Computation of the Topology Types of the Level Curves of Real Algebraic Surfaces

Juan Gerardo Alcazar [a,1], Josef Schicho [b,2]
Juan Rafael Sendra [a,1],

[a] *Departamento de Matemáticas, Universidad de Alcalá, E-28871-Madrid, Spain*
[b] *RICAM, Austrian Academy of Sciences, A-4040 Linz, Austria*

Abstract

In this paper, we address the problem of determining the z-values where the topology type of the level curves of an algebraic surface may change. In the case when the surface is bounded and non-singular, this question is solved by *Morse Theory*. However, here we consider the problem for the more general case of algebraic surfaces without further restrictions, i.e. not necessarily bounded or smooth. Our results allow to algorithmically compute these z-values by analyzing the real roots of a univariate polynomial; namely, the double discriminant of the implicit equation of the surface. Once this has been done, the different topology types of the level curves of the surface can be computed by means of well-known algorithms.

1 Introduction

In order to determine the topological features of a given real surface S over the real Euclidean space, it may be useful to analyze the topology of the real part of its level curves, i.e. the slices obtained when intersecting S with real planes parallel to the xy coordinate plane.

Email addresses: juangealcazar@yahoo.es (Juan Gerardo Alcazar), josef.schicho@oeaw.ac.at (Josef Schicho), rafael.sendra@uah.es (Juan Rafael Sendra).
[1] This work is partially supported by BMF2002-04402-C02-01 (*Curvas y Superficies: Fundamentos, Algoritmos y Aplicaciones*), and GAIA II (IST-2002-35512).
[2] Supported by the Austrian Science Fund (FWF) in frame of the Special Research Area F1303

For example, consider the algebraic surface S defined by $F(x, y, z) = x^2 + 2y^2 + z^5$. It is obvious that for $z > 0$ the level curves of S are empty; for $z = 0$, the level curve consists of only one real point, and for $z < 0$ the level curves are real ellipses. Thus, in this case there is only one z-value where the topology of the level curves changes, namely $z = 0$; above $z = 0$ one has nothing (in the real Euclidean space), while below $z = 0$ one has real ellipses. This simple analysis allows to make a good mental picture of the surface.

In this paper we address the problem of algorithmically determining the z-values where the topology type of the level curves of a given real algebraic surface S may change; two algebraic plane curves have the same topology type if and only if there exists an homeomorphism of $\mathbb{R}^2$ into itself such that one of the curves is mapped onto the other. Whenever S is bounded and non-singular, this question is solved by *Morse Theory* (see e.g. (5), (13), (18), etc), which shows that these z-values are among the *critical values* (see (5)) of the polynomial F defining S. Furthermore, Morse Theory also explains how the homotopy type changes when one of these z-values is crossed. Nevertheless, if S is either non-bounded or singular, Morse Theory is not applicable. These restrictions, however, do not affect to our study.

We approach the problem by providing a finite set of real numbers which contains the z-values where the topology of the level curves may change. This finite set consists of the real roots of a univariate polynomial; namely, the double discriminant of the implicit equation of the surface. Once this set is computed, the different topology types of the level curves can be easily derived (see for example (2), (14), (15), (16)).

Morse Theory has already been successfully applied to several problems: for example, for the shape determination of real curves and surfaces (see (10)), for the computation of the topology type of non-singular algebraic surfaces (see (9), (11)), or in constructive solid modelling by means of the so-called *digital Morse Theory*, (see for example (1), (4), (6), (12)). The results in this paper, beside the possible applications to these topics, can be applied to the computation of the topology types of the members of a family of plane algebraic curves depending on a parameter; a clear example of this assertion is the analysis of *offset* curves (see (3), (7), (8), (21) for further information on offset curves) where one is interested in studying how the topology of an offset curve varies when the distance changes (see section 5). In addition, our results may have other applications , for instance, to the plotting of algebraic surfaces (for example one can determine a cube of $\mathbb{R}^3$ where the surface should be plotted in order to get a whole idea of its topological behavior), to computer graphics and constructive solid modelling, for the case of surfaces with singularities, to decide the compactness of an implicit algebraic surface, etc.

The paper is structured as follows. In Section 2, we formally present the problem we address, and we introduce some notions. The Section 3 is entirely devoted to the concept of *delineability*; notion that plays an important role in the proof of the main theorem. In Section 4, we provide the main result of the paper. Finally, in Section 5 we present an application of our results to offset curves.

2 Statement of the Problem

This section is devoted to formally state the problem we deal with, and to describe the solution that we provide. For this purpose, we start introducing some notation that will be used throughout the paper.

We assume that $F \in \mathbb{R}[x, y, z]$ is an square-free polynomial defining a real algebraic surface S. Also, we assume that F has no factor only depending on the variable z, and that the leading coefficient of F w.r.t. y does not depend on the variable x. In addition, we exclude the degenerate case where F only depends on the variable z. Note that in this situation, S consists of finitely many planes parallel to the XY plane, and hence it does not make sense to speak about the level curves of S. Furthermore, we do not consider, either, the case when $F \in \mathbb{R}[x, z]$ (i.e. $\deg_y(F) = 0$) because in that case the analysis is reduced to the curve defined by $F(x, z) = 0$ in the XZ plane.

The *topology type* of two plane algebraic curves $\mathcal{C}_1$ and $\mathcal{C}_2$ is the same if and only if there exists an homeomorphism of $\mathbb{R}^2$ into itself such that $\mathcal{C}_1$ is mapped onto $\mathcal{C}_2$. In this paper, we consider the problem of determining the topology type of the level curve of S at $z = z_0$, for every $z_0 \in \mathbb{R}$, i.e. the topology types of the algebraic plane curves $F(x, y, z_0)$ with $z_0 \in \mathbb{R}$. We observe that the two conditions that we have imposed above to the polynomial $F(x, y, z)$ can be assumed w.l.o.g. Indeed, if F has any factor only depending on z then one can write $F(x, y, z) = H(z) \cdot G(x, y, z)$. Thus, the surface S decomposes as the union of a finite family of horizontal planes and a new surface S' where the hypothesis is verified; note that in this case, the topological analysis is reduced to S'. On the other hand, if the leading coefficient of F w.r.t. y depends on the variable x, a suitable linear change of coordinates of the type $\{x = \alpha X + Y, y = Y, z = Z\}$ transforms F into a polynomial satisfying the condition; note that with this transformation the topology type of the level curves stays invariant.

In order to solve the problem, we find a finite set $\mathcal{A} \subset \mathbb{R}$ such that for every open interval $I \subset \mathbb{R}$, with $I \cap \mathcal{A} = \emptyset$, the topology type of the corresponding level curve stays the same, i.e. for every $z_1, z_2 \in I$ the topology type of $F(x, y, z_1)$ and $F(x, y, z_2)$ stays invariant. We will say that such a set $\mathcal{A}$ is a Critical Set of S. Therefore, once that some critical set has been found, one can be sure that the z-values where the topology type of the level curves of S changes are among the (finitely many) elements of $\mathcal{A}$. We will refer to these z-values as the Critical Level Values of S. Since any critical set contains the critical level values, our problem is equivalent to computing one critical set of the surface.

