The Positivity Set of a Recurrence Sequence

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Abstract

We consider real sequences (f_n) that satisfy a linear recurrence with constant coefficients. We show that the density of the positivity set of such a sequence always exists. In the special case where the sequence has no positive dominating characteristic root, we establish that the density is positive. Furthermore, we determine the values that can occur as density of such a positivity set, both for the special case just mentioned and in general.

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1 Introduction and Main Results

A sequence $(f_n)_{n\geq 0}$ of real numbers is called a *recurrence sequence* if it satisfies a linear recurrence

$$f_{n+h} = c_1 f_{n+h-1} + \cdots + c_{h-1} f_{n+1} + c_h f_n, \qquad n \geq 0,$$

with constant coefficients $c_k \in \mathbb{R}$. (Since we are concerned with questions of positivity, we restrict attention to real sequences.) One of the most charming and celebrated results in the theory of recurrence sequences is the Skolem-Mahler-Lech theorem. It asserts that the zero set

$$\{n \in \mathbb{N} : f_n = 0\}$$

of a recurrence sequence is the union of a finite set and finitely many arithmetic progressions. The recent comprehensive monograph by Everest et al. [3] contains references to the substantial literature devoted to this result and to related questions. However, not much seems to be known about the positivity set

$$\{n \in \mathbb{N} : f_n > 0\}. \tag{1}$$

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In the following section we establish that the density of the set (1) always exists, where the (natural) density of a set $A \subseteq \mathbb{N}$ is defined as

$$\boldsymbol{d}(A) := \lim_{x\to\infty} x^{-1}\sharp\{n \le x : n \in A\},$$

provided that the limit exists.

Theorem 1. *Let (f_n) be a recurrence sequence. Then the density of the set $\{n \in \mathbb{N} : f_n > 0\}$ exists.*

Recall that a recurrence sequence can be written as a generalized power sum

$$f_n = \sum_{k=0}^{m} P_k(n)\gamma_k^n, \qquad n \ge 0, \tag{2}$$

with non-zero polynomials $P_k \in \mathbb{C}[n]$ and *roots* $\gamma_k \in \mathbb{C}$ that are roots of the *characteristic polynomial*

$$z^h - c_1 z^{h-1} - \cdots - c_{h-1} z - c_h$$

of the recurrence. The roots of largest modulus are called *dominating roots* of (f_n).

It does not come as a surprise that recurrence sequences with no positive dominating root have oscillating behaviour. Indeed, in section 3 we prove that for such a sequence (f_n) the densities of (1) and the negativity set

$$\{n \in \mathbb{N} : f_n < 0\} \tag{3}$$

are always positive. This generalizes the following known result [1, 4]: If a recurrence sequence has at most four dominating roots, and none of them is real positive, then the sets (1) and (3) both have infinitely many elements.

Theorem 2. *Let (f_n) be a nonzero recurrence sequence with no positive dominating characteristic root. Then the sets $\{n \in \mathbb{N} : f_n > 0\}$ and $\{n \in \mathbb{N} : f_n < 0\}$ have positive density.*

In section 4 we investigate which numbers actually occur as density of the positivity set of some recurrence sequence. It turns out that all possible values occur, both for sequences with no dominating positive root and in general.

Finally (section 5), we return to the Skolem-Mahler-Lech theorem. Our approach yields the following weak version: The density of the zero set exists and is a rational number.

The conclusion hints at algorithmic aspects of the positivity of recurrence sequences.

2 The Density of the Positivity Set

Notation: We write $f_n \equiv 0$ if $f_n = 0$ for all $n \ge 0$. The Lebesgue measure of a set $B \subset \mathbb{R}^m$ is denoted by $\boldsymbol{\lambda}(B)$.

The goal of this section is to prove Theorem 1. Dividing f_n by $n^D|\gamma_1|^n$, where γ_1 is a dominating root of f_n and D is the maximal degree of the P_k with $|\gamma_k| = |\gamma_1|$, we obtain from (2)

$$n^{-D}|\gamma_1|^{-n} f_n = \sum_{i=1}^{d} a_i \cos(2\pi\theta_i n + \beta_i) + v - r_n,$$

where $r_n = \mathrm{O}(1/n)$ is a recurrence sequence, $\theta_1, \dots, \theta_d$ are in $]0, 1[$, and $a_i, \beta_i, v \in \mathbb{R}$. From now on we will assume w.l.o.g. $D = 0$ and $|\gamma_1| = 1$. As a first step we get rid of any integer relations that the θ_i's might satisfy.

