Local Shape of Offsets to Rational Algebraic Curves*

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Abstract

In this paper we deal with the problem of deciding whether the local shape of a rational algebraic plane curve, around a non-isolated point, is preserved when applying an offsetting process. We provide sufficient conditions for checking the invariance of the local shape when offsetting. Moreover, we give algorithms for testing it. Also, we characterize the situations where the local shape, around every non-isolated point on the rational curve, is preserved in the offset for every distance in a real interval of the type $(0, \alpha)$ or (γ, ∞). Moreover, we derive algorithms for deciding the existence of these intervals and for determining them when they exist.

1 Introduction

Informally speaking, an offset to an irreducible algebraic plane curve $\mathcal{C}$ is a curve "parallel" to $\mathcal{C}$ at a fixed distance d (see e.g. (2) for a formal definition). In fact, Leibnitz already dealt with this type of geometric object calling them parallel curves (see (9)). Offsets to curves and surfaces play an important role in many practical applications, as for instance in the frame of C.A.G.D. (see e.g. (8)). However, the main difficulty when dealing with offsets is that the computations involved are usually very hard, even when working with quite simple curves. In particular, the equation of

*Both authors supported by the Spanish " Ministerio de Educación y Ciencia" under the Project MTM2005-08690-C02-01 and by the "Dirección General de Universidades de la Consejería de Educación de la CAM y la Universidad de Alcalá" under the project CAM-UAH2005/053.

an offset is generally much more complicated than the original one; for instance, the defining polynomial of the offset to the Folium of Descartes $x^3 + y^3 - 3xy = 0$ has degree 14 and 74 terms. So, even in case that one is able to obtain this equation, very often, it is almost impossible to work with it. For this reason, a great deal of work has been done to derive aspects of the offset just from the original curve, i.e. without making use of the offset equation. Thus, problems like the determination of the genus of the offset (see (3)), parametrizing offsets (see (2),(10),(11),(12),(17)), or computing the degree of an offset (see (14)), etc. have already been addressed.

In this paper, we are interested in deriving some topological aspects of the offset just from the starting curve. More precisely, though the informal description of an offset as a parallel curve may lead to think that offsetting processes always yield curves with the topology of the original curve, it is well-known (see e.g. (5), (6)) that this is not necessarily true, since offsetting process may introduce significant topological modifications. The simplest example that one may provide for this behavior is the parabola $y - x^2 = 0$ (see Figure 1). Here, one may see that for $d \leq 1/2$ the topology of the offsets is good (in the sense that one is able to distinguish two connected components each one homeomorphic to the initial curve). However, for $d > 1/2$ this good behavior does not happen anymore.

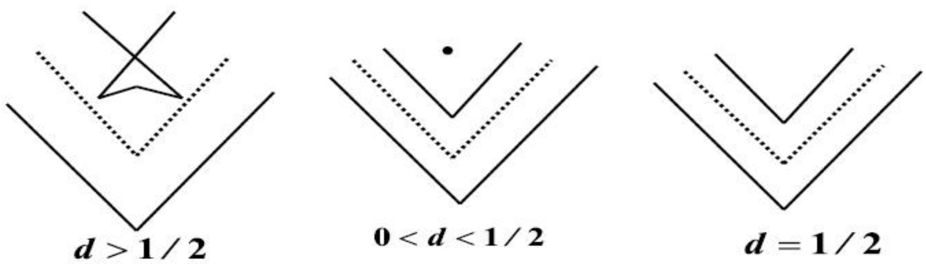

Figure 1: Topology Types of the Offsets to the parabola $y = x^2$

In general, if the implicit equation of the family of offset curves is known, the different topology types present in the offset family can be computed by applying the results in (1); the above example on the parabola has been derived using the results in (1). However, these results not only require to previously compute the equation of the offset family, but also to make resultant computations with it. As a consequence, in practice, the computations are heavy.

Thus, here we will focus not on global aspects of the topology of offsets, but on local questions. In this sense, one may observe (see e.g. (5)) that the "bad" behavior appearing in the offset to the parabola for $d > 1/2$ is in fact based on a local phenomenon, namely that the offsetting process transforms a smooth point of the parabola into a cusp of the offset. Clearly, from the point of view of the applications it is desirable that the offsetting process does not provide this kind of "local" modifications, so that every arc of the original curve is transformed into an arc of the offset with the same shape. This is exactly the problem that we study here.

Similar questions have already been addressed in (5) and (6), for the case when the original curve is defined by means of a regular parametrization. However, the present work, though inspired by these mentioned papers, uses a different approach and reaches more general results. More precisely, here we first define formally the notion of "shape" around a point by means of the concept of *local shape* (see Definition 4 in Section 3). Then, we analyze how the local shape is affected by the offsetting process (see Section 5). Finally, we apply the results to the case of rational algebraic curves, not necessarily regularly-parametrized (see Section 6 and Section 7). In this sense, we give algorithms (see Section 6) to check whether the offsetting process, for a given distance, preserves the shape around all the points of the original curve. Also, we address the existence of intervals I such that for any $d \in I$ the offsetting process behaves properly around every point, from a local point of view (see Section 7). Furthermore, although most of the results we present here are developed for rational curves, the framework that we provide seems to be also applicable to non-rational cases. This is currently ongoing work.

This work can be seen as a natural extension of the problem of analyzing globally the topology of offset curves by means of the results appearing in the joint paper of the authors with Josef Schicho, member of the SFB (see (1)).

Throughout this paper, we assume that $\mathcal{C}$ is a real irreducible algebraic plane curve. In some of our theoretical reasonings we need to work on algebraically close fields, hence we consider the real curve $\mathcal{C}$ immersed in $\mathbb{C}^2$. In addition, we assume that $\mathcal{C}$ is neither a line nor a circle; note that the results we present are straightforward for lines and circles. Lines are excluded because of Definition 2, and circles to ensure that the offset does not degenerate (see Subsection 2.1.), as well as in the proofs of Theorems 20 and 21. Therefore, $\mathcal{C}$ has infinitely many real points and it is defined by an irreducible polynomial in $\mathbb{R}[x, y]$ (see (18)). We also assume that the distances considered in the offsetting processes are real. Moreover, in Sections 6 and 7 we assume that $\mathcal{C}$ is rational.

