On the Number of Distinct Multinomial Coefficients

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Abstract

We study $M(n)$, the number of distinct values taken by multinomial coefficients with upper entry n, and some closely related sequences. We show that both $p_{\mathbb{P}}(n)/M(n)$ and $M(n)/p(n)$ tend to zero as n goes to infinity, where $p_{\mathbb{P}}(n)$ is the number of partitions of n into primes and $p(n)$ is the total number of partitions of n. To use methods from commutative algebra, we encode partitions and multinomial coefficients as monomials.

Key words: Factorials, binomial coefficients, combinatorial functions, partitions of integers; polynomial ideals, Gröbner bases.

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1 Introduction

The classical multinomial expansion is given by

$$(x_1 + x_2 + \cdots + x_k)^n = \sum \binom{n}{i_1, i_2, \ldots, i_k} x_1^{i_1} x_2^{i_2} \cdots x_k^{i_k}\,, \tag{1}$$

where the sum runs over all $(i_1, i_2, \ldots, i_k)$ such that $i_1 + i_2 + \cdots + i_k = n$ and $i_1, i_2, \ldots, i_k \geq 0$. Multinomial coefficients are defined by

$$\binom{n}{i_1, i_2, \ldots, i_k} := \frac{n!}{i_1! i_2! \cdots i_k!}. \tag{2}$$

It is natural to ask about $M_k(n)$, the number of different values of

$$\binom{n}{i_1, i_2, \ldots, i_k} \tag{3}$$

where $i_1 + i_2 + \cdots + i_k = n$. Obviously if the $i_1, i_2, \ldots, i_k$ are merely permuted, then the value of $\binom{n}{i_1, i_2, \ldots, i_k}$ is unchanged. However identical values do not necessarily arise only by permuting the $i_1, i_2, \ldots, i_k$. For example,

$$\binom{7}{3, 2, 2} = \binom{7}{4, 1, 1, 1} \tag{4}$$

and

$$\binom{236}{64,\ 55,\ 55,\ 52,\ 7,\ 3} = \binom{236}{62,\ 56,\ 54,\ 51,\ 13}. \tag{5}$$

We note that if $k \geq n$, then $M_k(n) = M_n(n)$, and we define $M(n) := M_n(n)$ to be the total number of distinct multinomial coefficients with upper entry n.

Since permuting its lower indices leaves the value of a multinomial coefficient unchanged it is immediately clear that

$$M_k(n) \leq p_k(n) \tag{6}$$

and

$$M(n) \leq p(n)\,, \tag{7}$$

where $p_k(n)$ is the number of partitions of n into at most k parts, and $p(n)$ is the total number of partitions of n respectively. Observing that the binomial coefficients $\binom{n}{k, n-k}$ are strictly increasing for $0 \leq k \leq \frac{n}{2}$, we deduce that, in fact,

$$M_2(n) = p_2(n). \tag{8}$$

However the inequality (6) seems to be stronger for large k. Indeed (Theorem 8),

$$\lim_{n\to\infty}\frac{M(n)}{p(n)}=0. \tag{9}$$

Bounding $M(n)$ from below we will prove (Theorem 1) that

$$M(n)\geq p_{\mathbb{P}}(n) \tag{10}$$

where $p_{\mathbb{P}}(n)$ is the number of partitions of n into parts belonging to the set of primes $\mathbb{P}$. Indeed (Theorem 12),

$$\lim_{n\to\infty}\frac{p_{\mathbb{P}}(n)}{M(n)}=0. \tag{11}$$

It is natural to generalize the problem from $M(n)$ to $M_S(n)$, the number of different multinomial coefficients with upper entry n whose lower entries belong to a given set S of natural numbers. Let

$$\mathcal{M}_S(q):=\sum_n M_S(n)q^n \tag{12}$$

and

$$\mathcal{P}_S(q):=\sum_n p_S(n)q^n \tag{13}$$

where $p_S(n)$ is the number of partitions of n into elements from S. Define $[s]:=\{1,2,\ldots,s\}$. Results of numerical calculations such as

$$\mathcal{M}_{[4]}(q)\,/\,\mathcal{P}_{[4]}(q)=1-q^7+O(q^{100}) \tag{14}$$

and

$$\mathcal{M}_{[7]}(q)\,/\,\mathcal{P}_{[7]}(q)=1-q^7-q^8-q^{10}+q^{12}+q^{13}+O(q^{100}) \tag{15}$$

suggest that $\mathcal{M}_S(q)\,/\,\mathcal{P}_S(q)$ is a polynomial for any finite S. This is indeed true (Theorem 5) and leads to an algorithm for computing a closed form for the sequence $M_S(n)$ for a given finite set S (Section 4).

