Georg Regensburger

Parametrizing compactly supported orthonormal wavelets by discrete moments

Abstract We discuss parametrizations of filter coefficients of scaling functions and compactly supported orthonormal wavelets with several vanishing moments. We introduce the first discrete moments of the filter coefficients as parameters. The discrete moments can be expressed in terms of the continuous moments of the related scaling function. To solve the resulting polynomial equations we use symbolic computation and in particular Gröbner bases. The cases of four to ten filter coefficients are discussed and explicit parametrizations are given.

Keywords Orthonormal wavelets · Parametrization · Filter Coefficients · Moments · Gröbner bases

1 Introduction

Over the last two decades wavelets have become a fundamental tool in many areas of applied mathematics and engineering ranging from signal and image processing to numerical analysis, see for example Daubechies [12], Mallat [24], and Strang and Nguyen [30]. A function $\psi \in L^2(\mathbb{R})$ is an *orthonormal wavelet* if the family

$$\psi_{jk}(x) = 2^{j/2}\psi(2^j x - k), \quad \text{for } j,k \in \mathbb{Z},$$

is an orthonormal basis of the Hilbert space $L^2(\mathbb{R})$. The first known example is the Haar wavelet [14]

$$\psi(x) = \begin{cases} 1, & \text{for } 0 \leq x < \frac{1}{2}, \\ -1, & \text{for } \frac{1}{2} \leq x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

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G. Regensburger
Johann Radon Institute for Computational and Applied Mathematics (RICAM),
Austrian Academy of Sciences, Altenbergerstraße 69, A-4040 Linz, Austria
E-mail: georg.regensburger@oeaw.ac.at

Daubechies [11] introduced a general method to construct compactly supported wavelets. It is based on *scaling functions* which satisfy a *dilation equation*

$$\phi(x) = \sum_{k=0}^{N} h_k \phi(2x-k) \tag{1}$$

given by a linear combination of real *filter coefficients* h_k and dilated and translated versions of the scaling function. We outline her construction in Section 2. The corresponding scaling function for the Haar wavelet is the box function

$$\phi(x) = \begin{cases} 1, & \text{for } 0 \leq x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

with the filter coefficients $h_0 = h_1 = 1$. In general, there is no closed analytic form for the scaling function and for computations with wavelets only the filter coefficients are used.

Conditions on the scaling function imply using the dilation equation (1) constraints on the filter coefficients. Orthonormality gives quadratic equations and vanishing moments of the associated wavelet and normalization linear constraints. For the existence of a wavelet at least one vanishing moment is necessary. Daubechies wavelets [11] have the maximal number of vanishing moments for a fixed number of filter coefficients and so there are only finitely many solutions. See Section 2 for details.

Parametrizing all possible filter coefficients that correspond to compactly supported orthonormal wavelets has been studied by several authors [18,23,25,27, 29,32–34]. For a discussion and illustrations of scaling functions with six filter coefficients depending on two parameters see also [3] and [16]. Applications of parametrized wavelets to compression are for example discussed in [15] and [26]. In all parametrizations the filter coefficients are expressed in terms of trigonometric functions and there is no natural interpretation of the angular parameters for the resulting scaling function. Furthermore, one has to solve transcendental constraints for the parameters to find wavelets with more than one vanishing moment.

In the proposed parametrization we introduce the first discrete moments of the filter coefficients as parameters. The discrete moments can be expressed in terms of the continuous moments of the scaling function, see Section 3. Moreover, we do not want to parametrize all possible filter coefficients but only such with a high number of vanishing moments. More precisely, we omit one vanishing moment condition from the construction of Daubechies wavelets. We also use the fact that the even discrete moments are determined by the odd up to the number of vanishing moments, see Section 3. We discussed a first parametrization using the same approach in [26]. In this paper we present new simplified parametrizations, discuss all computational aspects and different cases in detail, and give a parametrization for ten filter coefficients and at least four vanishing moments.

We solve the resulting parametrized polynomial equations for the filter coefficients using symbolic computation and for the more involved equations in particular Gröbner bases. Gröbner bases were introduced by Buchberger in [4], see also [5]. For further details on Gröbner bases we refer to [1,6,10]. Applications of Gröbner bases to the design of wavelets and filter coefficients are for example discussed in [8,21,22,28]. See in particular Chyzak et al. [8] where Gröbner bases are used to find closed form representations of filter coefficients of Daubechies wavelets. Their approach is related to the one presented in this paper but here we are interested in finding representations of parametrized families of filter coefficients.

In Sections 4 to 7 we describe in detail the cases of four to ten filter coefficients. We give explicit parametrizations and discuss several special parameter values, for example, for the Daubechies wavelets. The corresponding Maple worksheet with all computations, several MATLAB functions and a GUI to compute with and illustrate parametrized wavelets are available on request from the author.

2 Equations for the filter coefficients

We outline the construction of orthonormal wavelets based on scaling functions and recall the polynomial equations for the filter coefficients, see for example Daubechies [12] or Strang and Nguyen [30].

