A_n, \hat{\kappa}) &\le 2(1+4\gamma')\mu_n t_n(T_n t_n + 3\mu_n t_n) - \epsilon t_n^2 \\
&= \big((2+8\gamma')(\mu_n T_n + 3\mu_n^2) - \epsilon\big)t_n^2 < 0
\end{aligned}$$

yielding (2.6). Thus we can apply Theorem 2.4 to establish the existence of some $\tilde{x}_n \in \bar{x} + t_n z_n' + 3\mu_n t_n \mathcal{B}_X$ such that $h(\tilde{x}_n) \in C$ and the multifunction $h(\cdot) - C$ is metrically regular near $(\bar{x}, 0)$. This holds for each n and since $\tilde{x}_n \to \bar{x}$ we conclude from Theorem 2.7 that $\bar{x}$ is not a local weak minimizer, a contradiction. ◻

It is easy to show that if h is continuously differentiable in a neighborhood of $\bar{x}$, then for a sequence $(A_n) \subset L(X, Y)$ of linear operators satisfying condition (3.2) one has $\|h'(\bar{x} + t_n z_n) - A_n\| \le \varphi_n' t_n = o(t_n)$. The converse is also true, if h is sufficiently smooth near $\bar{x}$.

Lemma 3.2. *Suppose that h is continuously differentiable in some ball $\bar{x} + \rho\mathcal{B}_X$ around $\bar{x}$. Further suppose that either h is twice Fréchet differentiable at $\bar{x}$ or that $h'(\cdot)$ is Lipschitz continuous in $\bar{x} + \rho\mathcal{B}_X$. Then, for any sequences $(z_n) \subset \mathcal{S}_X$, $(t_n) \in \mathbb{T}$ and $(A_n) \subset L(X, Y)$ such that $\|h'(\bar{x} + t_n z_n) - A_n\| = o(t_n)$ there exist a sequence $(\varphi_n') \in \mathbb{T}$ and positive reals γ' and R' such that condition (3.2) holds for all n sufficiently large.*

Proof. Let $R' > 0$ be arbitrarily chosen and consider n be chosen so large that $t_n(z_n + R'\mathcal{B}_X) \subset \rho\mathcal{B}_X$. Consider an arbitrary linear functional $y^* \in \mathcal{B}_{Y^*}$. For every pair $x, x' \in \bar{x} + t_n(z_n + R'\mathcal{B}_X)$, by the mean-value theorem, there exists some element ξ belonging to the line segment $[x, x']$ such that $\langle y^*, h(x') - h(x)\rangle = \langle y^*, h'(\xi)(x' - x)\rangle$ and

$$\langle y^*, h(x') - h(\bar{x}) - A_n(x' - x)\rangle = \langle y^*, (h'(\xi) - A_n)(x' - x)\rangle \le \|y^*\|\|h'(\xi) - A_n\|\|x' - x\|$$

follows. Now, in order to prove the lemma it is sufficient to show the bound

$$\|h'(\xi) - A_n\| \le \varphi_n' t_n + \gamma' \max\{\|\bar{x} + t_n z_n - x'\|, \|\bar{x} + t_n z_n - x\|\} \tag{3.4}$$

for some constant γ' and some sequence $(\varphi_n') \in \mathbb{T}$. When h is twice Fréchet differentiable at $\bar{x}$ we have

$$\begin{aligned}
\|h'(\xi) - h'(\bar{x} + t_n z_n)\| \le{}& \|h'(\xi) - (h'(\bar{x}) + h''(\bar{x})(\xi - \bar{x}))\| + \|h''(\bar{x})(\xi - (\bar{x} + t_n z_n)\| \\
&+ \|h'(\bar{x}) + h''(\bar{x})t_n z_n - h'(\bar{x} + t_n z_n)\|.
\end{aligned}$$

Together with

$$\|h'(\xi) - A_n\| \leq \|h'(\xi) - h'(\bar{x} + t_n z_n)\| + \|h'(\bar{x} + t_n z_n) - A_n\|$$

and $\|\bar{x} + t_n z_n - \xi\| \leq \max\{\|\bar{x} + t_n z_n - x'\|, \|\bar{x} + t_n z_n - x\|\}$ condition (3.4) follows with $\varphi_n' := t_n^{-1}(2\sup\{\|h'(\bar{x}) + h''(\bar{x})(\eta - \bar{x}) - h'(\eta)\| : \eta \in \bar{x} + t_n(z_n + R'\mathcal{B}_X)\} + \|h'(\bar{x} + t_n z_n) - A_n\|) = o(1)$ and $\gamma' = \|h''(\bar{x})\|$. Similarly, when $h'(\cdot)$ is Lipschitz continuous in $\bar{x} + \rho\mathcal{B}_X$, condition (3.4) holds with $\varphi_n' := t_n^{-1}\|h'(\bar{x} + t_n z_n) - A_n\|) = o(1)$ and with γ' being the Lipschitz constant of $h'(\cdot)$. Thus the lemma is proved. □

Note, that a sequence $(A_n) \subset L(X, Y)$ satisfying (3.2) can also exist if h is not continuously differentiable. For instance, consider a twice continuously differentiable function $\Psi : \mathbb{R} \to \mathbb{R}$ with $\Psi(t) = o(t^2)$ for $t \to 0$, let $\tilde{y} \in Y$ be arbitrarily chosen, and set $h(x) := \Psi(\|x - \bar{x}\|)\tilde{y}$. Then it is easy to show that condition (3.2) holds with $A_n = 0\ \forall n$, but in general $h(\cdot)$ will be only continuously differentiable provided $\|\cdot - \bar{x}\|^2$ is.

In addition, this example together with the second part of Theorem 2.2 shows that the conclusion of Theorem 3.1 automatically holds at a point $\bar{x}$ which is non-degenerate for the problem (P) and where condition (2.1) is satisfied. On the other hand, when $\bar{x}$ is degenerate for the problem (P) (i.e. $\operatorname{int}(h'(\bar{x})X - C) = \emptyset$), then, as a consequence of Theorem 2.4, the necessary condition (2.1) is automatically satisfied regardless whether the point $\bar{x}$ is a local weak minimizer for the problem (P) or not. However, as we will see condition (3.3) of Theorem 3.1 may fail for non-optimal points $\bar{x}$. We will present a corresponding example in §5.

Further note that sequences $(z_n), (t_n), (\tau_n)$ satisfying the assumption (3.1) exist if and only if condition (2.2) does not hold, or equivalently, $\bar{x}$ is not an essential local minimizer of second order. Thus the necessary condition (3.3) reduces the gap between the necessary and sufficient conditions of Theorem 3.1 in the degenerate case.

4. Second-order necessary conditions for certain directions. We will now analyze Theorem 3.1 for the special case of convergent sequences $(z_n) \to z$ and we will rewrite condition (3.3) in terms of first- and second-order derivatives of h and first and second-order approximation sets for the convex set C. Since we deal with rather general sets C, there is an inherent non-smoothness when building second-order approximation sets. Hence it seems to be quite natural to assume a similar amount of smoothness on h only.

DEFINITION 4.1. *Let E, F be normed spaces and let an element $z \in E$ be given.*

1. *Let $k : E \to F$ be a mapping and $\bar{e} \in E$ so that k is differentiable at $\bar{e}$. We define the following second-order one-sided directional derivative to k at $\bar{e}$ with respect to z as*

$$k''(\bar{e}; z) := \{f \in F\ :\ \exists (t_n) \in \mathbb{T} \text{ such that } f = \lim_{n\to\infty} \frac{k(\bar{e} + t_n z) - k(\bar{e}) - t_n k'(\bar{e})z}{t_n^2/2}\}.$$

Further, for given $\vec{t} = (t_n) \in \mathbb{T}$ we write

$$k''_{\vec{t}}(\bar{e}; z) := \lim_{n\to\infty} \frac{k(\bar{e} + t_n z) - k(\bar{e}) - t_n k'(\bar{e})z}{t_n^2/2},$$

when the limit on the right hand side exists.

2. *Let S be a subset of F, let $A \in L(E, F)$ be a continuous linear operator and let $\bar{f} \in S$. Then for $z \in E$ the* second-order compound tangent set *to S at $(\bar{f}, z)$ (with respect to A) and a sequence $\vec{t} = (t_n) \in \mathbb{T}$ is the set*

$$S''_{A,\vec{t}}(\bar{f}; z) := \{w \in Y\ :\ \exists (z_n) \to z \text{ such that } d(\bar{f} + t_n A z_n + \frac{t_n^2}{2} w, S) = o(t_n^2)\}.$$

We also define

$$S''_A(\bar{f}, z) := \bigcup_{\vec{t} \in \mathbb{T}} S''_{A,\vec{t}}(\bar{f}; z),$$

which corresponds to the second-order compound tangent set introduced in [26] and which played a crucial role in [13].

In order to have $C''_{h'(\bar x),\vec t}(h(\bar x);z)\neq\emptyset$ there must necessarily hold

$$d(h(\bar x)+t_nh'(\bar x)z,C)=d(h(\bar x)+t_nh'(\bar x)z_n,C)+o(t_n)=O(t_n^2)+o(t_n)=o(t_n)$$

for some sequence $(z_n)\to z$, implying $h'(\bar x)z\in T_C(h(\bar x))$, i.e. z belongs to the the so-called *critical cone* $\mathcal{C}(\bar x)$ defined by

$$\mathcal{C}(\bar x):=\{z\in X\,:\,h'(\bar x)z\in T_C(h(\bar x))\}.$$

LEMMA 4.2. *For any element $z\in\mathcal{C}(\bar x)$ and any sequence $\vec t=(t_n)\in\mathbb{T}$ with $C''_{h'(\bar x),\vec t}(h(\bar x);z)\neq\emptyset$, the inclusions*

$$cl(C''_{h'(\bar x),\vec t}(h(\bar x);z)+T_C(h(\bar x))+\operatorname{Im}h'(\bar x))\subset C''_{h'(\bar x),\vec t}(h(\bar x);z)\subset\operatorname{cl}(T_C(h(\bar x))+\operatorname{Im}h'(\bar x))$$

hold. Moreover, $C''_{h'(\bar x),\vec t}(h(\bar x);z)$ is a closed convex set.