Partitions and multinomial coefficients can be written as monomials in a natural way: For instance, the monomial $q_4q_1^3$ represents the partition $4+1+1+1$, and $x_7x_5x_3x_2$ represents the multinomial coefficient $\binom{7}{4,1,1,1}$ whose factorization into primes is $7\cdot5\cdot3\cdot2$. This encoding serves as a link between our counting problem and Hilbert functions (Section 3). Sections 4, 5 and 6 are based on that link.

We call a pair of partitions of n that yield the same multinomial coefficient but have no common parts an *irreducible pair*. For example, the partitions $4+1+1+1$ and $3+2+2$ form an irreducible pair according to Equation (4). In Section 7, we study $i(n)$ the total number of irreducible pairs of partitions of n, and we prove (Theorem 13) that $i(n)>\frac{n}{56}-1$.

2 A Lower Bound for $M(n)$

We relate $M(n)$ to $p_{\mathbb{P}}(n)$ whose asymptotics is known by a theorem of Kerawala [1]:

$$\log p_{\mathbb{P}}(n) \sim \frac{2\pi}{\sqrt{3}} \sqrt{\frac{n}{\log n}}. \tag{16}$$

Theorem 1 $M(n) \geq p_{\mathbb{P}}(n)$.

Theorem 1 is implied by the following lemma:

Lemma 2 *Any two distinct partitions of the same natural number n into primes yield different multinomial coefficients.*

PROOF. [Proof of Lemma 2] It suffices to show that if

$$p_1!p_2! \cdots p_r! = q_1!q_2! \cdots q_s! \tag{17}$$

where $p_1 \leq p_2 \leq \cdots \leq p_r$ and $q_1 \leq q_2 \leq \cdots \leq q_s$ are all primes, then $r = s$ and $p_i = q_i$ for $i = 1, \ldots, s$. We proceed by mathematical induction on r.

If $r = 1$, then q_s must equal p_1 because if $q_s < p_1$ then p_1 divides the left side of the above equation but not the right side. If $q_s > p_1$ then q_s divides the right side but not the left. Hence $q_s = p_1$, and dividing both sides by $p_1!$ we see that there can be no other q_i. Hence $s = 1$ and $q_1 = p_1$.

Assume now that our result holds up to but not including a particular r. As in the case $r = 1$, we must have $q_s = p_r$. Cancel $p_r!$ from both sides and apply the induction hypothesis to conclude that $s - 1 = r - 1$ and $p_i = q_i$ for $i = 1, \ldots, s - 1$. Hence the lemma follows by mathematical induction. $\square$

Some values of $p_{\mathbb{P}}(n)$ and $M(n)$ are listed on page 15. We will refine Theorem 1 in Section 6.

3 The Algebraic Setting

Encoding partitions and multinomial coefficients as monomials allows us to apply constructive methods from commutative algebra to the problem of counting multinomial coefficients. Let us assume that $S \subseteq \mathbb{N}$ throughout the paper. We will see that $M_S(n)$ finds a natural interpretation as the Hilbert function of a certain graded ring (Lemma 4). In the case of finite S, it can be computed by the method of Gröbner bases [2–5].

We represent the partition $\lambda_0+\lambda_1+\cdots+\lambda_i$ of n by the monomial $q_{\lambda_0}q_{\lambda_1}\ldots q_{\lambda_i}$ whose degree is n if we define the degree of variables suitably by $\deg q_j := j$. For convenience, we will use the notions "partition of n" and "monomial of degree n" interchangeably.

Let k be a field of characteristic zero. We abbreviate the ring $k[q_i : i \in S]$ of polynomials in the variables q_i for $i \in S$ over k by $k[S]$. Define the degree of monomials by $\deg q_i := i$, and let $k[S]_n$ denote the subspace of all homogeneous polynomials of degree n. In other words, $k[S]_n$ is the k-vector space whose basis are the partitions of n into parts S. Note that $k[S]$ is graded by $k[S] = \bigoplus_n k[S]_n$. For instance,

$$k[\{1,\, 3, \ldots\}]_4 \;=\; k \cdot q_3 q_1 \oplus \, k \cdot {q_1}^4 \tag{18}$$

corresponding to the partitions $3+1$ and $1+1+1+1$ of 4 into odd parts.

