On the Number of Distinct Multinomial Coefficients

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Abstract

We study $M(n)$, the number of distinct values taken by multinomial coefficients with upper entry $n$, and some closely related sequences. We show that both $p_P(n)/M(n)$ and $M(n)/p(n)$ tend to zero as $n$ goes to infinity, where $p_P(n)$ is the number of partitions of $n$ into primes and $p(n)$ is the total number of partitions of $n$. To use methods from commutative algebra, we encode partitions and multinomial coefficients as monomials.

Key words: Factorials, binomial coefficients, combinatorial functions, partitions of integers; polynomial ideals, Gröbner bases.

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1 Introduction

The classical multinomial expansion is given by

\[(x_1 + x_2 + \cdots + x_k)^n = \sum \binom{n}{i_1, i_2, \ldots, i_k} x_1^{i_1} x_2^{i_2} \cdots x_k^{i_k}, \tag{1}\]

where the sum runs over all \((i_1, i_2, \ldots, i_k)\) such that \(i_1 + i_2 + \cdots + i_k = n\) and \(i_1, i_2, \ldots, i_k \geq 0\). Multinomial coefficients are defined by

\[\binom{n}{i_1, i_2, \ldots, i_k} := \frac{n!}{i_1! i_2! \cdots i_k!}. \tag{2}\]

It is natural to ask about \(M_k(n)\), the number of different values of

\[\binom{n}{i_1, i_2, \ldots, i_k} \tag{3}\]

where \(i_1 + i_2 + \cdots + i_k = n\). Obviously if the \(i_1, i_2, \ldots, i_k\) are merely permuted, then the value of \(\binom{n}{i_1, i_2, \ldots, i_k}\) is unchanged. However identical values do not necessarily arise only by permuting the \(i_1, i_2, \ldots, i_k\). For example,

\[\binom{7}{3, 2, 2} = \binom{7}{4, 1, 1, 1}. \tag{4}\]

and

\[\binom{236}{64, 55, 55, 52, 7, 3} = \binom{236}{62, 56, 54, 51, 13}. \tag{5}\]

We note that if \(k \geq n\), then \(M_k(n) = M_n(n)\), and we define \(M(n) := M_n(n)\) to be the total number of distinct multinomial coefficients with upper entry \(n\).

Since permuting its lower indices leaves the value of a multinomial coefficient unchanged it is immediately clear that

\[M_k(n) \leq p_k(n) \tag{6}\]

and

\[M(n) \leq p(n), \tag{7}\]

where \(p_k(n)\) is the number of partitions of \(n\) into at most \(k\) parts, and \(p(n)\) is the total number of partitions of \(n\) respectively. Observing that the binomial coefficients \(\binom{n}{k, n-k}\) are strictly increasing for \(0 \leq k \leq \frac{n}{2}\), we deduce that, in fact,

\[M_2(n) = p_2(n). \tag{8}\]
However the inequality (6) seems to be stronger for large \( k \). Indeed (Theorem 8),
\[
\lim_{n \to \infty} \frac{M(n)}{p(n)} = 0.
\]
(9)

Bounding \( M(n) \) from below we will prove (Theorem 1) that
\[
M(n) \geq p(P(n))
\]
(10)
where \( p(P(n)) \) is the number of partitions of \( n \) into parts belonging to the set of primes \( \mathbb{P} \). Indeed (Theorem 12),
\[
\lim_{n \to \infty} \frac{p(P(n))}{M(n)} = 0.
\]
(11)

It is natural to generalize the problem from \( M(n) \) to \( M_S(n) \), the number of different multinomial coefficients with upper entry \( n \) whose lower entries belong to a given set \( S \) of natural numbers. Let
\[
M_S(q) := \sum_n M_S(n)q^n
\]
(12)
and
\[
P_S(q) := \sum_n p_S(n)q^n
\]
(13)
where \( p_S(n) \) is the number of partitions of \( n \) into elements from \( S \). Define \([s] := \{1, 2, \ldots, s\}\). Results of numerical calculations such as
\[
M_4(q) / P_4(q) = 1 - q^7 + O(q^{100})
\]
(14)
and
\[
M_7(q) / P_7(q) = 1 - q^7 - q^8 - q^{10} + q^{12} + q^{13} + O(q^{100})
\]
(15)
suggest that \( M_S(q) / P_S(q) \) is a polynomial for any finite \( S \). This is indeed true (Theorem 5) and leads to an algorithm for computing a closed form for the sequence \( M_S(n) \) for a given finite set \( S \) (Section 4).