Lemma 3. *Let $\theta_1, \dots, \theta_d$ be real numbers. Then there is a basis $\{\tau_1, \dots, \tau_{m+1}\}$ of the $\mathbb{Z}$-module*

$$M = \mathbb{Z} + \mathbb{Z}\theta_1 + \dots \mathbb{Z}\theta_d$$

such that $1/\tau_{m+1}$ is a positive integer and $1, \tau_1, \dots, \tau_m$ are linearly independent over $\mathbb{Q}$.

Proof. M is finitely generated and torsion free, hence it is free [7, Theorem III.7.3]. Let $\{\alpha_1, \dots, \alpha_{m+1}\}$ be a basis. Since $1 \in M$, there are integers $e_1, \dots, e_{m+1}$ such that

$$e_1\alpha_1 + \dots + e_{m+1}\alpha_{m+1} = 1.$$

We complete $(e_1/g, \dots, e_{m+1}/g)$, where $g := \gcd(e_1, \dots, e_{m+1})$, to a unimodular integer matrix $\mathbf{C}$ with last row $(e_1/g, \dots, e_{m+1}/g)$ [7, §XXI.3]. Then

$$(\tau_1, \dots, \tau_{m+1})^T := \mathbf{C}(\alpha_1, \dots, \alpha_{m+1})^T$$

yields a basis of M with $\tau_{m+1} = 1/g \in \mathbb{Q}$. Now suppose

$$u_1\tau_1 + \dots + u_m\tau_m = u$$

for integers $u_1, \dots, u_m, u$. Since u has also the representation

$$ug\tau_{m+1} = u,$$

it follows $u_1 = \dots = u_m = u = 0$. □

Take $\tau_1, \dots, \tau_{m+1}$ as in Lemma 3, with $\tau_{m+1} = 1/g$. Roughly speaking, we have put all integer relations among the θ_i into the rational basis element τ_{m+1}. There are integers b_{ij} with

$$\theta_i = \sum_{j=1}^{m+1} b_{ij}\tau_j.$$

Now we split the sequence (f_n) into the subsequences $(f_{gn+k})_{n\geq 0}$ for $0 \leq k < g$. We have

$$f_{gn+k} = G_n - s_n,$$

where $s_n := r_{gn+k}$ and G_n is the dominant part. Defining the integer matrix

$$\mathbf{B} := (gb_{ij})_{\substack{1\leq i\leq d \\ 1\leq j\leq m}} \in \mathbb{Z}^{d\times m}$$

and the real vector $\mathbf{c} = (c_1, \dots, c_d)$ with

$$c_i := 2\pi k \sum_{j=1}^{m+1} b_{ij}\tau_j + \beta_i, \qquad 1 \leq i \leq d,$$

it can be written as (cos is applied component wise)

$$G_n = \mathbf{a}^T \cos(2\pi n\mathbf{B}\boldsymbol{\tau} + \mathbf{c}) + v.$$

We show that the density of $\{n \in \mathbb{N} : f_{gn+k} > 0\}$ exists for each k. Since s_n is a recurrence sequence with fewer characteristic roots than f_n, we may assume inductively that $\boldsymbol{d}(\{n \in \mathbb{N} : s_n < 0\})$ exists. Thus, if G_n is the zero sequence, we are done. Now let k be such that G_n is not the zero sequence. It is plain that $G_n = H(n\boldsymbol{\tau})$, where

$$H(\mathbf{t}) := \mathbf{a}^T \cos(2\pi \mathbf{B}\mathbf{t} + \mathbf{c}) + v, \quad \mathbf{t} \in [0,1]^m.$$

The following theorem shows that the function H can be used to evaluate the density of the positivity set of G_n, which equals, as we will see below, that of the set $\{n : f_{gn+k} > 0\}$.