To approach the problem, we consider the following two polynomials, where $D_w(G)$ denotes the discriminant of a polynomial G w.r.t. the variable w, i.e. $D_w(G) = \mathrm{Res}_w(G, \frac{\partial G}{\partial w})$, and where $\sqrt{G}$ denotes the square-free part of a polynomial G, i.e. the product of all the irreducible factors of G taken with multiplicity 1:

$$M(x, z) := \sqrt{D_y(F)}$$
$$R(z) := \begin{cases} 0 & \text{if } \deg_x(M) = 0 \\ D_x(M(x, z)) & \text{otherwise} \end{cases}$$

Furthermore, we will assume that R is not identically zero. The case $R = 0$ (see the last subsection in Section 4) leads to a special situation that can be treated in an easier way.

Under this assumption, we prove that the set

$$\mathcal{A} = \{\xi \in \mathbb{R} \,|\, R(\xi) = 0\}.$$

consisting of the real roots of $R(z)$, is a critical set of S. Therefore, writing $\mathcal{A} = \{\xi_1, \ldots, \xi_r\}$, we can decompose the z-axis as

$$(-\infty, \xi_1) \cup \{\xi_1\} \cup (\xi_1, \xi_2) \cup \cdots \cup (\xi_{r-1}, \xi_r) \cup \{\xi_r\} \cup (\xi_r, \infty).$$

Thus, taking a particular value in each open interval of the partition, and applying the existing algorithms for computing the topology type of a plane algebraic curve, one determines the topology type of all the level curves associated with the interval. The remaining finitely many level curves, that is $F(x, y, \xi_i)$, $i = 1, \ldots, r$, are also analyzed with the same strategy; note that in these cases, the topology may agree with either its left or its right intervals, or it may create a new type.

The topology type of a plane algebraic curve can be described by means of a planar *graph* associated with the curve (see (5)). The problem of computing this graph has been addressed by many authors (see for example (2), (14), (15), (16) and many others). The vertices of this graph correspond to the intersection points of the curve with the vertical lines passing through its critical points; furthermore, each edge of the graph corresponds to a branch of the curve joining two vertices.

Also, throughout the paper we will use the term *graph* for denoting the graph associated with a curve in the sense described above, and we will use the expression *function graph* to refer to the graph of a given function.

In addition to the notation introduced at the beginning of this section, namely the surface S and its defining polynomial $F(x, y, z)$, we fix throughout this paper the following notation and terminology:

- $\mathcal{M}$ is the affine plane algebraic curve defined by $M(x, z)$, whenever it is not a constant polynomial.
- for $a \in \mathbb{R}$, we denote by π_a the plane of equation $z - a = 0$.
- since $z - a$ is not a factor of F, it follows that the intersection of π_a and S defines a plane algebraic curve of equation $F(x, y, a) = 0$. We will refer to this plane curve as the *level curve of* S at $z = a$, and we will denote it by S_a.
- for $a \in \mathbb{R}$, we will write $F_a(x, y) = F(x, y, a)$.
- we will use the standard notions of critical, singular and regular point of an affine plane algebraic curve $\mathcal{C}$ without multiple components (see e.g. (5)), namely if $f(x, y)$ is the defining polynomial of $\mathcal{C}$, then: $P \in \mathcal{C}$, is a critical point of $\mathcal{C}$, if $f(P) = \frac{\partial f}{\partial y}(P) = 0$; P is a singular point of $\mathcal{C}$, if $f(P) = \frac{\partial f}{\partial x}(P) = \frac{\partial f}{\partial y}(P) = 0$; and P is a regular point of $\mathcal{C}$, if it is non-critical.

3 Delineability

The concept of *analytic delineability* is a fundamental tool of our reasoning. In this section, we recall the definition of delineability as well as some theoretical results.

The notion of delineability appears in the context of Cylindrical Algebraic Decomposition (see, e.g. (5), (17)). More precisely, one has the following definition (see (17) pp. 245)

Definition 1 *Let $\breve{x}$ denote the $(r-1)$-tuple $(x_1, \ldots, x_{r-1})$. An r-variate polynomial $f(\breve{x}, x_r)$ over the reals is said to be* (analytic) delineable *on a submanifold $\mathcal{T}$ of $\mathbb{R}^{r-1}$, if it holds that:*

1. *the portion of the real variety of f that lies in the cylinder $\mathcal{T} \times \mathbb{R}$ over $\mathcal{T}$*

consists of the union of the function graphs of some $k \geq 0$ analytic functions $\vartheta_1 < \cdots < \vartheta_k$ from $\mathcal{T}$ into $\mathbb{R}$,

2. *there exist positive integers $m_1, \ldots, m_k$ such that for every $a \in \mathcal{T}$, the multiplicity of the root $\vartheta_i(a)$ of $f(a, x_r)$ (considered as a polynomial in x_r alone) is m_i.*

Furthermore, the ϑ_i in the condition 1 of the definition above are called real root functions *of f on $\mathcal{T}$, the function graphs of the ϑ_i are called f-*sections *over $\mathcal{T}$, and the regions between successive f-sections are called f-*sectors *over $\mathcal{T}$.*

It can be proved (see (17), theorems 2.2.3. and 2.2.4.) that each f-section and f-sector of an analytic delineable polynomial are connected submanifolds.

Observe that, intuitively speaking, if a polynomial $G(x, y)$ is delineable on a subset $T \subset \mathbb{R}$, this means that over that subset, the real part of the curve defined by G consists of the union of finitely many *non-intersecting* curves, which correspond to the analytic functions ϑ_i of Definition 1. Similarly, if $F(x, y, z)$ is delineable on a subset $R \subset \mathbb{R}^2$, this implies that the real part of the surface defined by F over this subset is the union of finitely many *non-intersecting surfaces*, corresponding to the ϑ_i. These ideas are illustrated in Figure 1.