2 Preliminary notions and results

In this section we briefly recall some preliminary notions and results on offsets curves, as well as on normal parametrizations and places of curves. Also, we fix the notation to be used throughout the paper.

2.1 Offset Curves

We recall some basic facts and notions on offsets seen from the algebraic geometry perspective (see (2) for further details). The offset curve to $\mathcal{C}$ at distance d, that we denote as $\mathcal{O}_d(\mathcal{C})$, is "essentially" the envelope of the system of circles centered at the points of $\mathcal{C}$ with fixed radius d. More formally, $\mathcal{O}_d(\mathcal{C})$ is defined as follows. Let $\mathcal{C}_0 \subset \mathcal{C}$ be the set of all regular points of $\mathcal{C}$ having nonzero isotropic normal vectors to $\mathcal{C}$, and let $\mathcal{A}_d(\mathcal{C}_0) \subset \mathbb{C}^2$ be the constructible set of all the intersection points of the circles of radius $d \in \mathbb{C}$ centered at each point $P \in \mathcal{C}_0$ and the normal line to $\mathcal{C}$ at P. Then, $\mathcal{O}_d(\mathcal{C})$ is the the Zariski closure in $\mathbb{C}^2$ of $\mathcal{A}_d(\mathcal{C}_0)$.

In our case, since $\mathcal{C}$ is not a circle, the offset does not degenerate (see Corollary 1 in (16)), i.e. $\mathcal{O}_d(\mathcal{C})$ is an algebraic curve with at most two components being also algebraic curves. Moreover, since $\mathcal{C}$ is real, $\mathcal{C}_0$ is not empty and every component of $\mathcal{O}_d(\mathcal{C})$ is a real curve, for real values of d (see Proposition 1 in (16)). Indeed, in this case, it holds that $\mathcal{C}_0$ is the set of all regular points of $\mathcal{C}$. Now, let $P \in \mathcal{C}$ be regular, and let $\mathcal{N}(P)$ be a unitary vector normal to $\mathcal{C}$ at P. P generates two points $P_{+d}, P_{-d} \in \mathcal{O}_d(\mathcal{C})$, namely $P_{+d} = P + d \cdot \mathcal{N}(P)$, and $P_{-d} = P - d \cdot \mathcal{N}(P)$. We call the external offset of $\mathcal{C}$ to the set of all the points of $\mathcal{O}_d(\mathcal{C})$ which are of the form P_{+d}, for some point $P \in \mathcal{C}_0$. The internal offset of $\mathcal{C}$ is similarly introduced. We denote the external offset by $\mathcal{O}_{+d}(\mathcal{C})$, and the internal offset by $\mathcal{O}_{-d}(\mathcal{C})$. Note that, in general, $\mathcal{O}_{+d}(\mathcal{C})$ and $\mathcal{O}_{-d}(\mathcal{C})$ are not algebraic curves, though the algebraic closure of its union, which is $\mathcal{O}_d(\mathcal{C})$, obviously is.

2.2 Normal Parametrizations

In this subsection, we recall the notion of normal parametrizations and we relate them to the existence of isolated points. For further details we refer to (15). Therefore, throughout this subsection we assume that the real irreducible algebraic affine curve $\mathcal{C}$ we work with, is rational.

Let $\varphi(t)$ be a rational parametrization of $\mathcal{C}$ over $\mathbb{C}$. Then we say that $\varphi(t)$ is $\mathbb{C}$–normal if $\varphi(\mathbb{C}) = \mathcal{C}$; i.e. if for every $P \in \mathcal{C}$ there exists $t_0 \in \mathbb{C}$ such that $\varphi(t_0) = P$. Moreover, we say that $\mathcal{C}$ is $\mathbb{C}$-normal, or that it can be $\mathbb{C}$-normally parametrized if there exists a $\mathbb{C}$-normal rational parametrization of $\mathcal{C}$. Similarly, if $\mathcal{C}$ is a real curve (which is, by assumption, our case) and $\varphi(t)$ is real, then we say that $\varphi(t)$ is $\mathbb{R}$-normal if $\varphi(\mathbb{R}) = \mathcal{C} \cap \mathbb{R}^2$, i.e. if every real point of $\mathcal{C}$ is the image of a real value of the parameter by means of φ. Analogously, one introduces the notion of $\mathbb{R}$-normal or $\mathbb{R}$-normally parametrized curve.

The main difference between $\mathbb{C}$-normality and $\mathbb{R}$-normality is that every rational curve over $\mathbb{C}$ is $\mathbb{C}$-normal, but there exist real rational curves that are not $\mathbb{R}$-normal (see (15)). This phenomenon is related to the existence of isolated points. This claim is intuitively described in (15). In the following, we prove it formally.

Proposition 1 *Let $\mathcal{C}$ be real and rational. The following statements are equivalent:*
(i) $\mathcal{C}$ is not $\mathbb{R}$-normal.
(ii) $\mathcal{C}$ has isolated singularities.

Proof: Let us see that (ii) implies (i). Let $P \in \mathcal{C}$ be isolated, and let us assume that there exists an $\mathbb{R}$-normal real rational parametrization $\varphi(t) = (u(t), v(t))$ of $\mathcal{C}$. Then, there exists $t_0 \in \mathbb{R}$ such that $\varphi(t_0) = P$. Since $u(t), v(t)$ are defined at t_0, and they are real rational functions, there exists $\epsilon > 0$ such that $\varphi(t)$ is continuous in the interval $I_\epsilon = (t_0 - \epsilon, t_0 + \epsilon)$. Therefore, $\varphi(I_\epsilon)$ is connected. Furthermore, since $u(t), v(t)$ are rational functions, not both constant, one has that $\varphi(I_\epsilon)$ cannot degenerate into a point, i.e. $\varphi(I_\epsilon)$ consists of infinitely many points of $\mathcal{C}$. Then, since $P \in \varphi(I_\epsilon)$, we deduce that P is not an isolated point of $\mathcal{C}$, which is a contradiction. Hence $\mathcal{C}$ is not $\mathbb{R}$-normal.