Partitions and multinomial coefficients can be written as monomials in a natural way: For instance, the monomial \( q_1^7 q_2^3 \) represents the partition \( 4+1+1+1 \), and \( x_7 x_5 x_3 x_2 \) represents the multinomial coefficient \( \left( \begin{smallmatrix} 7 \\ 4,1,1,1 \end{smallmatrix} \right) \) whose factorization into primes is \( 7 \cdot 5 \cdot 3 \cdot 2 \). This encoding serves as a link between our counting problem and Hilbert functions (Section 3). Sections 4, 5 and 6 are based on that link.

We call a pair of partitions of \( n \) that yield the same multinomial coefficient but have no common parts an irreducible pair. For example, the partitions \( 4+1+1+1 \) and \( 3+2+2 \) form an irreducible pair according to Equation (4). In Section 7, we study \( i(n) \) the total number of irreducible pairs of partitions of \( n \), and we prove (Theorem 13) that
\[
i(n) > \frac{n^4}{56} - 1.
\]
2 A Lower Bound for $M(n)$

We relate $M(n)$ to $p_P(n)$ whose asymptotics is known by a theorem of Kerawala [1]:

$$\log p_P(n) \sim \frac{2\pi}{\sqrt{3}} \sqrt{\frac{n}{\log n}}.$$  \hspace{1cm} (16)

**Theorem 1** $M(n) \geq p_P(n)$.

Theorem 1 is implied by the following lemma:

**Lemma 2** Any two distinct partitions of the same natural number $n$ into primes yield different multinomial coefficients.

**PROOF.** [Proof of Lemma 2] It suffices to show that if

$$p_1!p_2!\cdots p_r! = q_1!q_2!\cdots q_s!$$ \hspace{1cm} (17)

where $p_1 \leq p_2 \leq \cdots \leq p_r$ and $q_1 \leq q_2 \leq \cdots \leq q_s$ are all primes, then $r = s$ and $p_i = q_i$ for $i = 1, \ldots, s$. We proceed by mathematical induction on $r$.

If $r = 1$, then $q_s$ must equal $p_1$ because if $q_s < p_1$ then $p_1$ divides the left side of the above equation but not the right side. If $q_s > p_1$ then $q_s$ divides the right side but not the left. Hence $q_s = p_1$, and dividing both sides by $p_1!$ we see that there can be no other $q_i$. Hence $s = 1$ and $q_1 = p_1$.

Assume now that our result holds up to but not including a particular $r$. As in the case $r = 1$, we must have $q_s = p_r$. Cancel $p_r!$ from both sides and apply the induction hypothesis to conclude that $s - 1 = r - 1$ and $p_i = q_i$ for $i = 1, \ldots, s - 1$. Hence the lemma follows by mathematical induction. \Box

Some values of $p_P(n)$ and $M(n)$ are listed on page 15. We will refine Theorem 1 in Section 6.

3 The Algebraic Setting

Encoding partitions and multinomial coefficients as monomials allows us to apply constructive methods from commutative algebra to the problem of counting multinomial coefficients. Let us assume that $S \subseteq \mathbb{N}$ throughout the paper. We will see that $M_S(n)$ finds a natural interpretation as the Hilbert function
of a certain graded ring (Lemma 4). In the case of finite $S$, it can be computed by the method of Gröbner bases [2–5].

We represent the partition $\lambda_0 + \lambda_1 + \cdots + \lambda_i$ of $n$ by the monomial $q_{\lambda_0}q_{\lambda_1}\cdots q_{\lambda_i}$, whose degree is $n$ if we define the degree of variables suitably by $\deg q_j := j$. For convenience, we will use the notions “partition of $n$” and “monomial of degree $n$” interchangeably.

Let $k$ be a field of characteristic zero. We abbreviate the ring $k[q_i : i \in S]$ of polynomials in the variables $q_i$ for $i \in S$ over $k$ by $k[S]$. Define the degree of monomials by $\deg q_i := i$, and let $k[S]_n$ denote the subspace of all homogeneous polynomials of degree $n$. In other words, $k[S]_n$ is the $k$-vector space whose basis are the partitions of $n$ into parts belonging to $S$. Note that $k[S]$ is graded by $k[S] = \bigoplus_n k[S]_n$. For instance,

$$k[\{1, 3, \ldots \}]_4 = k \cdot q_3 q_1 \oplus k \cdot q_4$$

(18)
corresponding to the partitions $3 + 1$ and $1 + 1 + 1 + 1$ of 4 into odd parts.

