SIMPLIFYING SUMS IN $\Pi\Sigma^*$-EXTENSIONS

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Abstract. We present algorithms which split a rational expression in terms of indefinite nested sums and products into a summable part which can be summed by telescoping and into a non-summable part which is degree-optimal with respect to one of the most nested sums or products. If possible, our algorithms find a non-summable part where all these most nested sums and products are eliminated.

1. Introduction

Indefinite summation can be described by the following telescoping problem: *Given* f where f belongs to some domain of sequences $\mathbb{E}$, *find* $g \in \mathbb{E}$ such that

$$f(k) = g(k+1) - g(k). \tag{1}$$

Then given such a solution g, we get the closed form evaluation

$$\sum_{k=a}^{b} f(k) = g(b+1) - g(a),$$

i.e., the sum $\sum_{k=a}^{b} f(k)$ can be simplified in terms of the sequences given in $\mathbb{E}$. E.g., there are algorithms for the rational case, see [Abr71], for hypergeometric terms, see [Gos78, PS95], for q-hypergeometric terms, see [Koo93, PR97], or more generally, for $\Pi\Sigma^*$-terms, see [Kar81, Sch04b]. Here arbitrarily nested indefinite sums and products are represented in the difference field setting of $\Pi\Sigma^*$-fields. Typical examples of such sums and products are d'Alembertian solutions [AP94, Sch01], a subclass of Liouvillian solutions [HS99] of linear recurrences.

We consider the following refined problem: *Given* $f \in \mathbb{E}$, *find* $(f', g) \in \mathbb{E}^2$ such that

$$f(k) = g(k+1) - g(k) + f'(k) \tag{2}$$

where f' is as "small" as possible. Since we consider $f' = 0$ as the "smallest" possible choice, f' is also called the non-summable part. Then given such a solution (f', g), we obtain

$$\sum_{k=a}^{b} f(k) = g(b+1) - g(a) + \sum_{k=a}^{b} f'(k),$$

i.e., the sum $\sum_{k=a}^{b} f(k)$ can be simplified in terms of the sequences given in $\mathbb{E}$ and by the sum $\sum_{k=a}^{b} f'(k)$. In a nutshell, one tries to solve the classical telescoping problem in $\mathbb{E}$ ($f' = 0$), and if this is not possible, tries to keep the non-summable part $f'(k)$ as small as possible.

For the rational case this refined telescoping approach has been considered in [Abr75]; here the minimality of f' is defined by the degree of the denominator polynomial. Theoretical inside and different algorithms have been derived in [Pau95].

For the $\Pi\Sigma^*$-field case the following variation has been considered in [Sch04c, Sch05b]: find a summand $f'(k)$ where the depth of the nested sums and products is optimal.

Based on the algorithmic theory given in [Kar81] we shall develop a framework which combines both versions: choose one of the most nested sums or products in $f(k)$ and find $f'(k)$ such that the degrees of its polynomial and fractional part are optimal w.r.t. to the

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selected sum or product. Applying this strategy recursively, we can eliminate, if possible, all such most nested sums and products in $f'(k)$. Typical examples are

$$\sum_{k=2}^{n} \frac{-kH_k^5+H_k^4-kH_k+2}{H_k-kH_k^2} = (n+1)H_n^3 - (2n+1)\big(\tfrac{3}{2}H_n^2 - 3H_n + \tfrac{3}{2}\big) + \tfrac{1}{H_n} + \sum_{k=2}^{n} \tfrac{k^2+H_k}{k^2H_k}, \quad (3)$$

$$\sum_{k=0}^{n}\left(\sum_{i=0}^{k}\binom{x}{i}\right)^2 = (x-n)\binom{x}{n}\sum_{i=0}^{n}\binom{x}{i} - \frac{x-2n-2}{2}\left(\sum_{i=0}^{n}\binom{x}{i}\right)^2 - \frac{x}{2}\sum_{k=0}^{n}\binom{x}{k}^2, \quad (4)$$

$$\sum_{k=1}^{n-1} H_k^2 H_k^{(2)} = -\tfrac{H_n^3}{3} + \left(nH_n^{(2)}+1\right)H_n^2 + (2n+1)\left(H_n^{(2)} - H_n^{(2)}H_n\right) - 2H_n + \tfrac{1}{3}H_n^{(3)} \quad (5)$$

where $H_k = \sum_{j=1}^{k} 1/j$ denote the harmonic numbers and $H_k^{(r)} = \sum_{j=1}^{k} 1/j^r$, $r > 1$, are its generalized versions. In (3) we simplify the sum on the left-hand side by finding $f'(k) = \frac{k^2+H_k}{k^2H_k}$ where the degrees w.r.t H_k are optimal. Moreover, in (4), which is a generalized version from [AP99, Page 9], we compute $f'(k) = \binom{n}{k}^2$ which is free of $\sum_{i=0}^{k}\binom{n}{i}$. In (5) we simplify the sum on the left-hand side by finding $f'(k) = \frac{1}{k^3}$ which is free of H_k and $H_k^{(2)}$.

The algorithms under consideration are illustrated by various concrete examples; some of them pop up in [Zha99, PS03, Sch04b, DPSW05]. All these ideas are implemented in the summation package Sigma [Sch04b].

The general structure is as follows. In Section 2 we formulate the refined telescoping problem *RT* in difference fields and supplement it by examples. In Section 3 we split problem *RT* in the two subproblems *PP* and *RP* which we solve in Sections 4 and 5. Using these results, we get an algorithm which can eliminate, if possible, all the extensions which are most nested, see Section 6. In Section 7 we show how problem (2) is related to the theory of $\Pi\Sigma^*$-extensions.

2. The problem in $\Pi\Sigma^*$-extensions

We describe the domain of sequences $\mathbb{E}$ in problem (2) by *difference fields*, i.e., by a field $\mathbb{E}$ and a field automorphism $\sigma : \mathbb{E} \to \mathbb{E}$; in short we write $(\mathbb{E}, \sigma)$. All fields in this paper are understood as having characteristic 0. The *constant field* of $(\mathbb{E}, \sigma)$ is defined by $\mathbb{K} := \{c \in \mathbb{E} \,|\, \sigma(c) = c\}$. It is easy to see that $\mathbb{K}$ is a subfield of $\mathbb{E}$; this implies that $\mathbb{Q} \subseteq \mathbb{K}$. Then problem (2) can be formulated as follows. Given $f \in \mathbb{E}$, find $(f', g) \in \mathbb{E}^2$ such that

$$\sigma(g) - g + f' = f \quad (6)$$

where f' is as simple as possible. We call $(f', g) \in \mathbb{E}$ a *Σ-pair* for f if it fulfills (6).

Subsequently, we restrict to difference fields which can be obtained by certain *difference field extensions* called $\Pi\Sigma^*$-extensions. A difference field $(\mathbb{E}, \sigma)$ is called a *difference field extension* of $(\mathbb{F}, \sigma')$, if $\mathbb{F}$ is a sub-field of $\mathbb{E}$ and $\sigma|_{\mathbb{F}} = \sigma'$ (since σ and σ' agree on $\mathbb{F}$, we do not distinguish them anymore). A difference field extension $(\mathbb{E}, \sigma)$ of $(\mathbb{F}, \sigma)$ is called a *$\Pi\Sigma^*$-extension*, if $\mathbb{E} = \mathbb{F}(t)$ is a rational function field extension, the field automorphism $\sigma : \mathbb{F} \to \mathbb{F}$ is extended to $\sigma : \mathbb{F}(t) \to \mathbb{F}(t)$ by $\sigma(t) = a\,t$ or $\sigma(t) = t + a$ for some $a \in \mathbb{F}^*$, and the constant field remains unchanged, i.e., $\mathrm{const}_\sigma\mathbb{F}(t) = \mathrm{const}_\sigma\mathbb{F} = \mathbb{K}$. If $\sigma(t) = a\,t$, we call the extension also a *Π-extension*; if $\sigma(t) = t + a$, we call it a *Σ^*-extension*.

Remark 2.1. Note that there are decision procedures which enable one to test if a given extension is a $\Pi\Sigma^*$-extension. For Σ^*-extensions we refer to Section 7. For general Π-extensions we refer to [Kar81]. Here we mention only that a hypergeometric term, like $\binom{n}{k}$ or $k!$, can be always represented by a Π-extension; only objects like $(-1)^k$ cannot be handled, see [Sch05c].

For such a $\Pi\Sigma^*$-extension $(\mathbb{F}(t), \sigma)$ of $(\mathbb{F}, \sigma)$ we are interested in the following problem:

RT: Refined Telescoping with optimal degree

Given $f \in \mathbb{F}(t)$; **find** a Σ-pair (f', g) for f where among the possible f' with

$$f' = p + \frac{q}{d} \quad \text{where} \quad p, q, d \in \mathbb{F}[t], \text{ and } \deg(q) < \deg(d) \tag{7}$$

the degree of d and the degree of p are minimal; we set $\deg(0) = -\infty$.

Remark. **(1)** The constraint that $\deg(p)$ is minimal does not restrict the constraint that $\deg(d)$ is minimal and vice versa. For further explanations we refer to Section 3.
(2) In [Sch05c] we consider the analogue problem for products: given $f \in \mathbb{F}(t)$, find $(f', g) \in \mathbb{F}(t)^2$ with $\frac{\sigma(g)}{g} f' = f$ where the degrees of the numerator and denominator of f' are minimal.

In this article we develop algorithms for problem *RT* where $\mathbb{F}$ is a $\Pi\Sigma^*$-field. This means that we start with the constant field $\mathbb{K}$ and adjoin step by step either a Π- or a Σ^*-extension t_i on top. Following [Kar81] we call such a tower of $\Pi\Sigma^*$-extensions $\mathbb{K}(t_1)\dots(t_e)$ a $\Pi\Sigma^*$*-field*. Usually, one chooses for t in *RT* a sum or product which is most nested.

We illustrate problem *RT* by various concrete examples. In Examples 2.1–2.5 we focus on the problem to obtain non-summable parts where the degree of p is reduced. In Examples 2.6–2.9 we compute non-summable parts where the degree of d reduced. In Example 2.10 (see identity (5)) we compute a non-summable part where the degrees in p and d are optimal.

Example 2.1. Consider the rational case, i.e., take the difference field $(\mathbb{K}(k), \sigma)$ with $\sigma(k) = k+1$ and $\text{const}_\sigma\mathbb{K}(k) = \mathbb{K}$; note that this is a Σ^*-extension of $(\mathbb{K}(k), \sigma)$. Then for any $f \in \mathbb{K}[k]$ we can compute a $g \in \mathbb{K}[k]$ with $\sigma(g) - g = f$; see e.g. [GKP94, (6.10),(6.11),(2.45)]. For the q-rational case we have the same result: Take the constant field $\mathbb{K}(q)$ with a parameter q and consider the $\Pi\Sigma^*$-extension $(\mathbb{K}(q)(P), \sigma)$ with $\sigma(P) = q\,P$. Then we can find for any $f \in \mathbb{K}(q)[P]$ a $g \in \mathbb{K}(q)[P]$ with $\sigma(g) - g = f$; note that $(0, P^i/(q-1))$ is a Σ-pair for P^i.