Proof. The fact that $C''_{h'(\bar x),\vec t}(h(\bar x);z)$ is a closed convex set follows easily from the definition. To show the inclusions let $y\in C''_{h'(\bar x),\vec t}(h(\bar x);z)$ be arbitrarily fixed. By the definition we can find sequences $(z_n)\to z$ and $(y_n)\to y$ such that $h(\bar x)+t_nh'(\bar x)z_n+\frac12t_n^2y_n\in C$ for each n and $y_n\in 2t_n^{-2}(C-h(\bar x))+h'(\bar x)(-t_n^{-1}z_n)\subset T_C(h(\bar x))+\operatorname{Im}h'(\bar x)$ follows. Hence $y\in\operatorname{cl}(T_C(h(\bar x))+\operatorname{Im}h'(\bar x))$. Now consider arbitrary elements $w\in T_C(h(\bar x))$ and $v=h'(\bar x)s\in\operatorname{Im}h'(\bar x)$. Then we can find a convergent sequence $(w_n)\to w$ such that $h(\bar x)+t_nw_n\in C$ for all n. For all n sufficiently large we have $t_n<2$ and by using the convexity of C we conclude

$$\begin{aligned}&\frac{t_n}{2}(h(\bar x)+t_nw_n)+(1-\frac{t_n}{2})(h(\bar x)+t_nh'(\bar x)z_n+\frac12t_n^2y_n)\\&\qquad=h(\bar x)+t_nh'(\bar x)(z_n-\frac{t_n}{2}(z_n+s))+\frac{t_n^2}{2}(y_n+w_n+v)\in C.\end{aligned}$$

Since the sequence $(z_n-\frac{t_n}{2}(z_n+s))$ converges to z we obtain $y+w+v=\lim_n y_n+w_n+v\in C''_{h'(\bar x),\vec t}(h(\bar x);z)$ and since $C''_{h'(\bar x),\vec t}(h(\bar x);z)$ is closed, the proposed inclusion follows. □

LEMMA 4.3. *Suppose Assumption 1 is satisfied. Let $(z_n)\to z$ be a convergent sequence in X and let $\vec t=(t_n)\in\mathbb{T}$, such that $h''_{\vec t}(\bar x;z)$ exists. Then*

$$h''_{\vec t}(\bar x;z)=\lim_{n\to\infty}\frac{h(\bar x+t_nz_n)-h(\bar x)-t_nh'(\bar x)z_n}{t_n^2/2}$$

also holds. Moreover, there exists a sequence $\tau_n\to0$ such that (3.1) holds if and only if $h''_{\vec t}(\bar x;z)\in C''_{h'(\bar x),\vec t}(h(\bar x);z)$.

Proof. Using Assumption 1, the first assertion follows immediately from the estimate

$$\|h(\bar x+t_nz_n)-h(\bar x+t_nz)-t_nh'(\bar x)(z_n-z)\|\le\eta t_n^2\max\{\|z_n\|,\|z\|\}\|z_n-z\|=o(t_n^2).$$

Now assume $h''_{\vec t}(\bar x;z)\in C''_{h'(\bar x),\vec t}(h(\bar x);z)$. By the definition, there exists a sequence $(z_n')\to z$ such that $d(h(\bar x)+t_nh'(\bar x)z_n'+(t_n^2/2)h''_{\vec t}(\bar x;z),C)=o(t_n^2)$. Hence,

$$\begin{aligned}&d(h(\bar x+t_nz_n)+t_nh'(\bar x)(z_n'-z_n),C)\\&\qquad=d(h(\bar x)+t_nh'(\bar x)z_n+\frac{t_n^2}{2}h''_{\vec t}(\bar x;z)+t_nh'(\bar x)(z_n'-z_n),C)+o(t_n^2)=o(t_n^2)\end{aligned}$$

and (3.1) follows from Theorem 2.4 with $\tau_n=\|z_n'-z_n\|$. Now let $(\tau_n)\in\mathbb{T}$ be a sequence such that (3.1) holds. Then, as in the proof of Theorem 3.1, we can find some sequence $(z_n')\to z$ with $z_n'=z_n+\tau_nt_n\delta z_n$, $\delta z_n\in\mathcal{B}_X$ such that

$$d(h(\bar x+t_nz_n'),C)=d(h(\bar x)+t_nh'(\bar x)z_n'+\frac{t_n^2}{2}h''_{\vec t}(\bar x;z),C)+o(t_n)^2=o(t_n^2),$$

showing $h''_{\vec{t}}(\bar{x};z) \in C''_{h'(\bar{x}),\vec{t}}(h(\bar{x});z)$. □

The second-order derivative $h''_{\vec{t}}(\bar{x};z)$ is useful for building second-order approximations of h in the direction z. But we need also another type of second-order derivatives, namely in the sense of first order approximations of first derivatives. We know from the discussion following Theorem 3.1, that, if h is sufficiently smooth near $\bar{x}$, for given sequences $(z_n) \subset \mathcal{S}_X$ and $(t_n) \in \mathbb{T}$ a sequence of linear operators $(A_n) \subset L(X,Y)$ satisfies condition (3.2) if and only if $A_n = h'(\bar{x}+t_nz_n)+o(t_n)$ holds. We use condition (3.2) to define this other type of second-order derivative without assuming existence of $h'(\cdot)$ near $\bar{x}$.

Lemma 4.4. *Let E,F be normed spaces, let $k : E \to F$ be a mapping and let $\bar{e} \in E$ such that k is differentiable at $\bar{e}$. Further let an element $z \in E$ and a sequence $(t_n) \in \mathbb{T}$ be given. Then there exists at most one continuous linear operator $K \in L(E,F)$ such that, together with some positive scalars $\tilde{\gamma}$, $\tilde{R}$ and some sequence $(\tilde{\varphi}_n) \in \mathbb{T}$, one has*

$$\begin{aligned}\|k(e') - k(e) - (k'(\bar{e}) + t_nK)(e'-e)\| \\ \le \big(\tilde{\varphi}_nt_n + \tilde{\gamma}\max\{\|\bar{e}+t_nz-e'\|, \|\bar{e}+t_nz-e\|\}\big)\|e'-e\|\end{aligned} \tag{4.1}$$

for all $e,e' \in \bar{e} + t_n(z + \tilde{R}\mathcal{B}_E)$ and all n.

Proof. By contraposition. Assume that there exist two continuous linear operators $K_1 \neq K_2$ satisfying (4.1) with parameters $\tilde{\gamma}_1, \tilde{R}_1, (\tilde{\varphi}_{1n})$ and $\tilde{\gamma}_2, \tilde{R}_2, (\tilde{\varphi}_{2n})$, respectively. Set $\tilde{\gamma} := \max\{\tilde{\gamma}_1,\tilde{\gamma}_2\}$, $\tilde{\varphi}_n := \max\{\tilde{\varphi}_{1n},\tilde{\varphi}_{2n}\}$, $\forall n$ and $\tilde{R} = \min\{\tilde{R}_1,\tilde{R}_2\}$. By the triangle inequality we obtain

$$\begin{aligned}\|t_n(K_1-K_2)(e'-e)\| &\le \|k(e') - k(e) - (k'(\bar{e}) + t_nK_1)(e'-e)\| \\ &\quad + \|k(e') - k(e) - (k'(\bar{e}) + t_nK_2)(e'-e)\| \\ &\le 2\big(\tilde{\varphi}_nt_n + \tilde{\gamma}\max\{\|\bar{e}+t_nz-e'\|, \|\bar{e}+t_nz-e\|\}\big)\|e'-e\|\end{aligned}$$

for all $e,e' \in \bar{e} + t_n(z + \tilde{R}\mathcal{B}_E)$ and all n. Let $d \in E$ denote a direction with $(K_1-K_2)d \neq 0$ and $\|d\| \le R'$. Applying the above estimate successively with $e = \bar{e} + t_nz$, $e' = e + \frac{t_n}{n}d$ for each n we obtain

$$\|t_n(K_1-K_2)\frac{t_n}{n}d\| \le 2(\tilde{\varphi}_nt_n + \tilde{\gamma}\|\frac{t_n}{n}d\|)\|\frac{t_n}{n}d\| = 2\frac{t_n^2}{n}(\tilde{\varphi}_n + \frac{\tilde{\gamma}\|d\|}{n})\|d\|.$$

Dividing by $\frac{t_n^2}{n}$ and passing to the limit yields $\|(K_1-K_2)d\| \le 0$, a contradiction. □

Definition 4.5. *Under the assumptions of Lemma 4.4, if the unique continuous linear operator $K \in L(E,F)$ satisfying condition (4.1) exists, we will denote it by $(k')'_{\vec{t}}(\bar{e};z)$.*

If k is differentiable near $\bar{e}$, then $\|k'(\bar{e}+t_nz) - (k'(\bar{e}) + t_n(k')'_{\vec{t}}(\bar{e};z))\| = o(t_n)$ follows easily from condition (4.1). Thus, $(k')'_{\vec{t}}(\bar{e};z)$ is an element of the so-called *contingent derivative*, also called *graphical derivative* or *Bouligand derivative*, (see, e.g. [22]) which in our case is given by

$$Ck'(\bar{e})z := \{K \in L(E,F) \,:\, \exists (t_n) \in \mathbb{T}, z_n \to z \text{ with } K = \lim_{n\to\infty}\frac{k'(\bar{e}+t_nz_n) - k'(\bar{e})}{t_n}\}.$$

If $k'(\cdot)$ is Lipschitz continuous near $\bar{e}$, using similar arguments as in Lemma 3.2 we can conclude that

$$\bigcup_{\vec{t}\in T}(k')'_{\vec{t}}(\bar{e};z) = Ck'(\bar{e})z$$

and if k is twice Fréchet differentiable at $\bar{e}$, then $(k')'_{\vec{t}}(\bar{e};z) = k''(\bar{e})z\ \forall\vec{t} \in \mathbb{T}$ holds.