Theorem 4 (Kronecker-Weyl). *Let $\tau_1, \dots, \tau_m$ be real numbers such that $1, \tau_1, \dots, \tau_m$ are linearly independent over $\mathbb{Q}$. Then for every Jordan measurable set $A \subseteq [0,1]^m$ we have*

$$\boldsymbol{d}(\{n \in \mathbb{N} : n\boldsymbol{\tau} \bmod 1 \in A\}) = \boldsymbol{\lambda}(A).$$

Proof. We refer to Cassels [2, Theorems IV.I and IV.II]. □

We define

$$L_\varepsilon := \{n \in \mathbb{N} : G_n \geq \varepsilon\} \qquad \text{and} \qquad S_\varepsilon := \{n \in \mathbb{N} : |G_n| < \varepsilon\}. \tag{4}$$

The corresponding sets for the function H are defined as

$$\widetilde{L}_\varepsilon := \{\mathbf{t} \in [0,1]^m : H(\mathbf{t}) \geq \varepsilon\} \qquad \text{and} \qquad \widetilde{S}_\varepsilon := \{\mathbf{t} \in [0,1]^m : |H(\mathbf{t})| < \varepsilon\}.$$

Since for all $\varepsilon \geq 0$

$$L_\varepsilon = \{n \in \mathbb{N} : n\boldsymbol{\tau} \bmod 1 \in \widetilde{L}_\varepsilon\},$$

we have $\boldsymbol{d}(L_\varepsilon) = \boldsymbol{\lambda}(\widetilde{L}_\varepsilon)$ for all $\varepsilon \geq 0$ by Theorem 4. Similarly,

$$\boldsymbol{d}(S_\varepsilon) = \boldsymbol{\lambda}(\widetilde{S}_\varepsilon), \quad \varepsilon > 0. \tag{5}$$

Note that the boundary of the bounded set $\widetilde{S}_\varepsilon$ (respectively $\widetilde{L}_\varepsilon$) is a Lebesgue null set (as seen by applying the following lemma with $F(\mathbf{t}) = H(\mathbf{t}) - \varepsilon$), hence $\widetilde{S}_\varepsilon$ and $\widetilde{L}_\varepsilon$ are Jordan measurable, and Theorem 4 is indeed applicable. Lemma 5 seems to be known [6], but we could not find a complete proof in the literature.

Lemma 5. *Let $F : \mathbb{R}^m \to \mathbb{R}$ be a real analytic function. Then the zero set of F has Lebesgue measure zero, unless F vanishes identically.*

The proof of Lemma 5 is postponed to the end of this section. Since G_n is not the zero sequence, the function H does not vanish identically on $[0,1]^m$. By the Lebesgue dominated convergence theorem and Lemma 5 we thus find

$$\lim_{\varepsilon \to 0} \boldsymbol{\lambda}(\widetilde{S}_\varepsilon) = 0 \qquad \text{and} \qquad \lim_{\varepsilon \to 0} \boldsymbol{\lambda}(\widetilde{L}_\varepsilon) = \boldsymbol{\lambda}(\widetilde{L}_0).$$

This yields $\boldsymbol{d}(\{n \in \mathbb{N} : G_n > s_n\}) = \boldsymbol{\lambda}(\widetilde{L}_0)$ by the following lemma, which completes the proof of Theorem 1.

Lemma 6. *Let G_n and s_n be real sequences with $s_n = \mathrm{o}(1)$ and let L_ε, S_ε be as in* (4). *Suppose that $\boldsymbol{d}(L_\varepsilon)$ and $\boldsymbol{d}(S_\varepsilon)$ exist for all $\varepsilon \geq 0$, and that*

$$\lim_{\varepsilon \to 0} \boldsymbol{d}(L_\varepsilon) = \boldsymbol{d}(L_0) \qquad \textit{and} \qquad \lim_{\varepsilon \to 0} \boldsymbol{d}(S_\varepsilon) = 0.$$