Example 2.2. Given $\sum_{k=1}^n H_k^4$, we derive the identity

$$\sum_{k=1}^n H_k^4 = H_n^2\big((n+1)H_n^2 - 2(2n+1)H_n + 12n\big) + \sum_{k=1}^n \frac{12k^2-8k-1-2kH_k\big(12k^2-6k-1\big)}{k^3} \tag{8}$$

as follows. Take the difference field $(\mathbb{Q}(k), \sigma)$ with $\sigma(k) = k+1$, and extend it with the Σ^*-extension $(\mathbb{Q}(k)(H), \sigma)$ where $\sigma(H) = H + \frac{1}{k+1}$. Note that the shift of H_k in k is reflected by the automorphism σ acting on H. Then we compute the Σ-pair $(f', g) = \big(\frac{12k^2-8k-1-2kH(12k^2-6k-1)}{k^3}, \frac{(Hk-1)^2((H^2-4H+12)k^2-8k-1)}{k^3}\big)$ for $f = H^4$; for further details see Example 4.2. This delivers (2) with $f(k) = H_k^4$, $f'(k) = \frac{12k^2-8k-1-2kH_k(12k^2-6k-1)}{k^3}$ and $g(k) = \frac{(H_kk-1)^2((H_k^2-4H_k+12)k^2-8k-1)}{k^3}$. Summing (2) over k gives (8). Note that $\sum_{k=1}^n f'(k) = 12\sum_{k=1}^n \frac{H_k}{k} + 2\sum_{k=1}^n \frac{H_k}{k^2} - 24\sum_{k=1}^n H_k + 12H_n - 8H_n^{(2)} - H_n^{(3)}$. With the identities $\sum_{k=1}^n H_k = H_n(n+1) - n$ and $\sum_{k=1}^n \frac{H_k}{k} = H_n^2 + H_n^{(2)}$, which we can also find with our machinery, we get

$$\sum_{k=1}^n H_k^4 = (n+1)H_n^4 - (2n+1)\big(2H_n^3 - 6H_n^2 + 12H_n\big) + 24n - H_n^{(3)} - 2H_n^{(2)} + 2\sum_{k=1}^n \frac{H_k}{k^2}. \tag{9}$$

Example 2.3. Given $\sum_{k=1}^n H_k^3$, we take $(\mathbb{Q}(k)(H), \sigma)$ from Example 2.2 and compute the Σ-pair $(f', g) = \big(-\frac{12k^2-6k-1}{2k^2}, \frac{(Hk-1)(2H^2k^2-6Hk^2+12k^2-Hk-6k-1)}{2k^2}\big)$ for H^3; see Example 4.6. Summing the result over k and using $\sum_{k=1}^n -\frac{12k^2-6k-1}{2k^2} = \frac{1}{2}(-12n + 6H_n + H_n^{(2)})$ gives

$$\sum_{k=1}^n H_k^3 = \frac{1}{2}\Big(2(n+1)H_n^3 - 3(2n+1)H_n^2 + 6(2n+1)H_n - 12n + H_n^{(2)}\Big). \tag{10}$$

Example 2.4. We find (4), a generalization given in [AP99, Page 9], as follows. Take the difference field $(\mathbb{Q}(x)(k),\sigma)$ with constant field $\mathbb{Q}(x)$ and $\sigma(k)=k+1$ and extend it with the Π-extension $(\mathbb{Q}(x)(k)(B),\sigma)$ with $\sigma(B)=\frac{x-k}{k+1}B$. Afterwards, extend it with the Σ^*-extension $(\mathbb{Q}(x)(k)(B)(S),\sigma)$ with $\sigma(S)=S+\sigma(B)$; note that the shift of $\binom{x}{k}$ and $\sum_{i=0}^{k}\binom{x}{i}$ in k is reflected by the automorphism σ acting on B and S. Then we compute the Σ-pair $(f',g)=(-\frac{x}{2}B^2,-\frac{1}{2}(B-S)(xB+(2k-x)S))$ for $f=S^2$. This gives $f'(k)=-\frac{x}{2}\binom{x}{k}^2$ and $g(k)=-\frac{1}{2}(\binom{x}{k}-\sum_{i=0}^{k}\binom{x}{i})(x\binom{x}{k}+(2k-x)\sum_{i=0}^{k}\binom{x}{i})$ for (2). Summing (2) over k gives (4). With the same mechanism we find the identities

$$\sum_{k=0}^{n}(-1)^k\Big(\sum_{j=0}^{k}\binom{x}{j}\Big)^2=(-1)^n\Big(2(x-n)\binom{x}{n}\sum_{j=0}^{n}\binom{x}{j}+x\Big(\sum_{j=0}^{n}\binom{x}{j}\Big)^2\Big)-\sum_{k=0}^{n}(x-2k)\binom{x}{k}^2(-1)^k,$$

$$\sum_{k=1}^{n}\frac{p(k)}{(1-3k)^2(2-3k)^2(1-2k)^2k^2}\sum_{j=1}^{k}\tfrac{108j^3-153j^2+68j-10}{j(2j-1)(3j-2)(3j-1)}=2\Big(\sum_{j=1}^{n}\tfrac{108j^3-153j^2+68j-10}{j(2j-1)(3j-2)(3j-1)}\Big)^2+$$
$$-\sum_{k=1}^{n}\tfrac{289656k^7-842886k^6+1001583k^5-622368k^4+213418k^3-38207k^2+2720k+20}{k^2(2k-1)^2(3k-2)^2(3k-1)^2}$$

where $p(k)=\big(-289656k^7+819558k^6-935487k^5+546174k^4-167482k^3+22839k^2+4\left(1944k^6-5670k^5+6759k^4-4221k^3+1460k^2-266k+20\right)k-220\big)$. The first identity is a generalization given in [Zha99]. Note that in this identity $(-1)^k$ occurs which cannot be expressed in $\Pi\Sigma^*$-extensions; see Remark 2.1 – nevertheless the machinery under consideration can be adapted for this case, see Section 8. The second identity has been used in [DPSW05].

Example 2.5. For (5) we take $(\mathbb{Q}(k)(H^{(2)})(H),\sigma)$ with $\sigma(k)=k+1$, $\sigma(H^{(2)})=H^{(2)}+\frac{1}{(k+1)^2}$ and $\sigma(H)=H+\frac{1}{k+1}$, and compute the Σ-pair $(-\frac{6k^2-3k-1}{3k^3},-\frac{H^3}{3}+(H^{(2)}k+1)H^2-H^{(2)}(2k+1)H+\frac{6H^{(2)}k^4-6k^2+3k+1}{3k^3})$ for $f=H^2H^{(2)}$; see Example 6.1. This gives (5).

Example 2.6. In [Sch04b, Page 381] we needed the simplification

$$\sum_{k=1}^{n}\frac{k+1}{k(k+2)}=-\frac{n(3n+5)}{4(n+1)(n+2)}+\sum_{k=1}^{n}\frac{1}{k}. \tag{11}$$

Given $(\mathbb{Q}(k),\sigma)$ with $\sigma(k)=k+1$, we can use any of the algorithms from [Abr75, Pau95] to compute the Σ-pair $(f',g)=(\frac{1}{k},\frac{2k+1}{2k(k+1)})$ for $f=\frac{k+1}{k(k+2)}$; in Example 5.5 we will apply our generalized method. Then summing (2) over k yields (11).

Example 2.7. In order to find the identity $\sum_{j=0}^{n}jH_j\binom{n}{j}=-\frac{1}{2}+2^{n-1}(1+nH_n-n\sum_{j=1}^{n}\frac{1}{j2^j}$ in [PS03, Page 370] we needed the identity

$$\sum_{k=2}^{n}\frac{1}{k(k-1)2^k}=-\frac{1}{n2^{n+1}}+\frac{1}{4}-\frac{1}{2}\sum_{k=2}^{n}\frac{1}{k2^k}. \tag{12}$$

Extend $(\mathbb{Q}(k),\sigma)$ with the Π-extension $(\mathbb{Q}(k)(P),\sigma)$ where $\sigma(P)=2\,P$, and compute the Σ-pair $(-\frac{1}{2kP},\frac{-1}{(k-1)P})$ for $\frac{1}{(k-1)kP}$; see Example 5.6. This produces (2) with $f(k)=\frac{1}{(k-1)kP}$, $f'(k)=-\frac{1}{2k2^k}$ and $g(k)=\frac{-1}{(k-1)2^k}$. Summing (2) over k gives (12).

Example 2.8. We find the right-hand side of

$$\sum_{k=1}^{n}\frac{k!\left(k^2+k+k!\left(k(k+1)^2+k!\left(k(k+1)^2+\left(2k^2-1\right)k!-3\right)-2\right)+1\right)+1}{(k!)^3(k!+1)((k+1)k!+1)}$$
$$=\frac{3(n+1)(n!)^3+(3-2n)(n!)^2-2(n+2)n!-2}{2(n!)^2((n+1)n!+1)}+\sum_{k=1}^{n}\frac{k(k!)^3+k!+1}{(k!)^3(k!+1)} \tag{13}$$

as follows. Take the Π-extension $(\mathbb{Q}(k)(F),\sigma)$ with $\sigma(F)=(k+1)F$ and represent the summand with $f=\frac{F\big(k^2+k+F\big(k(k+1)^2+F\big(k(k+1)^2+F\big(2k^2-1\big)-3\big)-2\big)+1\big)+1}{F^3(F+1)(kF+F+1)}$. Then we compute the Σ-pair $(f',g)=\{\frac{kF^3+F+1}{F^3(F+1)},-\frac{kF^2-F^2+k^2F+kF+k^2}{F^2(F+1)}\}$ for f; the details can be found in Examples 5.1, 5.2, 5.3, 5.4, and 5.8. Reinterpreting (f',g) in terms of $k!$ gives the closed form (13).

Example 2.9. Starting with the left-hand side of

$$\sum_{k=2}^{n}\frac{(k+1)\left(k(k+1)^2(k+2)H_k^3+k\left(3k^2+8k+5\right)H_k^2-(k+2)H_k-k-2\right)}{H_k\left(k(k+1)^2(k+2)H_k^3+2\left(k^3+2k^2-1\right)H_k^2-\left(k^2+5k+5\right)H_k-2k-3\right)}$$
$$=\frac{-6(n+1)(n+2)H_n^2-6(2n+3)H_n+11(n+1)(n+2)}{11H_n\left(2n+(n+1)(n+2)H_n+3\right)}+\sum_{k=2}^{n}\frac{k(k+1)}{kH_k-1}\quad(14)$$

we take the difference field $(\mathbb{Q}(k)(H),\sigma)$ from Example 2.2 and compute the Σ-pair $(f',g)=\left(\frac{k(k+1)}{Hk-1},\frac{k(k+1)}{(Hk-1)(kH+H+1)}\right)$ for $f=\frac{(k+1)\left(k(k+1)^2(k+2)H^3+k\left(3k^2+8k+5\right)H^2-(k+2)H-k-2\right)}{H(k(k+1)^2(k+2)H^3+2(k^3+2k^2-1)H^2-(k^2+5k+5)H-2k-3)}$; see Example 5.9. This gives the right hand side of (14).

Example 2.10. We derive identity (3) as follows. Take $(\mathbb{Q}(k)(H),\sigma)$ from Example 2.2 and compute the Σ-pair $(f',g)=\big(-\frac{12Hk^2-2k^2-6Hk-H}{2Hk^2},kH^3-\frac{3}{2}(2k+1)H^2+6kH+\frac{3}{k}+\frac{k}{Hk-1}+\frac{1}{2k^2}-6\big)$ for $f=\frac{-kH^5+H^4-kH+2}{H-H^2k}$; see Example 3.1. This produces (3).

The following simple facts are heavily used throughout this article.

Lemma 2.1. *Let $(\mathbb{F},\sigma)$ be a difference field.*

(1) If $(f_i',g_i)\in\mathbb{F}^2$ are Σ-pairs for $f_i\in\mathbb{F}$, $(f_0'+f_1',g_0+g_1)$ is a Σ-pair for f_0+f_2.
(2) If $(f',g)\in\mathbb{F}^2$ is a Σ-pair for f and $(\phi,\gamma)\in\mathbb{F}^2$ is a Σ-pair for f', $(\phi,\gamma+g)$ is one for f.
(3) Let $i\in\mathbb{Z}$ and $f\in\mathbb{F}$. Then (f,g) is a Σ-pair for $\sigma^i(f)$ where $g=\sum_{j=0}^{i-1}\sigma^j(f)$ if $i\geq 0$, and $g=-\sum_{j=0}^{-i-1}\sigma^{j+i}(f)$ if $i<0$.