In general, when k is not twice differentiable at $\bar{e}$, we can have $(k')'_{\vec{t}}(\bar{e};z)z \neq k''_{\vec{t}}(\bar{e};z)$. However, $(k')'_{\vec{t}}(\bar{e};z)$ acts like a derivative for the mapping $\frac{1}{2}k''_{\vec{t}}(\bar{e};\cdot)$ at z for given $\vec{t} \in \mathbb{T}$. Indeed, from condition (4.1) the estimate

$$\|\frac{1}{2}k''_{\vec{t}}(\bar{e};s) - \frac{1}{2}k''_{\vec{t}}(\bar{e};z) - (k')'_{\vec{t}}(\bar{e};z)(s-z)\| \le \tilde{\gamma}\|s-z\|^2,$$

being valid for all $s \in z + \tilde{R}\mathcal{B}_X$ such that $k''_{\vec{t}}(\bar{e}; s)$ exists, easily follows.

When $(h')'_{\vec{t}}(\bar{x}; z)$ exists for some $z \in X$, $\vec{t} \in \mathbb{T}$, then it is easy to see that for any convergent sequence $(z_n) \to z$ condition (3.2) holds with $A_n = h'(\bar{x}) + t_n(h')'_{\vec{t}}(\bar{x}; z)$, $\varphi'_n = \tilde{\varphi}_n + \tilde{\gamma}\|z_n - z\|$, $\gamma' = \tilde{\gamma}$, $R' = \tilde{R}/2$ for all n sufficiently large, such that $\|z_n - z\| \le \tilde{R}/2$.

We are now in a position to state the main result of this section. We state this result under the following technical assumption which will be analyzed separately.

Assumption 2. *Each sequence $(y^*_n) \subset \mathcal{S}_{Y^*}$ with*

$$\lim_{n\to\infty} t_n^{-1}(\langle y^*_n, h(\bar{x})\rangle - \sigma_C(y^*_n)) = \lim_{n\to\infty} t_n^{-1}\|(h'(\bar{x}) + t_n(h')'_{\vec{t}}(\bar{x}; z))^* y^*_n\| = 0 \tag{4.2}$$

has at least one weak-$*$ accumulation point which is not equal to 0.

Theorem 4.6. *Suppose that $\bar{x}$ is a local weak minimizer for the problem (P) and suppose that Assumption 1 holds and suppose we are given an element $z \in \mathcal{C}(\bar{x}) \cap \mathcal{S}_X$ and a sequence $(t_n) = \vec{t} \in \mathbb{T}$ such that the second-order directional derivatives $h''_{\vec{t}}(\bar{x}; z)$ and $(h')'_{\vec{t}}(\bar{x}; z)$ exist, the inclusion $h''_{\vec{t}}(\bar{x}; z) \in C''_{h'(\bar{x}),\vec{t}}(h(\bar{x}); z)$ holds and Assumption 2 is satisfied. Then there exist a multiplier $\bar{y}^* \in \Lambda_{FJ}$ and an element $\bar{\mu}^* \in (\operatorname{Ker} h'(\bar{x}))^\perp$ such that*

$$(h')'_{\vec{t}}(\bar{x}; z)^* \bar{y}^* + \bar{\mu}^* = 0 \tag{4.3}$$

and for each pair (s, y) with $s \in \mathcal{C}(\bar{x})$ and $y \in C''_{h'(\bar{x}),\vec{t}}(h(\bar{x}); s)$ one has

$$\langle \bar{\mu}^*, z - s\rangle + \frac{1}{2}\langle \bar{y}^*, h''_{\vec{t}}(\bar{x}; z) - y\rangle \ge 0. \tag{4.4}$$

Proof. From the preceding discussion we know that the assumptions of Theorem 3.1 hold with $z_n := z$, $A_n := h'(\bar{x}) + t_n(h')'_{\vec{t}}(\bar{x}; z)$ and some sequence $(\tau_n) \in \mathbb{T}$ given by Lemma 4.3. Hence, applying Theorem 3.1 with $T = 1$ we have

$$\liminf_{n\to\infty} \sup_{y^*\in\mathcal{S}_{Y^*}} \left\{ \frac{\langle y^*, h(\bar{x} + t_n z)\rangle - \sigma_C(y^*)}{t_n^2} - \frac{1}{t_n}\|(h'(\bar{x}) + t_n(h')'_{\vec{t}}(\bar{x}; z))^* y^*\| \right\} \ge 0$$

Now, for each n we can find some $y^*_n \in \mathcal{S}_{Y^*}$ approaching the supremum sufficiently accurate, such that

$$\liminf_{n\to\infty} \left\{ \frac{\langle y^*_n, h(\bar{x} + t_n z)\rangle - \sigma_C(y^*_n)}{t_n^2} - \frac{1}{t_n}\|(h'(\bar{x}) + t_n(h')'_{\vec{t}}(\bar{x}; z))^* y^*_n\| \right\} \ge 0.$$

Since $\limsup_{n\to\infty} t_n^{-2}\hat{d}_C(h(\bar{x} + t_n z), h'(\bar{x}), \tau_n t_n) \le 0$ we also have

$$\begin{aligned}
0 &\ge \limsup_{n\to\infty} \sup_{y^*\in\mathcal{S}_{Y^*}} \left\{ \frac{\langle y^*, h(\bar{x} + t_n z)\rangle - \sigma_C(y^*)}{t_n^2} - \frac{\tau_n}{t_n}\|(h'(\bar{x})^* y^*\| \right\} \\
&= \limsup_{n\to\infty} \sup_{y^*\in\mathcal{S}_{Y^*}} \left\{ \frac{\langle y^*, h(\bar{x} + t_n z)\rangle - \sigma_C(y^*)}{t_n^2} - \frac{\tau_n}{t_n}\|h'(\bar{x})^* y^*\| + \tau_n\|(h')'_{\vec{t}}(\bar{x}; z)^* y^*\| \right\} \\
&\ge \limsup_{n\to\infty} \sup_{y^*\in\mathcal{S}_{Y^*}} \left\{ \frac{\langle y^*, h(\bar{x} + t_n z)\rangle - \sigma_C(y^*)}{t_n^2} - \frac{\tau_n}{t_n}\|(h'(\bar{x}) + t_n(h')'_{\vec{t}}(\bar{x}; z))^* y^*\| \right\}
\end{aligned}$$

and $\lim_{n\to\infty} t_n^{-1}\|(h'(\bar{x}) + t_n(h')'_{\vec{t}}(\bar{x}; z))^* y^*_n\| = 0$ follows. In particular we obtain

$$\mu^*_n := \frac{h'(\bar{x})^* y^*_n}{t_n} = O(1), \tag{4.5}$$

$$\lim_{n\to\infty} t_n^{-1}(h'(\bar{x}) + t_n(h')'_{\vec{t}}(\bar{x}; z))^* y^*_n) = \lim_{n\to\infty} (h')'_{\vec{t}}(\bar{x}; z)^* y^*_n + \mu^*_n = 0, \tag{4.6}$$

$$\liminf_{n\to\infty} \frac{\langle y^*_n, h(\bar{x} + t_n z)\rangle - \sigma_C(y^*_n)}{t_n^2} \ge 0. \tag{4.7}$$

Because of (4.7) and the relation $h(\bar{x}+t_nz)=h(\bar{x})+t_nh'(\bar{x})z+\frac{t_n^2}{2}h''_{\vec{t}}(\bar{x};z)+o(t_n^2)$ we have

$$\liminf_{n\to\infty}\left\{\frac{\langle y_n^*,h(\bar{x})\rangle-\sigma_C(y_n^*)}{t_n^2}+\frac{\langle y_n^*,h'(\bar{x})z\rangle}{t_n}+\frac{1}{2}\langle y_n^*,h''_{\vec{t}}(\bar{x};z)\rangle\right\}\geq 0.$$

Together with $\langle y_n^*,h'(\bar{x})z\rangle=t_n\langle\mu_n^*,z\rangle=O(t_n)$ and $\langle y_n^*,h(\bar{x})\rangle-\sigma_C(y_n^*)\leq 0$ we obtain

$$\sigma_C(y_n^*)-\langle y_n^*,h(\bar{x})\rangle=O(t_n^2). \tag{4.8}$$

Together with (4.6) we may conclude from Assumption 2 that the sequence (y_n^*) has a nonzero accumulation point $\tilde{y}^*$. Then there exists some element $\tilde{y}\in Y$ with $\langle\tilde{y}^*,\tilde{y}\rangle=1$ and we can choose a subsequence $(y_{k_n}^*)$ such that $\langle y_{k_n}^*,\tilde{y}\rangle\to 1$. $(y_{k_n}^*,\mu_{k_n}^*)$ is a bounded sequence in $Y^*\times X^*=(Y\times X)^*$ and by the Alaoglu-Bourbaki Theorem, at least one weak-$*$ accumulation point, say $(\bar{y}^*,\bar{\mu}^*)$, exists. Of course, $\bar{y}^*$ is also a weak-$*$ accumulation point of the sequence $(y_{k_n}^*)$. Hence $\langle\bar{y}^*,\tilde{y}\rangle=1$ implying $\bar{y}^*\neq 0$. Note that $(\bar{y}^*,\bar{\mu}^*)$ is also a a weak-$*$ accumulation point of the entire sequence (y_n^*,μ_n^*). Since $\mu_n^*\in\operatorname{Im}h'(\bar{x})^*\subset(\operatorname{Ker}h'(\bar{x}))^\perp$ and the annihilator $(\operatorname{Ker}h'(\bar{x}))^\perp$ is weakly-$*$ closed in X^*, we have $\bar{\mu}^*\in(\operatorname{Ker}h'(\bar{x}))^\perp$. Further, equation (4.3) follows easily from (4.6).