Orthonormality of the integer translates $\{\phi(x-l)\}_{l\in\mathbb{Z}}$ in $L^2(\mathbb{R})$, that is,

$$\int \phi(x)\phi(x-l)dx = \delta_{0,l}$$

implies using the dilation equation (1) the quadratic equations

$$\sum_{k\in\mathbb{Z}} h_k h_{k-2l} = 2\delta_{0,l}, \quad \text{for } l \in \mathbb{Z}, \tag{2}$$

where we set $h_k = 0$ for $k < 0$ and $k > N$. We can assume that $h_0 h_N \neq 0$. Then with Equation (2) we see that N must be odd and the number of filter coefficients even. We have one nonhomogeneous equation

$$\sum_{k=0}^{N} h_k^2 = 2 \tag{3}$$

and the homogeneous equations

$$\sum_{k=0}^{N} h_k h_{k-2l} = 0, \quad \text{for } l = 1,\ldots,(N-1)/2. \tag{4}$$

If the filter coefficients satisfy the necessary conditions for orthogonality (2) and the normalization

$$\sum_{k=0}^{N} h_k = 2 \tag{5}$$

then there exists a unique solution of the dilation equation (1) in $L^2(\mathbb{R})$ with support $[0,N-1]$ and for which $\int \phi = 1$, see Lawton [19]. For almost all such scaling functions the integer translates $\{\phi(x-l)\}_{l\in\mathbb{Z}}$ are orthogonal and then

$$\psi(x) = \sum_{k=0}^{N} (-1)^k h_{N-k} \phi(2x-k) \tag{6}$$

is an orthonormal wavelet. Necessary and sufficient conditions for orthonormality were given by Cohen [9] and Lawton [20], see also Daubechies [12, ch. 6.3.]. The only example with four filter coefficients that satisfies the Equations (2) and (5) and where the integer translates of the corresponding scaling are not orthogonal is $h_0 = h_3 = 1$ and $h_1 = h_2 = 0$ with the scaling function

$$\phi(x) = \begin{cases} 1/3, & \text{for } 0 \leq x < 3, \\ 0, & \text{otherwise.} \end{cases} \tag{7}$$

Vanishing moments of the associated wavelet are related to several properties of the scaling function and wavelet. For example, to the smoothness, the polynomial reproduction and the approximation order of the scaling function, and the decay of the wavelet coefficients for smooth functions, see Strang and Nguyen [30] and the survey [31] by Unser and Blu for details. The condition that the first p moments of the wavelet ψ vanish, that is,

$$\int x^l \psi(x)\, dx = 0, \quad \text{for } l = 0, \ldots, p-1$$

are using Equation (6) equivalent to the *sum rules*

$$\sum_{k=0}^{N} (-1)^k k^l h_k = 0, \quad \text{for } l = 0, \ldots, p-1. \tag{8}$$

We say that ψ has p vanishing moments. Since the vector space of all polynomials with degree less then p is invariant under translation and dilation we can equivalently require vanishing moments of $\psi(x+n-1)$ with $N = 2n-1$. This corresponds to Daubechies choice [11,12] where the wavelet has support $[1-n, n]$. For the computations we use the resulting linear equations since they have smaller coefficients

$$\sum_{k=0}^{2n-1} (-1)^{n-k} h_k (n-k)^l = 0, \quad \text{for } l = 0, \ldots, p-1. \tag{9}$$

Notice that the normalization of the filter coefficients (5) and the first sum rule

$$\sum_{k=0}^{N} (-1)^k h_k = 0 \tag{10}$$

are equivalent to

$$\sum_{\substack{k=0 \\ k \text{ even}}}^{N} h_k = \sum_{\substack{k=0 \\ k \text{ odd}}}^{N} h_k = 1. \tag{11}$$

The following proposition is a consequence of the first Newton identities, which give a relation between power sums and elementary symmetric functions, see Bourbaki [2, A.IV. 70] and Knuth [17, p. 497].

Proposition 1 *Let $x_0, \ldots, x_n$ be variables of a polynomial ring over a commutative ring. Then*

$$\left(\sum_{k=0}^{n} x_k^2\right) = \left(\sum_{k=0}^{n} x_k\right)^2 - 2\left(\sum_{\substack{0\le i<j\le n \\ j-i \text{ even}}} x_i x_j\right) - 2\left(\sum_{\substack{k=0 \\ k \text{ even}}}^{n} x_k\right)\left(\sum_{\substack{k=0 \\ k \text{ odd}}}^{n} x_k\right). \tag{12}$$

Proof The Newton identities tell us in particular that

$$\left(\sum_{k=0}^{n} x_k^2\right) = \left(\sum_{k=0}^{n} x_k\right)^2 - 2\left(\sum_{0\le i<j\le n} x_i x_j\right).$$