Let us see that (i) implies (ii). Since $\mathcal{C}$ is not $\mathbb{R}$-normal, by Corollary 7 and Definition 7 in (15), for every real proper rational parametrization $\varphi(t)$ of $\mathcal{C}$ there exists a non-empty finite set $\mathcal{E}_\varphi$ of points in $\mathcal{C} \cap \mathbb{R}^2$ that are generated only by parameter values in $\mathbb{C} \setminus \mathbb{R}$, and such that $\lim_{t \to \pm\infty} \varphi(t) \neq P$ where $P \in \mathcal{E}_\varphi$. Let $\varphi(t) = (u(t), v(t))$ be a real proper rational parametrization of $\mathcal{C}$ (observe that, by (13), $\varphi(t)$ always exists) and let $P \in \mathcal{E}_\varphi$. Let us see that P is isolated. For this purpose, we assume that P is not isolated. Now, let $I_i = (a_i, a_{i+1})$ for $i = 0, \ldots, r$, where $a_0 = -\infty$, $a_{r+1} = \infty$, and $\{a_1, \ldots, a_r\}$ are the real roots of the least common multiple of the denominators in $\varphi(t)$. Then, there exists at least one real branch of $\mathcal{C}$ passing through P, and hence there exists a sequence $\{Q_n\}_{n \in \mathbb{N}}$ of real points on this branch such that $\lim_{n \to \infty} Q_n = P$. Moreover, since $\mathrm{Card}((\mathcal{C} \cap \mathbb{R}^2) \setminus \varphi(\mathbb{R})) < \infty$, and since for every I_i $\varphi(I_i)$ is connected, we may assume w.l.o.g. that all the points in the sequence are generated by real parameter values and that all these parameter values belong to the same interval I_i. Let us say that $t_n \in I_i$ and $\varphi(t_n) = Q_n$. We distinguish two cases:

(i) If $i \in \{1, \ldots, r-1\}$ then $\{t_n\}_{n \in \mathbb{N}}$ is bounded, and therefore there exists a convergent subsequence of $\{t_n\}_{n \in \mathbb{N}}$. Moreover, since the rational map induced by

$\varphi(t)$ from $\mathbb{R}$ on $\mathcal{C}$ is finite, we may assume w.l.o.g. that $\{t_n\}_{n\in\mathbb{N}}$ is indeed convergent; say to $\alpha \in [a_i, a_{i+1}]$. However, if $\alpha \in I_i$ then $\varphi(\alpha) = \lim_{t\to\alpha} \varphi(t) = \lim_{n\to\infty} \varphi(t_n) = \lim_{n\to\infty} Q_n = P$, which is impossible because P is not reachable by φ with a real parameter value. Similarly, if $\alpha \in \{a_i, a_{i+1}\}$ one reaches a contradiction because in this case $\lim_{t\to\alpha} \varphi(t) \notin \mathbb{R}^2$.

(ii) If $i \in \{0, r\}$, say $i = r$ and similarly for $i = 0$, then $\{t_n\}_{n\in\mathbb{N}}$ is either bounded or not. If it is bounded reasoning as in (i) one gets a contradiction, and if it is not bounded one may take a subsequence of $\{t_n\}_{n\in\mathbb{N}}$ convergent to $+\infty$, and this also yields to a contradiction because $\lim_{t\to\infty} \varphi(t) \neq P$. □

In (15) it is proved that, in case that $\mathcal{C}$ is not $\mathbb{R}$-normal, there are only finitely many real points of $\mathcal{C}$ that are not reached by any real value of the parameter. Moreover, these points can be algorithmically computed. We will use this fact in subsequent sections.

2.3 Places of a Curve

In this subsection we recall some basic fact on places. For a complete treatment on places see (19). For this purpose, let $\mathbb{C}[[h]]$ be the domain of formal power series over $\mathbb{C}$ in the variable h; similarly for $\mathbb{R}[[h]]$. Also, if $x(h) = a_r h^r + a_{r+1} h^{r+1} + \cdots \in \mathbb{C}[[h]]$, where $a_r \neq 0$, we denote by $\mathrm{ord}(x(h))$ the order of $x(h)$, i.e. $\mathrm{ord}(x(h)) = r$.

A place $\mathcal{P}(h)$ of $\mathcal{C}$ is an equivalence class of irreducible local parametrizations of $\mathcal{C}$ around a point $P \in \mathcal{C}$, which is called the center of the place. In the sequel, by an abuse of notation, we write $\mathcal{P}(h) = (x(h), y(h))$ to denote that $(x(h), y(h))$ is a representative of $\mathcal{P}(h)$. The order of $\mathcal{P}(h)$ is defined as the minimum of the orders at $\mathcal{P}(h)$ of all the lines passing through the center of the place. In (19), it is proved that $\mathcal{P}(h)$ can be written as $(x(h), y(h))$ where $x(h), y(h) \in \mathbb{C}[[h]]$, and

$$x(h) = \alpha_0 + \alpha_{r_1} h^{r_1} + \alpha_{r_2} h^{r_2} + \cdots, \quad y(h) = \beta_0 + \beta_{s_1} h^{s_1} + \beta_{s_2} h^{s_2} + \cdots$$

with $0 < r_1 < r_2 < \cdots$, $0 < s_1 < s_2 < \cdots$, and $(x(0), y(0))$ being the center of the place. Hence, $\mathrm{ord}(\mathcal{P}(h)) = \min\{r_1, s_1\}$. Also, we denote r_1 by $\mathrm{ord}_x(\mathcal{P}(h))$ and s_1 by $\mathrm{ord}_y(\mathcal{P}(h))$. Note that $\mathrm{ord}_x(\mathcal{P}(h)) = \mathrm{ord}(x(h) - \alpha_0)$ and $\mathrm{ord}_y(\mathcal{P}(h)) = \mathrm{ord}(y(h) - \beta_0)$. Moreover, we say that $\mathcal{P}(h)$ is real if there exists a representative of the class where all the coefficients are real. In the sequel, we always work with real places. It is well known that any point of $\mathcal{C}$ is the center of at least one complex place. Furthermore, any non-isolated real point of $\mathcal{C}$ is the center of at least one real place.