The multinomial coefficients with upper entry n into parts belonging to S are the numbers $n!/\prod_j j!^{a_j}$ where $\prod_j q_j^{a_j}$ ranges over the monomials in $k[S]_n$. Since the numerator $n!$ of these fractions is fixed, it suffices to count the set of all denominators:

$$M_S(n) = |\{\prod_j j!^{a_j} \;:\; \prod_j q_j^{a_j} \in k[S]_n\}|. \tag{19}$$

To count the values taken by $\prod_j j!^{a_j}$, we look at their factorization into primes. Let $h(q_j)$ be the factorization of $j!$ into primes, written as a monomial in $k[x] := k[x_p \;:\; p \text{ prime}]$, multiplied by q^j. For example, $h(q_5) = q^5 x_2^3 x_3 x_5$ corresponding to $5! = 2^3 \cdot 3 \cdot 5$. An elementary counting argument [6] shows that the prime p occurs in the factorization of $j!$ with exponent $\sum_{l=1}^{\infty} \lfloor j/p^l \rfloor$, where $\lfloor x \rfloor$ denotes the largest integer that does not exceed the real number x. Therefore,

$$h(q_j) = q^j \prod_{p \text{ prime}} {x_p}^{\sum_{l=1}^{\infty} \lfloor j/p^l \rfloor}. \tag{20}$$

Since factorization into primes is unique, (19) can be written as

$$M_S(n) = |\{\prod_j h(q_j)^{a_j} \;:\; \prod_j q_j^{a_j} \in k[S]_n\}|. \tag{21}$$

Extending h to a k-algebra homomorphism $k[S] \to k[x, q]$ allows us to reformulate (21) as

Lemma 3

$$M_S(n) = \dim_k h(k[S]_n). \tag{22}$$

Example: Since there are 10 partitions of 7 into parts $1, 2, 3$ and 4, the dimension of

$$k[\{1,2,3,4\}]_7 = k\, q_4 q_3 \oplus k\, q_4 q_2 q_1 \oplus k\, q_4 q_1^3 \oplus k\, q_3 q_2^2 \oplus \cdots \oplus k\, q_1^7 \tag{23}$$

over k is 10. However, the dimension of its image

$$h(k[\{1,2,3,4\}]_7) = k\, q^7 x_2^4 x_3^2 \oplus k\, q^7 x_2^4 x_3 \oplus k\, q^7 x_2^3 x_3 \oplus \cdots \oplus k\, q^7 \tag{24}$$

under h is only 9 and so $M_{[4]}(7) = 9$. The defect is due to $h(q_4 q_1^3) = h(q_3 q_2^2)$ which is nothing but a restatement of (4).

To use Lemma 3 for effective computation (in the case of finite S), we express $\dim_k h(k[S]_n)$ as the value (at n) of the Hilbert function of a certain elimination ideal. This method is taken from [2]; the result in our case is Lemma 4 below.

First we make the map h degree-preserving (graded) by defining $\deg q := 1$ and $\deg x_p := 0$ in the ring $k[x_p : p \text{ prime}][q]$. (This is why we introduced the extra factor of q^j in the defining equation (20) of h.) Second, note that

$$h(k[S]_n) \cong k[S]_n / (k[S]_n \cap \ker h) \tag{25}$$

as k-vector spaces, since h is a k-linear map on $k[S]_n$. In particular, dimensions agree. Therefore,

$$M_S(n) = \dim_k k[S]_n / (k[S]_n \cap \ker h). \tag{26}$$

Recall that the (projective) Hilbert function H_R of a graded k-algebra $R = \bigoplus_n R_n$ is defined by $H_R(n) := \dim_k R_n$. Thus (26) relates M_S to the Hilbert function of $k[S]/\ker h$:

$$M_S(n) = H_{k[S]/\ker h}(n). \tag{27}$$

By Theorem 2.4.2 of [2], $\ker h$ can be computed by elimination:

$$\ker h = I \cap k[S] \tag{28}$$

where the ideal I of $k[S][q][x_p : p\, \text{prime}]$ is defined by

$$I := \langle q_j - h(q_j) \,:\, j \in S \rangle. \tag{29}$$

Summarizing this section, we have proved the following Lemma:

Lemma 4 *Let $k[S]$ be graded by $\deg q_i := i$. Define a k-algebra homomorphism from $k[S]$ to $k[q, x]$ by*

$$h(q_j) := q^j \prod_p x_p{}^{\sum_{l=1}^{\infty} \lfloor j/p^l \rfloor}. \tag{30}$$