The multinomial coefficients with upper entry $n$ into parts belonging to $S$ are the numbers $n! / \prod_j j^{a_j}$, where $\prod_j q_j^{a_j}$ ranges over the monomials in $k[S]_n$. Since the numerator $n!$ of these fractions is fixed, it suffices to count the set of all denominators:

$$M_S(n) = \left| \left\{ \prod_j j^{a_j} : \prod_j q_j^{a_j} \in k[S]_n \right\} \right|.$$  

(19)

To count the values taken by $\prod_j j^{a_j}$, we look at their factorization into primes. Let $h(q_j)$ be the factorization of $j!$ into primes, written as a monomial in $k[x] := k[x_p : p \text{ prime}]$, multiplied by $q_j^i$. For example, $h(q_5) = q^4 x_3 x_5$ corresponding to $5! = 2^3 \cdot 3 \cdot 5$. An elementary counting argument [6] shows that $h(q_j)$ includes all prime $p$ in the factorization of $j!$ with exponent $\sum_{i=1}^{\infty} \lfloor j/p^i \rfloor$, where $\lfloor x \rfloor$ denotes the largest integer that does not exceed the real number $x$. Therefore,

$$h(q_j) = q_j^i \prod_{p \text{ prime}} x_p^{\sum_{i=1}^{\infty} \lfloor j/p^i \rfloor}.$$  

(20)

Since factorization into primes is unique, (19) can be written as

$$M_S(n) = \left| \left\{ \prod_j h(q_j)^{a_j} : \prod_j q_j^{a_j} \in k[S]_n \right\} \right|.$$  

(21)

Extending $h$ to a $k$-algebra homomorphism $k[S] \to k[x, q]$ allows us to reformulate (21) as

**Lemma 3**

$$M_S(n) = \dim_k h(k[S]_n).$$  

(22)
Example: Since there are 10 partitions of 7 into parts 1, 2, 3 and 4, the dimension of 
\[ k\{1, 2, 3, 4\}_7 = k q_4 q_3 \oplus k q_4 q_2 q_1 \oplus k q_4 q^3_1 \oplus k q^2_3 q^2_2 \oplus \cdots \oplus k q^7_1 \] (23)
over \( k \) is 10. However, the dimension of its image 
\[ h(k\{1, 2, 3, 4\}_7) = k q^7 x^2 x^3_2 \oplus k q^7 x^4 x^3_2 \oplus k q^7 x^3 x^3_3 \oplus \cdots \oplus k q^7 \] (24)
under \( h \) is only 9 and so \( M_S(7) = 9 \). The defect is due to \( h(q^3_4 q^3_2) \) which is nothing but a restatement of (4).

To use Lemma 3 for effective computation (in the case of finite \( S \)), we express \( \dim_k h(k[S]_n) \) as the value (at \( n \)) of the Hilbert function of a certain elimination ideal. This method is taken from [2]; the result in our case is Lemma 4 below.

First we make the map \( h \) degree-preserving (graded) by defining \( \deg q := 1 \) and \( \deg x_p := 0 \) in the ring \( k[x_p : p \text{ prime}][q] \). (This is why we introduced the extra factor of \( q^i \) in the defining equation (20) of \( h \).) Second, note that 
\[ h(k[S]_n) \cong k[S]_n/(k[S]_n \cap \ker h) \] (25)
as \( k \)-vector spaces, since \( h \) is a \( k \)-linear map on \( k[S]_n \). In particular, dimensions agree. Therefore,
\[ M_S(n) = \dim_k k[S]_n/(k[S]_n \cap \ker h). \] (26)
Recall that the (projective) Hilbert function \( H_R \) of a graded \( k \)-algebra \( R = \bigoplus_n R_n \) is defined by \( H_R(n) := \dim_k R_n \). Thus (26) relates \( M_S \) to the Hilbert function of \( k[S]/\ker h \):
\[ M_S(n) = H_{k[S]/\ker h}(n). \] (27)
By Theorem 2.4.2 of [2], \( \ker h \) can be computed by elimination:
\[ \ker h = I \cap k[S] \] (28)
where the ideal \( I \) of \( k[S][q][x_p : p \text{ prime}] \) is defined by
\[ I := \langle q_j - h(q_j) : j \in S \rangle. \] (29)
Summarizing this section, we have proved the following Lemma:

**Lemma 4** Let \( k[S] \) be graded by \( \deg q_i := i \). Define a \( k \)-algebra homomorphism from \( k[S] \) to \( k[q, x] \) by
\[ h(q_j) := q^j \prod_p x_p^{\sum_{i=1}^n j/i^j}. \] (30)
Let the ideal $I$ of $k[S,q,x]$ be defined by

$$I := \langle q_j - h(q_j) \, : \, j \in S \rangle \quad (31)$$

and let

$$J := I \cap k[S]. \quad (32)$$

Then $M_S$ is the (projective) Hilbert function of the $k$-algebra $k[S]/J$:

$$M_S(n) = H_{k[S]/J}(n). \quad (33)$$

Example: If $S = [4]$, then $I = \langle q_1 - q, q_2 - q^2x_2, q_3 - q^3x_2x_3, q_4 - q^4x_2^3x_3 \rangle$ and $J = \langle q_4q_3^4 - q_3q_2^5 \rangle$. For $M_{[4]}(n)$, see (41) on page 9.

4 Explicit Answers

Let $S$ be a given finite set throughout this section. Lemma 4 allows to compute a closed form for the sequence $M_S(n)$ by well-known methods from computational commutative algebra. For the sake of completeness, let us briefly review them:

1. Fix a term order $\preceq$ on $k[S,q,x]$ that allows the elimination of the variable $q$ and the variables $x_p$ in step 2 below. Compute a Gröbner basis $F$ for the (toric) ideal $I = \langle q_j - h(q_j) \, : \, j \in S \rangle$ with respect to this term order using Buchberger’s algorithm [3,4].

2. Let $G := F \cap k[S]$. By the elimination property of Gröbner bases with respect to a suitable elimination order $\preceq$, the set $G$ is a Gröbner basis for the elimination ideal $J = I \cap k[S]$.

3. Let $L := I_{\preceq}(G)$ be the set of leading terms of polynomials in $G$.

4. Compute $M_S(q)$ using

$$M_S(q) = H_{k[S]/J}(q) = H_{k[S]/I_{\preceq}(J)}(q) = H_{k[S]/\langle{L}\rangle}(q). \quad (34)$$

The first equality holds by Lemma 4. The second equality is an identity of Macaulay [7]. Since $G$ is a Gröbner basis, its initial terms $L$ generate the initial term ideal of $\langle G \rangle$ with respect to $\preceq$, which explains the third equation sign. A naive method for computing the Hilbert-Poincaré series of $k[S]/\langle{L}\rangle$ is to apply the inclusion-exclusion relation

$$H_{k[S]/\langle{L}\rangle}(q) = H_{k[S]/\langle{L}\rangle}(q) - q^\deg t H_{k[S]/\langle{L}\rangle}(q), \quad (35)$$

recursively until the base case

$$H_{k[S]/\langle{0}\rangle}(q) = H_{k[S]}(q) = \frac{1}{\prod_{j \in S}(1 - q_j)} \quad (36)$$
is reached. For better (faster) algorithms, see [8].

(5) Extract a closed form expression for $H_{k[S]/(L)}(n)$ from its generating function $\mathcal{H}_{k[S]/(L)}(q)$. (Use partial fraction decomposition and the binomial series.) It is the desired answer $M_S(n)$.

One of the authors computed $1 - 4$ for several finite $S$ using different computer algebra systems. It turned out that CoCaA[9] was fastest for that purpose.

**Theorem 5** Let $S$ be a finite subset of the positive natural numbers. Then

(1) $M_S(q)$ can be written as

\[ M_S(q) = \frac{f_S(q)}{\prod_{j \in S}(1 - q^j)} \]  

where $f_S(q)$ is a polynomial with integer coefficients.

(2) There exists $n_0$ such that $M_S(n)$ can be written as a quasipolynomial [10] for $n \geq n_0$. Moreover, it suffices to use periods which are divisors of elements of $S$.