Furthermore, given a place $\mathcal{P}(h) = (x(h), y(h))$ we denote by $x^{(k)}(h)$ the k-th derivative of $x(h)$ w.r.t. h, and similarly for $y^{(k)}(h)$. Also, we use the notation $\mathcal{P}^{(k)}(h) = (x^{(k)}(h), x^{(k)}(h))$.

3 Local shape of a place

In order to study how the offsetting process locally affects to the shape of $\mathcal{O}_d(\mathcal{C})$ w.r.t. the shape of $\mathcal{C}$, we introduce in this section the notion of local shape of a place. Afterwards, the local behavior of $\mathcal{C}$ around a point P will be described by putting together the information provided by the local shape of all the places that are centered at the point P. We start with the following definition.

Definition 2 *Let $\mathcal{P}(h)$ be a real place of $\mathcal{C}$. The* signature *of $\mathcal{P}(h)$ is defined as a pair (p, q) where p is the first non-zero natural number such that $\mathcal{P}^{(p)}(0) \neq (0, 0)$, and $q > p$ is the first natural number such that $\mathcal{P}^{(p)}(0), \mathcal{P}^{(q)}(0)$ are linearly independent. We denote by* $\operatorname{sign}(\mathcal{P}(h))$ *the signature of $\mathcal{P}(h)$.*

Remark 1 *We observe that:*

(1) Since $\mathcal{C}$ is not a line, the natural numbers p, q in Definition 2 always exist. Moreover, using well-known properties of places (see (19)), one can deduce that the signature of a place is independent of the representative.

(2) If $(p, q) = \operatorname{sign}(\mathcal{P}(h))$ then $p = \operatorname{ord}(\mathcal{P}(h))$. Moreover, if $p = \operatorname{ord}_x(\mathcal{P}(h))$ and $\operatorname{ord}_x(\mathcal{P}(h)) < \operatorname{ord}_y(\mathcal{P}(h))$ then $q = \operatorname{ord}_y(\mathcal{P}(h))$. Similarly, if $p = \operatorname{ord}_y(\mathcal{P}(h))$ and $\operatorname{ord}_y(\mathcal{P}(h)) < \operatorname{ord}_x(\mathcal{P}(h))$ then $q = \operatorname{ord}_x(\mathcal{P}(h))$.

(3) If $\operatorname{sign}(\mathcal{P}(h)) = (1, q)$ then $\mathcal{P}(h)$ is regular *or* linear *(compare to Section 5.3. in (19)) Otherwise we will say that the place is* singular. □

The next proposition states the existence of a particular representative of the place that will be used in our theoretical reasonings.

Proposition 3 *Let P be a non-isolated real point of $\mathcal{C}$. Then, in a suitable coordinate system, P is the center of a real place $\mathcal{P}(h) = (x(h), y(h))$ of $\mathcal{C}$ of the form:*

$$x(h) = \alpha_p h^p, \ y(h) = \beta_q h^q + \beta_{q+1} h^{q+1} + \cdots$$

where $(p, q) = \operatorname{sign}(\mathcal{P}(h))$, $\alpha_i, \beta_i \in \mathbb{R}$, and $\alpha_p \cdot \beta_q \neq 0$.

Proof: Since P is real and non-isolated, it is the center of a real place of $\mathcal{C}$, say $\mathcal{P}(h)$. Applying if necessary a translation, we can assume that $P = (0,0)$. Therefore, $\mathcal{P}(h)$ can be written as $\mathcal{P}(h) = (\bar{x}(h), \bar{y}(h))$, where

$$\bar{x}(h) = \bar{\alpha}_{n_1} h^{n_1} + \bar{\alpha}_{n_1+1} h^{n_1+1} + \cdots, \ \bar{y}(h) = \bar{\beta}_{n_2} h^{n_2} + \bar{\beta}_{n_2+1} h^{n_2+1} + \cdots$$

with $\bar{\alpha}_i, \bar{\beta}_j \in \mathbb{R}$, and $\bar{\alpha}_{n_1}\bar{\beta}_{n_2} \neq 0$. Now, since $p = \mathrm{ord}(\mathcal{P}(h))$, it is either n_1 or n_2. We assume w.l.o.g. that $p = n_1$, otherwise one may interchange the axes. In this situation, one has that $n_2 \geq n_1$. Let us see that we can also assume w.l.o.g. that $n_2 > n_1$. Indeed, if $n_2 = n_1$, we apply the orthogonal linear change of coordinates

$$X = \frac{\bar{\alpha}_{n_1}}{\sqrt{\bar{\alpha}_{n_1}^2 + \bar{\beta}_{n_2}^2}} x + \frac{\bar{\beta}_{n_2}}{\sqrt{\bar{\alpha}_{n_1}^2 + \bar{\beta}_{n_2}^2}} y, \ Y = \frac{\bar{\beta}_{n_2}}{\sqrt{\bar{\alpha}_{n_1}^2 + \bar{\beta}_{n_2}^2}} x - \frac{\bar{\alpha}_{n_1}}{\sqrt{\bar{\alpha}_{n_1}^2 + \bar{\beta}_{n_2}^2}} y,$$

so the resulting form of $\mathcal{P}(h)$, after this change, is $\mathcal{P}(h) = (\tilde{x}(h), \tilde{y}(h))$ where

$$\tilde{x}(h) = \tilde{\alpha}_{n_1} h^{n_1} + \tilde{\alpha}_{n_1+1} h^{n_1+1} + \cdots, \ \tilde{y}(h) = \tilde{\beta}_{n_2^*} h^{n_2^*} + \tilde{\beta}_{n_2^*+1} h^{n_2^*+1} + \cdots$$

with $\tilde{\alpha}_i, \tilde{\beta}_j \in \mathbb{R}, \tilde{\alpha}_{n_1}\tilde{\beta}_{n_2^*} \neq 0$ and $n_2^* > n_1$. Thus, we may assume that $n_2 > n_1$, and therefore $(p,q) = (n_1, n_2)$ Finally, applying a change of parameter (see the proof of Theorem 2.2. in page 95 of (19)), one gets the result. □