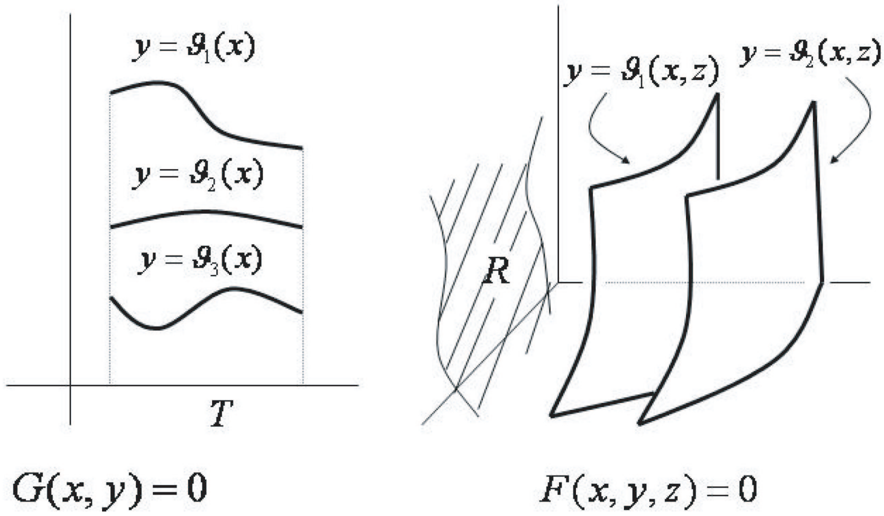

Fig. 1. Delineability

In order to state sufficient conditions for an r-variate polynomial to be delineable on a subset of $\mathbb{R}^{r-1}$, we need to introduce some additional concepts.

Definition 2 *A polynomial $g(x_1, \ldots, x_r)$ has* order *m at $p \in \mathbb{R}^r$, if m is the* *least non-negative integer such that some partial derivative of g of total order m does not vanish at p. If g does not vanish at p, we say that the order of g at p is 0. Furthermore, if g and all its derivatives vanish at p, we say that the order of g at p is ∞. Also, we say that g is* order-invariant *over a subset S of $\mathbb{R}^n$ if the order of g is the same for all $p \in S$. In addition, an r-variate polynomial $f(\breve{x}, x_r)$ over $\mathbb{R}$, with $\breve{x} = (x_1, \ldots, x_{r-1})$, is said* degree-invariant *on a subset $\mathcal{T}$ of $\mathbb{R}^{r-1}$ if the degree of $f(p, x_r)$ (as a polynomial in x_r) is the same for every point $p \in \mathcal{T}$.*

In this situation the following theorem holds (see (17) pp.246).

Theorem 3 *Let $r \geq 2$, and let $f(\breve{x}, x_r)$ be a polynomial in $\mathbb{R}[\breve{x}, x_r]$ of positive degree in x_r. Let $D(\breve{x})$ be the discriminant of $f(\breve{x}, x_r)$ and suppose that $D(\breve{x})$ is a nonzero polynomial. Let $\mathcal{T}$ be a connected submanifold of $\mathbb{R}^{r-1}$ on which f is degree-invariant and does not vanish identically, and over which D is order-invariant. Then, f is analytic delineable on $\mathcal{T}$ and is order-invariant over each f-section over $\mathcal{T}$.*

4 The Main Result

In this section, we will prove that, assuming that the polynomial $R(z)$ is not identically 0, the set of its real roots is a critical set of the surface, so it contains the z-values where the topology type of the level curves of S may change. More precisely, the main result of the section can be stated as follows.

Theorem 4 *If R is not identically zero, then the set*

$$\mathcal{A} := \{\xi \in \mathbb{R} \,|\, R(\xi) = 0\}$$

is a critical set of S. That is, if $z_1, z_2 \in \mathbb{R}$, $z_1 < z_2$, verify that no element in $[z_1, z_2]$ is a root of $R(z)$, then S_{z_1} and S_{z_2} have the same topology type.

The case when $R = 0$ is addressed at the end of the section (see the last subsection here)[3]. In order to prove Theorem 4, we will distinguish the following three phases:

Phase 1 (Delineability of F): using the results of Section 3, we prove the delineability of F, from where several analytic functions are defined.

[3] I have eliminated the section on special cases and I have left the case $R = 0$ as a subsection here

Phase 2 (Proper behavior of the real roots of F): we prove that the analytic functions, obtained in the Phase 1, behave properly.

Phase 3 (Graph Equality): finally, using the results of Phase 2, we prove that the graphs of the level curves can be taken equal, so the topology type of the level curves is the same.

Each of these phases will be addressed along the following subsections. For this purpose, in the rest of the section we assume that z_1 and z_2 are two distinct real numbers satisfying the hypotheses of Theorem 4; i.e. $z_1 < z_2$, and no element in $[z_1, z_2]$ is a root of $R(z)$. Furthermore, we will denote by $J = (q_1, q_2)$, where $q_1, q_2 \in \mathbb{R}$, an open real interval containing $[z_1, z_2]$ and verifying that $J \bigcap \mathcal{A} = \emptyset$.

Phase 1 of the Proof: Delineability of F

Here, we proceed to analyze the delineability of F on certain sets. For this purpose, first the delineability of M on the open interval J must be stated. In order to do this, the following previous lemma is required. [4]

Lemma 5 *If $a \in \mathbb{R}$ verifies anyone of the following conditions:*

(i) a is a root of the leading coefficient of F w.r.t. y (we recall that by hypothesis this leading coefficient is in $\mathbb{R}[z]$),
(ii) a is a root of the leading coefficient of $M(x, z)$ w.r.t. x,
(iii) the polynomial $F(x, y, a)$ has multiple factors,

then $R(a) = 0$; i.e. $a \in \mathcal{A}$.

PROOF. It follows from standard properties of resultants. □

This lemma allows to prove that M verifies the hypotheses of Theorem 3 over the set J, so the following result follows.

Theorem 6 *The polynomial $M(x, z)$, seen as a univariate polynomial in x, is analytic delineable on J. Moreover, M is order-invariant over each M-section*

[4] In the previous version we had two previous lemmas, but in fact the second one just made more clear the geometric meaning of M; thus, I have eliminated it

over J.[5]

From Theorem 6, one has that there exist analytic real functions

$$X_1 < \cdots < X_r$$

which are the real root functions of M over J. Let $\mathcal{X}_k$, with $k = 1, \ldots, r$, denote the function graph of X_k; i.e. the $\mathcal{X}_k$ are the M-sections over J. In addition, we introduce the following regions in the XZ plane (see Figure 2):

$$R_1 = \{(x, z) \in \mathbb{R}^2 \mid z \in J, x < X_1(z)\},$$

$$R_k = \{(x, z) \in \mathbb{R}^2 \mid z \in J, X_{k-1}(z) < x < X_k(z)\} \text{ for } k \in \{2, \ldots, r\},$$
$$R_{r+1} = \{(x, z) \in \mathbb{R}^2 \mid z \in J, x > X_r(z)\}.$$

Note that the sets R_k, for $k \in \{2, \ldots, r\}$, are the M-sectors over J.