Let the ideal I of $k[S,q,x]$ be defined by

$$I := \langle q_j - h(q_j) \,:\, j \in S \rangle \tag{31}$$

and let

$$J := I \cap k[S]. \tag{32}$$

Then M_S is the (projective) Hilbert function of the k-algebra $k[S]/J$:

$$M_S(n) = H_{k[S]/J}(n). \tag{33}$$

Example: If $S = [4]$, then $I = \langle q_1 - q,\, q_2 - q^2 x_2,\, q_3 - q^3 x_2 x_3,\, q_4 - q^4 x_2^3 x_3 \rangle$ and $J = \langle q_4 q_1^3 - q_3 q_2^2 \rangle$. For $M_{[4]}(n)$, see (41) on page 9.

4 Explicit Answers

Let S be a given finite set throughout this section. Lemma 4 allows to compute a closed form for the sequence $M_S(n)$ by well-known methods from computational commutative algebra. For the sake of completeness, let us briefly review them:

(1) Fix a term order $\preceq$ on $k[S,q,x]$ that allows the elimination of the variable q and the variables x_p in step 2 below. Compute a Gröbner basis F for the (toric) ideal $I = \langle q_j - h(q_j) \,:\, j \in S \rangle$ with respect to this term order using Buchberger's algorithm [3,4].
(2) Let $G := F \cap k[S]$. By the elimination property of Gröbner bases with respect to a suitable elimination order $\preceq$, the set G is a Gröbner basis for the elimination ideal $J = I \cap k[S]$.
(3) Let $L := I_{\preceq}(G)$ be the set of leading terms of polynomials in G.
(4) Compute $\mathcal{M}_S(q)$ using

$$\mathcal{M}_S(q) = \mathcal{H}_{k[S]/J}(q) = \mathcal{H}_{k[S]/I_{\preceq}(J)}(q) = \mathcal{H}_{k[S]/\langle L \rangle}(q). \tag{34}$$

The first equality holds by Lemma 4. The second equality is an identity of Macaulay [7]. Since G is a Gröbner basis, its initial terms L generate the initial term ideal of $\langle G \rangle$ with respect to $\preceq$, which explains the third equation sign. A naive method for computing the Hilbert-Poincaré series of $k[S]/\langle L \rangle$ is to apply the inclusion-exclusion relation

$$\mathcal{H}_{k[S]/\langle \{t\} \cup L \rangle}(q) = \mathcal{H}_{k[S]/\langle L \rangle}(q) - q^{\deg t} \mathcal{H}_{k[S]/\langle L \rangle : t}(q), \tag{35}$$

recursively until the base case

$$\mathcal{H}_{k[S]/\langle\rangle}(q) = \mathcal{H}_{k[S]}(q) = \frac{1}{\prod_{j \in S}(1 - q_j)} \tag{36}$$

is reached. For better (faster) algorithms, see [8].

(5) Extract a closed form expression for $H_{k[S]/\langle L\rangle}(n)$ from its generating function $\mathcal{H}_{k[S]/\langle L\rangle}(q)$. (Use partial fraction decomposition and the binomial series). It is the desired answer $M_S(n)$.

One of the authors computed 1 – 4 for several finite S using different computer algebra systems. It turned out that CoCoA [9] was fastest for that purpose.

Theorem 5 *Let S be a finite subset of the positive natural numbers. Then*

(1) $\mathcal{M}_S(q)$ can be written as

$$\mathcal{M}_S(q) = \frac{f_S(q)}{\prod_{j\in S}(1-q^j)} \tag{37}$$

where $f_S(q)$ is a polynomial with integer coefficients.

(2) There exists n_0 such that $M_S(n)$ can be written as a quasipolynomial [10] for $n \geq n_0$. Moreover, it suffices to use periods which are divisors of elements of S.