Then

$$\boldsymbol{d}(\{n \in \mathbb{N} : G_n > s_n\}) = \boldsymbol{d}(L_0).$$

Proof. For any set $A \subseteq \mathbb{N}$ we write $A(x) := \{n \leq x : n \in A\}$. Define

$$P := \{n \in \mathbb{N} : G_n > s_n\}.$$

Let $\varepsilon > 0$ be arbitrary. Take n_0 such that $|s_n| < \varepsilon$ for $n > n_0$. It follows

$$\begin{aligned}\sharp P(x) = \sharp\{n \leq n_0 : G_n > s_n\} + \sharp\{n_0 < n \leq x : G_n \geq \varepsilon\} \\ + \sharp\{n_0 < n \leq x : s_n < G_n < \varepsilon\},\end{aligned}$$

hence

$$|\sharp P(x) - \sharp L_\varepsilon(x)| \leq \sharp S_\varepsilon(x) + \mathrm{o}(x)$$

as $x \to \infty$. Thus we have

$$\begin{aligned}|x^{-1}\sharp P(x) - \boldsymbol{d}(L_0)| &\leq |x^{-1}\sharp P(x) - x^{-1}\sharp L_\varepsilon(x)| + |x^{-1}\sharp L_\varepsilon(x) - \boldsymbol{d}(L_0)| \\ &\leq x^{-1}\sharp S_\varepsilon(x) + |x^{-1}\sharp L_\varepsilon(x) - \boldsymbol{d}(L_0)| + \mathrm{o}(1).\end{aligned}$$

The right hand side goes to

$$\boldsymbol{d}(S_\varepsilon) + |\boldsymbol{d}(L_\varepsilon) - \boldsymbol{d}(L_0)|$$

as $x \to \infty$. By assumption, this can be made arbitrarily small, which implies $\boldsymbol{d}(P) = \boldsymbol{d}(L_0)$. □

Proof of Lemma 5. For $m = 1$ this is clear, since then the zero set is countable. Now assume that we have established the result for $1, \ldots, m-1$. Put

$$V := \{(t_2, \ldots, t_m) \in \mathbb{R}^{m-1} : F(\cdot, t_2, \ldots, t_m) \text{ vanishes identically }\}.$$

Take a real number s such that $F(s, \cdot, \ldots, \cdot)$ is not identically zero. Clearly, $F(s, t_2, \ldots, t_m) = 0$ for all $(t_2, \ldots, t_m) \in V$. By the induction hypothesis, this implies $\boldsymbol{\lambda}(V) = 0$. Note that V is closed, hence measurable. Since F is real analytic in the first argument, we have

$$\int_{\mathbb{R}} \chi_Z(t_1, \ldots, t_m) \mathrm{d}\boldsymbol{\lambda}(t_1) = 0$$

for all $(t_2, \ldots, t_m) \notin V$, where χ_Z is the characteristic function of the zero set

$$Z := \{(t_1, \ldots, t_m) \in \mathbb{R}^m : F(t_1, \ldots, t_m) = 0\}.$$

Since V has measure zero, this implies

$$\int_{\mathbb{R}} \cdots \int_{\mathbb{R}} \chi_Z(t_1, \ldots, t_m) \mathrm{d}\boldsymbol{\lambda}(t_1) \cdots \mathrm{d}\boldsymbol{\lambda}(t_m) = 0.$$

This argument works for any order of integration, hence we obtain $\int_{\mathbb{R}^m} \chi_Z = 0$ by Tonelli's theorem. □

3 Sequences with no Positive Dominating Root

In this section we prove Theorem 2. We begin by settling the special cases where the θ_i are all irrational or all rational, and then put them together.

Lemma 7. *Let $\theta_1, \ldots, \theta_d$ be irrational numbers, and let a_i, β_i be real numbers such that the sequence*

$$u_n = \sum_{i=1}^{d} a_i \cos(2\pi\theta_i n + \beta_i)$$

is not identically zero. Let further r_n be a recurrence sequence with $r_n = \mathrm{o}(1)$. Then the set $\{n \in \mathbb{N} : u_n > r_n\}$ has positive density.