Now let us show $\bar{y}^*\in\Lambda_{FJ}$. Since $\sigma_C(\cdot)-\langle\cdot,h(\bar{x})\rangle$ is weakly-$*$ lower semicontinuous, we obtain $\sigma_C(\bar{y}^*)-\langle\bar{y}^*,h(\bar{x})\rangle\leq 0$ from condition (4.8). Because of $h(\bar{x})\in C$ we also have $\sigma_C(\bar{y}^*)-\langle\bar{y}^*,h(\bar{x})\rangle\geq 0$ showing $\bar{y}^*\in N_C(h(\bar{x}))$. From condition (4.5) it follows that $h'(\bar{x})y_n^*\to 0$ and consequently, $h'(\bar{x})^*\bar{y}^*=0$. Hence $\bar{y}^*\in\Lambda_{FJ}$. It remains to show that (4.4) holds. Let the pair $(s,y)\in\mathcal{C}(\bar{x})\times C''_{h'(\bar{x}),\vec{t}}(h(\bar{x});s)$ be arbitrarily fixed. Then, by the definitions of the support function σ_C and the set $C''_{h'(\bar{x}),\vec{t}}(h(\bar{x});s)$ we have $\sigma_C(y_n^*)\geq\langle y_n^*,h(\bar{x})+t_nh'(\bar{x})s_n+\frac{t_n^2}{2}y\rangle+o(t_n^2)$ for some sequence $(s_n)\to s$. Together with condition (4.7) and the second-order expansion for $h(\bar{x}+t_nz)$ it follows that

$$\begin{aligned}0&\leq\liminf_{n\to\infty}\frac{\langle y_n^*,h(\bar{x})+t_nh'(\bar{x})z+\frac{t_n^2}{2}h''_{\vec{t}}(\bar{x};z)\rangle-\langle y_n^*,h(\bar{x})+t_nh'(\bar{x})s_n+\frac{t_n^2}{2}y\rangle}{t_n^2}\\&=\liminf_{n\to\infty}\left\{\langle\mu_n^*,z-s_n\rangle+\frac{1}{2}\langle y_n^*,h''_{\vec{t}}(\bar{x};z)-y\rangle\right\}\leq\langle\bar{\mu}^*,z-s\rangle+\frac{1}{2}\langle\bar{y}^*,h''_{\vec{t}}(\bar{x};z)-y\rangle\end{aligned}$$

and this completes the proof. □

Of course, Assumption 2 holds when Y is finite dimensional. But there are also other situations when this assumption holds.

DEFINITION 4.7. *Let E,F be Banach spaces, let $S\subset F$ be a closed convex subset of F, let $k:E\to F$ be a mapping, which is differentiable at $\bar{e}\in E$, with $k(\bar{e})\in S$, and let the multifunction $\Psi:E\rightrightarrows F$ be given by $\Psi(e):=k(e)-S$. Then, k is said to be 2-non-degenerate at the point $\bar{e}$ in the direction $z\in E$ with respect to the set C and the sequence $\vec{t}\in\mathbb{T}$ if the following conditions are satisfied:*

1. *$(k')'_{\vec{t}}(\bar{e};z)$ exists.*
2. *The set $k(\bar{e})+k'(\bar{e})E-S$ has nonempty relative interior.*
3. *The interior of the set $KE-(S-k(\bar{E}))\times\{0\}$ is nonempty in $Q\times F/Q$, where $Q:=\operatorname{aff}(k(\bar{e})+k'(\bar{e})E-S)\subset F$ is the affine hull of the set $k(\bar{e})+k'(\bar{e})E-S$, the continuous linear operator $K:E\to Q\times F/Q$ is given by $Ks:=(k'(\bar{e})s,\pi(k')'_{\vec{t}}(\bar{e};z)s)$ and where π denotes the quotient map from F onto the quotient space F/Q.*

In what follows let Q denote the affine hull of the set $h(\bar{x})+h'(\bar{x})X-C$ and let π denote the quotient map from Y onto the quotient space Y/Q.

THEOREM 4.8. *Let the mapping h be 2-non-degenerate at $\bar{x}$ in the direction $z\in X$ with respect to C and the sequence $\vec{t}\in\mathbb{T}$. Then it is sufficient for Assumption 2 to hold that either the quotient space Y/Q or the subspace $\operatorname{Im}h'(\bar{x})\cap\operatorname{aff}(C-h(\bar{x}))$ is finite dimensional.*

Proof. Let $(y_n^*)\subset\mathcal{S}_{Y^*}$ be a sequence satisfying condition (4.2). In order to prove the theorem we have to show that at least one nonzero weak-$*$ accumulation point of the sequence (y_n^*) exists. Since Q is a closed subspace of the Banach space Y, the quotient space $P:=Y/Q$ and hence also $Q\times P$ are Banach spaces. Let $H:X\to Q\times P$ be the continuous linear operator according to Definition 4.7, i.e. $Hs = (h'(\bar x)s, \pi(h')'_t(\bar x; z)s)\ \forall s$. Now choose $(q,p) \in Q\times P$, $x\in X$ and $\bar q \in \hat C := C - h(\bar x) \subset Q$ such that $(q,p) = Hx - (\bar q, 0) \in \mathrm{int}\,(HX - \hat C\times\{0\})$. Application of the Generalized Open Mapping Theorem (see [29, Theorem 1]) yields $(q,p)\in \mathrm{int}\,(H(x+\mathcal{B}_X) - \hat C\times\{0\})$ and it follows that

$$\rho\mathcal{B}_{Q\times P} \subset \mathrm{int}\,((\bar q,0) + H\mathcal{B}_X - \hat C\times\{0\}) \tag{4.9}$$

for some $\rho>0$. Using Theorem 2.4 we obtain that

$$\sup_{(q^*,p^*)\in\mathcal{S}_{(Q\times P)^*}} \left\{\langle q^*,\bar q\rangle - \sigma_{\hat C}(q^*) - \|H^*(q^*,p^*)\|\right\} \le -\rho$$

and therefore

$$\langle q^*,\bar q\rangle - \sigma_{\hat C}(q^*) - \|H^*(q^*,p^*)\| \le -\rho(\|q^*\|+\|p^*\|),\ \forall (q^*,p^*)\in Q^*\times P^*. \tag{4.10}$$

Now let the linear operators $A\in L(X,Q)$ and $B\in L(X,P)$ be given by $As = h'(\bar x)s$ and $Bs = \pi(h')'_t(\bar x;z)s$, respectively. Further let $i_Q: Q\to Y$ denote the natural embedding from Q into Y. Consequently, $h'(\bar x) = i_Q\circ A$. For each n, let $q_n^* := i_Q^* y_n^* \in Q^*$ be the restriction of the linear functional y_n^* to Q. By the Hahn-Banach Theorem we can extend the linear form $q_n^*\in Q^*$ to a linear functional $y_{Q,n}^*$ over Y such that $\|y_{Q,n}^*\| = \|q_n^*\|$. Setting $y_{P,n}^* := y_n^* - y_{Q,n}^*$ we have $i_Q^* y_{P,n}^* = i_Q^* y_n^* - i_Q^* y_{Q,n}^* = 0$, i.e. $y_{P,n}^*$ belongs to the annihilator $Q^\perp := \{y^*\in Y^* : \langle y^*, y\rangle = 0, \forall y\in Q\}$ of the subspace $Q\subset Y$. The mapping $\pi^*: P^*\to Y^*$ is isometric and allows us to identify the dual space $P^* = (Y/Q)^*$ with $Q^\perp$. For each n let $p_n^*\in P^*$ denote the linear functional uniquely given by the relation $\pi^* p_n^* = y_{P,n}^*$. Finally, let $\bar y_Q := i_Q\bar q$. Then we have

$$\langle y_n^*, \bar y_Q\rangle = \langle q_n^*, \bar q\rangle \tag{4.11}$$

$$\sigma_{\hat C}(q_n^*) = \sigma_{i_Q(\hat C)}(y_n^*) = \sigma_C(y_n^*) - \langle y_n^*, h(\bar x)\rangle \tag{4.12}$$

$$\begin{aligned}(h'(\bar x)^* + t_n(h')'_t(\bar x;z))^* y_n^* &= A^* q_n^* + t_n B^* p_n^* + t_n (h')'_t(\bar x;z)^* y_{Q,n}^*\\ &= H^*(q_n^*, t_n p_n^*) + t_n(h')'_t(\bar x;z)^* y_{Q,n}^*\end{aligned} \tag{4.13}$$

and (4.10) implies

$$\begin{aligned}&\langle y_n^*, \bar y_Q\rangle + \langle y_n^*, h(\bar x)\rangle - \sigma_C(y_n^*) - \|(h'(\bar x)^* + t_n(h')'_t(\bar x;z))^* y_n^*\|\\ &\quad\le -\rho(\|q_n^*\| + t_n\|p_n^*\|) + t_n\|(h')'_t(\bar x;\bar z)^* y_{Q,n}^*\|.\end{aligned} \tag{4.14}$$

Now let us assume that $\limsup_{n\to\infty}\|q_n^*\| > 0$. By passing to a subsequence if necessary we may also assume that $\liminf_{n\to\infty}\|q_n^*\| = \epsilon > 0$. The sequence (y_n^*) has at least one weak-$*$ accumulation point $\bar y^*$ by the Alaoglu-Bourbaki Theorem. Since $t_n\to 0$ and the sequence (y_n^*) satisfies condition (4.2) we obtain from condition (4.14) that $\langle \bar y^*, \bar y_Q\rangle \le -\rho\epsilon < 0$. Hence, $\bar y^*\ne 0$ and the theorem is proved in case $\limsup_{n\to\infty}\|q_n^*\| > 0$.