Proof. (1) and (2) are obvious. Take f,f',g from (3). If $i\geq 0$, $\sigma(g)-g=\sum_{j=1}^{i}\sigma^j(h)-\sum_{j=0}^{i-1}\sigma^j(h)=\sigma^i(h)-h$. If $i<0$, $\sigma(g)-g=\sum_{j=0}^{-i-1}\sigma^{j+i}(f)-\sum_{j=1}^{-i}\sigma^{j+i}(f)=\sigma^i(f)-f$. □

3. Problem reduction

Subsequently, let $(\mathbb{F}(t),\sigma)$ be a ΠΣ*-extension of $(\mathbb{F},\sigma)$, $\mathbb{K}=\text{const}_\sigma\mathbb{F}$, and $f\in\mathbb{F}(t)$. By polynomial division we get $f=f_0+f_1$ with $f_0\in\mathbb{F}[t]$ and $f_1\in\mathbb{F}(t)_{(r)}$ where

$$\mathbb{F}(t)_{(r)}:=\{\frac{a}{b}\,|\,a,b\in\mathbb{F}[t]\text{ and }\deg(a)<\deg(b)\}.$$

In short we write $f=f_0+f_1\in\mathbb{F}[t]\oplus\mathbb{F}(t)_{(r)}$ and say that f_0 is the *polynomial part* and f_1 is the *fractional part*. The following lemma tells us how we can continue.

Lemma 3.1. *Let $(\mathbb{F}(t),\sigma)$ be a ΠΣ*-extension of $(\mathbb{F},\sigma)$. Let $f,f',g\in\mathbb{F}(t)$ and write $f=f_0+f_1\in\mathbb{F}[t]\oplus\mathbb{F}(t)_{(r)}$, $f'=f_0'+f_1'\in\mathbb{F}[t]\oplus\mathbb{F}(t)_{(r)}$ and $g=g_0+g_1\in\mathbb{F}[t]\oplus\mathbb{F}(t)_{(r)}$. Then (f',g) is a Σ-pair for f iff (f_0',g_0) is a Σ-pair for f_0 and (f_1',g_1) is a Σ-pair for f_1.*

Proof. For the direction from right to left follows by Lemma 2.1.1. Suppose that (f',g) is a Σ-pair for f. Then $\big[\sigma(g_0)-g_0+f_0'-f_0\big]+\big[\sigma(g_1)-g_1+f_1'-f_1\big]=0$. Since $\sigma(g_0)-g_0+f_0'-f_0\in\mathbb{F}[t]$, $\sigma(g_1)-g_1+f_1'-f_1\in\mathbb{F}(t)_{(r)}$ and $\mathbb{F}(t)=\mathbb{F}[t]\oplus\mathbb{F}(t)_{(r)}$ is a direct sum ($\mathbb{F}[t],\mathbb{F}(t)_{(r)}$ are considered as subspaces of $\mathbb{F}(t)$ over $\mathbb{F}$), we have $\sigma(g_i)-g_i+f_i'-f_i=0$ for $i\in\{0,1\}$. □

This motivates us to consider the following problems separately.

PP: Polynomial Problem

Given $f \in \mathbb{F}[t]$; **find** from all the Σ-pairs $(f', g) \in \mathbb{F}[t]^2$ for f a pair where $\deg(f')$ is minimal.

RP: Rational Problem

Given $f \in \mathbb{F}(t)_{(r)}$; **find** from all the Σ-pairs $(f', g) \in \mathbb{F}(t)_{(r)}^2$ for f a pair where the degree of the denominator of f' is minimal.

This explains, why we can impose simultaneously optimal degrees of p and d in problem *RT*.

Example 3.1. (Cont. Example 2.10) Given f from Example 2.10 we compute the polynomial part $f_0 = H^3$ and the fractional part $f_1 = \frac{Hk-2}{H(Hk-1)}$ with $f = f_0 + f_1$. Denote with (f_0', g_0) the computed Σ-pair from Example 2.3. Next, we compute a solution of problem *RP*, namely the Σ-pair $(f_1', g_1) = (\frac{1}{H}, \frac{k}{kH-1})$ for f_1, see Example 5.7 (as byproduct we get $\sum_{k=2}^{n} \frac{kH_k-2}{H_k(kH_k-1)} = \frac{1}{H_n} - 1 + \sum_{k=2}^{n} \frac{1}{H_k}$). Combining the Σ-pairs, see Lemma 2.1.1, we get the Σ-pair $(f', g) = (f_0 + f_1, g_0 + g_1)$ for f which we used in Example 2.10.

Based on the previous considerations we propose the following algorithm.

Algorithm 3.1. `RefinedTelescoping`$((\mathbb{F}(t), \sigma), f)$

Input: A $\Pi\Sigma^*$-extension $(\mathbb{F}(t), \sigma)$ of $(\mathbb{F}, \sigma)$ and algorithms for *PP* and *RP*; $f \in \mathbb{F}[t]$.
Output: A solution of problem *RT*.

(1) Split $f = f_0 + f_1$ with $f_0 \in \mathbb{F}[t]$ and $f_1 \in \mathbb{F}(t)_{(r)}$ by polynomial division.
(2) Let $(f_0', g_0) \in \mathbb{F}[t]^2$ be a solution of problem *PP* for f_0.
(3) Let $(f_1', g_1) \in \mathbb{F}(t)_{(r)}^2$ be a solution of problem *RP* for f_1.
(4) RETURN $(f_0' + f_1, g_0 + g_1)$.

In Sections 4 and 5 we will solve problems *PP* and *RP* under the assumption that the two subproblems *PLDE* and *SEF* can be solved. Namely, we suppose that we can deal with

Problem *PLDE*: Solving First order-Parameter Linear Difference Equations

Given $a_1, a_2 \in \mathbb{F}^*$ and $f, \phi \in \mathbb{F}$; **find** $g \in \mathbb{F}$ and $c \in \mathbb{K}$ with $a_1\,\sigma(g) + a_2\,g = f + c\,\phi$.

Moreover, we must be able to factorize a polynomial $f \in \mathbb{F}[t]$ into its irreducible factors. Furthermore, we must be able to solve problem *SEF*; here we need the following definition: we say that $f, g \in \mathbb{F}[t]^*$ are *σ-prime*, in short, $h \perp_\sigma f$, if $\gcd(h, \sigma^k(f)) = 1$ for all $k \in \mathbb{Z}$.

Problem *SEF*: Separate Equivalent Factors

Given $q \in \mathbb{F}[t]^*$ and an irreducible $h \in \mathbb{F}[t]$; **find** $m_i \geq 0$ and $c \in \mathbb{F}[t]$ with

$$q = c \prod_i \sigma^i(h^{m_i}), \quad c \perp_\sigma h. \tag{15}$$

The following remarks are in place: If $f \in \mathbb{F}[t]^*$ is irreducible and $m \in \mathbb{Z}$, then $\sigma^m(f) \in \mathbb{F}[t]$ is irreducible. Hence, on the set of all irreducible polynomials from $\mathbb{F}[t]$ we get an equivalence relation $f \sim g$ (the shift-equivalence) iff $f \not\perp_\sigma g$. Thus, solving problem *SEF* means to separate the irreducible polynomials in q into the factors which are all shift-equivalent with h, i.e., $\prod_i \sigma^i(h^{m_i})$, and into the factors which are not shift-equivalent to h, i.e., c. Expanding this refined factorization on c, i.e., collecting it into shift-equivalent classes, gives Karr's σ-factorization introduced in [Kar81].

Summarizing, we will obtain the following results.

Corollary 3.1. *Let $(\mathbb{F}(t), \sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{F}, \sigma)$ where one can solve problems PLDE and SEF; let $f \in \mathbb{F}[t]$. Then Algorithm 3.1 is applicable and the output (f', g) is a solution of RT. Moreover, we have:* **(1)** *If there is a Σ-pair $(\phi', \gamma) \in \mathbb{F} \times \mathbb{F}(t)$ for f, then $f' \in \mathbb{F}$.*
(2) *If there is a $\gamma \in \mathbb{F}(t)$ with $\sigma(\gamma) - \gamma = f$, then $f' = 0$.*

To this end, we emphasize that there are algorithms for problems *PLDE* and *SEF* if $\mathbb{F}$ is a ΠΣ*-field. For problem *PLDE* see [Kar81, Section 3]; a simplified version is given in [Sch05d, Thm. 4.7] which uses results from [Bro00, Sch04a, Sch05a]. For problem *SEF* see [Kar81, Thm. 9]. Hence Algorithm 3.1 can be applied for any ΠΣ*-field $(\mathbb{F}(t), \sigma)$.

4. The Polynomial Problem

We reduce problem *PP* to problem *PLDE*. Here we consider two cases.

4.1. The Π-extension case.

The solution of problem *PP* is immediate with Lemma 4.1; the proof follows by coefficient comparison.

Lemma 4.1. *Let* $(\mathbb{F}(t), \sigma)$ *be a* Π*-extension of* $(\mathbb{F}, \sigma)$ *with* $\sigma(t) = \alpha\, t$*, and suppose that* $f, f', g \in \mathbb{F}[t]$ *with* $f = \sum_{i=0}^{n} f_i t^i$*,* $f' = \sum_{i=0}^{n} f'_i t^i$*, and* $g = \sum_{i=0}^{n} g_i t^i \in \mathbb{F}[t]$*. Then* (f', g) *is a* Σ*-pair for* f *iff* $(f'_i t^i, g_i t^i)$ *are* Σ*-pairs for* $f_i t^i$ *for all* $0 \leq i \leq n$*.*

Start with the Σ-pair (f', g) given by $f' := f$ and $g := 0$. Then we can eliminate a monomial $f_i t^r \neq 0$ in f' iff there is a $g_r \in \mathbb{F}$ with $\sigma(g_r t^r) - g_r t^r = f_r t^r$ or equivalently if

$$\alpha^r \sigma(g_r) - g_r = f_r. \tag{16}$$

Consequently, if we find such a solution g_r with (16), we can adapt the Σ-pair (f', g) with $f' := f - f_r t^r$ and $g := g + g_r t^r$. In this way we can eliminate all terms of highest degree in f' and get a Σ-pair (f', g) where $\deg(f')$ is minimal. Summarizing, we get

Algorithm 4.1. `OptimalPolyΠExtension`$((\mathbb{F}(t), \sigma), f)$

Input: A Π-extension $(\mathbb{F}(t), \sigma)$ of $(\mathbb{F}, \sigma)$ and $f = \sum_i f_i t^i \in \mathbb{F}[t]$; an algorithm for problem *PLDE*.
Output: A solution of problem *PP*.
(1) Set $g := 0$, $f' := f$, $r := \deg(f)$. DO
(2) If $f_r \neq 0$ and if there is no g_r with (16), STOP and RETURN (f', g).
(3) Otherwise, take such a g_r and set $g := g + g_r t^r$ and $f' := f - f_r t^r$. Set $r := r - 1$.
(4) UNTIL $r = -1$.
(5) RETURN (f', g).

Remark. If one continues the DO-loop although one fails to find a g_r one removes all possible terms in f. In this case the number of non-zero terms in f' is minimal.