The last sum in this equation is

$$\left(\sum_{0\le i<j\le n} x_i x_j\right) = \left(\sum_{\substack{0\le i<j\le n \\ j-i \text{ even}}} x_i x_j\right) + \left(\sum_{\substack{0\le i<j\le n \\ j-i \text{ odd}}} x_i x_j\right)$$

and the proposition follows by observing that

$$\left(\sum_{\substack{0\le i<j\le n \\ j-i \text{ odd}}} x_i x_j\right) = \left(\sum_{\substack{k=0 \\ k \text{ even}}}^{n} x_k\right)\left(\sum_{\substack{k=0 \\ k \text{ odd}}}^{n} x_k\right).$$

□

If the filter coefficients satisfy the homogeneous equations (4) from the orthonormality conditions then

$$\sum_{\substack{0\le i<j\le n \\ j-i \text{ even}}} h_i h_j = 0.$$

Therefore we see with the identity (12) that the normalization and the first sum rule, see Equations (5), (10) and (11) together with (4) imply the nonhomogeneous equation (3). So we can replace the quadratic equation (3) by the linear equation (10) which simplifies the computations.

3 Discrete and continuous moments

In this section we discuss relations between the *discrete moments*

$$m_n = \sum_{k=0}^{N} h_k k^n$$

of the filter coefficients on the *continuous moments* of the scaling function

$$M_n = \int x^n \phi(x)\, dx.$$

We first recall a well-known recursive relation between discrete and continuous moments, see for example Strang and Nguyen [30, p. 396].

Let ϕ be a scaling function satisfying $M_0 = \int \phi = 1$. Then $m_0 = 2$ and

$$M_n = \frac{1}{2^{n+1}-2} \sum_{i=1}^{n} \binom{n}{i} m_i M_{n-i},$$
$$m_n = \left(2^{n+1}-2\right) M_n - \sum_{i=1}^{n-1} \binom{n}{i} m_i M_{n-i}, \quad \text{for } n > 0.$$

Using the recursion we obtain for the first moments

$$\begin{aligned} M_1 &= 1/2\, m_1 \\ M_2 &= 1/6\, m_1^2 + 1/6\, m_2 \\ M_3 &= 1/28\, m_1^3 + 1/7\, m_1 m_2 + 1/14\, m_3 \end{aligned}$$

and

$$\begin{aligned} m_1 &= 2M_1 \\ m_2 &= -4M_1^2 + 6M_2 \\ m_3 &= 12M_1^3 - 24M_1M_2 + 14M_3. \end{aligned}$$

Explicit formulas expressing the discrete moments in terms of the continuous and vice versa are given in [26].

For the parametrization of the filter coefficients we use the fact that the even moments are determined by the odd moments up to the number of vanishing moments, see [26]. In more detail, if the first two moments of the associated wavelet vanish then

$$m_2 = m_1^2/2 \tag{13}$$

and if the first four moments vanish we additionally have

$$m_4 = -1/2\, m_1^4 + 2m_1^2 m_2 + 2m_1 m_3 - 7/2\, m_2^2 = -3/8\, m_1^4 + 2m_1 m_3. \tag{14}$$

4 Four filter coefficients

In the case of four filter coefficients we have the following system equations (normalization, first sum rule, parameter $m = m_1$, and orthogonality):

$$\begin{aligned} h_0 + h_1 + h_2 + h_3 &= 2 \\ h_0 - h_1 + h_2 - h_3 &= 0 \\ h_1 + 2h_2 + 3h_3 &= m \\ h_0 h_2 + h_1 h_3 &= 0. \end{aligned}$$

We solve the three linear equations for h_0, substitute the solution into the quadratic equation, and obtain

$$-2{h_0}^2 + (5-m)\, h_0 - 1/4\, m^2 + 2m - 15/4. \tag{15}$$

We first consider the solution

$$h_0 = 5/4 - 1/4\,m - 1/4\,\sqrt{-m^2 + 6m - 5}.$$

Since

$$-m^2 + 6m - 5 = -(m-1)(m-5) \tag{16}$$

we can choose $m \in [1,5]$ to get real filter coefficients. We set $m = a+3$ to obtain parameter values symmetrically around zero. This correspond to a Tschirnhaus transformation for the polynomial (16) and simplifies the expression for the filter coefficients. Substituting the solution for h_0 into the solution for the linear equations we get:

$$\begin{aligned} h_0 &= 1/2 - 1/4\,a - 1/4\,w \\ h_1 &= 1/2 - 1/4\,a + 1/4\,w \\ h_2 &= 1/2 + 1/4\,a + 1/4\,w \\ h_3 &= 1/2 + 1/4\,a - 1/4\,w \end{aligned} \tag{17}$$

with $w = \sqrt{4-a^2}$ and $a = m-3 \in [-2,2]$.

Notice that for $a = -a$ we obtain the flipped filter coefficients.

4.1 Special parameter values

For $a = 0$ we get the filter coefficients $(0,1,1,0)$ which correspond to a translated Haar scaling function and wavelet. The parameter values $a = -2,2$ give also Haar scaling functions with the filter coefficients $(1,1,0,0)$ and $(0,0,1,1)$.