Now let us assume that $\limsup_{n\to\infty}\|q_n^*\| = 0$. If we denote by $\bar p^*$ an arbitrary weak-$*$ accumulation point of the sequence (p_n^*), then $\bar y^* := \pi^*\bar p^*$ is a weak-$*$ accumulation point both of the sequence $(y_{P,n}^*)$ and the sequence (y_n^*), the latter because of $\|y_{Q,n}^*\| = \|y_n^* - y_{P,n}^*\|\to 0$. Further, $\bar y^*\ne 0$ if and only if $\bar p^*\ne 0$. Since $\|y_{P,n}^*\| = \|p_n^*\|\to 1$, the assertion of the theorem now follows immediately in case that P and hence also P^* are finite dimensional spaces.

It remains to prove the theorem in case the subspace $W := \mathrm{Im}\, h'(\bar x)\cap \mathrm{aff}\,(C - h(\bar x))$ is finite dimensional. To do this we will first show the inclusion

$$\rho\mathcal{B}_{Q\times P} \subset (\bar q, 0) + (\frac{A}{t}, B)\mathcal{B}_X - (\hat C + \frac{\rho + \|A\|}{t}\mathcal{B}_W)\times\{0\},\ \forall t\in]0, \frac{\rho}{\|A\|}[. \tag{4.15}$$

Let the element $(q,p)\in\rho\mathcal{B}_{Q\times P}$ and the scalar $t\in]0, \|A\|^{-1}\rho[$ be arbitrarily fixed. We observe that condition (4.9) together with $\bar q\in\hat C$ imply $\gamma\rho\mathcal{B}_{Q\times P}\subset (\bar q,0) + \gamma H\mathcal{B}_X - \hat C\times\{0\}$ for all $\gamma\in]0,1]$. Hence we can find some $x_1\in\rho^{-1}\|(p,q)\|\mathcal{B}_X$ and some $c_1\in\hat C$ such that $(q,p) = (\bar q + Ax_1 - c_1, Bx_1)$.

We have $\|t(q-\bar q+c_1)\| = \|tAx_1\| \le t\|A\|\|x_1\| \le \|(q,p)\|$ and $\|(t(q-\bar q+c_1),p)\| = \max\{\|t(q-\bar q+c_1)\|,\|p\|\} \le \|(q,p)\| \le \rho$ follows. Consequently, we can find some $x_2 \in \rho^{-1}\|(p,q)\|\mathcal{B}_X$ and some $c_2 \in \hat C$ with $(t(q-\bar q+c_1),p) = (\bar q + Ax_2 - c_2, Bx_2)$. Thus

$$(q,p) = (\bar q + \frac{\bar q - c_2}{t} + \frac{1}{t}Ax_2 - c_1, Bx_2)$$

and $\bar q - c_2 = A(tx_1 - x_2) \in W$. Moreover, we have

$$\|\bar q - c_2\| \le t\|Ax_1\| + \|A\|\|x_2\| \le \|(p,q)\|(1+\frac{\|A\|}{\rho}) \le \rho + \|A\|.$$

Therefore, $(p,q) \in (\bar q,0) - (\hat C + t^{-1}(\rho+\|A\|)\mathcal{B}_W)\times\{0\} + (t^{-1}A,B)\mathcal{B}_X$ and, since $(p,q)\in\rho\mathcal{B}_{Q\times P}$ has been chosen arbitrarily, the inclusion (4.15) follows. Now, by Theorem 2.4 we obtain

$$\sup_{(q^*,p^*)\in\mathcal{S}^*_{Q\times P}}\left\{\langle q^*,\bar q\rangle - \sigma_{\hat C}(q^*) - (\rho+\|A\|)\frac{\sigma_{\mathcal{B}_W}(q^*)}{t} - \|A^*\frac{q^*}{t} + B^*p^*\|\right\} \le -\rho.$$

Consequently, for all n sufficiently large such that $t_n\|A\| < \rho$ we have

$$\langle q_n^*,\bar q\rangle - \sigma_{\hat C}(q_n^*) - (\rho+\|A\|)\frac{\sigma_{\mathcal{B}_W}(q_n^*)}{t_n} - \|A^*\frac{q_n^*}{t_n} + B^*p_n^*\| \le -\rho(\|q_n^*\| + \|p_n\|^*).$$

Taking into account conditions (4.2), (4.11)–(4.13) and $\|q_n^*\| = \|y^*_{Q,n}\| \to 0$ we obtain

$$\lim_{n\to\infty}\langle q_n^*,\bar q\rangle = 0,\ \lim_{n\to\infty}\sigma_{\hat C}(q_n^*) = 0,\ \lim_{n\to\infty}\|A^*\frac{q_n^*}{t_n} + B^*p_n^*\| = 0.$$

Since we also have $\|p_n^*\| \to 1$,

$$\liminf_{n\to\infty}\frac{\sigma_{\mathcal{B}_W}(q_n^*)}{t_n} \ge \frac{\rho}{\rho+\|A\|}$$

follows. Now, if W is finite dimensional and extracting if necessary a subsequence, there exists some $\tilde x \in X$ such that $\bar w := A\tilde x \in \mathcal{B}_W$ satisfies

$$\langle q_n^*,\bar w\rangle/t_n \ge \frac{\rho}{2(\rho+\|A\|)} := \tilde\rho > 0$$

for all n. Let $\bar p$ denote an arbitrary weak-$*$ accumulation point of the sequence (p_n^*). Extracting if necessary a subsequence we have

$$\begin{aligned}\langle \bar p^*, B\tilde x\rangle &= \lim_{n\to\infty}\langle p_n^*, B\tilde x\rangle = \lim_{n\to\infty}\langle B^*p_n^*,\tilde x\rangle = -\lim_{n\to\infty}\langle A^*\frac{q_n^*}{t_n},\tilde x\rangle\\ &= -\lim_{n\to\infty}\langle\frac{q_n^*}{t_n},\bar w\rangle \le -\tilde\rho.\end{aligned}$$

and $\bar p^* \ne 0$ follows. Then $\bar y^* = \pi^*\bar p^*$ is a nonzero weak-$*$ accumulation point of the sequence (y_n^*) and this completes the proof. □

Let us discuss the assumptions of Theorem 4.8 in further detail. In our case, the set C has the form $(-L)\times K$ with $\operatorname{int} L \ne \emptyset$ and so the assumption that $h(\cdot)$ is 2-non-degenerate in direction z with respect to C and $\vec t$ can be reduced to an assumption on the constraints, namely that $g(\cdot)$ is is 2-non-degenerate in direction z with respect to K and $\vec t$. To be a little bit more general, let us consider the case when the Banach space Y and the set C can be decomposed in the form $C = C_1\times C_2 \subset Y_1\times Y_2 = Y$ with $\operatorname{int} C_1 \ne \emptyset$. Similarly, we denote by h_1, h_2 respectively $(h_1')'_{\vec t}(\bar x;z), (h_2')'_{\vec t}(\bar x;z)$ the components of h respectively $(h')'_{\vec t}(\bar x;z)$. Then there holds $Q = Y_1\times Q_2$ with $Q_2 = \operatorname{aff}(h_2(\bar x) + h_2'(\bar x)X - C_2)$ and it follows that $h(\cdot)$ is 2-non-degenerate in direction z and with respect to C and $\vec t$ if and only if $h_2(\cdot)$ is of such a kind with respect to C_2.

Finally let us mention that the remaining assumption of Theorem 4.8 that either the quotient space Y/Q or the space $\operatorname{Im} h'(\bar x)\cap\operatorname{aff}(C - h(\bar x))$ is finite dimensional, is satisfied in a variety of cases, e.g. when X or Y is finite dimensional or in the case of the scalar mathematical programming problem. Further, we know a lot of special cases where this assumption can be replaced, but we do not want to go down to the last detail here.

5. Second-order necessary conditions for the scalar mathematical programming problem. We consider here the results of the preceding section for the special case of the scalar mathematical programming problem

$$\begin{aligned} (MP)\quad & \min f(x) \\ s.t.\quad & g_i(x) \le 0,\ i = 1, \dots, m, \\ & G(x) = 0, \end{aligned}$$

where $f : X \to \mathbb{R}$, $g_i : X \to \mathbb{R}$ for $i = 1, \dots, m$, $G : X \to \hat{V}$ and X and $\hat{V}$ are Banach spaces. Then $Y = \mathbb{R} \times \mathbb{R}^m \times \hat{V}$ and for a given feasible point $\bar{x}$ the mapping h and the set C according to (1.5) are given by $h(x) = (f(x) - f(\bar{x}), g_1(x), \dots, g_m(x), G(x))$ and $C = R_- \times \mathbb{R}^m_- \times \{0\}$. We will assume throughout this section that Assumption 1 holds. The set of multipliers Λ_{FJ} satisfying the first order conditions of Fritz-John type consists of all multipliers $(\alpha, \lambda, v^*) \in \mathbb{R} \times \mathbb{R}^m \times \hat{V}^*$, such that

$$\begin{aligned} \mathcal{L}'_x(\bar{x}, \alpha, \lambda, v^*) &= 0, \\ \alpha &\ge 0, \\ \lambda_i \ge 0,\ \lambda_i g_i(\bar{x}) &= 0,\ i = 1, \dots, m, \\ (\alpha, \lambda, v^*) &\ne (0, 0, 0), \end{aligned}$$

where the generalized Lagrangian $\mathcal{L}$ is given in the usual way by $\mathcal{L}(x, \alpha, \lambda, v^*) := \alpha f(x) + \sum_{i=1}^m \lambda_i g_i(x) + \langle v^*, G(x)\rangle$ and $\mathcal{L}'_x$ is the partial derivative of the Lagrangian with respect to x. For partial second-order directional dervatives of the Lagrangian with respect to the first variable we use the following notation:

$$\begin{aligned} \mathcal{L}''_{x\vec{t}}(\bar{x}, \alpha, \lambda, v^*; z) &:= \alpha f''_{\vec{t}}(\bar{x}; z) + \sum_{i=1}^m \lambda_i g_i{}''_{\vec{t}}(\bar{x}; z) + \langle v^*, G''_{\vec{t}}(\bar{x}; z)\rangle, \\ (\mathcal{L}'_x)'_{\vec{t}}(\bar{x}, \alpha, \lambda, v^*; z) &:= \alpha (f')'_{\vec{t}}(\bar{x}; z) + \sum_{i=1}^m \lambda_i (g'_i)'_{\vec{t}}(z; +) (G')'_{\vec{t}}(\bar{x}; z)^* v^*. \end{aligned}$$

Note, that the multipliers (α, λ, v^*) form a linear functional $y^* \in Y^*$ and we have $\mathcal{L}'_x(\bar{x}, \alpha, \lambda, v^*) = h'(\bar{x})^* y^*$, $\mathcal{L}''_{x\vec{t}}(\bar{x}, \alpha, \lambda, v^*; z) = h''_{\vec{t}}(\bar{x}; z)$ and $(\mathcal{L}'_x)'_{\vec{t}}(\bar{x}, \alpha, \lambda, v^*; z) = (h')'_{\vec{t}}(\bar{x}; z)^* y^*$.