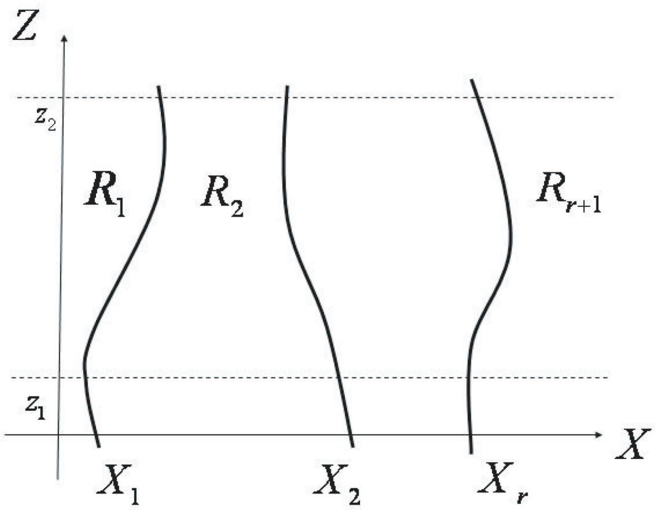

Fig. 2. Regions R_k

In this situation, one has the following theorem on the delineability of F

Theorem 7

(i) The polynomial $F(x, y, z)$, seen as a univariate polynomial in y, is analytic delineable on R_k for every $k = 1, \ldots, r + 1$.

(ii) The polynomial $F(x, y, z)$, seen as a univariate polynomial in y, is analytic delineable on $\mathcal{X}_k$ for every $k = 1, \ldots, r$.

[5] This last statement, i.e. the order-invariance of M on J, did not appear in the previous version, but it has to be used in the proof of Theorem 8

PROOF.

In order to prove the theorem, we check the conditions in Theorem 3. For this purpose, first we observe that since the polynomial $R(z)$ is not identically 0, then $\deg_y(F) > 0$. Also, note that $H := D_y(F)$ is non-zero, because otherwise we would have that $M = 0$ and hence $R = 0$. Thus, let us see that F is degree invariant in R_k and in $\mathcal{X}_k$. In fact, F is degree invariant in $\mathbb{R} \times J$ because of Lemma 5, and the degree of F w.r.t. y in $\mathbb{R} \times J$ is $\deg_y(F)$. Since $\deg_y(F) > 0$, in particular F does not vanish identically on R_k and $\mathcal{X}_k$. Hence, it remains to prove that H is order-invariant, and in order to do this we distinguish the two cases in the statement of the theorem:

(i) For any point in R_k the order of H is 0, because the order of M is 0 by definition of R_k.

(ii) Let us see that H is order-invariant on $\mathcal{X}_k$. From Theorem 6, M is order-invariant on the M-sections over J, i.e. on each $\mathcal{X}_k$. Moreover, since M is square-free, the plane curve $\mathcal{M}$ does not have multiple components, and therefore it has finitely many singularities. Thus, since M is order-invariant on each $\mathcal{X}_k$, one deduces that all points on $\mathcal{X}_k$ are simple points of M, and hence the order of M on $\mathcal{X}_k$ is 1. Now, let M be expressed as $M = A_1 \cdots A_s$, where the A_i are square-free and relatively prime, and let $H = A_1^{n_1} \cdots A_s^{n_s}$. In this situation, let us take any point $p := (x_0, z_0) \in \mathcal{X}_k$, and let us see that the order of H at p is invariant. For this purpose, we observe that if $k \neq j$ then $\mathcal{X}_k \cap \mathcal{X}_j = \emptyset$. Therefore, p belongs only to one of the components of the $\mathcal{M}$, and hence only one factor of M vanishes at p, say A_i. Then, the order of h at p is n_i. $\square$

Now, taking into account the delineability of F, we introduce some additional notation.

- Since F is delineable in R_k, for $k \in \{1, \ldots, r+1\}$, we denote by

$$V_{1,k} < \cdots < V_{s_k,k}$$

  the real root functions of F over R_k, and by $\mathcal{V}_{i,k}$ the function graph of the analytic function $V_{i,k}$; i.e. the F-sections over R_k.

  In the sequel, for simplicity of notation, whenever we are working on a fixed R_k, we will write V_i instead of $V_{i,k}$, and $\mathcal{V}_i$ instead of $\mathcal{V}_{i,k}$.

- Since F is delineable in $\mathcal{X}_k$, for $k \in \{1, \ldots, r\}$, we denote by

$$Y_{1,k} < \ldots < Y_{\ell_k,k}$$

the real root functions of F over $\mathcal{X}_k$, and by $\mathcal{Y}_{j,k}$ the function graph of the analytic function $Y_{j,k}$; i.e. the F-sections over $\mathcal{X}_k$.

In the sequel, for simplicity of notation, whenever we are working on a fixed $\mathcal{X}_k$, we will write Y_j instead of $Y_{j,k}$, and $\mathcal{Y}_j$ instead of $\mathcal{Y}_{j,k}$.

In Figure 3 we give a geometrical interpretation of these functions; note, however, that in Figure 3 it is implicitly assumed that the $\mathcal{V}_i$ and the $\mathcal{Y}_j$ join properly (i.e. for a given $\mathcal{V}_i$ there is only one $\mathcal{Y}_j$ that lies in the closure of $\mathcal{V}_i$). This property corresponds to the *"proper behavior"* of the real roots of F which is rigorously established in Phase 2 (see Lemma 11). In fact, this will be the key for proving Theorem 4.

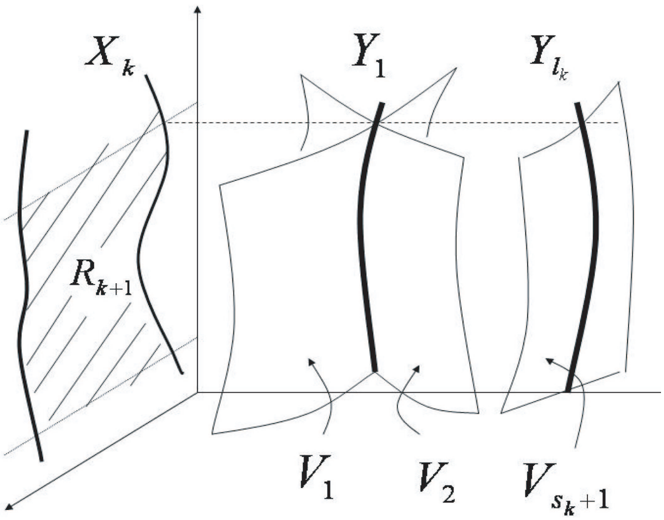

Fig. 3. Functions V_i and Y_j

Phase 2 of the Proof: Proper behavior of real roots of F

In this subsection, we will prove that the functions V_i and Y_j, introduced in Phase 1, behave properly in the following sense: the function graphs $\mathcal{V}_i$ and $\mathcal{Y}_j$ join properly, i.e. for each $\mathcal{V}_i$ there is only one $\mathcal{Y}_j$ in the closure of $\mathcal{V}_i$ (in the topological space $\mathbb{R}^3$ with the usual topology). Therefore, we will rigorously establish that Figure 3 is correct.

For this purpose, let us introduce the following notion: throughout this subsection, if $W \subset \mathbb{R}^n$, we will refer to the set $W \times \mathbb{R} \subset \mathbb{R}^{n+1}$ as the cylinder of W, and we will denote it by $\mathrm{Cyl}(W)$. Moreover, we assume that z_1, z_2 are as in the statement of Theorem 4, and that they are fixed.