Example 4.1. Take $(\mathbb{Q}(k)(F), \sigma)$ from Example 2.8 and let $f = \big(F^3 + (kH+1)(kH + 2H+1)F^2 + (k^2+k+1)F\big)$. The subproblems are $(k+1)^i \sigma(g_i) - g_i = f_i$ with $f_0 = 0$, $f_1 = \big(k^2+k+1\big)$, $f_2 = (kH+1)\,(kH+2H+1)$, and $f_3 = 1$. The solutions are $g_2 = H^2$, $g_1 = k$, and $g_0 = 0$; there is no such $g_3 \in \mathbb{Q}(k)(H)$. Hence $(f', g) = (F^3, H^2F^2 + kF)$ is a Σ-pair for f which solves *PP* and is optimal w.r.t. the number of terms in F.

4.2. The Σ*-extension case.

We solve problem *PP* by refining Karr's algorithm. First we bound the degree of the possible solutions; see also [Sch05a, Cor. 6].

Lemma 4.2. [Kar81, Cor. 1] *Let* $(\mathbb{F}(t), \sigma)$ *be a* ΠΣ**-extension of* $(\mathbb{F}, \sigma)$ *and* $f \in \mathbb{F}[t]^*$*. If there is a* $g \in \mathbb{F}[t]$ *with* $\sigma(g) - g = f$*, then* $\deg(g) \leq \deg(f) + 1$*.*

Then we try to compute step by step the coefficients of the polynomial solution $g = \sum_{k=0}^{b} g_i t^i$ with $b := \deg(f) + 1$. If this fails, i.e., if there does not exist a telescoping solution, we can extract a solution of problem *PP*. The following example illustrates these ideas.

Example 4.2. (Cont. Example 2.2) Take the ΠΣ*-field $(\mathbb{Q}(k)(H), \sigma)$ with $\sigma(k) = k + 1$ and $\sigma(H) = H + \frac{1}{k+1}$. We look for a $g \in \mathbb{Q}(k)[H]$ such that

$$\sigma(g) - g = H^4; \tag{17}$$

for convenience we set $f_4 := H^4$. Since $\deg(g) \leq 5$ by Lemma 4.2, we can set up the solution as $g = \sum_{i=0}^{5} g_i H^i$ with $g_i \in \mathbb{Q}(k)$. By plugging in g into (17) we get by coefficient comparison the constraint $\sigma(g_5) - g_5 = 0$ for the leading coefficient g_5. It follows that $g_5 := c_4 \in \mathbb{Q}$ for a constant c_4 which is not determined yet. Using this information it remains to look for $c_4 \in \mathbb{Q}$ and $\sum_{i=0}^{4} g_i H^i$ such that

$$\sigma(\sum_{i=0}^{4} g_i H^i) - \sum_{i=0}^{4} g_i H^i = f_4 - c_4\,\psi_4$$

where $\psi_4 := \sigma(H^5) - H^5$, i.e., $\psi_4 = \frac{1+5(1+k)H+10(1+k)^2H^2+10(1+k)^3H^3+5(1+k)^4H^4}{(1+k)^5}$. Coefficient comparison gives the constraint $\sigma(g_4) - g_4 = 1 + c_4\,\frac{-5}{k+1}$ for g_4. The only possible solution is $c_4 = 0$ and $g_4 = k + c_3$ for a new parameter $c_3 \in \mathbb{Q}$. Thus, we have to find $g_i \in \mathbb{Q}(k)$ and $c_3 \in \mathbb{Q}$ with

$$\sigma(\sum_{i=0}^{3} g_i H^i) - \sum_{i=0}^{3} g_i H^i = f_3 - c_3\psi_3$$

where $f_3 := f_4 - c_4\,\psi_4 - (\sigma(kH^3) - kH^3) = -\frac{1+4(1+k)H+6(1+k)^2H^2+4(1+k)^3H^3}{(1+k)^3}$ and $\psi_3 := \sigma(H^3) - H^3 = \frac{1+4(1+k)H+6(1+k)^2H^2+4(1+k)^3H^3}{(1+k)^4}$. Coefficient comparison gives the constraint $\sigma(g_3) - g_3 = -4 + c_3\frac{-4}{k+1}$. The only possible solution for $g_3 \in \mathbb{Q}(k)$ and $c_3 \in \mathbb{Q}$ is $g_3 = -4k + c_2$ with a new parameter $c_2 \in \mathbb{Q}$ and $c_3 = 0$. Therefore, it remains to look for $g_i \in \mathbb{Q}(k)$ and $c_2 \in \mathbb{Q}$ such that

$$\sigma(\sum_{i=0}^{2} g_i H^i) - \sum_{i=0}^{2} g_i H^i = f_2 - c_2\psi_2$$

where $f_2 := f_3 - c_3\,\psi_3 - (\sigma(-4kH^3) + 4kH^3) = \frac{3+4k+4(2+5k+3k^2)H+6(1+k)^2(1+2k)H^2}{(1+k)^3}$ and $\psi_2 := \sigma(H^3) - H^3 = \frac{1+3(1+k)H+3(1+k)^2H^2}{(1+k)^3}$. We obtain the constraint $\sigma(g_2) - g_2 = 6\frac{1+2k}{k+1} + c_2\frac{-3}{k+1}$. The solution is $g_2 = 12k + c_1$ with $c_1 \in \mathbb{Q}$ and $c_2 = -2$. To this end, we have to look for $g_i \in \mathbb{Q}(k)$ and $c_1 \in \mathbb{Q}$ such that

$$\sigma(\sum_{i=0}^{1} g_i H^i) - \sum_{i=0}^{1} g_i H^i = f_1 - c_1\psi_1$$

where $f_1 = f_2 - c_2\,\psi_2 - (\sigma(12kH^2) - 12kH^2) = \frac{-7-20k-12k^2+(-10-46k-60k^2-24k^3)H}{(1+k)^3}$ and $\psi_1 = \sigma(H^2) - H^2 = \frac{1+2(1+k)H}{(1+k)^2}$. This time we obtain the constraint $\sigma(g_1) - g_1 = \frac{-2(5+18k+12k^2)}{(1+k)^3} + c_1\frac{-2}{(1+k)^2}$ which does not have any solution for $g_1 \in \mathbb{Q}(k)$ and $c_1 \in \mathbb{Q}$. Here Karr's algorithm stops with the answer: there is no $g \in \mathbb{Q}(k)[H]$ with (17). Note that there is the following sub-result. Define $\gamma_r := \sum_{i=r}^{5} g_i H^i$ for $1 \leq r \leq 4$ by the given $g_r \in \mathbb{Q}(k)$. Then

$$\sigma(\gamma_r) - \gamma_r = f - f_r,$$

i.e., (f_r, γ_r) is a Σ-pair for f. As it turns out (f_1, γ_1) solves problem *PP* for H^4.

In general, let $(\mathbb{F}(t), \sigma)$ be a Σ^*-extension of $(\mathbb{F}, \sigma)$ with $\sigma(t) = t + \beta$ and $\mathbb{K} := \mathrm{const}_\sigma\mathbb{F}$. Then we can solve problem *PP* for $f \in \mathbb{F}[t]$ with $s := \deg(f)$ in the following way.
We start with the trivial Σ-pair (f', g) with $f' := f$ and $g := 0$. Given (f', g), we check if f' has already the minimal degree; this will be possible by Lemma 4.3.1. If yes, we are done. If no, Lemma 4.3.2 explains how we can construct a Σ-pair $(\phi', \gamma) \in \mathbb{F}[t]^2$ for f with $\deg(\phi) < \deg(f')$. Applying this degree reduction at most s times we find a Σ-pair $(\phi', \gamma) \in \mathbb{F}[t]^2$ where $\deg(\phi')$ is minimal.

Lemma 4.3. *Let* $(\mathbb{F}(t),\sigma)$ *be a* Σ^**-extension of* $(\mathbb{F},\sigma)$ *and* $\mathbb{K}:=\text{const}_\sigma\mathbb{F}(t)$*. Let* $(f',g)\in\mathbb{F}[t]^2$ *be a* Σ*-pair for* $f\in\mathbb{F}[t]$ *with* $s:=\deg(f')$ *and define* $\psi:=\sigma(t^{s+1})-t^{s+1}$*. Then:*
(1) If there are no $w\in\mathbb{F}$ *and* $c\in\mathbb{K}$ *with*[1]

$$\sigma(w)-w=\mathit{coeff}(f',s)-c\,\mathit{coeff}(\psi,s), \tag{18}$$

then (f',g) *is a* Σ*-pair for* f *where* $\deg(f')$ *is minimal.*
(2) If there are $w\in\mathbb{F}$ *and* $c\in\mathbb{K}$ *with* (18)*, then we get the* Σ*-pair* (ϕ,γ) *for* f *with*

$$\phi:=\sigma(w\,t^s)-w\,t^s+c\,\psi-f',\qquad\qquad \gamma:=g+c\,t^{s+1}+w\,t^s \tag{19}$$

where $\deg(\psi)<\deg(f')$*.*

Proof. **(1)** Suppose there is a Σ-pair $(\phi,\gamma)\in\mathbb{F}[t]$ with $\deg(\phi)<s$. Then $\sigma(g-\gamma)-(g-\gamma)=f'-\phi$ with $\deg(f'-\phi)=s$. By Lemma 4.2 it follows that $\deg(g-\gamma)\leq s+1$. Consequently, $g-\gamma=c\,t^{s+1}+wt^s+v$ with $c\in\mathbb{K}$, $w\in\mathbb{F}$ and $v\in\mathbb{F}[t]$ with $\deg(v)<s$. Therefore

$$\sigma(w\,t^s+v)-(w\,t^s+v)=f'-\phi-c\,\psi.$$

Note that $\deg(\psi)\leq s$ (we even have equality by Lemma 4.2). By coefficient comparison of the leading coefficient we get (18).
(2) Conversely, suppose there are such $w\in\mathbb{F}$ and $c\in\mathbb{K}$ with (18). Then take $\gamma:=g+c\,t^{s+1}+w\,t^s$. We have $\phi:=f-(\sigma(\gamma)-\gamma)=f'-(\sigma(w\,t^s)-w\,t^s)-c\,\psi$ with $\deg(\phi)\leq s$. By (18), $\deg(\phi)<s$. By construction (ϕ,γ) is a Σ-pair for f. □

Corollary 4.1. *Let* $(\mathbb{F}(t),\sigma)$ *be a* Σ^**-extension of* $(\mathbb{F},\sigma)$ *and* $\mathbb{K}:=\text{const}_\sigma\mathbb{F}(t)$*. Let* $(f',g)\in\mathbb{F}[t]^2$ *be a* Σ*-pair for* $f\in\mathbb{F}[t]$ *with* $s:=\deg(f')$ *and define* $\psi:=\sigma(t^{s+1})-t^{s+1}$*. Then* (f',g) *is a solution of problem PP iff there are no* $w\in\mathbb{F}$ *and* $c\in\mathbb{K}$ *with* (18)*.*

Summarizing, we reduce problem *PP* to problem *PLDE* as follows.

Algorithm 4.2. `OptimalPolyΣExtension`$((\mathbb{F}(t),\sigma),f)$
Input: A Σ^*-extension $(\mathbb{F}(t),\sigma)$ of $(\mathbb{F},\sigma)$ with $\mathbb{K}:=\text{const}_\sigma\mathbb{F}$, $f\in\mathbb{F}[t]$; an algorithm for problem *PLDE*.
Output: A solution of problem *PP*.
(1) Set $(f',g):=(f,0)$.
(2) WHILE $f'\neq 0$ DO
(3) Define $s:=\deg(f)$ and set $\psi:=\sigma(t^{s+1})-t^{s+1}$. Decide if there are $w\in\mathbb{F}$ and $c\in\mathbb{K}$ with (18).
(4) IF not, STOP and RETURN (f',g).
(5) Otherwise, take such a w and c, and define (ϕ,γ) as in (19). Set $(f',g):=(\phi,\gamma)$.
(6) OD
(7) RETURN (f',g)

Example 4.3. (Cont. Example 3.1) With Algorithm 4.2 we compute for $f=H^3$ the Σ-pairs $(H^3,0)$, $\big(-\frac{3k^2H^2+6kH^2+3H^2+3kH+3H+1}{(k+1)^2},H^3k\big)$, $\big(\frac{6Hk^2+9Hk+3k+3H+2}{(k+1)^2},(H-3)H^2k\big)$ and $\big(-\frac{12k^2+18k+5}{2(k+1)^2},\frac{1}{2}H(2kH^2-6kH-3H+12k)\big)$. Since there are no $g\in\mathbb{Q}(k)$ and $c\in\mathbb{Q}$ with $\sigma(g)-g=-\frac{12k^2+18k+5}{2(k+1)^2}+\frac{c}{k+1}$, the last Σ-pair solves problem *PP* by Corollary 4.1.