In case of the mathematical programming problem the second-order compound tangent sets have the property that

$$0 \in C''_{h'(\bar{x}),\vec{t}}(h(\bar{x}); s), \forall s \in \mathcal{C}(\bar{x}), \vec{t} \in \mathbb{T}. \tag{5.1}$$

This follows easily from the definition and the observation, that for arbitrary $s \in \mathcal{C}(\bar{x})$ and $\vec{t} \in \mathbb{T}$ we have $h(\bar{x}) + t_n s \in C$ for all n sufficiently large. Condition (5.1) together with Lemma 4.2 implies

$$C''_{h'(\bar{x}),\vec{t}}(h(\bar{x}); s) = \mathrm{cl}\Big(T_C(h(\bar{x})) + \mathrm{Im}\, h'(\bar{x})\Big).$$

Hence, $C''_{h'(\bar{x}),\vec{t}}(h(\bar{x}); s)$ does not depend on the choice of the direction $s \in \mathcal{C}(\bar{x})$ and the sequence $\vec{t} \in \mathbb{T}$. Moreover, $C''_{h'(\bar{x}),\vec{t}}(h(\bar{x}); s)$ is a cone and its polar cone can be written as

$$C''_{h'(\bar{x}),\vec{t}}(h(\bar{x}); s)^\circ = N_C(h(\bar{x})) \cap (\mathrm{Im}\, h'(\bar{x}))^\perp = \Lambda_{FJ} \cup \{0\}.$$

It follows that $y \in C''_{h'(\bar{x}),\vec{t}}(h(\bar{x}); s)$ holds if and only if one has $\langle y^*, y\rangle \le 0$ for each $y^* \in \Lambda_{FJ}$.

In a next step, for fixed $z \in \mathcal{C}(\bar{x}) \cap \mathcal{S}_X$ and $\vec{t} \in \mathbb{T}$ with $h''_{\vec{t}}(\bar{x}; z) \in C''_{h'(\bar{x}),\vec{t}}(h(\bar{x}); z)$ we will analyze the conclusions (4.3) and (4.4) of Theorem 4.6 under condition (5.1). Using condition (4.4) with $s = z$ and $y = 0$ yields, together with the above characterization for $h''_{\vec{t}}(\bar{x}; z) \in C''_{h'(\bar{x}),\vec{t}}(h(\bar{x}); z)$,

$$\langle \bar{y}^*, h''_{\vec{t}}(\bar{x}; z)\rangle = \max_{y^* \in \Lambda_{FJ}} \langle y^*, h''_{\vec{t}}(\bar{x}; z)\rangle = 0.$$

Since $h''_{\vec{t}}(\bar{x};z) \in C''_{h'(\bar{x}),\vec{t}}(h(\bar{x});z)$ implies $h''_{\vec{t}}(\bar{x};z) \in C''_{h'(\bar{x}),\vec{t}}(h(\bar{x});s)$ for all $s \in \mathcal{C}(\bar{x})$ we obtain $\langle \bar{\mu}^*, z-s\rangle \geq 0, \forall s \in \mathcal{C}(\bar{x})$. Thus $\bar{\mu}^* \in N_{\mathcal{C}(\bar{x})}(z)$ or equivalently, since $\mathcal{C}(\bar{x})$ is a cone, $\bar{\mu}^* \in \{\mu^* \in \mathcal{C}(\bar{x})^\circ : \langle \mu^*, z\rangle = 0\}$, where $\mathcal{C}(\bar{x})^\circ$ denotes the polar cone of the critical cone $\mathcal{C}(\bar{x})$. Now in case of (MP) the critical cone is given by

$$\mathcal{C}(\bar{x}) = \left\{ z \in X \; : \; \begin{array}{rcl} \langle f'(\bar{x}), z\rangle & \leq & 0 \\ \langle g_i'(\bar{x}), z\rangle & \leq & 0, \; \forall i \in \bar{I}, \\ G'(\bar{x})z & = & 0 \end{array} \right\},$$

where $\bar{I} := \{i \in \{1,\ldots,m\} : g_i(x) = 0\}$ denotes the index set of active inequality constraints. If $\operatorname{Im} G'(\bar{x})$ is closed, then by the Generalized Farkas lemma (see, e.g. [9, Proposition 2.201]) the polar cone $\mathcal{C}(\bar{x})^\circ$ is given by the formula

$$\mathcal{C}(\bar{x})^\circ = \{\alpha f'(x) + \sum_{i\in\bar{I}} \lambda_i g_i'(x)^* + G'(\bar{x})^* v^* \; : \; \alpha \geq 0, \lambda_i \geq 0, i \in \bar{I}, v^* \in \hat{V}^*\}.$$

Hence, $\bar{\mu}^* \in \{\mu^* \in \mathcal{C}(\bar{x})^\circ : \langle \mu^*, z\rangle = 0\}$ has the following representation:

$$\bar{\mu}^* = \tilde{\alpha} f'(x) + \sum_{i=1}^m \tilde{\lambda}_i g_i'(x) + G'(\bar{x})^* \tilde{v}^*,$$

where $\tilde{\alpha} \geq 0$, $\tilde{\lambda}_i \geq 0$, $\tilde{\lambda}_i g_i(\bar{x}) = 0$, $i = 1,\ldots,m$ and $\tilde{v}^* \in \hat{V}^*$ are such that $\tilde{\alpha}\langle g_i'(\bar{x}), z\rangle + \sum_{i=0}^m \tilde{\lambda}_i \langle g_i'(\bar{x}), z\rangle + \langle \tilde{v}^*, G'(\bar{x})z\rangle = 0$, or equivalently, since $z \in \mathcal{C}(\bar{x})$,

$$\tilde{\alpha}\langle f'(\bar{x}), z\rangle = 0, \quad \tilde{\lambda}_i g_i(\bar{x}) = \tilde{\lambda}_i \langle g_i'(\bar{x}), z\rangle = 0, \; i = 1,\ldots,m.$$

Next let us consider Assumption 2. It surely holds if the space $\hat{V}^*$ is finite dimensional. In case of infinite dimensional $\hat{V}$ let us examine the assumptions of Theorem 4.8. Note that $\operatorname{aff}(C - h(\bar{x})) = \mathbb{R} \times \mathbb{R}^m \times \{0\}$ is always finite-dimensional for the problem (MP). From the discussion at the end of the preceding section we know that to verify 2-non-degeneracy of $h(\cdot)$ with respect to C we only have to consider the equality constraints $G(x) = 0$. It is straightforward to see that the mapping $G(\cdot)$ is 2-non-degenerate in direction z with respect to $\{0\}$ and the sequence $\vec{t} \in \mathbb{T}$, provided that $(G')'_{\vec{t}}(\bar{x};z)$ exists, if and only if $\operatorname{Im} G'(\bar{x})$ is closed in $\hat{V}$ and the mapping $s \to (G'(\bar{x})s, \pi G''_{\vec{t}}(\bar{x};z)s)$ carries X onto $\operatorname{Im} G'(\bar{x}) \times \hat{V}/\operatorname{Im} G'(\bar{x})$, where π here denotes the quotient mapping onto the quotient space $\hat{V}/\operatorname{Im} G'(\bar{x})$. For twice respectively three times continuously differentiable mappings G this condition already appears in a lot of papers on optimality conditions for problems with degenerate equality constraints, see [3], [4], [23]. Further, in a slightly different version it is also known as the property of *2-regularity* of the mapping G (see [5], [31]). We refer also to [17] where the theory of 2-regularity was applied to once differentiable mappings having a locally Lipschitzian derivative.

We summarize these considerations in the following Corollary:

COROLLARY 5.1. *Let $\bar{x}$ be a local minimizer for the mathematical programming problem (MP) and let Assumption 1 hold. Then for each element $z \in \mathcal{C}(\bar{x})$ and each sequence $\vec{t} \in \mathbb{T}$ such that the second-order directional derivatives $h''_{\vec{t}}(\bar{x};z)$ and $(h')'_{\vec{t}}(\bar{x};z)$ exist, such that*

$$\sup_{(\alpha,\lambda,\hat{v}^*)\in\Lambda_{FJ}} \mathcal{L}''_{x\vec{t}}(\bar{x},\alpha,\lambda,v^*;z) \leq 0 \tag{5.2}$$

and that either $\dim \hat{V} < \infty$ or $G(\cdot)$ is 2-non-degenerate in direction z with respect to $\{0\}$ and $(\vec{t})$, there exist multipliers $(\alpha,\lambda,v^*) \in \Lambda_{FJ}$ and $(\tilde{\alpha},\tilde{\lambda},\tilde{v}^*) \in \mathbb{R}_+ \times \mathbb{R}^m_+ \times \hat{V}^*$ such that

$$\mathcal{L}''_{x\vec{t}}(\bar{x},\alpha,\lambda,v^*;z) = 0 \tag{5.3}$$

$$(\mathcal{L}'_x)'_{\vec{t}}(\bar{x},\alpha,\lambda,v^*;z) + \mathcal{L}'_x(\bar{x},\tilde{\alpha},\tilde{\lambda},\tilde{v}^*) = 0, \tag{5.4}$$

$$\tilde{\lambda}_i g_i(\bar{x}) = \tilde{\lambda}_i \langle g_i'(\bar{x}), z\rangle = 0, \; i = 1,\ldots,m \tag{5.5}$$

$$\tilde{\alpha}\langle f'(\bar{x}), z\rangle = 0. \tag{5.6}$$

Let us compare Corollary 5.1 with the standard second-order conditions (see, e.g. [14]) for (MP). For sake of simplicity let us assume that f, g_i, $i = 1, \ldots, m$, and G are twice continuously differentiable at $\bar{x}$ and that the range $\operatorname{Im} G'(\bar{x})$ is closed. If $\Lambda_{FJ} \neq \emptyset$ and if there is some $\beta > 0$ such that

$$\max_{(\alpha,\lambda,v^*)\in\Lambda_{FJ}\cap\mathcal{S}_{Y^*}} \mathcal{L}''_x(\bar{x},\alpha,\lambda,v^*)(z,z) \geq \beta$$

for all $z \in \mathcal{C}(\bar{x}) \cap S_X$, then $\bar{x}$ is a strict local minimizer and in fact one can show [13] that this condition is also equivalent for $\bar{x}$ to be an essential local minimizer of second order. On the other hand, the standard second-order necessary conditions state that at a local minimizer $\bar{x}$ the set Λ_{FJ} is not empty and for each $z \in \mathcal{C}(\bar{x})$ there is some multiplier $(\alpha, \lambda, v^*) \in \Lambda_{FJ} \cap \mathcal{S}_{Y^*}$ such that

$$\mathcal{L}''_x(\bar{x},\alpha,\lambda,v^*)(z,z) \geq 0.$$

It is also well-known that on the one hand this necessary condition is equivalent to condition (2.1) for non-degenerate points $\bar{x}$, i.e. when $\operatorname{Im} G'(\bar{x}) = \hat{V}$, and on the other hand, that it is always satisfied when $\bar{x}$ is degenerate, whether $\bar{x}$ is a local minimizer or not. Now, for degenerate points $\bar{x}$ Corollary 5.1 states the additional necessary conditions (5.4)-(5.6) for exactly those directions $z \in \mathcal{C}(\bar{x}) \cap S_X$, where the sufficient conditions fail to hold, and thus reduces the gap between the standard necessary and sufficient conditions for the mathematical programming problem (MP).

Now let us compare our work with Avakov's results [3], [4]: For the sake of simplicity we consider only the case when no inequality constraints are present, i.e. $m = 0$. Avakov shows, assuming that f is Fréchet differentiable at $\bar{x}$ and G is twice Fréchet differentiable at a local minimizer $\bar{x}$ and the range $\operatorname{Im} G'(\bar{x})$ is closed, the following "first-order" conditions hold: For each $z \in X$ such that $G'(\bar{x})z = 0$, $G''(\bar{x})(z,z) \in \operatorname{Im} G'(\bar{x})$ and the operator $\mathcal{G}(\bar{x}; z) : X \to \operatorname{Im} G'(\bar{x}) \times \hat{V}/\operatorname{Im} G'(\bar{x})$ given by $\mathcal{G}(\bar{x}, z)s \to (G'(\bar{x})s, \pi G''(\bar{x})(z,s))$ has closed range there exist multipliers $(0,0) \neq (\alpha, v^*) \in \mathbb{R}_+ \times \operatorname{Ker} G'(\bar{x})^*$ and $\tilde{v}^* \in \hat{V}^*$ such that

$$\alpha f'(\bar{x}) + G'(\bar{x})^*\tilde{v}^* + G''(\bar{x})(z,\cdot)^*v^* = 0. \tag{5.7}$$

Now let us demonstrate how this result follows from our work in case the equality constraints are degenerate, i.e. $\operatorname{Im} G'(\bar{x}) \neq \hat{V}$, under the assumption that f is strictly differentiable at $\bar{x}$. Let $z \in X$ be fixed, satisfying $G'(\bar{x})z = 0$, $G''(\bar{x})(z,z) \in \operatorname{Im} G'(\bar{x})$. W.l.o.g. we can assume $\langle f'(\bar{x}), z\rangle \leq 0$, since otherwise we can take the direction $-z$. When $\mathcal{G}(\bar{x}; z)$ is not surjective but has closed range, equation (5.7) with $\alpha = 0$ follows from the existence of a nontrivial functional in $(\operatorname{Im} \mathcal{G}(\bar{x}; z))^\perp$ by using standard arguments.

Now assume that $\mathcal{G}(\bar{x}; z)$ is surjective, i.e. G is 2-non-degenerate in direction z. It follows that $\pi G''(\bar{x})(z,\cdot) \neq 0$ and hence $z \neq 0$, w.l.o.g. $\|z\| = 1$. Let $p^* \in X^*$ denote a continuous linear functional with $\langle p^*, z\rangle = 1$ and consider the problem

$$(MP_{p^*}) \quad \min_{x\in X} \varphi(x) := \langle p^*, x - \bar{x}\rangle(f(x) - f(\bar{x})) \text{ s.t. } \langle -p^*, x - \bar{x}\rangle \leq 0,\ G(x) = 0.$$

Then, $\bar{x}$ is also a local minimizer for this problem and one can show that Assumption 1 is satisfied and also the second-order directional derivatives exist for every direction and every sequence $\vec{t} \in T$. In particular, we have $\varphi'(\bar{x}) = 0$, $\varphi''_{\vec{t}}(\bar{x}; z) = 2\langle f'(\bar{x}), z\rangle\langle p, z\rangle$ and $(\varphi')'_{\vec{t}}(\bar{x}; z) = \langle p^*, z\rangle f'(\bar{x}) + \langle f'(\bar{x}), z\rangle p^*$. It follows immediately that z belongs to the critical cone of problem (MP_{p^*}).

Now let $(0,0,0) \neq (\alpha, \lambda, v^*) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \hat{V}^*$ be an arbitrary multiplier satisfying the Fritz-John-conditions for (MP_{p^*}). We have $-\lambda p^* + G'(\bar{x})^*v^* = 0$ and consequently $\lambda = \lambda\langle p^*, z\rangle = \langle v^*, G'(\bar{x})z\rangle = 0$ and $G'(\bar{x})^*v^* = 0$. Due to our assumption on z we have $G''(\bar{x})(z,z) = G'(\bar{x})w$ for some $w \in X$ and therefore

$$\alpha\varphi''_{\vec{t}}(\bar{x}; z) + 0 + \langle v^*, G''(\bar{x})(z,z)\rangle = 2\alpha\langle f'(\bar{x}), z\rangle\langle p^*, z\rangle + \langle v^*, G'(\bar{x})w\rangle = 2\alpha\langle f'(\bar{x}), z\rangle \leq 0.$$

This shows that condition (5.2) is satisfied and application of Corollary 5.1 proves the existence of multipliers $(0,0,0) \neq (\alpha, 0, v^*) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \operatorname{Ker} G'(\bar{x})^*$ and $(\tilde{\alpha}, \tilde{\lambda}, \tilde{v}^*) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \hat{V}^*$ such that we have $-\tilde{\lambda} = \tilde{\lambda}\langle -p^*, z\rangle = 0$ by condition (5.5), $\alpha\langle f'(\bar{x}), z\rangle = 0$ by condition (5.3) and also

$$\alpha(\varphi')'_{\bar{t}}(\bar{x}; z)G''(\bar{x})(z,\cdot)^*v^* + G'(\bar{x})^*\tilde{v}^* + = \alpha f'(\bar{x}) + G'(\bar{x})^*\tilde{v}^* + G''(\bar{x})(z,\cdot)^*v^* = 0,$$

by condition (5.4). Hence, condition (5.7) holds.

Note that condition (5.7) is called a "first-order" condition, although is contains the second derivative $G''(\bar{x})$. Similarly, Avakov and others presented "second-order" conditions (see [3],[4], [16], [23]), where third derivatives of the mapping G are involved. Of course, our results cannot cover such "second-order" conditions.

There are also other second-order conditions known from the literature, see for instance the monograph of Arutyunov [2] and the references cited therein.

Example 5.1. *This is a very easy example which can be treated by the results of this paper but not by results in the literature. Consider the problem*

$$\min_{(x_1,x_2,x_3)\in\mathbb{R}^3} -x_1^2 + x_3 \quad s.t. \quad x_1x_2 + x_1^{5/2} = 0, \quad x_3 = 0$$

at $\bar{x} = (0,0,0)$. *Then* $\bar{x} = (0,0,0)$ *is not a local minimum and this follows also from Corollary 5.1 since the second-order conditions (5.3)-(5.6) are not satisfied for the direction* $z = (-1,0,0)$. *However, the necessary "first-order" conditions (5.7) hold and the "second-order" conditions from [3], [23], [24] do not apply since the constraints are not three times differentiable at* $\bar{x}$. *Also the necessary conditions of Arutyunov [1, Theorem 3.1, Theorem 3.2] and Belash and Tret'yakov [5, Theorem 3] are either satisfied or cannot be used since* $f'(\bar{x}) \neq 0$ *and* $G'(\bar{x}) \neq 0$.