PROOF. Relations (35) and (36) prove the first statement. The second statement follows from the first easily. □

Let us follow the algorithm in the case $S = [4]$, which is the simplest nontrivial case. We have $I = \langle q_1 - q,\, q_2 - q^2x_2,\, q_3 - q^3x_2x_3,\, q_4 - q^4x_2^3x_3\rangle$. To eliminate the variables x_3, x_2 and q we choose the lexical term order where $x_3 \succ x_2 \succ q \succ q_4 \succ q_3 \succ q_2 \succ q_1$. The corresponding reduced Gröbner basis of I is $F = \{q_1^3q_4 - q_2^2q_3,\, q - q_1,\, q_1^2x_2 - q_2,\, q_2q_3x_2 - q_1q_4,\, q_1q_3x_2^2 - q_4,\, q_1q_2x_3 - q_3,\, q_2^2x_3 - q_1q_3x_2,\, q_1^2q_4x_3 - q_3^2x_2,\, q_2q_4x_3 - q_3^2x_2^2,\, q_1q_4^2x_3 - q_3^3x_2^3,\, q_4^3x_3 - q_3^4x_2^5\}$. By the elimination property of Gröbner bases $G := F \cap k[q_1, q_2, q_3, q_4] = \{q_1^3q_4 - q_2^2q_3\}$ is a Gröbner basis for the elimination ideal $J = I \cap k[q_1, q_2, q_3, q_4]$. Collecting leading terms of G gives $L = \{q_1^3q_4\}$. Since G is a Gröbner basis of J we know that $I_{\preceq}(J) = \langle q_1^3q_4\rangle$. The Hilbert-Poincaré series of $k[q_1, q_2, q_3, q_4]/\langle q_1^3q_4\rangle$ gives

$$\mathcal{M}_{[4]}(q) = \frac{1-q^7}{(1-q)(1-q^2)(1-q^3)(1-q^4)}. \tag{38}$$

It is clear that we may replace any occurrence of the partition $4 + 1 + 1 + 1$ in a multinomial coefficient by $3 + 2 + 2$ without changing the value of the multinomial coefficient. Therefore, there are at most as many multinomial coefficients as there are partitions avoiding $4+1+1+1$. Equation (38) states that this upper bound gives in fact the exact number in the case of $S = \{1, 2, 3, 4\}$.

Note that all denominators in the partial fraction decomposition

$$\mathcal{M}_{[4]}(q) = -\frac{7}{24}\frac{1}{(q-1)^3} - \frac{77}{288}\frac{1}{(q-1)} + \frac{1}{16}\frac{1}{(q+1)^2} + \\ + \frac{1}{32}\frac{1}{(q+1)} + \frac{1}{9}\frac{(q+2)}{(q^2+q+1)} + \frac{1}{8}\frac{(q+1)}{(q^2+1)} \quad (39)$$

of (38) are powers of cyclotomic polynomials $C_j(q)$ where j divides an element of $S = \{1, 2, 3, 4\}$. We rewrite this as

$$\mathcal{M}_{[4]}(q) = \frac{7}{24}\frac{1}{(1-q)^3} + \frac{77}{288}\frac{1}{(1-q)} + \frac{1}{16}\frac{(1-q)^2}{(1-q^2)^2} + \\ + \frac{1}{32}\frac{(1-q)}{(1-q^2)} + \frac{1}{9}\frac{(2-q-q^2)}{(1-q^3)} + \frac{1}{8}\frac{(1+q-q^2-q^3)}{(1-q^4)}. \quad (40)$$

in order to use the binomial series $(1-z)^{-a-1} = \sum_{n=0}^{\infty} \binom{a+n}{a} z^n$. The result is

$$M_{[4]}(n) = \frac{7}{48}n^2 + \left(\frac{1}{16}[1, -1](n) + \frac{7}{16}\right)n + \\ + \frac{1}{8}[1, 1, -1, -1](n) + \frac{1}{9}[2, -1, -1](n) + \frac{3}{32}[1, -1](n) + \frac{161}{288} \quad (41)$$

where $[a_0, a_1, \ldots, a_m](n) := a_j$ for $n \equiv j(m)$. Similar computations show that

$$\mathcal{M}_{[5]}(q) = \frac{1-q^7}{(1-q)(1-q^2)\ldots(1-q^5)}, \quad (42)$$

$$\mathcal{M}_{[6]}(q) = \frac{1-q^7-q^8-q^{10}+q^{12}+q^{13}}{(1-q)(1-q^2)\ldots(1-q^6)}, \quad (43)$$

and

$$\mathcal{M}_{[7]}(q) = \frac{1-q^7-q^8-q^{10}+q^{12}+q^{13}}{(1-q)(1-q^2)\ldots(1-q^7)}. \quad (44)$$

It is no coincidence that the numerators of (43) and (44) agree (Theorem 11).

5 Upper Bounds

Trivially, $M(n) \leq p(n)$. Our goal is to find sharper upper bounds.