**PROOF.** Relations (35) and (36) prove the first statement. The second statement follows from the first easily. □

Let us follow the algorithm in the case $S = \{4\}$, which is the simplest non-trivial case. We have $I = \langle q_1 - q, q_2 - q^2x_2, q_3 - q^3x_2x_3, q_4 - q^4x_3^2x_4 \rangle$. To eliminate the variables $x_3, x_2$ and $q$ we choose the lexical term order where $x_3 \succ x_2 \succ q \succ q_4 \succ q_3 \succ q_2 \succ q_1$. The corresponding reduced Gröbner basis of $I$ is $F = \{q_1^3q_4 - q_2^2q_3, q - q_1, q^2x_2 - q_2, q_2q_3x_2 - q_1q_4, q_1q_3x_2^2 - q_4, q_1q_2x_3 - q_3, q_2^2x_3^2 - q_1q_3x_2, q_2^2q_4x_3 - q_3^2x_2, q_2^2q_4x_3 - q_3^2x_2, q_1q_4^2x_3 - q_3^3x_2^2, q_3^2x_3 - q_3^3x_2^2 \}$. By the elimination property of Gröbner bases $G := F \cap k[q_1, q_2, q_3, q_4] = \{q_1^3q_4 - q_2^2q_3 \}$ is a Gröbner basis for the elimination ideal $J = I \cap k[q_1, q_2, q_3, q_4]$. Collecting leading terms of $G$ gives $L = \{q_1^3q_4 \}$. Since $G$ is a Gröbner basis of $J$ we know that $I_{\geq}(J) = \langle q_1^3q_4 \rangle$. The Hilbert-Poincaré series of $k[q_1, q_2, q_3, q_4]/\langle q_1^3q_4 \rangle$ gives

\[ M_{\{4\}}(q) = \frac{1 - q^7}{(1 - q)(1 - q^2)(1 - q^3)(1 - q^4)}. \]  

(38)

It is clear that we may replace any occurrence of the partition $4 + 1 + 1 + 1$ in a multinomial coefficient by $3 + 2 + 2$ without changing the value of the multinomial coefficient. Therefore, there are at most as many multinomial coefficients as there are partitions avoiding $4 + 1 + 1 + 1$. Equation (38) states that this upper bound gives in fact the exact number in the case of $S = \{1, 2, 3, 4\}$.
Note that all denominators in the partial fraction decomposition

\[
\mathcal{M}_4(q) = -\frac{7}{24} \frac{1}{(q-1)^3} - \frac{77}{288} \frac{1}{(q-1)} + \frac{1}{16} \frac{1}{(q+1)^2} + \\
+ \frac{1}{32} \frac{1}{(q+1)} + \frac{1}{9} \frac{(q+2)}{(q^2 + q + 1)} + \frac{1}{8} \frac{(q+1)}{(q^2 + 1)}
\]

of (38) are powers of cyclotomic polynomials \( C_j(q) \) where \( j \) divides an element of \( S = \{1, 2, 3, 4\} \). We rewrite this as

\[
\mathcal{M}_4(q) = \frac{7}{24} \frac{1}{(1-q)^3} + \frac{77}{288} \frac{1}{(1-q)} + \frac{1}{16} \frac{(1-q)^2}{(1-q^2)^2} + \\
+ \frac{1}{32} \frac{(1-q)}{(1-q^2)} + \frac{1}{9} \frac{(2-q-q^2)}{(1-q^3)} + \frac{1}{8} \frac{(1+q-q^2-q^3)}{(1-q^4)}
\]

in order to use the binomial series \((1-z)^{-a-1} = \sum_{n=0}^{\infty} \left(\frac{a+n}{a}\right) z^n\). The result is

\[
\mathcal{M}_4(n) = \frac{7}{48} n^2 + \left(\frac{1}{16} [1, -1](n) + \frac{7}{16}\right) n + \\
+ \frac{1}{8} [1, 1, -1, -1](n) + \frac{1}{9} [2, -1, -1](n) + \frac{3}{32} [1, -1](n) + \frac{161}{288}
\]

where \([a_0, a_1, \ldots, a_m](n) := a_j \text{ for } n \equiv j(m)\). Similar computations show that

\[
\mathcal{M}_5(q) = \frac{1 - q^7}{(1-q)(1-q^2) \cdots (1-q^5)},
\]

\[
\mathcal{M}_6(q) = \frac{1 - q^7 - q^8 - q^{10} + q^{12} + q^{13}}{(1-q)(1-q^2) \cdots (1-q^6)},
\]

and

\[
\mathcal{M}_7(q) = \frac{1 - q^7 - q^8 - q^{10} + q^{12} + q^{13}}{(1-q)(1-q^2) \cdots (1-q^7)}.
\]

It is no coincidence that the numerators of (43) and (44) agree (Theorem 11).