Proof. Proceeding as in the proof of Theorem 1, we can write

$$G_n := u_{gn+k} = \mathbf{a}^T \cos(2\pi n \mathbf{B}\boldsymbol{\tau} + \mathbf{c}),$$

where $\mathbf{B}$ is an integer matrix no row of which is zero, $\mathbf{c}$ is a real vector and $1, \tau_1, \ldots, \tau_m$ are linearly independent over $\mathbb{Q}$. If k is such that $G_n = u_{gn+k} \equiv 0$, then the density of $\{n \in \mathbb{N} : G_n > s_n\}$, where $s_n = r_{qn+k}$, exists by Theorem 1, but may be zero. Now choose a k_0 such that the corresponding sequence $G_n = u_{gn+k_0}$ is not the zero sequence. We have $G_n = H(n\boldsymbol{\tau})$, where

$$H(\mathbf{t}) := \mathbf{a}^T \cos(2\pi \mathbf{B}\mathbf{t} + \mathbf{c}).$$

Moreover, with the notation of the proof of Theorem 1, we have

$$\boldsymbol{d}(\{n \in \mathbb{N} : G_n > s_n\}) = \boldsymbol{\lambda}(\widetilde{L}_0).$$

The function H is not identically zero on $[0,1]^m$. But

$$\int_0^1 \cdots \int_0^1 H(t_1, \ldots, t_m) \mathrm{d}t_1 \cdots \mathrm{d}t_m = 0, \tag{6}$$

because no row of $\mathbf{B}$ is the zero vector. Hence H has a positive value on $[0,1]^m$, and since it is continuous, we have $\boldsymbol{\lambda}(\widetilde{L}_0) > 0$. $\square$

Observe that the integral in (6) need not vanish if $\mathbf{B}$ has a zero row, which can only happen if the θ_i corresponding to this row is a rational number. This is the reason why we consider rational θ_i's separately.

Lemma 8. *Let $\theta_1, \ldots, \theta_d$ be rational numbers in $]0,1[$, and let a_i, β_i be real numbers such that the purely periodic sequence*

$$u_n = \sum_{i=1}^{d} a_i \cos(2\pi\theta_i n + \beta_i)$$

is not identically zero. Then u_n has a positive and a negative value.

Proof. By the identity

$$\sum_{k=0}^{q-1} \cos \tfrac{2\pi k p}{q} + \mathrm{i} \sum_{k=0}^{q-1} \sin \tfrac{2\pi k p}{q} = \sum_{k=0}^{q-1} \mathrm{e}^{2\pi \mathrm{i} k p/q} = 0,$$

valid for integers $0 < p < q$, and the addition formula of cos we obtain

$$u_0 + \cdots + u_{q-1} = 0,$$

where q is a common denominator of $\theta_1, \ldots, \theta_d$. $\square$

Proof of Theorem 2. It suffices to consider the positivity set. We may write

$$f_n = u_n + v_n - r_n,$$

where $r_n = \mathrm{o}(1)$ is a recurrence sequence,

$$\begin{aligned} u_n &= \sum_{i=1}^{d} a_i \cos(2\pi\theta_i n + \beta_i), \\ v_n &= \sum_{i=d+1}^{e} a_i \cos(2\pi\theta_i n + \beta_i), \end{aligned}$$

$\theta_1, \ldots, \theta_d$ are irrational, $\theta_{d+1}, \ldots, \theta_e$ are rational numbers in $]0,1[$ with common denominator $q > 0$, and $u_n + v_n \not\equiv 0$. If $v_n \equiv 0$, then the result follows from Lemma 7. Now suppose $v_n \not\equiv 0$. Then for each k the density of the set $\{n \in \mathbb{N} : f_{qn+k} > 0\}$ exists by Theorem 1. By Lemma 8 there is k_0 such that $v_{qn+k_0} = v > 0$. It suffices to show that the set $\{n \in \mathbb{N} : f_{qn+k_0} > 0\}$ has positive density. This is clear if $u_{qn+k_0} \equiv 0$. Otherwise, notice that

$$\{n \in \mathbb{N} : f_{qn+k_0} > 0\} \supseteq \{n \in \mathbb{N} : u_{qn+k_0} > r_{qn+k_0}\},$$

and the latter set has positive density by Lemma 7. $\square$

4 The Possible Values of the Density

In this section we investigate which values from $[0,1]$ occur as density of some recurrence sequence. In its basic form, the question is readily answered:

Example 9. *Let w be a real number and define*

$$f_n := \sin(2\pi n\sqrt{2}) - w.$$

Then, by Theorem 4,

$$\begin{aligned} \boldsymbol{d}(\{n \in \mathbb{N} : f_n > 0\}) &= \boldsymbol{\lambda}(\{t \in [0,1] : \sin(2\pi t) > w\}) \\ &= \begin{cases} 1 & w \leq -1 \\ \frac{1}{2} - \frac{1}{\pi}\arcsin w & -1 \leq w \leq 1 \\ 0 & w \geq 1 \end{cases}. \end{aligned}$$