The Daubechies wavelet has two vanishing moments so we have one more sum rule

$$2\,h_0 - h_1 + h_3 = 0.$$

Substituting the parametrized filter coefficients into this equations and solving for a we get the two solutions $a = -\sqrt{3}, \sqrt{3}$ with the first discrete moments $m = 3-\sqrt{3}, 3+\sqrt{3}$. The first solution gives the famous Daubechies filters [11]

$$1/4\,(1+\sqrt{3}, 3+\sqrt{3}, 3-\sqrt{3}, 1-\sqrt{3}) \tag{18}$$

and the second the flipped version.

For $a = -8/5$ we get the rational filters $(3/5, 6/5, 2/5, -1/5)$. These rational filter coefficients give the smoothest scaling function with respect to the Hölder continuity, see Daubechies [12, p. 242].

4.2 Second root

If we choose the second root

$$h_0 = 5/4 - 1/4\,m + 1/4\,\sqrt{-m^2 + 6m - 5}$$

for the quadratic equation (15) and apply again the Tschirnhaus transformation $m = a + 3$ we obtain the parametrized filter coefficients:

$$\begin{aligned} h_0 &= 1/2 - 1/4\,a + 1/4\,w \\ h_1 &= 1/2 - 1/4\,a - 1/4\,w \\ h_2 &= 1/2 + 1/4\,a - 1/4\,w \\ h_3 &= 1/2 + 1/4\,a + 1/4\,w \end{aligned}$$

with $w = \sqrt{4 - a^2}$ and $a = m - 3 \in [-2, 2]$.

Comparing this solution with the parametrized filter coefficients (17) we see that w is replaced by $-w$ and so the two first and the two last filter coefficients are swapped. Notice that again for $a = -a$ we obtain the flipped filters.

For $a = 0$ we now get the filter coefficients $(1, 0, 0, 1)$ which give the scaling function (7) where the integer translates of the scaling function are not orthogonal. The parameter values $a = -2, 2$ also give Haar scaling functions with the filter coefficients $(1, 1, 0, 0)$ and $(0, 0, 1, 1)$. This parametrization does not contain filter coefficients with a second vanishing moment. The corresponding scaling functions are, compared to the parametrization (17), discontinuous.

5 Six filter coefficients

For six filter coefficients we have two vanishing moments and we can use the relation $m_2 = m_1^2/2$, see Equation (13). This gives an additional linear constraint and we have the following linear equations with $m = m_1$:

$$\begin{aligned} h_0 + h_1 + h_2 + h_3 + h_5 + h_4 &= 2 \\ -h_0 + h_1 - h_2 + h_3 - h_4 + h_5 &= 0 \\ -3\,h_0 + 2\,h_1 - h_2 + h_4 - 2\,h_5 &= 0 \\ h_1 + 2\,h_2 + 3\,h_3 + 4\,h_4 + 5\,h_5 &= m \\ h_1 + 4\,h_2 + 9\,h_3 + 16\,h_4 + 25\,h_5 &= m^2/2 \end{aligned}$$

and the quadratic equations

$$\begin{aligned} h_0h_2 + h_1h_3 + h_2h_4 + h_3h_5 &= 0 \\ h_0h_4 + h_1h_5 &= 0. \end{aligned}$$

We solve the linear equations for h_0, substitute the solution into the quadratic equations and obtain:

$$\begin{aligned} -8\,{h_0}^2 + (1/2\,m^2 - 7\,m + 21)h_0 - \frac{1}{64}m^4 + \frac{3}{8}m^3 - \frac{13}{4}m^2 + 12\,m - \frac{253}{16} &= 0 \\ 2\,{h_0}^2 + (-1/8\,m^2 + \frac{7}{4}m - \frac{21}{4})h_0 + \frac{1}{256}m^4 - \frac{3}{32}m^3 + \frac{13}{16}m^2 - 3\,m + \frac{253}{64} &= 0. \end{aligned} \tag{19}$$

Since the first equation is minus four times the second equation we have, as in the case of four filter coefficients, only one quadratic equation to solve. We first consider the solution

$$h_0 = \frac{21}{16} - \frac{7}{16}m + \frac{1}{32}m^2 - \frac{1}{32}\sqrt{-m^4+20m^3-136m^2+360m-260}.$$

The Tschirnhaus transformation $m = a+5$ for the polynomial

$$-m^4+20m^3-136m^2+360m-260$$

yields

$$-a^4+14a^2+15 = -\left(a^2-15\right)\left(a^2+1\right).$$

So we get real filter coefficients for $a \in [-\sqrt{15}, \sqrt{15}]$ or the first discrete moment $m \in [5-\sqrt{15}, 5+\sqrt{15}]$. Substituting the solution for h_0 into the solution for the linear equations we get the following parametrized filter coefficients with at least two vanishing moments:

$$\begin{aligned}
h_0 &= -3/32 - 1/8\,a + 1/32\,a^2 - 1/32\,w \\
h_1 &= 5/32 - 1/8\,a + 1/32\,a^2 + 1/32\,w \\
h_2 &= 15/16 - 1/16\,a^2 + 1/16\,w \\
h_3 &= 15/16 - 1/16\,a^2 - 1/16\,w \\
h_4 &= 5/32 + 1/8\,a + 1/32\,a^2 - 1/32\,w \\
h_5 &= -3/32 + 1/8\,a + 1/32\,a^2 + 1/32\,w
\end{aligned} \tag{20}$$

with $w = \sqrt{-a^4+14a^2+15}$ and $a = m-5 \in [-\sqrt{15}, \sqrt{15}]$.