Remark 2 *Observe that from the proof of the proposition, one gets that in order to set the place into the desired form, one may only need to apply translations and orthogonal transformations. This fact is crucial because offsets behave properly w.r.t. this type of transformations (see (16)), i.e. the offset of the transformed curve is the transformed of the offset.* □

Now, based on the notion of signature, we describe the shape of $\mathcal{C}$ in the neighborhood of a given real point $P \in \mathcal{C}$. First of all, we note that, since P may be the center of several different real places of $\mathcal{C}$, each of these places must be analyzed. For this purpose, we proceed as in Differential Geometry of planar curves. There, one analyzes the local behavior of a parametric curve around a point by considering the terms of lowest order in the Taylor's expansion, around the point, of the parametric equations defining the curve. Thus, in order to analyze the local behavior of a place $\mathcal{P}(h)$ of $\mathcal{C}$ around its center P, we can proceed in the same way. In this sense, we observe that in our case the term of lowest order of $x(h)$ is $\alpha_p h^p$, while the term of lowest order of $y(h)$ is $\beta_q h^q$ (see Proposition 3). Then, we have four different possibilities depending on whether p, q are even or odd. We formalize this in the following definition. Finally, the local behavior of $\mathcal{C}$ around a point is described by collecting all the information corresponding to the places that are centered at the point.

Definition 4 *We say that a real place $\mathcal{P}(h)$ of signature (p, q), centered at $P \in \mathcal{C}$, has* local shape (I), (II), (III) *or* (IV), *according to the following criterion:*
(I): *If both p, q are even.* (II): *If p is odd, and q is even.*
(III): *If p is even, and q is odd.* (IV): *If both p, q are odd.*
In Figure 2 one can see the shape corresponding to each local shape up to rotations. In each case, the center of the place is the intersection point of the two dotted lines. Furthermore, in all cases the horizontal dotted line is tangent to $\mathcal{C}$ in the direction of $\mathcal{P}^{(p)}(0)$. Observe that (I) and (III) correspond to cusps of $\mathcal{C}$. Moreover, in (I) the curve $\mathcal{C}$ is not crossed by the tangent, while in (III) it is. Also, in (II), one has a local maximum or minimum in a reference system whose OX-axis is parallel to $\mathcal{P}^{(p)}(0)$. Finally, in (IV) a flex point is reached. We also note that if $\mathcal{P}(h)$ is regular, then $p = 1$, and therefore the only possibilities for the local shape of $\mathcal{P}(h)$ are (II) or (IV).

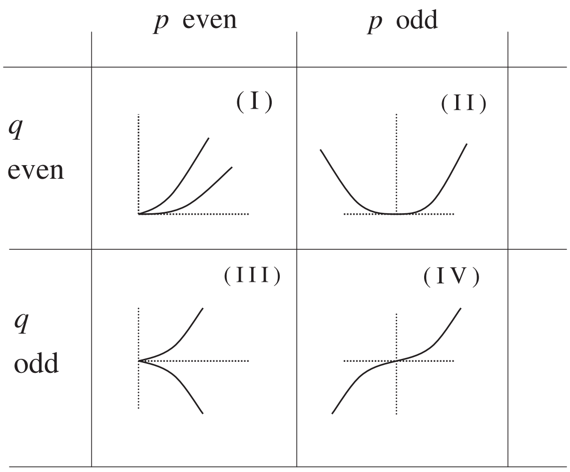

Figure 2: Local shapes

4 Offsetting real places

In this section we see how the offsetting processes can be generalized to real places; a similar reasoning can be done for complex places. In this sense, if $\mathcal{P}(h)$ is a real place of $\mathcal{C}$, we see that it gives rise to two real places $\mathcal{P}_{+d}(h)$ and $\mathcal{P}_{-d}(h)$ of $\mathcal{O}_d(\mathcal{C})$. If the center of $\mathcal{P}(h)$, P, is a regular point (i.e. if $P \in \mathcal{C}_0$, recall the notation $\mathcal{C}_0$ from Subsection 2.1.), the places $\mathcal{P}_{+d}(h)$ and $\mathcal{P}_{-d}(h)$ are centered at the points P_{+d} and P_{-d}, respectively, which are generated in the offset by the point P. If P is singular, then one may consider a sequence $\{P_n\}_{n\in\mathbb{N}}$ of regular real points on the branch defined by $\mathcal{P}(h)$ and converging to P, and then the centers of the places $\mathcal{P}_{+d}(h)$ and $\mathcal{P}_{-d}(h)$ are the limits of the corresponding sequences of points $\{P_{-d,n}\}_{n\in\mathbb{N}}$ and $\{P_{-d,n}\}_{n\in\mathbb{N}}$ generated by $\{P_n\}_{n\in\mathbb{N}}$ on the external and internal of offset, respectively.