5 Upper Bounds

Trivially, \( M(n) \leq p(n) \). Our goal is to find sharper upper bounds.

Lemma 6 Assume \( S' \subseteq S \).

Let \( \tilde{I} \) be the ideal of \( k[S, q, x] \) generated by the set of polynomials \( \{q_j - h(q_j) : j \in S'\} \). Let \( \tilde{J} \) be the ideal generated by \( \tilde{I} \cap k[S'] \) in the ring \( k[S] \). Let \( U_{S,S'}(n) := H_{k[S]/\tilde{J}}(n) \). Then

9
(1) \( M_S(n) \leq U_{S,S'}(n) \).

(2) We have

\[
\sum_n U_{S,S'}(n)q^n = \frac{f_{S'}(q)}{\prod_{j \in S'}(1 - q^j)}
\]

where \( f_{S'}(q) \) is defined by

\[
\sum_n M_{S'}(n)q^n = \frac{f_{S'}(q)}{\prod_{j \in S'}(1 - q^j)}.
\]

**Proof.** We prove the first statement. Let \( I \) be the ideal of \( k[S,q,x] \) generated by the set of polynomials \( \{q_j - h(q_j) : j \in S\} \) and let \( J = I \cap k[S] \). Since \( \tilde{J} \) is a \( k \)-vector subspace of \( J \) we have

\[
\dim_k k[S]_n \cap \tilde{J} \geq \dim_k k[S]_n \cap J
\]

and therefore

\[
\dim_k (k[S]/J)_n \leq \dim_k (k[S]/\tilde{J})_n
\]

i.e.

\[
M_S(n) \leq U_{S,S'}(n).
\]

To prove the second statement, let \( I' \) be the ideal generated by \( \{q_j - h(q_j) : j \in S'\} \) in the ring \( k[S',q,x] \) and let \( J' := I' \cap k[S'] \). Since the ideals \( \tilde{J} \) and \( J' \) are generated by the same set of polynomials (albeit in different rings), the Hilbert functions \( U_{S,S'}(n) = H_{k[S]/J}(n) \) and \( M_S'(n) = H_{k[S']/J'}(n) \) correspond in the way claimed by (45) and (46). □

To get upper bounds for \( M(n) \), we use the preceding Lemma in the special case \( S = \mathbb{N} \) getting:

**Theorem 7** For any \( S' \) we have

\[
M(n) \leq [q^n] \frac{f_{S'}(q)}{\prod_{j=1}^{\infty}(1 - q^j)}
\]

(where \([q^n]A(q)\) denotes the coefficient of \( q^n \) in the power series expansion of \( A(q) \)). For instance, the cases \( S' = [4] \) and \( S' = [6] \) yield the bounds

\[
M(n) \leq p(n) - p(n - 7),
\]

and

\[
M(n) \leq p(n) - p(n - 7) - p(n - 8) - p(n - 10) + p(n - 12) + p(n - 13).
\]
Note that a direct proof of $M(n) \leq p(n) - p(n - 7)$ could be given by exploiting the equivalence of the partitions $4 + 1 + 1 + 1$ and $3 + 2 + 2$ in the sense of Equation (4).

The bound $M(n) \leq p(n) - p(n - 7)$ is good enough to imply:

**Theorem 8** $M(n) = o(p(n))$, i.e. $\lim_{n \to \infty} M(n)/p(n) = 0$.

**PROOF.** Due to the monotonicity of $p(n)$ and the fact that the unit circle is the natural boundary for

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1 - q^n},$$

(53)

we see that

$$\lim_{n \to \infty} \frac{p(n - 7)}{p(n)} = 1.$$  (54)

Hence

$$0 \leq \lim_{n \to \infty} \frac{M(n)}{p(n)} \leq \lim_{n \to \infty} \frac{p(n) - p(n - 7)}{p(n)} = 1 - 1 = 0,$$

(55)

which proves Theorem 8.  \(\square\)

### 6 Lower Bounds

Recall that $M(n) \geq p_p(n)$ (Theorem 1). The numbers given on page 15 suggest that $M(n)$ grows much faster than $p_p(n)$. We will prove that this is indeed the case: $\lim_{n \to \infty} p_p(n)/M(n) = 0$ (Theorem 12) and we will give better lower bounds for $M(n)$.