Lemma 6 *Assume $S' \subseteq S$.*

Let $\tilde{I}$ be the ideal of $k[S, q, x]$ generated by the set of polynomials $\{q_j - h(q_j) : j \in S'\}$. Let $\tilde{J}$ be the ideal generated by $\tilde{I} \cap k[S']$ in the ring $k[S]$. Let $U_{S,S'}(n) := H_{k[S]/\tilde{J}}(n)$. Then

(1) $M_S(n) \leq U_{S,S'}(n)$.
(2) We have

$$\sum_n U_{S,S'}(n) q^n = \frac{f_{S'}(q)}{\prod_{j \in S}(1-q^j)} \tag{45}$$

where $f_{S'}(q)$ *is defined by*

$$\sum_n M_{S'}(n) q^n = \frac{f_{S'}(q)}{\prod_{j \in S'}(1-q^j)}. \tag{46}$$

PROOF. We prove the first statement. Let I be the ideal of $k[S,q,x]$ generated by the set of polynomials $\{q_j - h(q_j) : j \in S\}$ and let $J = I \cap k[S]$. Since $\tilde{J}$ is a k-vector subspace of J we have

$$\dim_k k[S]_n \cap J \geq \dim_k k[S]_n \cap \tilde{J} \tag{47}$$

and therefore

$$\dim_k (k[S]/J)_n \leq \dim_k (k[S]/\tilde{J})_n \tag{48}$$

i.e.

$$M_S(n) \leq U_{S,S'}(n). \tag{49}$$

To prove the second statement, let I' be the ideal generated by $\{q_j - h(q_j) : j \in S'\}$ in the ring $k[S', q, x]$ and let $J' := I' \cap k[S']$. Since the ideals $\tilde{J}$ and J' are generated by the same set of polynomials (albeit in different rings), the Hilbert functions $U_{S,S'}(n) = H_{k[S]/\tilde{J}}(n)$ and $M'_S(n) = H_{k[S']/J'}(n)$ correspond in the way claimed by (45) and (46). $\square$

To get upper bounds for $M(n)$, we use the preceding Lemma in the special case $S = \mathbb{N}$ getting:

Theorem 7 *For any* S' *we have*

$$M(n) \leq [q^n] \frac{f_{S'}(q)}{\prod_{j=1}^{\infty}(1-q^j)} \tag{50}$$

(where $[q^n]\mathcal{A}(q)$ *denotes the coefficient of* q^n *in the power series expansion of* $\mathcal{A}(q)$*). For instance, the cases* $S' = [4]$ *and* $S' = [6]$ *yield the bounds*

$$M(n) \leq p(n) - p(n-7), \tag{51}$$

and

$$M(n) \leq p(n) - p(n-7) - p(n-8) - p(n-10) + p(n-12) + p(n-13). \tag{52}$$

Note that a direct proof of $M(n) \le p(n) - p(n-7)$ could be given by exploiting the equivalence of the partitions $4+1+1+1$ and $3+2+2$ in the sense of Equation (4).

The bound $M(n) \le p(n) - p(n-7)$ is good enough to imply:

Theorem 8 *$M(n) = o(p(n))$, i.e. $\lim_{n\to\infty} M(n)/p(n) = 0$.*

PROOF. Due to the monotonicity of $p(n)$ and the fact that the unit circle is the natural boundary for

$$\sum_{n=0}^{\infty} p(n) q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n}\,, \tag{53}$$

we see that

$$\lim_{n\to\infty} \frac{p(n-7)}{p(n)} = 1\,. \tag{54}$$

Hence

$$0 \le \lim_{n\to\infty} \frac{M(n)}{p(n)} \le \lim_{n\to\infty} \frac{p(n)-p(n-7)}{p(n)} = 1 - 1 = 0\,, \tag{55}$$

which proves Theorem 8. $\square$

6 Lower Bounds

Recall that $M(n) \ge p_{\mathbb{P}}(n)$ (Theorem 1). The numbers given on page 15 suggest that $M(n)$ grows much faster than $p_{\mathbb{P}}(n)$. We will prove that this is indeed the case: $\lim_{n\to\infty} p_{\mathbb{P}}(n)/M(n) = 0$ (Theorem 12) and we will give better lower bounds for $M(n)$.

Let us write $S < P$ if each element of S is less than each element of P. We need the following generalization of Lemma 2:

Lemma 9 *Assume $S < P$ where P is a set of primes. Let s and s' be any two power products in $k[S]$ and let p and p' be distinct power products in $k[P]$. Then $h(sp) \neq h(s'p')$.*

In the case $S = \emptyset$, Lemma 9 states that distinct partitions p and p' into primes yield different multinomial coefficients: $h(p) \neq h(p')$. Lemma 9 can be proved by the same induction argument as Lemma 2.