We start with some technical lemmas. We fix $k \in \{1, \ldots, r\}$, and we consider the real roots of F over the region R_k, which for sake of simplicity are denoted as $\{V_1, \ldots, V_{s_k}\}$; moreover, we consider the real root functions of F over $\mathcal{X}_k$, which also for simplicity are denoted as $\{Y_1, \ldots, Y_{\ell_k}\}$. Note that R_k lies at the left of $\mathcal{X}_k$ (see Figure 2). Thus, in this phase of the proof we study the behavior of the $\{V_1, \ldots, V_{s_k}\}$ and the $\{Y_1, \ldots, Y_{\ell_k}\}$. Similarly one might study the relationship between the real roots $\{Y_1, \ldots, Y_{\ell_k}\}$ and the real roots of F over R_{k+1}, where R_{k+1} lies in this case at the right of $\mathcal{X}_k$.

Now, let T_i, where $i \in \{1, \ldots, s_k\}$, be the intersection of the following three sets:

- the cylinder of $\mathcal{X}_k$; i.e. $\mathrm{Cyl}(\mathcal{X}_k) = \{(x, y, z) \,|\, (x, z) \in \mathcal{X}_k,\ y \in \mathbb{R}\}$,
- the set $\mathcal{Z} = \{(x, y, z) \in \mathbb{R}^3 |\ z \in [z_1, z_2]\}$,
- and the closure $\overline{\mathcal{V}}_i$ of $\mathcal{V}_i$

That is:

$$T_i = \mathrm{Cyl}(\mathcal{X}_k) \cap \mathcal{Z} \cap \overline{\mathcal{V}}_i.$$

In this situation, the following result on T_i holds.

Lemma 8

(i) The y-projection maps T_i surjectively onto the set $\mathcal{X}_k \bigcap \mathcal{Z}$.

(ii) T_i is contained in the union of the F-sections over $\mathcal{X}_k$; that is

$$T_i \subset \cup_{j=1}^{\ell_k} \mathcal{Y}_j.$$

PROOF. In order to prove (i), we consider $\bar{z} \in [z_1, z_2]$, and we find a point of T_i whose z-coordinate is $\bar{z}$. For this purpose, we first observe that the intersection of the horizontal plane $\pi_{\bar{z}}$ with $\mathcal{V}_i$ is a real branch of the level curve $S_{\bar{z}}$. Moreover, since $[z_1, z_2] \cap \mathcal{A} = \emptyset$ and $\bar{z} \in [z_1, z_2]$, one has that $\bar{z} \notin \mathcal{A}$. Furthermore, we recall that by hypothesis the leading coefficient of F w.r.t. y does not depend on x, and by Lemma 5, since $\bar{z} \notin \mathcal{A}$, it holds that the leading coefficient w.r.t. y of F does not vanish at $\bar{z}$. Therefore, the plane algebraic curve $S_{\bar{z}}$ has no asymptotes normal to the x-axis (see (19) for further information on the notion of asymptote). Thus, one can find a sequence (x_n, y_n) of real points of $\mathcal{V}_i$ (i.e. $y_n = V_i(x_n)$) where x_n converges to $\bar{x} = X_k(\bar{z})$, and such that y_n converges to $\bar{y} \in \mathbb{R}$, with $(\bar{x}, \bar{y}) \in S_{\bar{z}}$. In this situation, one has that the point $\bar{P} = (\bar{x}, \bar{y}, \bar{z})$ belongs to T_i. Therefore, (i) is proved. Now, in order to prove (ii) note that since $\mathcal{V}_i \subset S$ and S is closed, one has that $\overline{\mathcal{V}}_i \subset S$, and therefore $T_i \subset S$. Since $\cup_{j=1}^{\ell_k} \mathcal{Y}_j$ is the part of S projecting onto $\mathcal{X}_k$, the statement (ii) follows from statement (i). □

The next step consists in proving that the set T_i is connected. This fact can be proven by a topological argument, which could be an exercise in a textbook on topology. The proof of this statement, which follows from Urysohn's Lemma, is left to the reader.

Lemma 9 *Let $\{A_i\}_{i\in I}$ be a family of compact and connected non-empty subsets of $\mathbb{R}^n$ such that A_{i+1} is contained in A_i for each $i \in I$. Then, the intersection $A = \bigcap_{i\in I} A_i$ is also connected.*

This lemma provides the following result:

Lemma 10 *The set T_i is connected.*

PROOF. For $n \in \mathbb{N}$ we define the set (see Figure 4):

$$h_n = \left\{(x, z) \in \mathbb{R}^2 | z \in [z_1, z_2] \text{ and } X_k(z) - \frac{1}{n} < x < X_k(z))\right\}.$$

Now, we take a natural number $n_0 \in \mathbb{N}$ such that for $n \geq n_0$ it holds that $h_n \cap \mathcal{X}_{k-1} = \emptyset$ if $\mathcal{X}_k$ is not the most left M-section over J (i.e. such that $h_n \subset R_k$), otherwise we take $n_0 = 1$. Now, taking into account that for $n \geq n_0$

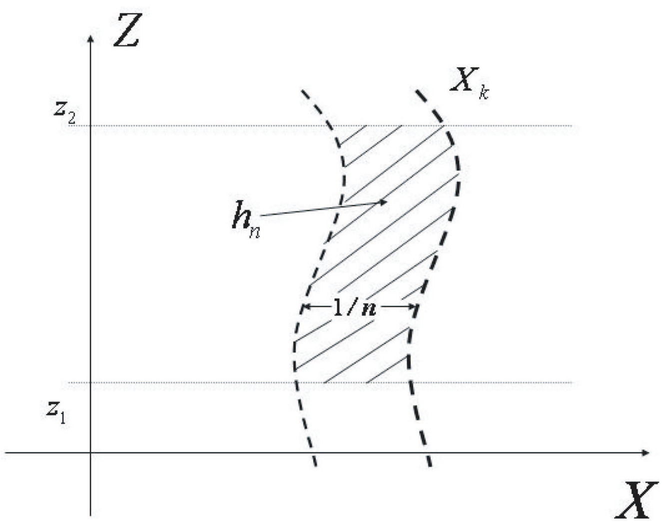

Fig. 4. The sets h_n

it holds that $h_n \subset R_k$, we define for $n \geq n_0$ the set A_n as the closure of the image of h_n by means of the function V_i (recall that V_i is the analytic function whose function graph appears in the definition of T_i), i.e.

$$A_n = \overline{V_i(h_n)}.$$

Let us see that the family $\{A_n\}_{n\geq n_0}$ verifies the hypotheses of Lemma 9. Indeed, it is clear that $A_{n+1} \subset A_n$, and $A_n \neq \emptyset$. Furthermore, since h_n is connected and V_i is continuous then $V_i(h_n)$ is connected, and hence A_n is connected. Thus, it rests to see that A_n is compact. For this purpose, observe that A_n is obviously closed and bounded w.r.t. the variables x and z, so it suffices to check that it is also bounded w.r.t. the variable y. But this follows from the fact that no level curve of S has any asymptote normal to the x-axis. [6]
□

Therefore, we see that the family $\{A_n\}_{n\geq n_0}$ verifies the hypotheses of Lemma 9. Hence, by Lemma 9 it follows that $\bigcap_{n\geq n_0} A_n$ is connected. Moreover, it is easy to check that $\bigcap_{n\geq n_0} A_n = T_i$. Hence, it follows that T_i is connected. □

The preceding results are used to state the following lemma, that certifies that the sets $\mathcal{V}_i$ and $\mathcal{Y}_j$ join properly.