Notice that for $a = -a$ the two coefficients h_2 and h_3 do not change.

5.1 Special parameter values

The Daubechies wavelet has one more vanishing moment, that is, it satisfies the sum rule

$$-9h_0 + 4h_1 - h_2 - h_4 + 4h_5.$$

Substituting the parametrized filter coefficients into this equations and solving for a we get one real solution $a = -\sqrt{5+2\sqrt{10}}$, which gives the filter coefficients

$$\begin{aligned}
1/16\,(1+\sqrt{10}+w, 5+\sqrt{10}+3w, 10-2\sqrt{10}+2w,& \\
10-2\sqrt{10}-2w, 5+\sqrt{10}-3w, 1+\sqrt{10}-w)&
\end{aligned} \tag{21}$$

with $w = \sqrt{5+2\sqrt{10}}$.

The Daubechies filters with four nonzero filter coefficients (18) satisfy two sum rules and are therefore contained in this parametrization. Their first discrete moment is $m = 3-\sqrt{3}$. So here the corresponding parameter $a = -2-\sqrt{3}$. We get a translated version for $a = -\sqrt{3}$.

For $a = -\sqrt{15}$ we obtain

$$1/8\,(3+\sqrt{15}, 5+\sqrt{15}, 0, 0, 5-\sqrt{15}, 3-\sqrt{15}).$$

The parameter $a = -1$ gives the first coiflet

$$1/16\,(1-\sqrt{7}, 5+\sqrt{7}, 14+2\sqrt{7}, 14-2\sqrt{7}, 1-\sqrt{7}, -3+\sqrt{7}),$$

see Daubechies [13] and [12, ch. 8.2.]. For $a = 0$ we get

$$1/32\,(-3-\sqrt{15}, 5+\sqrt{15}, 30+2\sqrt{15}, 30-2\sqrt{15}, 5-\sqrt{15}, -3+\sqrt{15}).$$

The corresponding scaling functions and wavelets for $a > 0$ become increasingly discontinuous.

5.2 Second root

If we choose the second solution for the quadratic equation (19) and apply the Tschirnhaus transformation $m = a+5$ we obtain:

$$\begin{aligned}
h_0 &= -3/32 - 1/8\,a + 1/32\,a^2 + 1/32\,w\\
h_1 &= 5/32 - 1/8\,a + 1/32\,a^2 - 1/32\,w\\
h_2 &= 15/16 - 1/16\,a^2 - 1/16\,w\\
h_3 &= 15/16 - 1/16\,a^2 + 1/16\,w\\
h_4 &= 5/32 + 1/8\,a + 1/32\,a^2 + 1/32\,w\\
h_5 &= -3/32 + 1/8\,a + 1/32\,a^2 - 1/32\,w
\end{aligned} \tag{22}$$

with $w = \sqrt{-a^4 + 14a^2 + 15}$ and $a = m-5 \in [-\sqrt{15}, \sqrt{15}]$.

Notice that compared to the parametrization (22) here w is replaced by $-w$ and substituting $a = -a$ gives the flipped filter coefficients. The parametrized filter coefficients (22) give smoother scaling functions and wavelets for $a > 0$.

6 Eight filter coefficients

For eight filter coefficients we have three vanishing moments and we can use as in the previous section the relation $m_2 = 1/2\,m_1^2$, see Equation (13). We have the following six linear equations with $m = m_1$:

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\
-1 & 1 & -1 & 1 & -1 & 1 & -1 & 1\\
3 & -2 & 1 & 0 & -1 & 2 & -3 & 4\\
-9 & 4 & -1 & 0 & -1 & 4 & -9 & 16\\
7 & 6 & 5 & 4 & 3 & 2 & 1 & 0\\
49 & 36 & 25 & 16 & 9 & 4 & 1 & 0
\end{pmatrix}
\begin{pmatrix} h_7\\ h_6\\ h_5\\ h_4\\ h_3\\ h_2\\ h_1\\ h_0 \end{pmatrix}
=
\begin{pmatrix} 2\\ 0\\ 0\\ 0\\ m\\ 1/2\,m^2 \end{pmatrix} \tag{23}$$

and the quadratic equations

$$\begin{aligned} h_0h_2 + h_1h_3 + h_3h_5 + h_2h_4 + h_4h_6 + h_5h_7 &= 0 \\ h_0h_4 + h_1h_5 + h_3h_7 + h_2h_6 &= 0 \\ h_0h_6 + h_1h_7 &= 0. \end{aligned}$$

We solve the linear equations for h_0 and h_1 and substitute the solutions into the quadratic equations. Then we compute a Gröbner basis with respect to the lexicographic order with $h_1 >_{\text{lex}} h_0$ treating m as a parameter, that is, we compute a Gröbner basis in $\mathbb{Q}(m)[h_1, h_0]$.