Lemma 10 *Assume $S < P$ where P is a set of primes. Define h on $k[S \cup P]$ by (20). Then $\ker h$ is generated, as an ideal of $k[S \cup P]$, by $\ker h \cap k[S]$.*

PROOF. Let $f \in \ker h$. Since $k[S \cup P] = k[S][P]$, we can f as a finite sum $f = \sum_s \sum_p c_{s,p}\, sp$ indexed by power products s and p from $k[S]$ and $k[P]$ respectively, with coefficients $c_{s,p} \in k$. As $f \in \ker h$, $\sum_s \sum_p c_{s,p}\, h(sp) = 0$. By Lemma 9, this implies $\sum_s c_{s,p}\, h(sp) = 0$ for arbitrary but fixed p. Canceling $h(p)$ from this equation shows that $h(f_p) = 0$ where $f_p := \sum_s c_{s,p}\, s$. In this way we succeed in writing f as $f = \sum_p f_p p$ where each f_p is in $\ker h \cap k[S]$. □

As an immediate consequence of Lemma 10 we get:

Theorem 11 *Assume $S < P$ where P is a set of primes. Then*

$$\mathcal{M}_{S\cup P}(q) = \mathcal{M}_S(q) / \prod_{j\in P} (1 - q^j). \tag{56}$$

As a first application of Theorem 11, we count multinomial coefficients with lower entries which are either prime or equal to 1:

$$\mathcal{M}_{\{1\}\cup\mathbb{P}}(q) = \frac{1}{(1-q)\prod_{j\in\mathbb{P}}(1-q^j)}, \tag{57}$$

which allows for improving Theorem 1:

Theorem 12 *We have*

$$\lim_{n\to\infty} p_{\mathbb{P}}(n) / M_{\{1\}\cup\mathbb{P}}(n) = 0 \tag{58}$$

and therefore $\lim_{n\to\infty} p_{\mathbb{P}}(n)/M(n) = 0$.

PROOF. Let $A(n) := M_{\{1\}\cup\mathbb{P}}(n)$. Due to the monotonicity of $A(n)$ and the fact that the unit circle is the natural boundary for we see that

$$\lim_{n\to\infty} A(n-1)/A(n) = 1. \tag{59}$$

By (57),

$$p_{\mathbb{P}}(n) = A(n) - A(n-1). \tag{60}$$

Therefore,

$$0 \le \lim_{n\to\infty} \frac{p_{\mathbb{P}}(n)}{A(n)} \le \lim_{n\to\infty} \frac{A(n) - A(n-1)}{A(n)} = 1 - 1 = 0\,, \tag{61}$$

which proves Theorem 12. □

Let $L_S(n) := M_{S\cup\mathbb{P}}(n)$; clearly, $L_S(n)$ is a lower bound for $M(n)$. Theorem 11 allows us deduce

$$\mathcal{L}_{[4]}(q) = \mathcal{L}_{[5]}(q) = \frac{1-q^7}{\prod_{j\in[4]\cup\mathbb{P}}(1-q^j)} \tag{62}$$

and

$$\mathcal{L}_{[6]}(q) = \mathcal{L}_{[7]}(q) = \frac{1-q^7-q^8-q^{10}+q^{12}+q^{13}}{\prod_{j\in[6]\cup\mathbb{P}}(1-q^j)} \tag{63}$$

from the Equations (38) – (44); some values of $L_{[4]}(n)$ are listed on page 15.

7 The Irreducible Pairs

An *irreducible pair* is a pair of partitions of n that yield the same multinomial coefficient but have no parts in common. For example,

$$(4,1,1,1) \text{ and } (3,2,2) \tag{64}$$

is an irreducible pair.

It turns out that there are infinitely many irreducible pairs of partitions. The following is a partial list: Generalizing (64) we see that

$$(2^m, \underbrace{1,1,\ldots,1}_{2m-1}) \text{ and } (2^m-1, \underbrace{2,2,\ldots,2}_{m}) \tag{65}$$

form an irreducible pair of partitions of 2^m+2m-1. More generally, for any integers $a\geq 2$ and $m\geq 1$ the partitions

$$(a^m, \underbrace{a-1,a-1,\ldots,a-1}_{m}, \underbrace{1,1,\ldots,1}_{m-1}) \text{ and } (a^m-1, \underbrace{a,a,\ldots,a}_{m}) \tag{66}$$

form an irreducible pair of partitions of a^m+am-1.