Example 4.4. (Cont. Example 4.2) The computed Σ-pairs (f_r,γ_r) for H^4 from Example 4.2 are the (f',g) in each iteration step. By Corollary 4.1 the output (f_1,γ_1) is a solution of *PP*.

Remark 4.1. Let (ϕ,γ) be a Σ-pair for f with $s:=\deg(\phi)$ minimal. The following remarks are in place. **(1)** The coefficients of the monomials t^i with $i>s+1$ in γ are uniquely determined. Namely, take any other Σ-pair $(\phi',\gamma')\in\mathbb{F}[t]^2$ for f with $\deg(\phi')=s$. Then $\sigma(\gamma-\gamma')-(\gamma-\gamma')=\phi'-\phi$, and therefore by Lemma 4.3 it follows that $\gamma-\gamma'\leq\deg(\phi'-\phi)+1\leq s+1$. Hence all coefficients of the monomials t^i with $i>s+1$ in γ and γ' must be equal.

[1] coeff(f,s) denotes the coefficient f_s in $\sum_i f_it^i\in\mathbb{F}[t]$.

(2) From Remark 4.1.1 we get the following additional consequence. If $(\phi', \gamma') \in \mathbb{F}[t]^2$ is a Σ-pair for f with $\deg(\phi') = s$, then there is a $w \in \mathbb{F}[t]$ with $\deg(w) \leq s+1$ such that

$$\sigma(w) - w + \phi' = \phi. \tag{20}$$

Hence for all degree optimal ϕ, ϕ' we have (20) for some $w \in \mathbb{F}[t]$ with $\deg(w) \leq s+1$.

E.g., with Lemma 2.1.3 in combination with Lemma 2.1.2 we can get a rather simple transformation: we can shift the non-summable part in positive or negative direction.

Example 4.5. (Cont. Example 4.4) Take for $f = H^4$ the already computed Σ-pair $(f', g) = (-\frac{12k^2+20k+2H(12k^3+30k^2+23k+5)+7}{(k+1)^3}, H^2\left(kH^2 - 2(2k+1)H + 12k\right))$. By Lemma 2.1.3 we get the Σ-pair (f', f') for $\sigma(f')$. Hence $(\sigma^{-1}(f'), \sigma^{-1}(f'))$ is a Σ-pair for f'. With Lemma 2.1.1 we get the Σ-pair $(\sigma^{-1}(f'), \sigma^{-1}(f') + g)$ for H^4 which we used in Example 2.2.

Example 4.6. (Cont. Example 4.3) Let $(f', g) = (-\frac{12k^2+18k+5}{2(k+1)^2}, \frac{1}{2}H(2kH^2 - 6kH - 3H + 12k))$ be the Σ-pair for H^3 from Example 4.3. Like in Example 4.5 we get the Σ-pair $(\sigma^{-1}(f'), \sigma^{-1}(f') + g)$ for H^3 which we used in Example 3.1.

5. The Rational Problem

Under the assumption that we can solve *SEF* and *PLDE* we reduce problem *RP* to problem *SFP* given below. The corresponding algorithms generalize the results in [Abr75, Pau95].

To accomplish this task, we proceed as follows. Write[2] $f = \frac{p}{q} \in \mathbb{F}(t)_{(r)} \setminus \{0\}$ and let $h \in \mathbb{F}[t]$ be an irreducible factor of q. Then solve *SEF* and compute $m_i \geq 0$ and $c \in \mathbb{F}[t]^*$ with (15); note that not all m_i are zero.

Example 5.1. (Cont. Example 2.8) Given $f = \frac{p}{q}$ from Example 2.8 with $q = F^3(F+1)(kF+F+1)$, we choose $h = F$ and get $q = F^3\, c$ with $c = (F+1)(kF+F+1)$ and $c \perp_\sigma F$.

Since c and $\prod_i \sigma^i(h^{m_i})$ are σ-prime, in particular co-prime, we can compute by the extended Euclidean algorithm, see [Win96, Corollary, p. 53], polynomials $a, b, c \in \mathbb{F}[t]$ such that

$$f = \frac{p}{\prod_i \sigma^i(h^{m_i}) c} = \frac{a}{\prod_i \sigma^i(h^{m_i})} + \frac{b}{c}$$

where $\frac{a}{\prod_i \sigma^i(h^{m_i})}, \frac{b}{c} \in \mathbb{F}(t)_{(r)}$.

Example 5.2. (Cont. Example 5.1) We get $f = f_0 + f_1$ with $f_0 := \frac{kF^2 - F^2 + k^2F - F + 1}{F^3}$ and $f_1 = \frac{b}{c} := \frac{k(Fk+1)}{(F+1)(kF+F+1)}$.

Given this representation of f the following lemma tells us how to proceed.

Lemma 5.1. *Let $(\mathbb{F}(t), \sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{F}, \sigma)$, let $f, f', g \in \mathbb{F}(t)_{(r)}$, and let $h \in \mathbb{F}[t]^*$ be irreducible. Write $f = f_0 + f_1$, $f' = f'_0 + f'_1$ and $g = g_0 + g_1$ with $f_i, f'_i, g_i \in \mathbb{F}(t)_{(r)}$ and*

$$f_0 = \frac{a}{\prod_i \sigma^i(h^{m_i})}, \qquad f_1 = \frac{b}{c} \qquad \text{for some } a, b, c \in \mathbb{F}[t] \text{ with } h \perp_\sigma c,\ m_i \geq 0, \tag{21}$$

$$f'_0 = \frac{a'}{\prod_i \sigma^i(h^{m'_i})} \qquad f'_1 = \frac{b'}{c'} \qquad \text{for some } a', b', c' \in \mathbb{F}[t] \text{ with } h \perp_\sigma c',\ m'_i \geq 0, \tag{22}$$

$$g_0 = \frac{\alpha}{\prod_i \sigma^i(h^{\mu_i})} \qquad g_1 = \frac{\beta}{\gamma} \qquad \text{for some } \alpha, \beta, \gamma \in \mathbb{F}[t] \text{ with } h \perp_\sigma \gamma,\ \mu_i \geq 0. \tag{23}$$

[2] If not stated differently, we suppose that $p, q \in \mathbb{F}[t]$, $q \neq 0$, and $\gcd(p, q) = 1$, whenever we write $f = \frac{p}{q}$; we define $\mathrm{den}(f) = q$ (up to a unit in $\mathbb{F}$).

(i) Then (f', g) is a Σ-pair for f iff (f'_i, g_i) are Σ-pairs for f_i with $i \in \{0, 1\}$.
(ii) Let (f', g) be a Σ-pair for f, and (f'_i, g_i) be Σ-pairs for f_i with $i \in \{0, 1\}$ where the f_i, f'_i and g_i are as above. Then (f', g) is a solution of problem RP for f iff (f'_1, g_1) is a solution of problem RP for f_1 and $\deg(\text{den}(f'_0))$ is minimal w.r.t. all Σ-pairs in $\mathbb{F}(t)^2_{(r)}$ where the denominators are of the form $\prod_i \sigma^i(h^{\nu_i})$ for some $\nu_i \geq 0$ (see problem SFP).

Proof. (i) The direction from left to right follows by Lemma 2.1.1. Now suppose that (f', g) is a Σ-pair for f. Then $0 = \sigma(g)-g+f'-f = [\sigma(g_0)-g_0+f'_0-f_0]+[\sigma(g_1)-g_1+f'_1-f_1] = h_0+h_1$ with $h_i := \sigma(g_i) - g_i + f'_i - f_i$. We have $h_0 = \frac{A}{\prod_i \sigma^i(h^{\nu_i})}$ and $h_1 = \frac{B}{C}$ for some $\nu_i \geq 0$, and $A, B, C \in \mathbb{F}[t]$ with $h \perp_\sigma C$. Suppose that $h_0, h_1 \neq 0$. Since $h_0 = -h_1$, $C = u \prod_i \sigma^i(h^{\nu_i})$ with $u \in \mathbb{F}^*$ and $\deg(C) > 0$. A contradiction that $h \perp_\sigma C$. Hence $h_0 = 0 = h_1$, and therefore (f'_i, g_i) are Σ-pairs for f_i with $i \in \{0, 1\}$.
(ii) Suppose that $\deg(\text{den}(f'_i))$ is not minimal for some $i \in \{0, 1\}$ as stated in the lemma. Let $j \in \{0, 1\} \setminus \{i\}$ and take $\psi, \gamma \in \mathbb{F}(t)_{(r)}$ such that $\sigma(\gamma) - \gamma + \psi = f_i$ and $\deg(\text{den}(\psi)) < \deg(\text{den}(f'_i))$. Then $(\psi + f'_j, \gamma + g_j)$ is a Σ-pair for f by Lemma 2.1.1. We have

$$\deg(\text{den}(\psi+f'_j)) \leq \deg(den(\psi))+\deg(den(f'_j)) < \deg(den(f'_0))+\deg(den(f'_1)) = \deg(\text{den}(f')).$$

Conversely, suppose that $\deg(\text{den}(f_0))$ and $\deg(\text{den}(f_1))$ are minimal as stated in the lemma, but (f', g) does not solve *RP*. Take a Σ-pair $(\psi, \gamma) \in \mathbb{F}(t)^2_{(r)}$ for f with $\deg(\text{den}(\psi)) < \deg(\text{den}(f'))$. By (i) there are $\psi = \psi_0 + \psi_1$ and $\gamma = \gamma_0 + \gamma_1$ such that (ψ_i, γ_i) are Σ-pairs for f_i with $i \in \{0, 1\}$ and where we can write $\psi_0 = \frac{A}{\prod_i \sigma^i(h^{\nu_i})}$ and $\psi_1 = \frac{B}{C}$ for some $\nu_i \geq 0$, and $A, B, C \in \mathbb{F}[t]$ with $h \perp_\sigma C$. Then it follows that

$$\deg(\text{den}(\psi_0)) + \deg(\text{den}(\psi_1)) = \deg(\text{den}(\psi)) < \deg(\text{den}(f')) = \deg(\text{den}(f_0)) + \deg(\text{den}(f_1)),$$

a contradiction to $\deg(\text{den}(f'_i)) \leq \deg(\text{den}(\psi_i))$ for $i = 0, 1$. □

This gives the following reduction. Write f in the representation $f = f_0 + f_1$ with (21), see above. Then find a Σ-pair (f'_0, g_0) for f_0 where the degree of the denominator of f'_0 is minimal. More precisely, solve problem *SFP* for f_0.