The Gröbner basis has two elements. The first element is a quadratic polynomial in h_0 and the second linear in h_1 and h_0. We consider the following solution for the quadratic equation from the Gröbner basis

$$h_0 = -\frac{1}{512}\,\frac{m^5 - 42m^4 + 684m^3 - 5416m^2 + 20840m - 31088 + w}{m^2 - 14m + 50}$$

with $w =$

$$\sqrt{-(m^8-56m^7+1336m^6-17696m^5+141792m^4-699328m^3+2049600m^2-3186176m+1891904)(m-8)^2}.$$

We set $m = a + 7$, which corresponds to a Tschirnhaus transformation for the first factor of the polynomial under the square root in w, and obtain

$$h_0 = -\frac{1}{512}\,\frac{a^5 - 7a^4 - 2a^3 + 30a^2 - 55a - 15 + w}{a^2 + 1}$$

with

$$w = \sqrt{-(a^8 - 36a^6 + 182a^4 - 1540a^2 + 945)(a-1)^2}. \tag{24}$$

To get real filter coefficients we can choose a in

$$[-\sqrt{\beta}, -\sqrt{\alpha}] \text{ or } [\sqrt{\alpha}, \sqrt{\beta}], \tag{25}$$

where α denotes the smaller and β the bigger real root of

$$x^4 - 36x^3 + 182x^2 - 1540x + 945,$$

or numerically

$$a \in [-5.636256559, -0.8113601077] \text{ or } [0.8113601077, 5.636256559].$$

We substitute the solution for h_0 into the linear equation from the Gröbner basis, solve for h_1 and obtain with w as in (24)

$$h_1 = -\frac{1}{512}\,\frac{a^6 - 10a^5 + 39a^4 - 28a^3 - 25a^2 + 86a - 63 - (1+a)w}{a^3 - a^2 + a - 1}.$$

The denominator

$$a^3 - a^2 + a - 1 = (a-1)(a^2+1)$$

is zero for $a = 1$. We first assume $a < 1$. Then we can also simplify the root (24) and obtain with the solution for the linear equations (23) the following parametrized filter coefficients with at least three vanishing moments:

$$\begin{aligned}
h_0 &= -\frac{1}{512}\,\frac{a^5 - 7a^4 - 2a^3 + 30a^2 - 55a - 15 + (1-a)w}{a^2+1}\\
h_1 &= -\frac{1}{512}\,\frac{a^5 - 9a^4 + 30a^3 + 2a^2 - 23a + 63 + (1+a)w}{a^2+1}\\
h_2 &= \frac{1}{512}\,\frac{3a^5 - 5a^4 - 102a^3 + 186a^2 - 261a + 35 + 3(1-a)w}{a^2+1}\\
h_3 &= \frac{1}{512}\,\frac{3a^5 - 11a^4 - 70a^3 + 358a^2 - 229a + 525 + 3(1+a)w}{a^2+1}\\
h_4 &= -\frac{1}{512}\,\frac{3a^5 + 11a^4 - 70a^3 - 358a^2 - 229a - 525 + 3(1-a)w}{a^2+1}\\
h_5 &= -\frac{1}{512}\,\frac{3a^5 + 5a^4 - 102a^3 - 186a^2 - 261a - 35 + 3(1+a)w}{a^2+1}\\
h_6 &= \frac{1}{512}\,\frac{a^5 + 9a^4 + 30a^3 - 2a^2 - 23a - 63 + (1-a)w}{a^2+1}\\
h_7 &= \frac{1}{512}\,\frac{a^5 + 7a^4 - 2a^3 - 30a^2 - 55a + 15 + (1+a)w}{a^2+1}
\end{aligned} \tag{26}$$

with

$$w = \sqrt{-a^8 + 36a^6 - 182a^4 + 1540a^2 - 945},$$

$a = m - 7 < 1$ and a in the intervals (25).

If we choose the second root for the quadratic equation from the Gröbner basis and perform the same computations as before with the assumption $a < 1$, then we obtain the filter coefficients (26) with w replaced by $-w$.

6.1 Different order on the variables

We now compute a Gröbner basis with respect to the lexicographic order with $h_0 >_{\text{lex}} h_1$. The Gröbner basis has again two elements. The first element is a quadratic polynomial in h_1 and the second linear in h_0 and h_1.

