Abstract. Let 
\[ \Delta_n(x) = P_n(x)^2 - P_{n-1}(x)P_{n+1}(x), \]
where \( P_n \) is the Legendre polynomial of degree \( n \). A classical result of Turán states that \( \Delta_n(x) \geq 0 \) for \( x \in [-1, 1] \) and \( n = 1, 2, 3, ... \). Recently, Constantinescu improved this result. He established
\[ \frac{h_n}{n(n+1)}(1 - x^2) \leq \Delta_n(x) \quad (-1 \leq x \leq 1; \; n = 1, 2, 3, ...), \]
where \( h_n \) denotes the \( n \)-th harmonic number. We present the following refinement. Let \( n \geq 1 \) be an integer. Then we have for all \( x \in [-1, 1] \):
\[ \alpha_n (1 - x^2) \leq \Delta_n(x) \]
with the best possible factor
\[ \alpha_n = \mu_{\lfloor n/2 \rfloor} \mu_{\lfloor (n+1)/2 \rfloor}. \]
Here, \( \mu_n = 2^{-2n} \binom{2n}{n} \) is the normalized binomial mid-coefficient.

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1. Introduction

The Legendre polynomial of degree \( n \) can be defined by

\[
P_n(x) = \frac{1}{n!} \frac{d^n}{dx^n} (x^2 - 1)^n \quad (n = 0, 1, 2, \ldots),
\]

which leads to the explicit representation

\[
P_n(x) = \frac{1}{2^n} \sum_{\nu=0}^{[n/2]} (-1)^\nu \frac{(2n - 2\nu)!}{\nu!(n-\nu)!(n-2\nu)!} x^{n-2\nu}.
\]

(As usual, \([x]\) denotes the greatest integer not greater than \( x \).) The most important properties of \( P_n(x) \) are collected, for example, in [1] and [14]. Legendre polynomials belong to the class of Jacobi polynomials, which are studied in detail in [3] and [12]. These functions have various interesting applications. For instance, they play an important role in numerical integration; see [11].

The following beautiful inequality for Legendre polynomials is due to P. Turán [13]:

\[
\Delta_n(x) = P_n(x)^2 - P_{n-1}(x)P_{n+1}(x) \geq 0 \quad \text{for } -1 \leq x \leq 1 \text{ and } n \geq 1. \quad (1.1)
\]

This inequality has found much attention and several mathematicians provided new proofs, far-reaching generalizations, and refinements of (1.1). We refer to [8], [10], and the references given therein.

In this paper we are concerned with a remarkable result published by E. Constantinescu [7] in 2005. He offered a new refinement and a converse of Turán’s inequality. More precisely, he proved that the double-inequality

\[
\frac{h_n}{n(n+1)} (1 - x^2) \leq \Delta_n(x) \leq \frac{1}{2} (1 - x^2) \quad (1.2)
\]

is valid for \( x \in [-1, 1] \) and \( n \geq 1 \). Here, \( h_n = 1 + 1/2 + \cdots + 1/n \) denotes the \( n \)-th harmonic number. It is natural to ask whether the bounds given in (1.2) can be improved. In the next section, we determine the largest number \( \alpha_n \) and the smallest number \( \beta_n \) such that we have for all \( x \in [-1, 1] \):

\[
\alpha_n (1 - x^2) \leq \Delta_n(x) \leq \beta_n (1 - x^2).
\]

We show that the right-hand side of (1.2) is sharp, but the left-hand side can be improved. It turns out that the best possible factor \( \alpha_n \) can be expressed in terms of the normalized binomial mid-coefficient

\[
\mu_n = 2^{-2n} \binom{2n}{n} = \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (2n)} \quad (n = 0, 1, 2, \ldots).
\]

We remark that \( \mu_n \) has been the subject of recent number theoretic research; see [2] and [5].

In our proof we reduce the desired refinement of Turán’s inequality to another inequality, which also depends polynomially on Legendre polynomials. This latter inequality is amenable to a recent computer algebra procedure [9, 10]. The procedure sets up a formula that encodes the induction step of an inductive proof of the inequality and, replacing the quantities \( P_n(x), P_{n+1}(x), \ldots \) by real variables \( Y_1, Y_2, \ldots \), transforms the induction step formula into a polynomial formula in finitely many variables. The recurrence relation of the Legendre polynomials translates into polynomial equations in the \( Y_k \), which are added to the induction step formula. The truth of the resulting formula for all real \( Y_1, Y_2, \ldots \) can be decided algorithmically and is a sufficient (in general not necessary!) condition for the truth of the initial inequality, if we assume that sufficiently many initial values have been checked.

\[\text{A nice anecdote about Turán reveals that he used (1.1) as his ‘visiting card’; see [4].}\]
2. Main result

The following refinement of (1.2) is valid.