Let us write $S < P$ if each element of $S$ is less than each element of $P$. We need the following generalization of Lemma 2:

**Lemma 9** Assume $S < P$ where $P$ is a set of primes. Let $s$ and $s'$ be any two power products in $k[S]$ and let $p$ and $p'$ be distinct power products in $k[P]$. Then $h(sp) \neq h(s'p')$.

In the case $S = \emptyset$, Lemma 9 states that distinct partitions $p$ and $p'$ into primes yield different multinomial coefficients: $h(p) \neq h(p')$. Lemma 9 can be proved by the same induction argument as Lemma 2.

**Lemma 10** Assume $S < P$ where $P$ is a set of primes. Define $h$ on $k[S \cup P]$ by (20). Then $\ker h$ is generated, as an ideal of $k[S \cup P]$, by $\ker h \cap k[S]$.
**PROOF.** Let \( f \in \ker h \). Since \( k[S \cup P] = k[S][P] \), we can \( f \) as a finite sum \( f = \sum s \sum p c_{s,p} sp \) indexed by power products \( s \) and \( p \) from \( k[S] \) and \( k[P] \) respectively, with coefficients \( c_{s,p} \in k \). As \( f \in \ker h \), \( \sum s \sum p c_{s,p} h(sp) = 0 \). By Lemma 9, this implies \( \sum s c_{s,p} h(sp) = 0 \) for arbitrary but fixed \( p \). Canceling \( h(p) \) from this equation shows that \( h(f_p) = 0 \) where \( f_p := \sum s c_{s,p} s \). In this way we succeed in writing \( f \) as \( f = \sum_p f_p p \) where each \( f_p \) is in \( \ker h \cap k[S] \). \( \square \)

As an immediate consequence of Lemma 10 we get:

**Theorem 11** Assume \( S < P \) where \( P \) is a set of primes. Then
\[
\mathcal{M}_{S \cup P}(q) = \mathcal{M}_S(q) / \prod_{j \in P} (1 - q^j).
\]  
(56)

As a first application of Theorem 11, we count multinomial coefficients with lower entries which are either prime or equal to 1:
\[
\mathcal{M}_{\{1\} \cup \mathbb{P}}(q) = \frac{1}{(1 - q) \prod_{j \in \mathbb{P}} (1 - q^j)},
\]  
(57)

which allows for improving Theorem 1:

**Theorem 12** We have
\[
\lim_{n \to \infty} p_p(n)/M_{\{1\} \cup \mathbb{P}}(n) = 0
\]  
(58)

and therefore \( \lim_{n \to \infty} p_p(n)/M(n) = 0 \).

**PROOF.** Let \( A(n) := M_{\{1\} \cup \mathbb{P}}(n) \). Due to the monotonicity of \( A(n) \) and the fact that the unit circle is the natural boundary for we see that
\[
\lim_{n \to \infty} A(n - 1)/A(n) = 1.
\]  
(59)

By (57),
\[
p_p(n) = A(n) - A(n - 1).
\]  
(60)

Therefore,
\[
0 \leq \lim_{n \to \infty} \frac{p_p(n)}{A(n)} \leq \lim_{n \to \infty} \frac{A(n) - A(n - 1)}{A(n)} = 1 - 1 = 0,
\]  
(61)

which proves Theorem 12. \( \square \)
Let \( L_S(n) := M_{S \cup P}(n) \); clearly, \( L_S(n) \) is a lower bound for \( M(n) \). Theorem 11 allows us deduce
\[
L_{[4]}(q) = L_{[5]}(q) = \frac{1 - q^7}{\prod_{j \in \{4\} \cup P}(1 - q^j)}
\]
and
\[
L_{[6]}(q) = L_{[7]}(q) = \frac{1 - q^7 - q^8 - q^{10} + q^{12} + q^{13}}{\prod_{j \in \{6\} \cup P}(1 - q^j)}
\]
from the Equations (38) – (44); some values of \( L_{[4]}(n) \) are listed on page 15.

7 The Irreducible Pairs

An irreducible pair is a pair of partitions of \( n \) that yield the same multinomial coefficient but have no parts in common. For example,
\[
(4, 1, 1, 1) \text{ and } (3, 2, 2)
\]
is an irreducible pair.