Example 5.2. *Now consider the problem*

$$\min_{(x_1,x_2)\in\mathbb{R}^2} x_1 \quad s.t. \quad x_2 \le 0, \; x_1x_2 = 0$$

at $\bar{x} = (0,0)$. *Again,* $\bar{x}$ *is not a local minimum, but the second-order conditions of Corollary 5.1 now hold. To verify these conditions we have to consider the directions* $z = (-1,0)$ *and* $z = (0,-1)$ *and in both cases the multipliers* $(\alpha,\lambda,v^*) \in \Lambda_{FJ}$ *satsfying the conditions (5.3)-(5.6) are given by* $(0,0,1)$. *Moreover, for any direction* $z \in \mathcal{C}(\bar{x}) \cap \mathcal{S}_X$ *we have* $\mathcal{L}''_{x\bar{t}}(\bar{x},\alpha,\lambda,v^*;z) \ge 0$ *for multipliers of the form* $(\alpha,\lambda,v^*) = (0,0,t)$, $t > 0$. The following Theorem states that in a situation as in Example 5.2 the second-order necessary conditions are sharp. We state this result in terms of the general problem (P):

Theorem 5.2. *Let the point* $\bar{x}$ *be feasible for the problem (P) and suppose that Assumption 1 holds, that* $\dim Y < \infty$ *and*

$$\lim_{t\to 0_+} d\Big(\frac{h(\bar{x}+tz) - h(\bar{x}) - th'(\bar{x})z}{t^2/2}, h''(\bar{x};z)\Big) = 0 \tag{5.8}$$

holds uniformly for all $z \in \mathcal{C}(\bar{x}) \cap \mathcal{B}_X$. *Further suppose that there is a pointed closed convex cone* $\bar{\Lambda} \subset \Lambda_{FJ} \cup \{0\}$ *such that for every* $z \in \mathcal{C}(\bar{x})$ *and every* $y \in h''(\bar{x};z)$ *there is some* $\bar{y}^* \in \bar{\Lambda} \cap \mathcal{S}_{Y^*}$ *with* $\langle \bar{y}^*, y\rangle \ge 0$. *Then there exists a mapping* $\delta h = (\delta f, \delta g)$ *with* $\delta h(x) = \psi(\|x - \bar{x}\|)y$, *where* $y \in Y$ *and* $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ *is a twice continuously differentiable function satisfying* $\psi(0) = \psi'(0) = \psi''(0) = 0$, *such that* $\bar{x}$ *is a strict local minimizer for (P) with* f *and* g *replaced by* $f + \delta f$ *and* $g + \delta g$, *respectively.*

Proof. Let $S := \bar{\Lambda}^\circ$ denote the polar cone of the pointed convex closed cone $\bar{\Lambda}$. Since $\dim Y < \infty$, we have $\operatorname{int} S \neq \emptyset$. Further we have $\operatorname{cl}(h'(\bar{x})X + T_C(h(\bar{x}))) = \Lambda^\circ_{FJ} \subset S$. Let the subspace Q be given by $Q := \operatorname{aff}(h'(\bar{x})X - T_C(h(\bar{x}))$. If $\dim Q < \dim Y$ then we can find $p := \dim Y - \dim Q$ linearly independent elements $y_i \in S \setminus Q$, $i = 1,\dots,p$, forming a basis for some topological complement Q^c to Q, such that $\operatorname{int}(h'(\bar{x})X - \bar{C}) \neq \emptyset$, where $\bar{C} := h(\bar{x}) + T_C(h(\bar{x})) + \hat{S}$ and the cone $\hat{S}$ is given by $\hat{S} := \{\sum_{i=1}^p \alpha_i y_i \,:\, \alpha_i \ge 0,\; i = 1,\dots,p\}$. Note that $\bar{C}$ is closed since $T_C(h(\bar{x})) \subset Q$ and $\hat{S} \subset Q^c$ are closed and $Y = Q \oplus Q^c$. On the other hand, if $\dim Q = \dim Y$ take $\bar{C} := h(\bar{x}) + T_C(h(\bar{x}) + \hat{S}$ with $\hat{S} = \{0\}$. In any case we have $\operatorname{int}(h'(\bar{x})X - \bar{C}) \neq \emptyset$ and

$C \subset h(\bar{x}) + T_C(h(\bar{x})) \subset \bar{C} \subset h(\bar{x}) + S$. By taking into account the remark following Theorem 2.2 we see that in order to prove the Theorem it is sufficient to show

$$\liminf_{\substack{x\to\bar{x}\\ \tau\to 0_+}} \frac{\hat{d}_{\bar{C}}(h(\bar{x}), h'(\bar{x}), \tau\|x-\bar{x}\|)}{\|x-\bar{x}\|^2} \geq 0.$$

Assume on the contrary that there are sequences $(z_n) \subset \mathcal{S}_X$, $\vec{t} = (t_n) \in \mathbb{T}$ and $(\tau_n) \in \mathbb{T}$ and a real $\beta > 0$ such that $t_n^{-2}\hat{d}_{\bar{C}}(h(\bar{x}+t_n z_n), h'(\bar{x}), \tau_n t_n) \leq -\beta\ \forall n$. Then, by Theorem 2.4 we have $0 \in h(\bar{x}+t_n z_n) + \tau_n t_n h'(\bar{x})\mathcal{B}_X - \bar{C}$ and using Assumption 1 we see $t_n^{-1}(h(\bar{x}+t_n z_n) - h(\bar{x})) = h'(\bar{x})z_n + O(t_n) \in t_n^{-1}(\bar{C} - h(\bar{x})) = T_C(h(\bar{x})) + \hat{S}$. Since $\dim Y < \infty$, $h'(\bar{x})z_n \subset Y$ is a bounded sequence and $\operatorname{Im} h'(\bar{x})$ is closed, we have, by passing to a subsequence if necessary, $h'(\bar{x})z_n \to h'(\bar{x})\bar{z}$ for some $\bar{z} \in X$. Then, $h'(\bar{x})\bar{z} \in T_C(h(\bar{x})) + \hat{S}$ and since $h'(\bar{x})\bar{z} \in Q$, $T_C(h(\bar{x}) \subset Q$ and $\hat{S} \cap Q = \{0\}$, $h'(\bar{x}) \in T_C(h(\bar{x}))$ follows, i.e. $\bar{z} \in \mathcal{C}(\bar{x})$. Further, since $\operatorname{Im} h'(\bar{x})$ is closed we can find another sequence, say (z_n'), such that $h'(\bar{x})z_n' = h'(\bar{x})\bar{z}$ and $\|z_n' - z_n\| \leq \gamma\|h'(\bar{x})\bar{z} - h'(\bar{x})z_n\|$ for some $\gamma > 0$. By taking $\tilde{z}_n := z_n'/\|z_n'\|$ we have found a sequence $(\tilde{z}_n) \subset \mathcal{C}(\bar{x}) \cap \mathcal{S}_X$ with $\tilde{z}_n - z_n \to 0$. By our assumptions, there exists a sequence (w_n) with $w_n \in h''(\bar{x};\tilde{z}_n)$ such that $\|h(\bar{x}+t_n\tilde{z}_n) - (h(\bar{x}) + t_n h'(\bar{x})\tilde{z}_n + \frac{t_n^2}{2}w_n)\| = o(t_n^2)$ and, together with Assumption 1,

$$\|h(\bar{x}+t_n z_n) - (h(\bar{x}) + t_n h'(\bar{x})z_n + \frac{t_n^2}{2}w_n)\| \leq \eta t_n^2\|z_n - \tilde{z}_n\| + o(t_n^2) = o(t_n^2)$$

follows. Now, for each n let $y_n^* \in \bar{\Lambda} \cap \mathcal{S}_{Y^*}$ be chosen such that $\langle y_n^*, w_n\rangle \geq 0$. Since $y_n^* \in \Lambda_{FJ}$ we have $h'(\bar{x})^* y_n^* = 0$ and because of $h(\bar{x}) \in \bar{C}$ we obtain $\sigma_{\bar{C}}(y_n^*) \geq \langle y_n^*, h(\bar{x})\rangle$. Hence we conclude that

$$\begin{aligned}
t_n^{-2}\hat{d}_{\bar{C}}(h(\bar{x}+t_n z_n), h'(\bar{x}), \tau_n t_n) &\leq t_n^{-2}(\langle y_n^*, h(\bar{x}+t_n z_n)\rangle - \sigma_{\bar{C}}(y_n^*) - \tau_n t_n\|h'(\bar{x})^* y_n^*\|)\\
&\leq t_n^{-2}\langle y_n^*, h(\bar{x}+t_n z_n) - h(\bar{x})\rangle\\
&= t_n^{-2}\langle y_n^*, h(\bar{x}+t_n z_n) - h(\bar{x}) - t_n h'(\bar{x})z_n\rangle\\
&= \frac{1}{2}\langle y_n^*, w_n\rangle + o(1) \geq o(1),
\end{aligned}$$

a contradiction. □ Note that, as a consequence of Assumption 1 and $\dim Y < \infty$, convergence in condition (5.8) is always uniform with respect to z in compact sets. Hence, besides the case when h is twice Fréchet differentiable at $\bar{x}$, condition (5.8) holds uniformly for all $z \in \mathcal{C}(\bar{x})$ when $\dim X < \infty$.

In case when f is scalar and K is a polyhedral cone, using the notion of *2-normal mappings*, Arutyunov [1],[2] presented conditions which are sufficient for the existence of a cone $\bar{\Lambda}$ satisfying the assumptions of Theorem 5.2. Further, Arutyunov showed 2-normal mappings to be generic under certain circumstances. However note that the constraint mapping of Example 5.2 is not 2-normal.

Arutyunov [1, Theorem 4.3] stated also a result which is very similar to Theorem 5.2 and from which he concluded that the "gap" between his necessary and sufficient second-order conditions is as minimal as possible. However, this is not quite correct. To see this, consider the case when X is finite dimensional and f is scalar. In our notation, with a slight modification, [1, Theorem 4.3] essentially says that there exists some $\tilde{v} \in V$ such that for any $\epsilon > 0$ the point $\bar{x}$ is a strict local minimum for the perturbed problem

$$\min f_\epsilon(x) := f(x) + \epsilon\|x-\bar{x}\|^2 \quad \text{s.t.} \quad g_\epsilon(x) := g(x) + \epsilon\|x-\bar{x}\|^2\tilde{v} \in K.$$

However, from the proof of [1, Theorem 4.3] together with [13, Theorem 5.6] it follows that for any $\epsilon > 0$ the point $\bar{x}$ is even an essential local minimizer of second order for the perturbed problem and so the necessary conditions of Theorem 3.1 (and hence all the other necessary conditions of this paper) would not follow from Arutyunov's result.

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