We consider the following solution for the quadratic equation from the Gröbner basis

$$h_1 = -\frac{1}{512}\,\frac{m^5 - 44m^4 + 772m^3 - 6704m^2 + 28712m - 48384 - w}{m^2 - 14m + 50}$$

with $w =$

$$\sqrt{-(m^8-56m^7+1336m^6-17696m^5+141792m^4-699328m^3+2049600m^2-3186176m+1891904)(m-6)^2}.$$

We set again $a = m + 7$ and obtain

$$h_1 = -\frac{1}{512}\,\frac{a^5 - 9a^4 + 30a^3 + 2a^2 - 23a + 63 - w}{a^2+1}$$

with

$$w = \sqrt{-(a^8 - 36a^6 + 182a^4 - 1540a^2 + 945)(a+1)^2}. \tag{27}$$

We get real filter coefficients for a in the same intervals (25) as in the previous section. We substitute the solution for h_1 into the linear equation from the second Gröbner basis, solve for h_0 and obtain with w as in (27)

$$h_0 = -\frac{1}{512}\,\frac{a^6 - 6a^5 - 9a^4 + 28a^3 - 25a^2 - 70a - 15 + (a-1)w}{a^3 + a^2 + a + 1}.$$

The denominator

$$a^3 + a^2 + a + 1 = (a+1)(a^2+1)$$

is zero for $a = -1$. We assume $a > -1$. Then we can also simplify the root (27) and obtain with the solution for the linear equations (23) the filter coefficients from Equation (26) with w replaced by $-w$. From the previous section we know that this parametrization is also valid for $a < 1$ and hence for a in the intervals (25). Notice that substituting $a = -a$ in this parametrization gives the flipped filter coefficients from Equation (26).

If we choose the second root for the quadratic equation from the Gröbner basis and perform the same computations as before with the assumption $a > -1$, then we obtain the filter coefficients (26). Therefore the parametrization (26) is also valid for a in the intervals (25).

6.2 Special parameter values

The Daubechies wavelet satisfies one more sum rule

$$64h_0 - 27h_1 + 8h_2 - h_3 + h_5 - 8h_6 + 27h_7.$$

Substituting the parametrized filter coefficients (26) into this equations and solving for a we get two real solution $a = -\sqrt{\beta}, -\sqrt{\alpha}$, where α denotes the smaller and β the bigger real root of

$$x^4 - 28x^3 + 126x^2 - 1260x + 1225$$

or numerically

$$a = -4.989213573, -1.029063869.$$

The first parameter gives the Daubechies wavelet with extremal phase [12, p. 195] and the second the “least asymmetric” [12, p. 198].

The Daubechies wavelet with six nonzero filter coefficients has the first discrete moment $m = 5 - \sqrt{5 + 2\sqrt{10}}$, so the corresponding parameter value for the parametrization (26) is $a = -2 - \sqrt{5 + 2\sqrt{10}}$.

7 Ten filter coefficients

For ten filter coefficients we require four vanishing moments. We can therefore use the two relations $m_2 = 1/2\,m_1^2$ and $m_4 = -3/8\,m_1^4 + 2\,m_1 m_3$, see Equation (13) and (14). This gives two additional linear constraints and we have the following linear equations with the two parameters $a = m_1$ and $c = m_3$:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ -4 & 3 & -2 & 1 & 0 & -1 & 2 & -3 & 4 & -5 \\ 16 & -9 & 4 & -1 & 0 & -1 & 4 & -9 & 16 & -25 \\ -64 & 27 & -8 & 1 & 0 & -1 & 8 & -27 & 64 & -125 \\ 9 & 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 \\ 81 & 64 & 49 & 36 & 25 & 16 & 9 & 4 & 1 & 0 \\ 729 & 512 & 343 & 216 & 125 & 64 & 27 & 8 & 1 & 0 \\ 6561 & 4096 & 2401 & 1296 & 625 & 256 & 81 & 16 & 1 & 0 \end{pmatrix} \begin{pmatrix} h_9 \\ h_8 \\ h_7 \\ h_6 \\ h_5 \\ h_4 \\ h_3 \\ h_2 \\ h_1 \\ h_0 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \\ a \\ 1/2\,a^2 \\ c \\ -\frac{3}{8}\,a^4 + 2\,ac \end{pmatrix} \tag{28}$$

and the quadratic equations

$$\begin{aligned} h_0h_2 + h_1h_3 + h_2h_4 + h_3h_5 + h_4h_6 + h_5h_7 + h_6h_8 + h_7h_9 &= 0 \\ h_0h_4 + h_1h_5 + h_2h_6 + h_3h_7 + h_4h_8 + h_5h_9 &= 0 \\ h_0h_6 + h_1h_7 + h_2h_8 + h_3h_9 &= 0 \\ h_0h_8 + h_1h_9 &= 0. \end{aligned}$$

We solve the linear equations for h_0 and substitute the solutions into the quadratic equations. We compute a Gröbner basis with respect to the lexicographic order with $h_0 >_{\text{lex}} c$ treating a as a parameter, that is, we compute a Gröbner basis in $\mathbb{Q}(a)[h_0, c]$.