SFP: **S**imple **F**ractional **Part**

Given $f = \frac{a}{\prod_i \sigma^i(h^{m_i})} \in \mathbb{F}(t)_{(r)} \setminus \{0\}$ for some $a \in \mathbb{F}[t]$ and $h \in \mathbb{F}[t]$ irreducible (not all m_i are zero);
find $f' = \frac{a'}{\prod_i \sigma^i(h^{m'_i})} \in \mathbb{F}(t)_{(r)}$ and $g = \frac{\alpha}{\prod_i \sigma^i(h^{\mu_i})} \in \mathbb{F}(t)_{(r)}$ for some $a', \alpha \in \mathbb{F}[t]$ and $m_i, \mu_i \in \geq 0$ with (6) where the degree of $\prod_i \sigma^i(h^{m'_i})$ is optimal w.r.t. all Σ-pairs in $\mathbb{F}(t)^2_{(r)}$ where the denominators are of the form $\prod_i \sigma^i(h^{\nu_i})$ for some $\nu_i \geq 0$.

Then continue to solve problem *RP* for f_1; note that the degree of the denominator of f_1 is reduced by $\deg(h) \sum_i m_i > 0$. If $f_1 = 0$, take the Σ-pair $(0, 0)$. Otherwise, apply the same reduction strategy to $f_1 \in \mathbb{F}(t)_{(r)} \setminus \{0\}$ as sketched above (for a new irreducible polynomial $h \in \mathbb{F}[t]$ in the denominator of f_1). This finally gives the solution (f'_1, g_1) of problem *RP* for f_1. By Lemma 5.1 we get the solution $(f'_0 + f'_1, g_0 + g_1)$ of problem *RP* for f.

Example 5.3. (Cont. Example 5.2) We solve problem *SFP* for f_0 and get the Σ-pair $(f'_0, g_0) = (\frac{1}{F^3}, -\frac{k^2}{F^2} - \frac{k}{F})$; see Example 5.4. As byproduct we get $\sum_{k=0}^n \frac{(k-1)k!^2+(k^2-1)k!-1}{(k!)^3} = \frac{2n!^2-n!-1}{n!^2} + \sum_{k=1}^n \frac{1}{k!}$. Next we solve problem *RP* for f_1. As result we get the Σ-pair $(f'_1, g_1) = (\frac{1}{F+1}, \frac{k}{F+1})$ for f_1; see Example 5.8. This results in $\sum_{k=1}^n \frac{k(k!k+1)}{(k!+1)(kk!+k!+1)} = \frac{-(n+1)n!+1}{2((n+1)n!+1)} + \sum_{k=1}^n \frac{k}{1+k!}$. Combining the results we get the solution $(f', g) = (f'_0 + f'_1, g_0 + g_1)$ of problem *RP* for f.

Summarizing, we can reduce problem *SEF* to problem *SFP* as follows.

Algorithm 5.1. `ReduceFractionalPart`$((\mathbb{F}(t),\sigma), f)$
Input: A $\Pi\Sigma^*$-extension $(\mathbb{F}(t),\sigma)$ of $(\mathbb{F},\sigma)$ and $f \in \mathbb{F}(t)_{(r)}$; algorithms for problems *SFP*, *SEF*.
Output: A solution of problem *RP*.
(1) Set $g := 0$ and $f' := f$. WHILE $f \neq 0$ DO
(2) Let $f = \frac{p}{q}$. Take an irreducible factor $h \in \mathbb{F}[t]^*$ of q and represent q in the form (15).
(3) By the extended Euclidean algorithm write $f = f_0 + f_1$ in the form (21).
(4) Compute a Σ-pair (f'_0, g_0) for f_0 which is a solution of problem *SFP*.
(5) Set $f := f - f_0$, $f' := f' + f'_0$ and $g := g + g_0$.
(6) OD
(7) RETURN (f', g)

To this end, we show how we can solve problem *SFP*. Here the following property is essential.

Lemma 5.2. *Let $(\mathbb{F}(t),\sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$ and $h \in \mathbb{F}[t]^*$ be irreducible. Suppose that $\frac{\sigma(t)}{t} \notin \mathbb{F}$ or $\frac{h}{t} \in \mathbb{F}$. Then $\gcd(\sigma^k(h), \sigma^l(h)) = 1$ for all integers k, l with $k \neq l$.*

Proof. Assume $\gcd(\sigma^k(h), \sigma^l(h)) \neq 1$. Since $\sigma^k(h), \sigma^l(h) \in \mathbb{F}[t]$ are irreducible, $\frac{\sigma^k(h)}{\sigma^l(h)} \in \mathbb{F}$. Hence $\frac{\sigma^{k-l}(h)}{h} \in \mathbb{F}$. By [Kar81, Thm. 4] (compare [Bro00, Cor. 1,2] or [Sch01, Thm. 2.2.4]) it follows that $\frac{\sigma(t)}{t} \in \mathbb{F}$ and $h/t \in \mathbb{F}$. □

Since Lemma 5.2 cannot be applied if $h = t$ and $\frac{\sigma(t)}{t} \in \mathbb{F}$, we do a case distinction.

5.1. **A special case.** Let $(\mathbb{F}(t),\sigma)$ be a Π-extension of $(\mathbb{F},\sigma)$ with $\sigma(t) = \alpha\, t$, let $h = t$, and let $f = \frac{a}{\prod_{i=1}^n \sigma^i(h^{m_i})} \neq 0$ as in problem *SFP*. Then for some $u \in \mathbb{F}^*$, $n > 0$ and $t \nmid a$ we have

$$f = \frac{a\, u}{t^n}, \quad 0 \leq \deg(a) < n.$$

Hence we can write $f = \sum_{i=1}^n f_i \frac{1}{t^i}$ for some $f_i \in \mathbb{F}$, i.e., $f \in \mathbb{F}[\frac{1}{t}]$. Similarly, the $f', g \in \mathbb{F}(t)_{(r)}$ in problem *SFP* are also elements from $\mathbb{F}[\frac{1}{t}]$. Thus, problem *SFP* boils down to find a Σ-pair $(f', g) \in \mathbb{F}[\frac{1}{t}]^2$ for $f \in \mathbb{F}[\frac{1}{t}]$ where in $f' = \sum_{i=0}^{n'} f'_i \frac{1}{t^i}$ the degree n' is minimal.

Now observe that the difference field $(\mathbb{F}(\frac{1}{t}),\sigma)$ with $\sigma(\frac{1}{t}) = \frac{1}{\alpha}\frac{1}{t}$ is a Π-extension of $(\mathbb{F},\sigma)$. This is a direct consequence of [Kar81, Thm. 2]; see also [Sch05c, Prop. 4.4]. Hence in $(\mathbb{F}(\frac{1}{t}),\sigma)$ problem *SFP* is nothing else than problem *PP* handled in Subsection 4.1. In a nutshell, we can apply Algorithm 4.1 with the function call `OptimalPolyΠExtension`$((\mathbb{F}(\frac{1}{t}),\sigma), f)$.

Example 5.4. (Cont. Example 5.3) Given $f = \frac{(k-1)(F)^2+(k^2-1)F-1}{F^3} = \frac{1}{F^3} + \frac{k^2-1}{F^2} + \frac{k-1}{F} = \sum_{i=1}^3 f_i \frac{1}{F^i}$ we have to solve the problems $\frac{1}{(k+1)^i}\sigma(g_i) - g_i = f_i$ with $f_3 = 1$, $f_2 = k^2 - 1$ and $f_1 = k - 1$. We get the solutions $g_2 = -k^2$, $g_1 = -k$; there is no solution $g_3 \in \mathbb{Q}(k)$. Hence we obtain $(f', g) = (\frac{1}{F^3}, \frac{-k^2}{F^2} + \frac{-k}{F})$ for problem *SFP*.

5.2. **The remaining cases.** The solution of problem *SFP* can be summarized in

Theorem 5.1. *Let $(\mathbb{F}(t),\sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$ and let $h \in \mathbb{F}[t]$ be irreducible with $h \neq t$ or $\frac{\sigma(t)}{t} \notin \mathbb{F}$. Let $f \in \mathbb{F}(t)_{(r)} \setminus \{0\}$ with $\mathrm{den}(f) = \prod_i \sigma^i(h^{m_i})$ for some $m_i \geq 0$. Then:*

(1) A Σ-pair $(f', g) \in \mathbb{F}(t)_{(r)}^2$ of f can be computed where the denominator of f' has the form

$$u\sigma^i(h)^m \quad \text{for some } u \in \mathbb{F}^*,\ i \in \mathbb{Z} \text{ and } m \geq 0. \tag{24}$$

(2) A Σ-pair $(f', g) \in \mathbb{F}(t)_{(r)}^2$ of f solves SFP iff the denominator of f' is of the form (24).

In the remaining part of the subsection we prove the theorem. Let $(\mathbb{F}(t),\sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$ where either $\frac{\sigma(t)}{t}\notin\mathbb{F}$ (a Σ^*-extension) or $h\neq t$. Moreover, consider $f=\frac{a}{\prod_{i=1}^n\sigma^i(h^{m_i})}$ with $m_i\geq 0$ as in problem *SFP*.

Proof of Theorem 5.1.1. By Lemma 5.2 all the $\sigma^i(h^{m_i})$ with $m_i\neq 0$ are pairwise coprime. Thus, we can invoke the extended Euclidean algorithm and compute polynomials $s_i\in\mathbb{F}[t]$ with

$$f=\frac{a}{\prod_{i=1}^n\sigma^i(h^{m_i})}=\sum_{i=0}^n\frac{s_i}{\sigma^i(h^{m_i})} \tag{25}$$

where $\deg(s_i)<\deg(h)m_i$. Equivalently, we can write

$$f=\sum_i\sigma^i(f_i)$$

with $f_i:=\frac{\sigma^{-i}(s_i)}{h^{m_i}}$. Then by Lemma 2.1.3 we can compute $g_i\in\mathbb{F}(t)_{(r)}$ with

$$\sigma^i(f_i)=\sigma(g_i)-g_i+f_i. \tag{26}$$

Therefore, with (25), (26) and Lemma 2.1.1 we get a Σ-pair (f',g) for f defined by

$$f':=\sum_{i=0}^n\frac{\sigma^{-i}(s_i)}{h^{m_i}}\quad\text{and}\quad g:=\sum_{i=0}^n g_i\quad\text{with } g_i=\begin{cases}\sum_{j=0}^{i-1}\sigma^j(\frac{\sigma^{-i}(s_i)}{h^{m_i}}) & \text{if } i\geq 0\\ -\sum_{j=0}^{-i-1}\sigma^{j+i}(\frac{\sigma^{-i}(s_i)}{h^{m_i}}) & \text{if } i<0;\end{cases} \tag{27}$$

f' and g are of the required form given in problem *SFP*. This proves Theorem 5.1.1. □

Example 5.5. (Cont. Example 2.6) Write $f=\frac{1+k}{k(k+2)}=\frac{1}{2k}+\frac{1}{2(k+1)}$. We apply Lemma 2.1.3 and get $\frac{1}{2(k+1)}=\frac{1}{2k}+\sigma(g)-g$ with $g=\frac{1}{2k}+\frac{1}{2(k+1)}=\frac{2k+1}{2k(k+1)}$. Hence we obtain $f=\frac{1}{2k}+\frac{1}{2k}+\sigma(g)-g$, and therefore $(\frac{1}{k},\frac{2k+1}{2k(k+1)})$ is a Σ-pair for f.

Example 5.6. (Cont. Example 2.7) Write $f=\frac{1}{k(k-1)P}=\frac{1}{(k-1)P}-\frac{1}{kP}$. By $\frac{1}{(k-1)P}=\sigma(\frac{1}{(k-1)P})+\sigma(g)-g$ with $g=\frac{-1}{(k-1)P}$ we get the Σ-pair $(-\frac{1}{2kP},\frac{-1}{(k-1)P})$ for f.

Example 5.7. (Cont. Example 3.1) Write $f=\frac{Hk-2}{H(Hk-1)}=\frac{2}{H}+\frac{-1}{\sigma^{-1}(H)}$. We have $\frac{-1}{\sigma^{-1}(H)}=\frac{-1}{H}+\sigma(g)-g$ with $g=\frac{1}{\sigma^1(H)}=\frac{k}{kH-1}$. Thus we get the Σ-pair $(f',g)=(\frac{1}{H},\frac{k}{kH-1})$ for f.