It turns out that there are infinitely many irreducible pairs of partitions. The following is a partial list: Generalizing (64) we see that
\[
(2^m, 1, 1, \ldots, 1) \text{ and } (2^m - 1, 2, 2, \ldots, 2)
\]
form an irreducible pair of partitions of \( 2^m + 2m - 1 \). More generally, for any integers \( a \geq 2 \) and \( m \geq 1 \) the partitions
\[
(a^m, a-1, a-1, \ldots, a-1, 1, 1, \ldots, 1) \text{ and } (a^m - 1, a, a, \ldots, a)
\]
form an irreducible pair of partitions of \( a^m + am - 1 \).

The pair
\[
(6, 1, 1) \text{ and } (5, 3)
\]
can be generalized to irreducible pairs
\[
(j!, 1, \ldots, 1) \text{ and } (j! - 1, j)
\]
of partitions of \( (j! + j - 1) \) for \( j \geq 3 \).

From any two irreducible pairs we can get a third one by combining them in a natural way. For instance, combining \( a \) copies of (67) with \( b \) copies of (64) gives the pair (70) which is used in the proof below.
The above examples show that \( i(n) \) is positive infinitely often. Indeed we have:

**Theorem 13** \( i(n) \geq \frac{n}{56} - 1 \).

**PROOF.** For each pair of non-negative integers \( a \) and \( b \) satisfying

\[
8a + 7b = n, \tag{69}
\]

we see that

\[
(6, \ldots, 6, 4, \ldots, 4, 1, \ldots, 1) \text{ and } (5, \ldots, 5, 3, \ldots, 3, 2, \ldots, 2) \tag{70}
\]

forms a new irreducible pair of partitions of \( n \). Consequently \( i(n) \) is at least as large as the number of non-negative solutions of the linear Diophantine equation (69).

Now the segment of the line \( 8a + 7b = n \) in the first quadrant is of length \( n\sqrt{113}/56 \). Furthermore from the full solution of the linear Diophantine equation we note that the integral solutions of (3.7) are points on this spaced a distance \( \sqrt{113} \) apart. Hence in the first quadrant there must be at least

\[
\left\lfloor \frac{n\sqrt{113}/56}{\sqrt{113}} \right\rfloor = \left\lfloor \frac{n}{56} \right\rfloor > \frac{n}{56} - 1 \tag{71}
\]

such points. Therefore

\[
i(n) > \frac{n}{56} - 1. \tag{72}
\]

\( \square \)

Theorem 13 shows that \( i(n) > 0 \) for all \( n \geq 56 \). Direct computation shows that \( i(n) > 0 \) for all \( n > 7 \) with the exception of \( n = 9, 11 \) and 12.

**8 Further Problems**

Clearly we have only scratched the surface concerning the order of magnitude of \( M_k(n), M(n) \) and \( i(n) \). We have computed tables of the functions, and based on that evidence we make the following conjectures.

**Conjecture 14** \( M(n) \geq p^*(n) \) for \( n \geq 0 \), where \( p^*(n) \) is the total number of partitions of \( n \) into parts that are either \( \leq 6 \) or multiples of 3 or both.
Conjecture 15 There exists a positive constant $C$ so that

$$\lim_{n \to \infty} \frac{\log M(n)}{\sqrt{n}} = C. \quad (73)$$

If $C$ exists and if Conjecture 14 is true, then [11, Th. 6.2, p.89]

$$\frac{\pi}{3} \sqrt{2} \leq C \leq \pi \sqrt{\frac{2}{3}}. \quad (74)$$

Conjecture 16 Let $C_k$ be the infimum of the quotients $M_k(n)/p_k(n)$ where $n$ ranges over the natural numbers. Then $C_k > 0$ for all natural numbers $k$. Moreover, $C_k$ is a strictly decreasing function of $k$ for $k \geq 3$ and $C_k \to 0$ as $k \to \infty$.

Conjecture 17 $M(n) \leq p^\#(n)$ for $n \geq 0$ where $p^\#(n)$ is the total number of partitions of $n$ into parts that are either $\leq 7$ or multiples of $3$ or both.

Conjecture 17 together with Conjecture 14 allows us to replace Conjecture 15 with

Conjecture 18

$$\lim_{n \to \infty} \frac{\log M(n)}{\sqrt{n}} = \frac{\pi}{3} \sqrt{2}. \quad (75)$$

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References