**Theorem.** Let \( n \) be a natural number. For all real numbers \( x \in [-1,1] \) we have
\[
\alpha_n (1 - x^2) \leq P_n(x)^2 - P_{n-1}(x)P_{n+1}(x) \leq \beta_n (1 - x^2)
\]
with the best possible factors
\[
\alpha_n = \mu_{[n/2]} \mu_{[(n+1)/2]} \quad \text{and} \quad \beta_n = \frac{1}{2}.
\]

**Proof.** We define for \( x \in (-1,1) \) and \( n \geq 1 \):
\[
f_n(x) = \frac{\Delta_n(x)}{1 - x^2}.
\]
We have \( f_1(x) \equiv \alpha_1 = \beta_1 = 1/2 \). First, we prove that \( f_n \) is strictly increasing on \((0,1)\) for \( n \geq 2 \). Differentiation yields
\[
f'_n(x) = \frac{2x\Delta_n(x) + (1 - x^2)\Delta'_n(x)}{(1 - x^2)^2}.
\]
Using the well-known formulas
\[
P'_n(x) = \frac{n+1}{1-x^2}(xP_n(x) - P_{n+1}(x))
\]
and
\[(n + 1)P_{n+1}(x) = (2n + 1)xP_n(x) - nP_{n-1}(x)
\]
we obtain the representation
\[
n(1 - x^2)^2f'_n(x) = (n - 1)xP_n(x)^2 - (2nx^2 + x^2 - 1)P_n(x)P_{n+1}(x) + (n + 1)xP_{n+1}(x)^2.
\]
We prove the positivity of the right-hand side of (2.3) on \((0,1)\) by typing
\[
\text{In[1]}:= << \text{SumCracker.m}
\]
\[
\text{SumCracker Package by Manuel Kauers – © RISC Linz – V 0.3 2006-05-24}
\]
\[
\text{In[2]}:= \text{ProveInequality[}
(n - 1)x\text{LegendreP}[n, x]^2
- (2nx^2 + x^2 - 1)\text{LegendreP}[n, x]\text{LegendreP}[n + 1, x]
+ (n + 1)x\text{LegendreP}[n + 1, x]^2
> 0,
\text{From} \rightarrow 2, \text{Using} \rightarrow \{0 < x < 1\}, \text{Variable} \rightarrow n]
\]
into Mathematica, obtaining, after a couple of seconds, the output
\[
\text{Out[2]}:= \text{True}
\]
It follows from this that \( f_n \) is strictly increasing on \((0,1)\) for \( n \geq 2 \). Since
\[
P_n(x) = (-1)^nP_n(-x),
\]
we conclude that \( f_n \) is even. Thus, we obtain
\[
f_n(0) < f_n(x) < f_n(1) \quad \text{for} \ -1 < x < 1, \ x \neq 0.
\]
We have
\[
P_n(1) = 1 \quad \text{and} \quad P'_n(1) = \frac{1}{2}n(n + 1).
\]
Therefore,
\[
\Delta_n(1) = 0 \quad \text{and} \quad \Delta'_n(1) = -1.
\]
Applying l'Hospital's rule gives

\[(2.5) \quad f_n(1) = \lim_{x \to 1} \frac{\Delta_n(x)}{1-x^2} = -\frac{1}{2} \Delta'_n(1) = \frac{1}{2}.\]

Since

\[P_{2k-1}(0) = 0 \quad \text{and} \quad P_{2k}(0) = (-1)^k \mu_k,\]

we get

\[(2.6) \quad f_{2k-1}(0) = \mu_{k-1} \mu_k \quad \text{and} \quad f_{2k}(0) = \mu_k^2.\]

Combining (2.4)–(2.6) we conclude that (2.1) holds with the best possible factors \(\alpha_n\) and \(\beta_n\) given in (2.2).

\[\square\]

**Remarks.**

(1) The proof of the Theorem reveals that for \(n \geq 2\) the sign of equality holds on the left-hand side of (2.1) if and only if \(x = -1, 0, 1\) and on the right-hand side if and only if \(x = -1, 1\).

(2) The numbers \(\mu_{p+q} \ (p, q = 0, 1, 2, \ldots; p \leq q)\) are the eigenvalues of Liouville's integral operator for the case of a planar circular disc of radius 1 lying in \(\mathbb{R}^3\); see [6].

(3) The automated proving procedure can be applied to (2.1) directly. However, owing to the computational complexity of the method, we did not obtain any output after a reasonable amount of computation time.

(4) The Mathematica package SumCracker used in the proof of the Theorem contains an implementation of the proving procedure described in [9]. It is available online at

\[http://www.risc.uni-linz.ac.at/research/combinat/software\]

(5) The normalized Jacobi polynomial of degree \(n\) is defined for \(\alpha, \beta > -1\) by

\[R_n^{(\alpha, \beta)}(x) = 2F_1(-n, n + \alpha + \beta + 1; \alpha + 1; (1-x)/2).\]

The special case \(\alpha = \beta\) leads to the normalized ultraspherical polynomial

\[R_n^{(\alpha, \alpha)}(x) = 2F_1(-n, n + 2\alpha + 1; \alpha + 1; (1-x)/2) = \frac{(-1)^n}{2^n \alpha_n} \frac{1}{(1-x^2)^{\alpha_n}} d^n x^n (1-x^2)^{n+\alpha},\]

where \((\alpha)_n\) denotes the Pochhammer symbol. Obviously, we have \(R_n^{(0,0)}(x) = P_n(x)\). We conjecture that the following extension of our Theorem holds.

**Conjecture.** Let \(\alpha > -1/2\) and \(n \geq 1\). For all \(x \in [-1, 1]\) we have

\[a_n^{(\alpha)} (1-x^2) \leq R_n^{(\alpha, \alpha)}(x)^2 - R_n^{(\alpha, \alpha)}(x) R_{n-1}^{(\alpha, \alpha)}(x) R_{n+1}^{(\alpha, \alpha)}(x) \leq b_n^{(\alpha)} (1-x^2)\]

with the best possible factors

\[a_n^{(\alpha)} = \mu_{[n/2]}^{(\alpha)} \mu_{[(n+1)/2]}^{(\alpha)} \quad \text{and} \quad b_n^{(\alpha)} = \frac{1}{2(\alpha + 1)}.\]

Here, \(\mu_{[n/2]}^{(\alpha)} = \mu_{n/2}^{(\alpha)}\).
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REFERENCES