For this purpose, we need to work with units and square roots of formal power series. We briefly recall these notions. Units in $\mathbb{R}[[h]]$ are characterized as those formal power series of order zero. Moreover, if $\phi(h) = a_0 + a_1 h + a_2 h^2 + \cdots \in \mathbb{R}[[h]]$ is a unit, then by performing elementary calculations one gets that its inverse is

$$\frac{1}{\phi(h)} = \frac{1}{a_0} - \frac{a_r}{a_0^2} h^r + \mathcal{O}(h^{r+1})$$

where $r = \operatorname{ord}(\phi(h) - a_0)$. On the other hand, if $\phi(h) \in \mathbb{R}[[h]]$, we say that $\psi(h) \in \mathbb{R}[[h]]$ is a square-root of $\phi(h)$ if it holds that $\psi(h)^2 = \phi(h)$. As in the case of real numbers, one may check that if $\phi(h)$ is not identically zero, and it has square-roots then there exist two square-roots and they differ in the sign. We will denote them as $\pm\sqrt{\phi(h)}$. Indeed, one can prove that $\phi(h)$ has square-roots if $\operatorname{ord}(\phi(h))$ is even and the coefficient of $h^{\operatorname{ord}(\phi)}$ is positive. More precisely, if $\phi(h) = a_0 + a_1 h + a_2 h^2 + \cdots \in \mathbb{R}[[h]]$ has order zero, then $\phi(h)$ has an square-root iff $a_0 > 0$. Furthermore if $a_0 > 0$, by performing elementary calculations, then $\phi(h)$ has two square-roots, maybe equal, namely:

$$\pm\sqrt{a_0} + \frac{a_r}{2 \cdot \pm\sqrt{a_0}} h^r + \mathcal{O}(h^{r+1}),$$

where $r = \operatorname{ord}(\phi(h) - a_0)$. In particular, we observe that the square-roots of a zero-order formal power series $\phi(h)$, if they exist, are units of $\mathbb{R}[[h]]$, and

$$\pm\frac{1}{\sqrt{\phi(h)}} = \frac{1}{\pm\sqrt{a_0}} - \frac{a_r}{\pm 2 a_0^{3/2}} h^r + \mathcal{O}(h^{r+1}).$$

More generally, if $\phi(h)$ has even order, say $2n$, then it can be written as $\phi(h) = h^{2n}\phi^*(h)$, where $\operatorname{ord}(\phi^*(h)) = 0$. Thus, in this case, $\phi(h)$ has an square-root iff $\phi^*(h)$ does have, and applying the above remarks to $\phi^*(h)$ one gets the square-roots of $\phi(h)$ as $\pm h^n\sqrt{\phi^*(h)}$.

Now, let us see how to offset the real place $\mathcal{P}(h) = (x(h), y(h))$, centered at $P \in \mathcal{C}$. We assume w.l.o.g. that $\mathcal{P}(h)$ is expressed as in Proposition 3. This implies that

P is the origin. Let $x(h) = \alpha_p h^p$ and $y(h) = \beta_q h^q + \beta_{q+1} h^{q+1} + \cdots$. We introduce the power series $\bar{\phi}(h) = x^{(1)}(h)^2 + y^{(1)}(h)^2$ (recall from Subsection 2.3. the notation $\mathcal{P}^{(k)}(h) = (x^{(k)}(h), x^{(k)}(h))$ for the derivatives of the place $\mathcal{P}(h)$). Note that, since $x^{(1)}(h), y^{(1)}(h) \in \mathbb{R}[[h]]$, and $x^{(1)}(h), y^{(1)}(h)$ are not identically zero, then $\bar{\phi}(h)$ is not identically zero. Then, we consider

$$\mathcal{P}_{\pm d}(h) = \mathcal{P}(h) \pm \frac{d}{\sqrt{\bar{\phi}(h)}}(-y^{(1)}(h), x^{(1)}(h)).$$

Then

$$\pm\sqrt{\bar{\phi}(h)} = \pm h^{p-1}\sqrt{\phi(h)}$$

where $\phi(h) := p^2\alpha_p^2 + q^2\beta_q^2 \cdot h^{2(q-p)} + \mathcal{O}(h^{2(q-p)+1})$; note that $p^2\alpha_p^2 > 0$. Moreover,

$$\frac{1}{\pm\sqrt{\phi(h)}} = \frac{1}{\pm p\alpha_p} - \frac{q^2\beta_q^2}{\pm 2p^3\alpha_p^3} \cdot h^{2(q-p)} + \cdots \in \mathbb{R}[[h]]$$

Thus, one gets that:

$$\mathcal{P}_{\pm d}(h) = (\alpha_p h^p \mp d(\frac{q\beta_q}{p\alpha_p}h^{q-p} + \cdots), \beta_q h^q \pm dp\alpha_p(\frac{1}{p\alpha_p} - \frac{q^2\beta_q^2}{2p^3\alpha_p^3} \cdot h^{2(q-p)} + \cdots)).$$

Note that, since $p > 0$ and $q > p$, then $\mathcal{P}_{+d}(h)$ and $\mathcal{P}_{-d}(h)$ are centered at the points $P_{+d} := (0, d)$ and $P_{-d} := (0, -d)$, respectively. Moreover, because of the construction, $\mathcal{P}_{+d}(h)$ and $\mathcal{P}_{-d}(h)$ are places of $\mathcal{O}_d(\mathcal{C})$, clearly real. Furthermore, taking into account the convergence properties of analytic functions, there exists an Euclidean neighborhood E of 0 such that for all $h_0 \in E$, but finitely many exceptions, $P_0 := \mathcal{P}(h_0) \in \mathcal{C}_0$ and $\mathcal{P}_{\pm d}(h_0)$ are the two points generated by P_0 in $\mathcal{O}_d(\mathcal{C})$. This offsetting construction of places provides the following relationship between places on $\mathcal{C}$ and $\mathcal{O}_d(\mathcal{C})$.