The pair

$$(6,1,1) \text{ and } (5,3) \tag{67}$$

can be generalized to irreducible pairs

$$(j!,\ \underbrace{1,\ldots,1}_{(j-1)}) \text{ and } (j!-1,\ j) \tag{68}$$

of partitions of $(j!+j-1)$ for $j\geq 3$.

From any two irreducible pairs we can get a third one by combining them in a natural way. For instance, combining a copies of (67) with b copies of (64) gives the pair (70) which is used in the proof below.

The above examples show that $i(n)$ is positive infinitely often. Indeed we have:

Theorem 13 $i(n) \geq \frac{n}{56} - 1$.

PROOF. For each pair of non-negative integers a and b satisfying

$$8a + 7b = n\,, \tag{69}$$

we see that

$$(\underbrace{6,\ldots,6}_{a},\underbrace{4,\ldots,4}_{b},\underbrace{1,\ldots,1}_{2a+3b}) \text{ and } (\underbrace{5,\ldots,5}_{a},\underbrace{3,\ldots,3}_{a+b},\underbrace{2,\ldots,2}_{2b}) \tag{70}$$

forms a new irreducible pair of partitions of n. Consequently $i(n)$ is at least as large as the number of non-negative solutions of the linear Diophantine equation (69).

Now the segment of the line $8a + 7b = n$ in the first quadrant is of length $n\sqrt{113}\big/56$. Furthermore from the full solution of the linear Diophantine equation we note that the integral solutions of (3.7) are points on this spaced a distance $\sqrt{113}$ apart. Hence in the first quadrant there must be at least

$$\left\lfloor \frac{n\sqrt{113}/56}{\sqrt{113}} \right\rfloor = \left\lfloor \frac{n}{56} \right\rfloor > \frac{n}{56} - 1 \tag{71}$$

such points. Therefore

$$i(n) > \frac{n}{56} - 1\,. \tag{72}$$

□

Theorem 13 shows that $i(n) > 0$ for all $n \geq 56$. Direct computation shows that $i(n) > 0$ for all $n > 7$ with the exception of $n = 9, 11$ and 12.

8 Further Problems

Clearly we have only scratched the surface concerning the order of magnitude of $M_k(n)$, $M(n)$ and $i(n)$. We have computed tables of the functions, and based on that evidence we make the following conjectures.

Conjecture 14 *$M(n) \geq p^*(n)$ for $n \geq 0$, where $p^*(n)$ is the total number of partitions of n into parts that are either ≤ 6 or multiples of 3 or both.*

n	$p_{\mathbb{P}}(n)$	$L_{[4]}(n)$	$p^*(n)$	$M(n)$	$p^{\#}(n)$	$U_{\mathbb{N}^+,[4]}(n)$	$p(n)$
0	1	1	1	1	1	1	1
10	5	30	36	36	39	39	42
20	26	232	357	366	445	526	627
30	98	1102	2064	2131	2875	4349	5604
40	302	4020	8853	9292	13549	27195	37338
50	819	12405	31639	33799	52321	140965	204226
60	2018	34016	99245	107726	175426	636536	966467
70	4624	85333	281307	310226	527909	2582469	4087968

Conjecture 15 *There exists a positive constant C so that*

$$\lim_{n\to\infty} \frac{\log M(n)}{\sqrt{n}} = C. \tag{73}$$

If C exists and if Conjecture 14 is true, then [11, Th. 6.2, p.89]

$$\frac{\pi}{3}\sqrt{2} \le C \le \pi\sqrt{\frac{2}{3}}\,. \tag{74}$$

Conjecture 16 *Let C_k be the infimum of the quotients $M_k(n)/p_k(n)$ where n ranges over the natural numbers. Then $C_k > 0$ for all natural numbers k. Moreover, C_k is a strictly decreasing function of k for $k \ge 3$ and $C_k \to 0$ as $k \to \infty$.*

Conjecture 17 *$M(n) \le p^{\#}(n)$ for $n \ge 0$ where $p^{\#}(n)$ is the total number of partitions of n into parts that are either ≤ 7 or multiples of 3 or both.*

Conjecture 17 together with Conjecture 14 allows us to replace Conjecture 15 with

Conjecture 18

$$\lim_{n\to\infty} \frac{\log M(n)}{\sqrt{n}} = \frac{\pi}{3}\sqrt{2}\,. \tag{75}$$

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