Lemma 11 *For a given $\mathcal{V}_i$, with $i \in \{1, \ldots, s_k\}$, there exists a unique $j \in \{1, \ldots, \ell_k\}$ such that*

$$\mathcal{Y}_j \subset \overline{\mathcal{V}_i}.$$

Furthermore, $\mathcal{Y}_j = T_i$.

PROOF. Let us fix $i \in \{1, \ldots, s_k\}$, and let us consider the set T_i corresponding to $\mathcal{V}_i$. Now, let us see that there exists a unique $j \in \{1, \ldots, \ell_k\}$ such that $T_i = \mathcal{Y}_j$; since $T_i \subset \overline{\mathcal{V}_i}$, then the result follows. Indeed, from Lemma 10, we know that T_i is a connected set, and from Lemma 8, we have that it is included in the union of $\mathcal{Y}_1, \ldots, \mathcal{Y}_{\ell_k}$. Since $\mathcal{Y}_1, \ldots, \mathcal{Y}_{\ell_k}$ are all connected and disjoint, it holds that there exists $j \in \{1, \ldots, \ell_k\}$ such that T_i is included in $\mathcal{Y}_j$. Finally, T_i must be equal to $\mathcal{Y}_j$ because otherwise the projection of T_i onto $\mathcal{X}_k \bigcap \mathcal{Z}$ would not be surjective (see Lemma 8). □

Proceeding in a similar way, an analogous result to Lemma 11 relating the real roots of F over $\mathcal{X}_k$ and the real roots of F over R_{k+1}, where $k \in \{1, \ldots, r\}$, might also be obtained.

In the previous development, we have simplified the notation so the real roots of F over R_k, i.e. the functions $V_{i,k}$ with $i \in \{1, \ldots, s_k\}$, were denoted by

[6] JPAA's refree mentioned that this argument was enough. I also think that it is sufficiently clear. Perhaps the argument which we used was more formal, but I think it may be a good idea to include this simple reasoning

V_i, assuming that we were working on a fixed R_k; similarly, the real roots of F over $\mathcal{X}_k$, i.e. the functions $Y_{j,k}$ with $j \in \{1,\ldots,l_k\}$, were also denoted by Y_j. Now, let us recover the notation with two subindexes. Thus, taking into account Lemma 11, there exist functions $\mathcal{R}_1,\ldots,\mathcal{R}_r$

$$\mathcal{R}_k : \{1,\ldots,s_k\} \longrightarrow \{1,\ldots,l_k\}$$

such that

$$\mathcal{Y}_{\mathcal{R}_k(i),k} \subset \bar{\mathcal{V}}_{i,k}.$$

Therefore, $\mathcal{R}_k$ maps a subindex i, which corresponds to a real root of F over R_k, onto a subindex $\mathcal{R}_k(i)$ that corresponds to the real root of F over $\mathcal{X}_k$ which lies in the closure. See, for example, Figure 5: here, one has that $\mathcal{R}_k(1) = \mathcal{R}_k(2) = 1$ because it holds that $\mathcal{Y}_{1,k} \subset \bar{\mathcal{V}}_{1,k}$ and also $\mathcal{Y}_{1,k} \subset \bar{\mathcal{V}}_{2,k}$.

Similarly, one may define functions $\mathcal{L}_1 \ldots, \mathcal{L}_r$

$$\mathcal{L}_k : \{1,\ldots,s_{k+1}\} \longrightarrow \{1,\ldots,l_k\}$$

such that

$$\mathcal{Y}_{\mathcal{L}_k(i),k} \subset \bar{\mathcal{V}}_{i,k+1}.$$

Thus, $\mathcal{L}_k$ maps a subindex i corresponding to a real root of F over R_{k+1}, onto a subindex $\mathcal{L}_k(i)$ corresponding to the real root of F over $\mathcal{X}_k$ which lies in the closure. For example, in Figure 5 one has that $\mathcal{L}_{k+1}(1) = \mathcal{L}_{k+1}(2) = 1$.

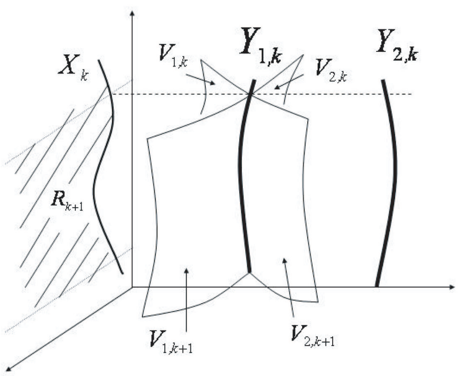

Fig. 5. Functions $\mathcal{R}_k$ and $\mathcal{L}_k$

Remark 12 *Note that the functions $\mathcal{L}_k$ and $\mathcal{R}_k$ need be neither surjective nor injective. In Figure 5 one may see this phenomenon.*[7]

[7] Originally the remark was longer

Phase 3 of the Proof: Graph Equality

In order to prove that the level curves S_{z_1} and S_{z_2} have the same topology type, we give a construction of a planar graph which[8] describes the topology type of any level curve in the interval $[z_1, z_2]$. This fact follows immediately from the results in the preceding subsection, specially Lemma 11. In other words, this planar graph is associated with any of these level curves in the sense of (16), (14), etc.

Let us see how the planar graph construction works. First, we introduce the vertices. Thus, for each $k \in \{1, \ldots, r\}$, we add vertices with coordinates (see Figure 6)

$$(k, 1), \ldots, (k, l_k).$$

We will refer to these vertices as *integer vertices*; furthermore, recall that l_k was the number of F-sections over $\mathcal{X}_k$. Moreover, for each $k \in \{1, \ldots, r+1\}$, we add vertices of coordinates (see Figure 7)

$$(k - \frac{1}{2}, 1), \ldots, (k - \frac{1}{2}, s_k);$$

recall that s_k was the number of F-sections over R_k.