Since the range of arcsin *is $[-\frac{\pi}{2}, \frac{\pi}{2}]$, for every $\kappa \in [0,1]$ this yields a recurrence sequence f_n such that*

$$\boldsymbol{d}(\{n \in \mathbb{N} : f_n > 0\}) = \kappa.$$

The following proposition generalizes this example. Note that the density of the zero set of a recurrence sequence is always a rational number by the Skolem-Mahler-Lech theorem.

Proposition 10. *Let κ be a real number and r be a rational number with $0 \le \kappa, r \le 1$ and $\kappa + r \le 1$. Then there is a recurrence sequence (f_n) such that*

$$\boldsymbol{d}(\{n \in \mathbb{N} : f_n > 0\}) = \kappa \qquad \textit{and} \qquad \boldsymbol{d}(\{n \in \mathbb{N} : f_n = 0\}) = r.$$

Proof. Suppose that $r = p/q$ for positive integers p and q. As seen in Example 9, there is a recurrence sequence (g_n) such that the density of the zero set of (g_n) is zero and the density of its positivity set is $\kappa/(1-r)$ (The case $r = 1$ is trivial). The interlacing sequence

$$f_{bn+k} := \begin{cases} 0 & 0 \le k < p \\ g_n & p \le k < q \end{cases}$$

is a recurrence sequence [3, section 4.1]. Clearly, the density of its zero set is r, and the density of its positivity set is

$$\boldsymbol{d}(\{n \in \mathbb{N} : f_n > 0\}) = \frac{q-p}{q} \times \frac{\kappa}{1-r} = \kappa,$$

as required. □

If we restrict attention to sequences without dominating real positive roots, then Theorem 2 tells us that the density of the positivity set can be neither zero nor one. Still, all values in between occur.

Theorem 11. *Let $\kappa \in]0,1[$. Then there is a recurrence sequence (f_n) with no positive dominating characteristic root and $\boldsymbol{d}(\{n \in \mathbb{N} : f_n > 0\}) = \kappa$.*

Proof. Let $\varepsilon > 0$ be arbitrary. We define a function H on $[0, \frac{1}{2}]$ by

$$H(t) := \begin{cases} \frac{(\varepsilon-1)^2}{\varepsilon}\left(1 - \frac{2t}{\varepsilon}\right) & 0 \le t \le \frac{\varepsilon}{2} \\ \varepsilon - 2t & \frac{\varepsilon}{2} \le t \le \frac{1}{2} \end{cases}$$

and extend it to an even, 1-periodic function H on $\mathbb{R}$ (see Figure 1). It is continuous and satisfies

$$\int_0^1 H(t)\mathrm{d}t = 0 \qquad \text{and} \qquad \boldsymbol{\lambda}(\{t \in [0,1] : H(t) > 0\}) = \varepsilon.$$

Expanding H into a Fourier series, we find that there are real a_j such that H is the pointwise limit of

$$H_m(t) := \sum_{j=1}^{m} a_j \cos(2\pi j t)$$

as $m \to \infty$. Since the zero set of H is a null set, the Lebesgue dominated convergence theorem yields

$$\lim_{m\to\infty} \boldsymbol{\lambda}(\{t \in [0,1] : H_m(t) > 0\}) = \varepsilon.$$

We fix an m such that

$$\boldsymbol{\lambda}(\{t \in [0,1] : H_m(t) > 0\}) \le 2\varepsilon.$$

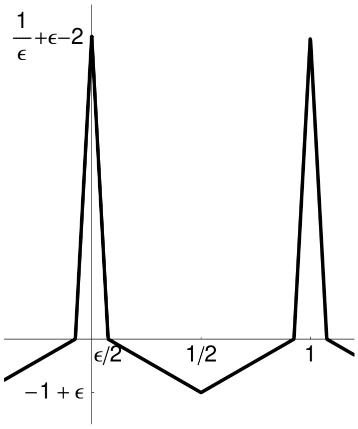

Figure 1: The function H

The function

$$\phi(A_1, \ldots, A_m) := \boldsymbol{\lambda}\Big(\Big\{t \in [0,1] : \sum_{j=1}^{m} A_j \cos(2\pi j t) > 0\Big\}\Big)$$