The Gröbner basis consist of two elements. The first is the polynomial

$$\begin{aligned} f = {} & 81\,a^{12} - 2916\,a^{11} + 40716\,a^{10} - 864\,a^9 c - 155520\,a^9 + 31104\,a^8 c - 2354328\,a^8 - 496512\,a^7 c + 2880\,a^6 c^2 \\ & + 31658688\,a^7 + 3768768\,a^6 c - 93312\,a^5 c^2 - 102669504\,a^6 - 4056192\,a^5 c + 1540224\,a^4 c^2 - 3072\,a^3 c^3 \\ & - 590398848\,a^5 - 176214528\,a^4 c - 15303168\,a^3 c^2 + 55296\,a^2 c^3 + 6210049216\,a^4 + 1512364544\,a^3 c \\ & + 97677312\,a^2 c^2 - 489472\,ac^3 + 1024\,c^4 - 22429995264\,a^3 - 5357366784\,a^2 c - 358511616\,ac^2 + 1419264\,c^3 \\ & + 41210318592\,a^2 + 8252955648\,ac + 548785152\,c^2 - 39607335936\,a - 4229148672\,c + 16394918400 \end{aligned} \tag{29}$$

in the two parameters a, c and has $\deg_a(f) = 12$ and $\deg_c(f) = 4$. All possible parameters must lie on the real algebraic curve defined by the polynomial f. This curve has genus eleven and two finite singular points with multiplicity two and coordinates

$$a = 9,\ c = 729/4 \pm 3/8\,\sqrt{210}. \tag{30}$$

We compute the discriminant f with respect to c. Approximating its zeros we see that we have real solutions for c if the first discrete moment

$$a \in [1.641693501, 16.35830649].$$

The number of real solutions for c is given in Table 1.

Table 1 Number of real solutions for f from (29)

parameter a	# real solutions for c
$(1.6417, 7.6167]$	two
$(7.6167, 9)$	four
9	two, singular point
$(9, 10.3832]$	four
$(10.3832, 16.3583)$	two

The second element in the Gröbner basis is linear in h_0. We solve this polynomial for h_0 and obtain with the solution for the linear equations (28) the following parametrized filter coefficients with at least four vanishing moments:

$$
\begin{aligned}
h_0 &= \tfrac{1}{36864}\,\frac{9a^6-180a^5+948a^4-48a^3c+9840a^3+960a^2c-116824a^2-9568ac+32c^2+384480a+31680c-482976}{a-9}\\
h_1 &= -\tfrac{1}{36864}\,\frac{9a^6-144a^5+624a^4-48a^3c+1536a^3+768a^2c+12824a^2-5728ac+32c^2-237312a+12672c+665280}{a-9}\\
h_2 &= -\tfrac{1}{9216}\,\frac{9a^6-180a^5+948a^4-48a^3c+8976a^3+960a^2c-99064a^2-9472ac+32c^2+257760a+30816c-151200}{a-9}\\
h_3 &= \tfrac{1}{9216}\,\frac{9a^6-144a^5+624a^4-48a^3c+2544a^3+768a^2c-9976a^2-5824ac+32c^2-53280a+13536c+120960}{a-9}\\
h_4 &= \tfrac{1}{6144}\,\frac{9a^6-180a^5+948a^4-48a^3c+8304a^3+960a^2c-88408a^2-9376ac+32c^2+216288a+29952c-151200}{a-9}\\
h_5 &= -\tfrac{1}{6144}\,\frac{9a^6-144a^5+624a^4-48a^3c+3360a^3+768a^2c-24904a^2-5920ac+32c^2+27072a+14400c+12096}{a-9}\\
h_6 &= -\tfrac{1}{9216}\,\frac{9a^6-180a^5+948a^4-48a^3c+7824a^3+960a^2c-82552a^2-9280ac+32c^2+202464a+29088c-151200}{a-9}\\
h_7 &= \tfrac{1}{9216}\,\frac{9a^6-144a^5+624a^4-48a^3c+3984a^3+768a^2c-34264a^2-6016ac+32c^2+65952a+15264c-34560}{a-9}\\
h_8 &= \tfrac{1}{36864}\,\frac{9a^6-180a^5+948a^4-48a^3c+7536a^3+960a^2c-79192a^2-9184ac+32c^2+195552a+28224c-151200}{a-9}\\
h_9 &= -\tfrac{1}{36864}\,\frac{9a^6-144a^5+624a^4-48a^3c+4416a^3+768a^2c-40360a^2-6112ac+32c^2+88704a+16128c-60480}{a-9}
\end{aligned}
$$

with $a \neq 9, c \in \mathbb{R}$ such that $f(a,c) = 0$ with f from (29).

To compute the filter coefficients for $a = 9$ we solve the linear equations (28) with the parameter values (30) for h_0 and substitute the solution into the quadratic equations. Then we solve the four univariate polynomials and obtain two solutions for h_0 which give two different filter coefficients. The second choice for c from (30) gives the flipped filter coefficients.

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