Example 5.8. (Cont. Example 5.3) Write $f=\frac{k(Fk+1)}{(F+1)(kF+F+1)}=\frac{k(Fk+1)}{(F+1)\sigma(F+1)}=\frac{k-1}{F+1}+\frac{1}{\sigma(F+1)}$. Then $\sigma(\frac{1}{F+1})=\sigma(g)-(g)+\frac{1}{F+1}$ with $g=\frac{1}{F+1}$. Hence $(f',g)=(\frac{1}{F+1},\frac{k}{F+1})$ is a Σ-pair for f.

Example 5.9. (Cont. Example 2.9) Take f from Example 2.9 and split it in the form

$$f=\frac{k(k+1)^2}{(2k+1)h}+\frac{(k+1)^2(k+2)}{(2k+3)\sigma(h)}+\frac{k(k+1)(k+2)}{(2k+1)\sigma^2(h)}-\frac{(k+1)(k+2)(k+3)}{(2k+3)\sigma^3(h)}=f_0+f_1+f_2+f_3.$$

Then by Lemma 2.1.3 we obtain (f_i',g_i) with $f_i'=f_i+\sigma(g_i)-g_i$ where

$$(f_1',g_1)=(\tfrac{k^2(k+1)}{(2k+1)h},\tfrac{k^2(k+1)}{(2k+1)(Hk-1)},\quad (f_2',g_2)=(\tfrac{(k-2)(k-1)k}{(2k-3)h},\tfrac{(k-1)k\left(-2k+3+H\left(4k^2-8k+2\right)\right)}{H(Hk-1)(4k^2-8k+3)})$$

$$(f_3',g_3)=(\tfrac{-(k-2)(k-1)k}{(2k-3)h},\tfrac{-k\left(-4k^3+8k^2-k-3-4H\left(4k^3-6k^2-k+2\right)+H^2\left(12k^5-12k^4-21k^3+12k^2+7k-2\right)\right)}{H(8k^3-12k^2-2k+3)(k(k+1)H^2-H-1)}).$$

This gives the Σ-pair $(f',g)=(f_0+f_1'+f_2'+f_3',g_1'+g_2'+g_3')=(\frac{k(k+1)}{Hk-1},\frac{k(k+1)}{(Hk-1)(kH+H+1)})$.

With Theorem 5.1.2 it follows that all Σ-pairs from the previous examples are solutions of problem *SFP*. To prove it, we need Remark 5.1 and Lemma 5.3 (compare [Bro00, Cor. 4]).

Remark 5.1. Let (f',g) be a Σ-pair for f which we get by (27). Then $\deg(\mathrm{den}(f'))\leq\deg(\mathrm{den}(f))$. If $m_im_j\neq 0$ in (25) for some $i\neq j$, then $\deg(\mathrm{den}(f'))<\deg(\mathrm{den}(f))$.

Lemma 5.3. *Let $(\mathbb{F}(t),\sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$, let $h\in\mathbb{F}[t]$ be irreducible and suppose that $\frac{\sigma(t)}{t}\notin\mathbb{F}$ or $t\nmid h$. Then there is no $g\in\mathbb{F}(t)$ with $\sigma(g)-g=\frac{c}{h^r}$ for $c\in\mathbb{F}[t]^*$ and $r>0$.*

Proof. Suppose that there is a solution $g=\frac{a}{b}\in\mathbb{F}(t)$. Define $d:=\gcd(b,\sigma(b))$. Then $v\,h^r=\operatorname{lcm}(b,\sigma(b))$ with $v\mid d$; see e.g. [Win96, Thm. 2.3.1]. Let $m\in\mathbb{Z}$ be maximal such that $\sigma^m(h)\mid b$. Then $\sigma^{m+1}(h)\nmid d$. Hence $\sigma^{m+1}(h)\nmid v$ and $\sigma^{m+1}(h)\mid\operatorname{lcm}(b,\sigma(b))$. Thus $\sigma^{m+1}(h)\mid h^r$, i.e., $m=-1$. Now take m' minimal with $\sigma^{m'}(h)\mid b$. Then $\sigma^{m'}(h)\nmid d$. Hence $\sigma^{m'}(h)\nmid v$ and $\sigma^{m'}(h)\mid\operatorname{lcm}(b,\sigma(b))$. Thus $\sigma^{m'}(h)\mid h^r$, i.e., $m'=0$; a contradiction. □

Proof of Theorem 5.1.2. "⇒" Let (f',g) be a Σ-pair of f with $\operatorname{den}(f')=\prod_{i=1}^{n'}\sigma^i(h^{m'_i})$ where $m'_i\,m'_j\neq 0$ for some $i\neq j$. By Remark 5.1 there is a Σ-pair (ϕ,γ) for f' where $\deg(\operatorname{den}(\phi))<\deg(\operatorname{den}(f'))$. Hence $(\phi,g+\gamma)$ is a Σ-pair for f by Lemma 2.1.2. Thus $\deg(\operatorname{den}(f'))$ is not minimal.
"⇐" Let (f',g) be a Σ-pair for f with $f'=\frac{p}{q}$ and $q=\sigma^i(h)^m$. If $m=0$, then $f'=0$, i.e., nothing has to be shown. Let $m>0$, and hence $p\neq 0$, and assume that there is a Σ-pair $(\frac{p'}{q'},g')$ for f where $\deg(q')$ is minimal and $\deg(q')<\deg(q)$. By the implication "⇒" and the minimality of $\deg(q')$ it follows that $q'=u\,\sigma^j(h)^{m'}$ for some $u\in\mathbb{F}^*$, $j\in\mathbb{Z}$ and $m'\geq 0$. Then by Lemma 2.1.3 there are $a\in\mathbb{F}[t]$ and $\gamma\in\mathbb{F}(t)$ with $\sigma(\gamma)-\gamma+\frac{a}{\sigma^i(h)^{m'}}=f$. Hence

$$\sigma(g-\gamma)-(g-\gamma)=\frac{a}{\sigma^i(h)^{m'}}-\frac{p}{\sigma^i(h)^m}=\frac{a\,\sigma^i(h)^{m-m'}-p}{\sigma^i(h)^m}\qquad(28)$$

where $\sigma^i(h)\in\mathbb{F}[t]$ is irreducible. Since $m'<m$, $p\neq 0$ and $\gcd(p,\sigma^i(h)))=1$, the right-hand side of (28) is non-zero; a contradiction to Lemma 5.3. This proves Theorem 5.1.2. □

Define the *dispersion* of $f\in\mathbb{F}[t]^*$ by

$$\operatorname{disp}(f)=\max\{m\geq 0\,|\,\gcd(\sigma^m(f),f)\neq 1\}.$$

By Lemma 5.1.2 (Algorithm 5.1) and Theorem 5.1 we get

Corollary 5.1. *Let $(\mathbb{F}(t),\sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$ and $f\in\mathbb{F}(t)_{(r)}$ where t is a Σ^*-extension or $t\nmid\operatorname{den}(f)$. Let $(\frac{p}{q},g)\in\mathbb{F}(t)^2_{(r)}$ be a Σ-pair for f. Then the following is equivalent:*
(1) $(\frac{p}{q},g)$ is a solution of problem RP.
(2) $q=u\prod_i h_i^{m_i}$ where $u\in\mathbb{F}^*$ and where the $h_i\in\mathbb{F}[t]$ are irreducible and pairwise σ-prime.
(3) $disp(q)=0$.

Corollary 5.1 is a generalized version of the rational case given in [Abr75] or [Pau95, Prop. 3.3].

Remark 5.2. A solution $(\frac{p}{q},g)$ of problem *RP* with q as in Corollary 5.1.2 is not uniquely determined. More precisely, by splitting $\frac{p}{q}$ in the form $f'=\sum_i\frac{p_i}{{h_i}^{m_i}}$ with $p_i\in\mathbb{F}[t]$ and $\deg(p_i)<\deg(h_i)m_i$ we can apply Lemma 2.1.3 and obtain all other Σ-pairs (ϕ,γ) where $\operatorname{den}(\phi)$ is of the form $\prod_i\sigma^{z_i}h_i^{m_i}$ with $z_i\in\mathbb{Z}$.

6. Eliminating several top extensions in a sum

As shown in Corollary 3.1 we can eliminate the top extension from the non-summable part, if possible; see Examples 2.3 and 2.4. More generally, we are interested to eliminate several extensions, like for identity (5) or

$$\sum_{k=1}^n\Big(\sum_{j=1}^k\binom{n}{j}\Big)\Big(\sum_{j=1}^k\binom{n}{j}^2\Big)=\frac{n+2}{2}\sum_{j=1}^n\binom{n}{j}\sum_{j=1}^n\binom{n}{j}^2-\frac{1}{n}\sum_{k=1}^n(n^2-nk+k^2)\binom{n}{k}^3.$$

Assume we have given several $\Pi\Sigma^*$-extensions $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ over $\mathbb{F}$ with $\sigma(t_i)=a_i\,t_i+b_i$ where $a_i,b_i\in\mathbb{F}$; in short we say that $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ is a $\Pi\Sigma^*$-*extension over* $\mathbb{F}$. Then we are interested in the following problem.

ET: Eliminate top extensions

Given $f\in\mathbb{F}(t_1)\dots(t_e)$; **find** a Σ-pair $(f',g)\in\mathbb{F}(t_1)\dots(t_r)\times\mathbb{F}(t_1)\dots(t_e)$ for f where r is minimal, i.e., eliminate as many extensions in f' as possible. In particular, choose $f'=0$, if possible.

In particular, we are interested in the following application: Let $\mathbb{F}$ be a $\Pi\Sigma^*$-field where all the maximal nested sums and products are the t_i's and all less nested sums and products are in $\mathbb{F}$. Then solving *ET* enables one to decide constructively if there is a Σ-pair (f',g) for f where f' is less nested than f.

Example 6.1. (Cont. Example 2.5) Given f from Example 2.5 we compute with Algorithm 3.1 the Σ-pair $(f_2,g_2)=\big(\frac{6H^{(2)}(k+1)^3+3k+4}{3(k+1)^3},-\frac{1}{3}H(H^2-3(H^{(2)}k+1)H+3H^{(2)}(2k+1))\big)$ for f. Since we managed to eliminate the extension H from the non-summable part f_2, we apply Algorithm 3.1 to f_2 and get as result the Σ-pair $(f_1,g_1)=\big(-\frac{6k^2+9k+2}{3(k+1)^3},2H^{(2)}k\big)$. Finally, we apply Algorithm 3.1 to f_1 and get the Σ-pair $\big(-\frac{6k^2+9k+2}{3(k+1)^3},0\big)$, i.e., f_1 cannot be simplified further in the degree (of the numerator or denominator). Combining all the steps by using Lemma 2.1.2 we obtain the Σ-pair $(f_1,g_1+g_2)=(-\frac{6k^2+9k+2}{3(k+1)^3},-\frac{H^3}{3}+(H^{(2)}k+1)H^2-(2kH^{(2)}+H^{(2)})H+2H^{(2)}k)$ for f. Finally, by using Lemma 2.1.3 we change the Σ-pair for f to $(\sigma^{-1}(f'),\sigma^{-1}(f')+g)$, see also Example 4.5. This result is used in Example 2.5.

As illustrated in the previous example, we can attack problem *ET* by running Algorithm 3.1 recursively and using Lemma 2.1.2. More precisely, we propose the following algorithm.