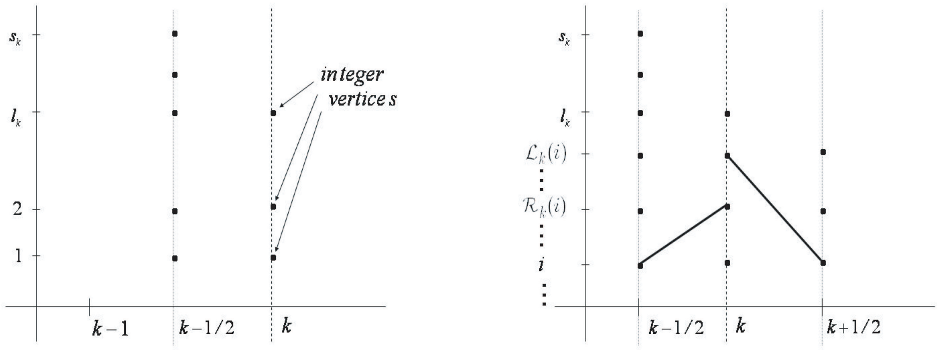

Fig. 6. Vertices (left), Edges (right)

Now, we introduce the edges; for this purpose, we use the functions $\mathcal{L}_k$ and $\mathcal{R}_k$ defined in Phase 2. Thus, for each $k \in \{1, \ldots, r\}$ and for each $i \in \{1, \ldots, s_k\}$, we join the vertices $(k - \frac{1}{2}, i)$ and $(k, \mathcal{R}_k(i))$ (see Figure 6), and we also connect

[8] I have eliminated some words

the vertices $(k, \mathcal{L}_k(i))$ and $(k+\frac{1}{2}, i)$ (see also Figure 6); note that the vertex $(k+\frac{1}{2}, i)$ can be written as $((k+1)-\frac{1}{2}, i)$, and so it has been added previously. [9]

We denote by $\mathcal{G}$ to the graph constructed by the process described above. By construction, it follows that this graph describes the topology type of the level curves. Hence, the following theorem holds.

Theorem 13 *For every $\bar{z} \in [z_1, z_2]$, the topology type of the level curve $S_{\bar{z}}$ is described by the graph $\mathcal{G}$. In other words, there is a homeomorphism of $\mathbb{R}^2$ into itself such that $S_{\bar{z}}$ is mapped onto $\mathcal{G}$.* [10]

Now, the main theorem, namely Theorem 4, follows directly from Theorem 13.

THE SPECIAL CASE $R = 0$ [11]

In the preceding subsections, we have assumed that the polynomial $R(z)$ was not identically 0. Here, we will consider the special case where $R = 0$. Now, taking into account the definition of the polynomial R in Section 2, $R = 0$ implies that $\deg_x(M) = 0$, i.e. $M = M(z)$. In this situation, the following theorem holds:

Theorem 14 *If $\deg_x(M) = 0$, then the set $\mathcal{A}$ of all the real roots of $D_y(F)$ is a critical set of S. That is, if $z_1, z_2 \in \mathbb{R}$, $z_1 < z_2$, verify that no element in $[z_1, z_2]$ is a root of $D_y(F)$, then S_{z_1} and S_{z_2} have the same topology type.*

PROOF. Let $\mathcal{A} = \{a_1, \ldots, a_r\}$, with $a_1 < \cdots < a_r$, be the set of real roots of M; i.e of $\sqrt{D_y(F)})$; note that M only depends on z. Also, let $a_0 = -\infty$, $a_{r+1} = +\infty$. Then, for $k = 1, \ldots, r+1$, we introduce the sets

$$R_k = \{(x, z) \in \mathbb{R}^2 | a_{k-1} < z < a_k\}.$$

Note that the R_k are horizontal, non-bounded, stripes of the xz plane. Reasoning similarly as in Theorem 7, one proves that F, seen as a univariate polynomial y, is delineable over R_k. This implies that there exist analytic real functions $V_{1,k} < \cdots < V_{s_k,k}$ (i.e. the real root functions of F over R_k), such that the real part of S over R_k consists of the union of the function graphs of these functions. Now, since the function graphs of the V_i are non-intersecting,

[9] In the previous version, there were a graphic showing the process in the situation of Figure 5. I have also eliminated it

[10] I have eliminated the proof of this theorem, as well as a previous lemma

[11] I have eliminated a graphic here

one has that the level curve S_{z_1} consists of s_k non-intersecting real branches, and the same happens for S_{z_2}. Therefore both level curves have the same topology type. □

5 Application to offsets

In this section, we show an application of the results in this paper to the topological behavior of offset curves. The notion of offset is directly related to the concept of envelope. More precisely, the offset curve, at distance d, to an irreducible curve $\mathcal{C}$ over $\mathbb{C}$ is "essentially" the envelope of the system of circles centered at the points of $\mathcal{C}$ with fixed radius d.

More formally, the *offset curve* to the plane curve $\mathcal{C}$ at distance d is defined as the Zariski closure in $\mathbb{C}^2$, of the constructible set $\mathcal{A}_d(\mathcal{C}_0)$ in $\mathbb{C}^2$ of the intersection points of the circles of radius $d \in \mathbb{C}$ centered at each point $P \in \mathcal{C}_0$ and the normal line to $\mathcal{C}$ at P, where $\mathcal{C}_0 \subset \mathcal{C}$ is the set of all regular points of $\mathcal{C}$ having nonzero isotropic normal vectors to $\mathcal{C}$ (for further details see (3) and (21)).

Offsets play an important role in many practical applications such as tolerance analysis, geometric control, robot path-planning and numerical-control machining problems, like the description of the curve that a cylindrical tool executes when it moves through a prescribed path. The study of offsets is an active research area. Indeed, as a consequence of this research, many interesting questions related to algebraic geometry, such as the study of the unirationality of the components of the offset, or the construction of rational parametrizations of the components of an offset, or the analysis of algebraic and geometric properties of the offset in terms of the corresponding properties of the initial variety (e.g., geometric genus offset curves, offset degree, etc), or the development of special implicitization techniques for offsets, have been treated.

Nevertheless, topological aspects for offsets have not been treated so extensively; some local results can be found in (7) and (8). The intuitive interpretation of the notion of offset curve suggests that the topology type of the original curve should be somehow "duplicated" by the offsetting process. However, this does not necessarily happen (see for example (7), (8)). In fact, the topological features of the offset curves of a given $\mathcal{C}$ may vary as d takes different values, and the study of this topological behavior was an open problem.

Now, with the results introduced in this paper, we see how to solve this problem. More precisely, if one computes the implicit equation $F(x, y, d)$ of the offset curve, where the distance d is treated as a new variable, one can see $F(x, y, d)$ as the implicit equation of an algebraic surface. In this situation, the topology type of the level curves of this new surface indicates how the topology of the offset behaves. Let us illustrate these ideas by two examples.