Theorem 5 *Every real place of $\mathcal{C}$ generates two real places of $\mathcal{O}_d(\mathcal{C})$, and every real place of $\mathcal{O}_d(\mathcal{C})$ is generated by at least one real place of $\mathcal{C}$.*

Proof: We have already proved the first part. Let us prove that every real place $\mathcal{Q}(h)$ of $\mathcal{O}_d(\mathcal{C})$ comes from at least one real place of $\mathcal{C}$. Let $\mathcal{M}$ be the real component of $\mathcal{O}_d(\mathcal{C})$ where the branch defined by $\mathcal{Q}(h)$ belongs to. We consider $\mathcal{O}_d(\mathcal{M})$. By Corollary 6 in (16), one has that $\mathcal{C}$ is a component of $\mathcal{O}_d(\mathcal{M})$. Let $\mathcal{Q}_{\pm d}(h)$ be the two real places generated by $\mathcal{Q}(h)$ on $\mathcal{O}_d(\mathcal{M})$. Then, one of them is a place of $\mathcal{C}$, and it is the one generating $\mathcal{Q}(h)$. $\square$

5 Local shape analysis of places on the offset

In this section we analyze how the offsetting process affects to the local shape of a real place $\mathcal{P}(h)$ of $\mathcal{C}$ (see Theorem 7). Moreover, we study the cases when this local shape is preserved (see Corollaries 8 and 9); we say that the local shape of $\mathcal{P}(h)$ is preserved for d if the local shapes of $\mathcal{P}(h)$, $\mathcal{P}_{\pm d}(h)$ are the same.

For this purpose, we consider $\mathcal{P}(h)$ expressed as $\mathcal{P}(h) = (\alpha_p h^p, \beta_q h^q + \cdots)$, where $\mathrm{sign}(\mathcal{P}(h)) = (p, q)$ (see Proposition 3). Now, we analyze the local shape by studying $\mathrm{sign}(\mathcal{P}_{\pm d}(h))$. From the subsection before, we observe that $\mathrm{ord}(\mathcal{P}_{\pm d}(h)) = \min\{p, q - p\} = \mathrm{ord}_x(\mathcal{P}_{\pm d}(h))$, and that $\mathrm{ord}_x(\mathcal{P}_{\pm d}(h)) < \min\{q, 2(q-p)\} = \mathrm{ord}_y(\mathcal{P}_{\pm d}(h))$. Therefore, by Remark 1, one has that $\mathrm{sign}(\mathcal{P}_{\pm d}(h)) = (\mathrm{ord}_x(\mathcal{P}_{\pm d}(h)), \mathrm{ord}_y(\mathcal{P}_{\pm d}(h)))$. Thus, different cases have to be considered depending on the value of $q - 2p$. Furthermore, if $q - 2p = 0$ it must also be considered whether $\alpha_p \mp d \cdot \frac{q\beta_q}{p\alpha_p}$ vanishes or not. This last distinction motivates the following definition.

Definition 6 *Let $\mathcal{P}(h) = (\alpha_p h^p, \beta_q h^q + \cdots)$ be a real place of $\mathcal{C}$ and $\mathrm{sign}(\mathcal{P}(h)) = (p, q)$. We say that $d_0 \in \mathbb{R}$ is an* (offsetting) critical distance for $\mathcal{P}(h)$ *if*

$$q - 2p = 0 \ \ \textit{and} \ \ d_0 = \left| \frac{\alpha_p^2}{2\beta_q} \right|.$$

Furthermore, we say that d_0 is an (offsetting) critical distance for $\mathcal{C}$ *if there exists a real place $\mathcal{P}(h)$ of $\mathcal{C}$ such that d_0 is critical for $\mathcal{P}(h)$.*

Now, we can state the main result of the section.

Theorem 7 *Let $\mathcal{P}(h) = (\alpha_p h^p, \beta_q h^q + \cdots)$ be a real place of $\mathcal{C}$ and $\mathrm{sign}(\mathcal{P}(h)) = (p, q)$. It holds that*

(1) If $q - 2p > 0$, $\forall\, d \in \mathbb{R}^*$ the local shapes of $\mathcal{P}(h)$, $\mathcal{P}_{+d}(h)$, $\mathcal{P}_{-d}(h)$ are the same.

(2) If $q - 2p = 0$, $\forall\, d \in \mathbb{R}^* \setminus \{\frac{\alpha_p^2}{2\beta_q}\}$ the local shapes of $\mathcal{P}(h)$ and $\mathcal{P}_{+d}(h)$ are the same. Similarly for $d \in \mathbb{R}^* \setminus \{\frac{-\alpha_p^2}{2\beta_q}\}$ and $\mathcal{P}(h)$, $\mathcal{P}_{-d}(h)$.

(3) If $q - 2p < 0$, $\forall\, d \in \mathbb{R}^*$ the local shapes of $\mathcal{P}_{\pm d}(h)$ follow this scheme:

	p is even	*p is odd*
q is even	$\mathcal{P}_{\pm d}(h)$ has (I)-shape	$\mathcal{P}_{\pm d}(h)$ has (II)-shape
q is odd	$\mathcal{P}_{\pm d}(h)$ has (II)-shape	$\mathcal{P}_{\pm d}(h)$ has (I)-shape

Proof: We prove the theorem for $\mathcal{P}_{+d}(h)$. The proof for $\mathcal{P}_{-d}(h)$ is similar. Let $(\tilde{p},\tilde{q}) = \text{sign}(\mathcal{P}_{+d}(h))$. Let us prove (1). Since $q-2p>0$, then $p<q-p$ and $q<2(q-p)$. Therefore, $\text{ord}_x(\mathcal{P}_{+d}(h))=p$, and $\text{ord}_y(\mathcal{P}_{+d}(h))=q$. Moreover, since $p<q$, one has that $p=\text{ord}(\mathcal{P}_{+d}(h))=\text{ord}_x(\mathcal{P}_{+d}(h))<\text{ord}_y(\mathcal{P}_{+d}(h))$ and therefore $(\tilde{p},\tilde{q})=(p,q)$. Now, let us see (2). Since $q=2p$, then $q-p=p$ and $q=2(q-p)$, and hence:

$$\mathcal{P}_{+d}(h) = (\alpha_p(1-2d\frac{\beta_q}{\alpha_p^2})h^p+\cdots, d+\beta_q(1-2d\frac{\beta_q}{\alpha_p^2})h^q+\cdots).$$