Algorithm 6.1. `EliminateExtensions`$((\mathbb{F}(t_1)\dots(t_e),\sigma),f)$

Input: A $\Pi\Sigma^*$-extension $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ over $\mathbb{F}$ with $e\geq 1$ where we can solve problems *PLDE* and *SEF* for all extensions t_i. $f\in\mathbb{F}(t_1)\dots(t_e)$.
Output: A solution of problem *ET*.

(1) If $e=0$, decide constructively, if there is a $g\in\mathbb{F}$ with $\sigma(g)-g=f$. If yes, RETURN $(0,g)$, otherwise RETURN $(f,0)$.
(2) Decide constructively, if there is a Σ-pair $(f',g)\in\mathbb{F}(t_1)\dots(t_{e-1})\times\mathbb{F}(t_1)\dots(t_e)$ for f.
(3) If no, THEN RETURN (f',g). Otherwise, take such an (f',g).
(4) Compute (ϕ,γ) :=`EliminateExtensions`$((\mathbb{F}(t_1)\dots(t_{e-1}),\sigma),f')$ and RETURN $(\phi,g+\gamma)$.

In order to prove the correctness of Algorithm 6.1, we need the following Lemma; see [Kar81, Thm. 24] or [Sch01, Prop. 4.1.3].

Lemma 6.1. *Let $(\mathbb{F}(t),\sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$. Let $g\in\mathbb{F}(t)$ with $\sigma(g)-g\in\mathbb{F}$. If $\frac{\sigma(t)}{t}\in\mathbb{F}$, then $g\in\mathbb{F}$. Otherwise, $g=c\,t+w$ for some $c\in\text{const}_\sigma\mathbb{F}$ and $w\in\mathbb{F}$.*

Theorem 6.1. *Let $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ be a $\Pi\Sigma^*$-extension over $\mathbb{F}$ $(e\geq 1)$ where one can solve problems PLDE and SEF for all extensions t_i. Then Algorithm 6.1 solves problem ET.*

Proof. If $e=0$, the output is correct. Now suppose that Algorithm 6.1 works correct for $e-1$ extensions with $e>1$. Consider $\mathbb{F}(t_1)\dots(t_e)$ with $\sigma(t_i)=a_it_i+b_i$ where $a_i,b_i\in\mathbb{F}$. Let $(F,G)\in\mathbb{F}(t_1)\dots(t_r)\times\mathbb{F}(t_1)\dots(t_e)$ be a Σ-pair for f where r is minimal. If $r=e$, we return the correct result in step (3) by Corollary 3.1. Now suppose that $r<e$. Hence, by Corollary 3.1 we can compute a Σ-pair (f',g) for f with $f'\in\mathbb{F}(t_1)\dots(t_{e-1})$ and $g\in\mathbb{F}(t_1)\dots(t_e)$. Thus, $\sigma(h)-(h)=f'-F$, where $h:=G-g\in\mathbb{F}(t_1)\dots(t_e)$. Note that $f'-F\in\mathbb{F}(t_1)\dots(t_{e-1})$. Now suppose that t_e is a Π-extension. By Lemma 6.1, $h\in\mathbb{F}(t_1)\dots(t_{e-1})$. Hence, $(F,h)\in\mathbb{F}(t_1)\dots(t_r)\times\mathbb{F}(t_1)\dots(t_{e-1})$ is a Σ-pair for f'. Otherwise, suppose that t_e is a Σ^*-extension. By Lemma 6.1, $h=c\,t+w$ for some $w\in\mathbb{F}(t_1)\dots(t_{e-1})$ and $c\in\text{const}_\sigma\mathbb{F}$.

Hence $\sigma(w) - w + (F + c\,a_e) = f'$, i.e., $(F + c\,a_e, w) \in \mathbb{F}(t_1)\dots(t_r) \times \mathbb{F}(t_1)\dots(t_{e-1})$ is a Σ-pair for f'. By the induction assumption Algorithm 6.1 computes a Σ-pair $(\phi,\gamma) \in \mathbb{F}(t_1)\dots(t_r) \times \mathbb{F}(t_1)\dots(t_{e-1})$ for f'. By Lemma 2.1.2, $(\phi, \gamma + g)$ is a Σ-pair for f. □

Concerning Algorithm 6.1 the following remarks are in place:

(1) Step (2) of Algorithm 6.1 can be accomplished by Algorithm 3.1, i.e., by the function call $(f', g) :=$`RefinedTelescoping`$((\mathbb{F}(t_1)\dots(t_e),\sigma), f)$; see Example 6.1. In particular, if one fails to eliminate the extension t_e from f', one obtains a Σ-pair (f', g) where the degrees in t_e are optimal. Hence we can combine problems RT and ET.

(2) We can improve the computation in step (2): Since we only have to eliminate the extension t_e, if possible, but we do not have to decide, if $f' = 0$ is possible, we can avoid unnecessary computations in Algorithm 3.1. More precisely, in Sub-algorithm 4.1 we can quit the do-loop when $r = 0$; in Sub-algorithm 4.2 we can quit the while-loop when $\deg(f') = 0$.

(3) The proposed algorithm might fail to find a sum extension where the depth is optimal. E.g., starting with the left-hand side of (29) we find the first simplification in

$$\sum_{j=1}^{n}\sum_{k=1}^{j}\frac{H_k}{k^2} = n\sum_{k=1}^{n}\frac{H_k}{k^2} - \sum_{k=1}^{j}\frac{H_k(k+1)}{k^2} = n\sum_{k=1}^{n}\frac{H_k}{k^2} - \Big(\sum_{k=1}^{n}\frac{H_k}{k^2} + \frac{1}{2}\big(H_n^2 + H_n^{(2)}\big)\Big). \quad (29)$$

But our algorithm fails to find $H_n^{(2)}$ in order to simplify $\sum_{k=1}^{j}\frac{H_k(k+1)}{k^2}$ further. Here we would need in addition the sum $\sum_{k=1}^{j}\frac{H_k(k+1)}{k^2}$ which we dropped in the reduction; see step (4) in Algorithm 6.1. In [Sch04c, Sch05b] this problem can be handled properly by using a rather complicated machinery.

7. Simplification of Σ^*-extensions

By [Kar81] there is the following result concerning the construction of Σ^*-extensions.

Theorem 7.1. *Let $(\mathbb{F}(t),\sigma)$ be a difference field extension of $(\mathbb{F},\sigma)$ with $\sigma(t) = t + f$ where $f \in \mathbb{F}$. Then this is a Σ^*-extension iff there is no $g \in \mathbb{F}$ with $\sigma(g) - g = f$.*

This result provides a constructive theory to represent sums, like

$$S(n) = \sum_{k=1}^{n} f(k),$$

in $\Pi\Sigma^*$-fields. More precisely, suppose that $f(k)$ can be written in a $\Pi\Sigma^*$-field, say $(\mathbb{F},\sigma)$ with $f \in \mathbb{F}$; for typical examples see Section 2. Two cases can occur: **(1)** One finds a $g \in \mathbb{F}$ with $\sigma(g) - g = f$. Then reconstruct from g a sequence $g(k)$ with $g(k+1) - g(k) = f(k)$ and derive, with some mild extra-conditions, the closed form $S(n) = g(n+1) - g(1)$. In particular, the sum $S(n)$ can be expressed by $t := \sigma(g) + c \in \mathbb{F}$ for some $c \in \mathbb{K}$ ($c = g(1)$) with

$$\sigma(t) = t + \sigma(f); \quad (30)$$

this reflects the shift behavior $S(n + 1) = S(n) + f(n + 1)$.
(2) One shows that there is no $g \in \mathbb{F}$ with $\sigma(g) - g = f$. Then by Theorem 7.1 adjoin the sum $S(n)$ formally in form of the Σ^*-extension $(\mathbb{F}(t),\sigma)$ with (30).

Our refined telescoping methods enable one to construct refined Σ^*-extensions. In general, suppose there is no $g \in \mathbb{F}$ with $\sigma(g) - g = f$ and let (f', g) be any Σ-pair for f. Then there is no $h \in \mathbb{F}$ with $\sigma(h) - h = f'$; otherwise, we would have $\sigma(g + h) - (g + h) = f$ for $g + h \in \mathbb{F}$. Hence, by Theorem 7.1 we can construct the Σ^*-extension $(\mathbb{F}(s),\sigma)$ with $\sigma(s) = s + \sigma(f')$. Moreover, for $T := s + \sigma(g) + c$ with some $c \in \mathbb{K}$ we have $\sigma(T) = \sigma(s) + \sigma^2(g) + c = s + \sigma(f') + \sigma^2(g) + c = s + \sigma(g) + c + \sigma(f) = T + \sigma(f)$, i.e., $\sigma(T) = T + \sigma(f)$. Thus, we can represent $S(n)$ by $T \in \mathbb{F}(s)$.

Remark. Note that the Σ^*-extensions $(\mathbb{F}(t), \sigma)$ and $(\mathbb{F}(s), \sigma)$ from above are isomorphic by the difference field isomorphism $\tau : \mathbb{F}(t) \to \mathbb{F}(s)$ with $\tau(f) = f$ for all $f \in \mathbb{F}$ and $\tau(t) = s + \sigma(g) + d$; $d \in \mathbb{K}$ is arbitrary, but fixed.

Summarizing, if we compute the Σ-pair (f', g) with Algorithms 3.1 or 6.1 we can get better Σ^*-extensions to represent the sum $S(n)$.

Example 7.1. (Cont. Example 2.6) Since there is no $g \in \mathbb{Q}(k)$ with $\sigma(g) - g = f$ for $f = \frac{k+1}{k(k+2)}$, we can construct the Σ^*-extension $\mathbb{Q}(k)(t)$ with $\sigma(t) = t + \frac{k+2}{(k+1)(k+3)}$ and can represent the sum $S(n) = \sum_{k=1}^{n} \frac{k^2+1}{k(k+1)(k+2)}$ by t. Given the Σ-pair $(\frac{1}{k}, \frac{2k+1}{2k(k+1)})$ from Example 2.6, we can represent the sum $S(n)$ with $T := H + \frac{2k+3}{2(k+1)(k+2)}$ in the Σ^*-extension $(\mathbb{Q}(k)(H), \sigma)$ with $\sigma(H) = H + \frac{1}{k+1}$. We get the difference field isomorphism $\tau : \mathbb{Q}(k)(t) \to \mathbb{Q}(k)(H)$ given by $\tau(t) = H + \frac{6k+2}{2(k+1)(k+2)} - \frac{7}{4}$. This is exactly reflected by the identity (11).

8. Conclusion

We developed algorithms that can express a given sum in terms of a sum $\sum f'(k)$ where $f'(k)$ is degree-optimal.

Here we restricted so far to the domain of $\Pi\Sigma^*$-fields. More generally, one can apply the underlying reductions also to difference rings, see Example 2.4. Here further investigations are necessary, in particular, one needs more general algorithms for problems *PLDE* and *SEF*; some first steps can be found in [Sch01]. Note that our algorithms can be applied for more general difference fields described in [KS06b, KS06a]

Carrying over Paule's greatest factorial factorization [Pau95] (the discrete analogue of greatest squarefree factorization) to the $\Pi\Sigma^*$-case might give further theoretical insight to our algorithmic results. Some steps in this direction can be found in [PR97, BP99].

Following [Pir95] one might refine our algorithms further: given $f(k)$, find $f'(k)$ and $g(k)$ with (2) where among all the degree optimal $f'(k)$ also $g(k)$ is "optimal"; see Remarks 4.1 and 5.2. Special cases have been considered in [AP02, ALP03] for the hypergeometric case.

We presented a simple algorithm in Section 6 that computes, if possible, a summand $f'(k)$ which is less nested than $f(k)$. More general, but also more complicated, algorithms have been proposed in [Sch04c, Sch05b] which find depth-optimal $f'(k)$, see e.g. identity (29). Using results from this article might simplify these general algorithms.

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