Example 1. The implicit equation of the offset to the parabola $y - x^2 = 0$ at distance d is:

$F(x,y,d) = -1/16y^2 + 2x^2d^2y^2 - 1/2x^2yd^2 + 1/16d^2 + 5/4x^2d^2 - 2x^2y^2 + 1/2d^2y^2 + 1/8yx^2 - 1/2yd^2 + 3x^4d^2 - x^4y^2 - 3x^2d^4 + 5/2x^4y + 2x^2y^3 - d^4y^2 - 2d^4y + 2d^2y^3 - 1/16x^4 + 1/2d^4 + 1/2y^3 - x^6 + d^6 - y^4$

We consider the surface S defined by $F(x, y, d)$. In order to study the level curves of S corresponding to $d > 0$ (note that for $d < 0$ one has an equivalent behavior), we compute the roots of the double discriminant $R(d)$, and we find that the only real root, greater than 0, is $1/2$.

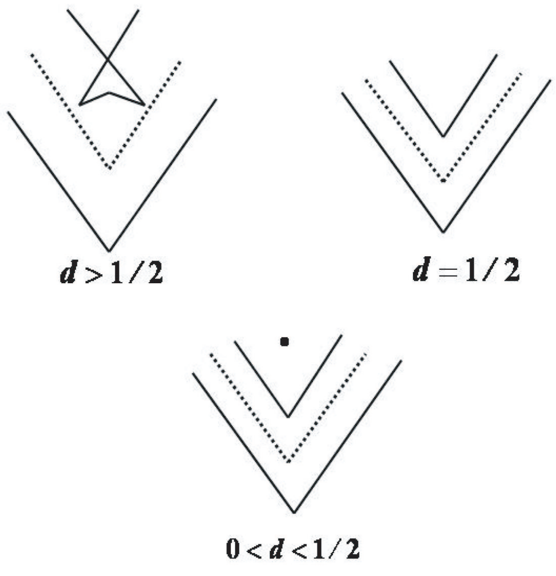

Fig. 7. Topology Types of the Offsets to the parabola $y = x^2$

Thus, we get that there are at most three topology types of the offsets to the parabola, corresponding to the cases $0 < d < 1/2$, $d = 1/2$ and $d > 1/2$, respectively. The graphs corresponding to these three cases are shown in Figure 7, where the graph of the parabola has also been included (in dotted lines).

Note that when $0 < d < 1/2$ and $d = 1/2$, one obtains what one expects, i.e. the offset "duplicates" the topology of the original curve (except for the isolated real point obtained for $0 < d < 1/2$). However, for $d > 1/2$ the picture is different (in fact, in this case two cusps appear).

Example 2. Now, let us consider the curve of equation $y^2 - x^3 = 0$. As in Example 4, the double discriminant $R(d)$ corresponding to the polynomial $F(x, y, d)$ (obtained from the implicit equation of the family of offset curves), has also just one real root d_0 greater than 0, than can be approached as $d_0 = 0.3009908371$. However, in this case the graphs corresponding to $0 < d < d_0$, $d = d_0$ and $d > d_0$, respectively, are all equal. Therefore, we get just one topology type for the offset curves, which is kept $\forall d \in \mathbb{R}$. The graph describing this topology type is shown in Figure 8; here, we have also included the graph of $y^2 - x^3 = 0$, in dotted lines.

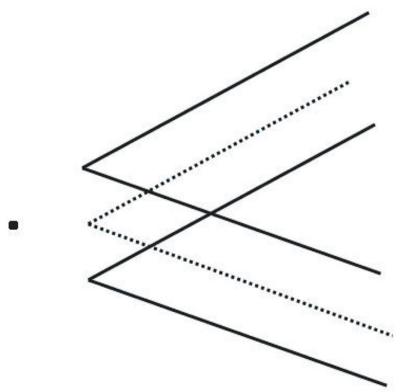

Fig. 8. Only one Topology Type in the Offsets to the Curve $y^2 - x^3 = 0$

References

[1] Afra, Z. (2001) *Computing and Comprehending Topology: persistence and hierarchical Morse complexes*, Ph.D. Thesis, University of Illinois. To be published by Cambridge University Press in 2004.

[2] Arnon D., MacCallum S. (1988). *A polynomial time algorithm for the topology type of a real algebraic curve*, Journal of Symbolic Computation, vol. 5 pp 213-236.

[3] Arrondo E., Sendra J., Sendra J.R., (1997). *Parametric Generalized Offsets to Hypersurfaces.* J. of Symbolic Computation vol. 23, pp 267-285.

[4] Axen, Ulrike (1998). *Topological Analysis using Morse Theory and Auditory Display*, Ph. D. Thesis, University of Illinois.

[5] Basu S., Pollack R., Roy M.F. (2003) *Algorithms in Real Algebraic Geometry* , Springer Verlag.

[6] Cox J., Karron D.B., Ferdous N. (2003) *Topological zone segmentation of scalar volume data*, Journal of Mathematical Imaging and Vision, vol. 18, pp. 95-117.

[7] Farouki R.T., Neff C.A. (1990). *Analytic properties of plane offset curves*, Computer Aided Geometric Design 7, pp. 83-99.

[8] Farouki R.T., Neff C.A. (1990). *Algebraic properties of plane offset curves*, Computer Aided Geometric Design 7, pp. 101-127.

[9] Fortuna R., Gianni P., Luminati D. (2004). *Algorithmical Determination of the Topology of a Real Algebraic Surface*. Journal of Symbolic Computation 38, pp. 1551–1567.

[10] Gianni P., Traverso C. (1983). *Shape determination of real curves and surfaces*, Ann. Univ. Ferrera Sez VII Sec. Math. XXIX pp 87-109.

[11] Gianni P., Fortuna E., Parenti P., Traverso C. (2002). *Computing the topology of real algebraic surfaces* Proc. ISSAC 2002 pp. 92-100, ACM Press.

[12] Hart, J.C. (1999) *Computational Topology for Shape modeling*, in Proceedings of Shape Modeling International '99.

[13] Hirsch M.W. (1976) *Differential topology*. Springer-Verlag, New York, 1976. Graduate Texts in Mathematics, No. 33.

[14] Hong H. (1996). *An effective method for analyzing the topology of plane real algebraic curves*, Math. Comput. Simulation 42 pp. 571-582

[15] Gonzalez-Vega L., El Kahoui M. (1996). *An improved upper complexity bound for the topology computation of a real algebraic plane curve*, J. Complexity 12 pp 527-544.

[16] Gonzalez-Vega L., Necula I. (2002). *Efficient topology determination of implicitly defined algebraic plane curves*, Computer aided geometric design, vol. 19 pp. 719-743.

[17] MacCallum S. (1998). *An Improved Projection Operation for Cylindrical Algebraic Decomposition*. In Quantifier Elimination and Cylindrical Algebraic Decomposition (Eds. B.F. Caviness, J.R. Johnson), Springer Verlag, pp.242–268.

[18] Milnor, J. (1963). *Morse Theory*. Princeton University Press, Princeton N.J.

[19] Reid M. (1988) *Undergraduate Algebraic Geometry*, Cambridge University Press.

[21] Sendra J.R., Sendra J. (2000) *Algebraic analysis of offsets to hypersurfaces*, Mathematische Zeitschrift 234, 697-719.