is continuous on $\mathbb{R}^m \backslash \{\mathbf{0}\}$. To see this, observe that ϕ is continuous at all points $(A_1, \ldots, A_m)$ for which $\sum_{j=1}^{m} A_j \cos(2\pi j t)$ is not identically zero and appeal to the uniqueness of the Fourier expansion. Since $\phi(1, 0, \ldots, 0) = \frac{1}{2}$ and $\phi(a_1, \ldots, a_m) \leq 2\varepsilon$, the function ϕ assumes every value from $[2\varepsilon, \frac{1}{2}]$ by the intermediate value theorem.

Hence the positivity sets of the sequences

$$f_n := \sum_{j=1}^{m} A_j \cos(2\pi j n \sqrt{2})$$

assume all densities from $[2\varepsilon, \frac{1}{2}]$ for appropriate choices of $(A_1, \ldots, A_m)$ by Theorem 4. Repeating the whole argument with $-H$ instead of H yields the desired result for $\kappa \in [\frac{1}{2}, 1 - 2\varepsilon]$. Since ε was arbitrary, the theorem is proved. $\square$

5 A Weak Version of Skolem-Mahler-Lech

Without using the Skolem-Mahler-Lech theorem, it follows from Theorem 1 that the density of the zero set of a recurrence sequence (f_n) exists. We can show a bit more with our approach. Recall, however, that we only deal with real sequences, whereas the Skolem-Mahler-Lech theorem holds for any field of characteristic zero.

Proposition 12. *The density of the zero set of a (real) recurrence sequence (f_n) is a rational number.*

Proof. Let k be a natural number, and let g, G_n and s_n be as in the proof of Theorem 1. If k is such that $G_n \equiv 0$, then the density of the zero set of f_{gn+k}

is rational, since we may assume inductively that the density of $\{n : s_n = 0\}$ is rational.

Now suppose $G_n \not\equiv 0$. The zero set of f_{gn+k} can be partitioned as

$$\{n \in \mathbb{N} : G_n = s_n\} = \{n : G_n = s_n, |G_n| < \varepsilon\} \cup \{n : G_n = s_n, |G_n| \geq \varepsilon\},$$

where $\varepsilon \geq 0$ is arbitrary. The latter set is finite, and the first one is contained in S_ε, defined in (4). Hence

$$\boldsymbol{d}(\{n \in \mathbb{N} : G_n = s_n\}) \leq \boldsymbol{d}(S_\varepsilon)$$

for all $\varepsilon \geq 0$. But we know that $\lim_{\varepsilon \to 0} \boldsymbol{d}(S_\varepsilon) = 0$ from the proof of Theorem 1, which yields

$$\boldsymbol{d}(\{n \in \mathbb{N} : G_n = s_n\}) = 0.$$

Thus, the zero sets of all subsequences $(f_{gn+k})_{n \geq 0}$, $0 \leq k < g$, have rational density, which proves the desired result. □

6 Conclusion

There is no algorithm known that decides, given a recurrence sequence (f_n), whether $f_n > 0$ for all n, nor has the problem been shown to be undecidable. When we are talking about algorithmics, it is natural to assume that the recurrence coefficients and the initial values are rational numbers. In this case Gourdon and Salvy [5] have proposed an efficient method for ordering the characteristic roots w.r.t. to their modulus. Thus, the dominating characteristic roots can be identified algorithmically. If none of them is real positive, then we know that the sequence oscillates by Theorem 2. On the other hand, sequences where a positive dominating root is accompanied by complex dominating roots seem to pose difficult Diophantine problems. For instance, we do not know if the sequence

$$f_n := \cos(2\pi\theta n) + 1 + \left(-\tfrac{1}{2}\right)^n \tag{7}$$

is positive for $\theta = \sqrt{2}$, say. It is positive for $n \leq 10^5$. It can be shown, however, that the set of θ's for which the corresponding sequence (f_n) (defined by (7)) is eventually positive has measure zero [4, Theorem 7.2].

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