APÉRY'S DOUBLE SUM IS PLAIN SAILING INDEED

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Abstract. We demonstrate that also the second sum involved in Apéry's proof of the irrationality of $\zeta(3)$ becomes trivial by symbolic summation.

In his beautiful survey [4], van der Poorten explained that Apéry's proof [1] of the irrationality of $\zeta(3)$ relies on the following fact: If

$$a(n)=\sum_{k=0}^{n}\binom{n+k}{k}^2\binom{n}{k}^2$$

and

$$b(n)=\sum_{k=0}^{n}\binom{n+k}{k}^2\binom{n}{k}^2\left(H_n^{(3)}+\sum_{m=1}^{k}\frac{(-1)^{m-1}}{2m^3\binom{n+m}{m}\binom{n}{m}}\right) \tag{1}$$

where $H_n^{(3)}=\sum_{i=1}^{n}\frac{1}{i^3}$ are the harmonic numbers of order three, then both sums $a(n)$ and $b(n)$ satisfy the same recurrence relation

$$(n+1)^3A(n)-(2n+3)\left(17n^2+51n+39\right)A(n+1)+(n+2)^3A(n+2)=0. \tag{2}$$

Van der Poorten points out that Henri Cohen and Don Zagier showed this key ingredient by "some rather complicated but ingenious explanations" [4, Section 8] based on the creative telescoping method.

Due to Doron Zeilberger's algorithmic breakthrough [9], the $a(n)$-case became a trivial exercise. Also the $b(n)$-case can be handled by skillful application of computer algebra: In [10] Zeilberger was able to generalize the Zagier/Cohen method in the setting of WZ-forms. Later developments for multiple sums [8, 7] together with holonomic closure properties [5, 3] enable alternative computer proofs of the $b(n)$-case; see, e.g., [2].

Nowadays, also the $b(n)$-case is completely trivialized: Using the summation package Sigma [6] we get plain sailing – instead of plane sailing, cf. van der Poorten's statement in [4, Section 8]. Namely, after loading the package into the computer algebra system Mathematica

In[1]:= **<< Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

we insert our sum `mySum` $=b(n)$

In[2]:= $$\mathbf{mySum}=\sum_{k=0}^{n}\binom{n+k}{k}^2\binom{n}{k}^2\left(H_n^{(3)}+\sum_{m=1}^{k}\frac{(-1)^{m-1}}{2m^3\binom{n+m}{m}\binom{n}{m}}\right);$$

and produce the desired recurrence with

In[3]:= **GenerateRecurrence[mySum]**

Out[3]= $\{(n+1)^3\mathrm{SUM}[n]-(2n+3)\left(17n^2+51n+39\right)\mathrm{SUM}[n+1]+(n+2)^3\mathrm{SUM}[n+2]==0\}$

Supported by the SFB-grant F1305 and the grant P16613-N12 of the Austrian FWF.

where $\mathrm{SUM}[n] = b(n) = \texttt{mySum}$. The correctness proof is immediate from the proof certificates delivered by Sigma.

Proof. Set $h(n,k) := \binom{n+k}{k}\binom{n}{k}$, $s(n,k) := \sum_{m=1}^{k} \frac{(-1)^{m-1}}{2m^3\binom{n+m}{m}\binom{n}{m}}$, and let $f(n,k)$ be the summand of (1), i.e., $f(n,k) = h(n,k)^2\big(H_n^{(3)} + s(n,k)\big)$. The correctness follows by the relation

$$s(n+1,k) = s(n,k) - \frac{1}{(n+1)^3} - \frac{(-1)^{k-1}}{(n+1)^2(n+k+1)h(n,k)} \tag{3}$$

and by the creative telescoping equation

$$c_0(n)f(n,k) + c_1(n)f(n+1,k) + c_2(n)f(n+2,k) = g(n,k+1) - g(n,k) \tag{4}$$

with the proof certificate given by $c_0(n) = (n+1)^3$, $c_1(n) = 17n^2+51n+39$, $c_2(n) = (n+2)^3$, and

$$g(n,k) = \frac{h(n,k)^2\Big[p_0(n,k)H_n^{(3)} + p_1(n,k)\sum_{m=1}^{k}\frac{(-1)^{m-1}}{2m^3\binom{n+m}{m}\binom{n}{m}}\Big] + (-1)^k h(n,k)p_2(n,k)}{(n+1)^2(n+2)(-k+n+1)^2(-k+n+2)^2}$$

where

$$\begin{aligned}
p_0(n,k) =&4k^4(n+1)^2(n+2)(2n+3)(2k^2-3k-4n^2-12n-8),\\
p_1(n,k) =&4k^4(n+1)^2(n+2)(2n+3)(2k^2-3k-4n^2-12n-8),\\
p_2(n,k) =&k(k+n+1)(2n+3)(-8n^4+24kn^3-48n^3-31k^2n^2+109kn^2\\
&-104n^2+13k^3n-100k^2n+159kn-96n+21k^3-81k^2+74k-32).
\end{aligned}$$

Relation (3) is straightforward to check: Take its shifted version in k, subtract the original version, and then verify equality of hypergeometric terms. To conclude that (4) holds for all $0 \le k \le n$ and all $n \ge 0$ one proceeds as follows: Express $g(n,k+1)$ in (4) in terms of $h(n,k)$ and $s(n,k)$ by using the relations $h(n,k+1) = \frac{(n-k)(n+k+1)}{(k+1)^2}h(n,k)$ and $s(n,k+1) = s(n,k) + \frac{(-1)^k}{2(k+1)^3h(n,k+1)}$. Similarly, express the $f(n+i,k)$ in (4) in terms of $h(n,k)$ and $s(n,k)$ by using the relations $b(n+1,k) = \frac{n+k+1}{n-k+1}h(n,k)$ and (3). Then verify (4) by polynomial arithmetic. Finally, summing (4) over k from 0 to n gives Out[3] or (2). □

In conclusion, we remark that the harmonic numbers $H_n^{(3)}$ in (1) are crucial to obtain the recurrence relation (2). More precisely, for the input sum

$$\sum_{k=0}^{n}\binom{n+k}{k}^2\binom{n}{k}^2\sum_{m=1}^{k}\frac{(-1)^{m-1}}{2m^3\binom{n+m}{m}\binom{n}{m}}$$

Sigma is only able to derive a recurrence relation of order four.

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