Therefore, since $\alpha_p\neq 0, \beta_q\neq 0$ and $d\neq \alpha_p^2/(2\beta_q)$, one has that $\text{ord}_x(\mathcal{P}_{+d}(h)) = p < \text{ord}_y(\mathcal{P}_{+d}(h))$. Thus, reasoning as in the previous case one also concludes that $(\tilde{p},\tilde{q})=(p,q)$. Finally, let us see (3). Since $q-2p<0$, it holds that $q-p<p$ and $2(q-p)<q$, so $\text{ord}_x(\mathcal{P}_{+d}(h))=q-p<2(q-p)=\text{ord}_y(\mathcal{P}_{+d}(h))$. Therefore $(\tilde{p},\tilde{q})=(q-p,2(q-p))$. From this fact, and taking into account Definition 4, the statement follows directly. □

The next corollary follows from Theorem 7.

Corollary 8 *Let $\mathcal{P}(h)$ be a real place of $\mathcal{C}$ of signature (p,q), and let $d\in\mathbb{R}^*$. If one of the following assertions holds:*

(1) $q-2p>0$,

(2) $q-2p=0$ and d is not critical for $\mathcal{P}(h)$,

(3) $q-2p<0$ and q is even,

the local shape of $\mathcal{P}(h)$ is preserved for d. Furthermore, if $q-2p<0$ and q is odd, the local shape of $\mathcal{P}(h)$ is not preserved for d.

Furthermore, if $\mathcal{P}(h)$, since $q>p=1$, it cannot happen that $q-2p<0$. Therefore, we deduce the following corollary from Theorem 7.

Corollary 9 *Let $\mathcal{P}(h)$ be a regular real place of $\mathcal{C}$. Then, the local shape of $\mathcal{P}(h)$ is preserved for all $d\in\mathbb{R}^*$ except, perhaps, if* $\text{sign}(\mathcal{P}(h))=(1,2)$ *and d is critical.*

Observe that, from an algorithmic point of view, given a particular place $\mathcal{P}(h)$ of $\mathcal{C}$ and a non-critical distance d, one can apply Theorem 7 for checking whether its local shape is preserved or not. Note that there exist algorithmic methods to compute places of curves (see (19) for a theoretical description, and (7) for a software to compute them). Moreover, in this section we have explicitly described how to lift places from the original curve to the offset. In case that we are interested in examining the problem for a critical distance, more terms of $\mathcal{P}_{\pm d}(h)$ need to be computed. In addition, in the following section (see Lemma 17) we present further results for analyzing the local shape of $\mathcal{P}_{+d}(h)$ or $\mathcal{P}_{-d}(h)$ when d is critical.

The preceding ideas and results are illustrated in the following examples.

Example 1 *We consider the curve of equation* $-x^5 + y^2 - 2yx^3 + x^6 = 0$. $\mathcal{P}(h) = (h^2, h^5 + h^6)$ *is a place of the curve centered at* $(0, 0)$. $\mathrm{sign}(\mathcal{P}(h)) = (2, 5)$. *Therefore, the local shape of* $\mathcal{P}(h)$ *is (III). Now since* $q - 2p = 1 > 0$, *by Theorem 7, the local shape of* $\mathcal{P}(h)$ *is preserved for every distance.*

Example 2 *We consider the ellipse of equation* $x^2 + \frac{(y+2)^2}{4} = 1$, *and the place* $\mathcal{P}(h) = (h, -h^2 - \frac{1}{4}h^4 - \frac{1}{8}h^6 + \cdots)$ *centered at* $(0, 0)$. $\mathrm{sign}(\mathcal{P}(h)) = (1, 2)$, *hence it has local shape (II). Moreover,* $q - 2p = 0$ *and the critical distance is* $\frac{1}{2}$. *Thus, from Corollary 8, we have that whenever* $d \neq \frac{1}{2}$, *the local shape of* $\mathcal{P}(h)$ *is preserved. For* $d = \frac{1}{2}$, *we apply the formulas in this section to get that* $\mathcal{P}_{+\frac{1}{2}}(h) = (2h - \frac{3}{2}h^3 + \cdots, \frac{1}{2} - 2h^2 + \cdots), \mathcal{P}_{-\frac{1}{2}}(h) = (\frac{3}{2}h^3 + \frac{5}{8}h^5 + \cdots, -\frac{1}{2} - \frac{1}{4}h^4 + \cdots)$. *Hence* $\mathrm{sign}(\mathcal{P}_{+\frac{1}{2}}(h)) = (1, 2)$, *and* $\mathrm{sign}(\mathcal{P}_{-\frac{1}{2}}(h)) = (3, 4)$. *Therefore, the local shape is also preserved for the critical distance.*

Example 3 *Consider the curve of equation* $y^3 - x^2 = 0$. $\mathcal{P}(h) = (h^2, h^3)$ *is a place of the curve centered at* $(0, 0)$. *Since* $\mathrm{sign}(\mathcal{P}(h)) = (2, 3)$ *its local shape is (III). Now, since* $q - 2p = -1 < 0$, *from Theorem 7 it follows that the local shape of* $\mathcal{P}_{+d}(h), \mathcal{P}_{-d}(h)$ *is (II) for every distance. Thus, the local shape of* $\mathcal{P}(h)$ *is never preserved. Indeed,* $\mathcal{P}_{+d}(h) = (-\frac{3}{2}dh + h^2 + \cdots, d - \frac{9}{8}dh^2 + \cdots)$, $\mathcal{P}_{-d}(h) = (\frac{3}{2}dh + h^2 + \cdots, -d + \frac{9}{8}dh^2 + \cdots)$. *Thus, for* $d \neq 0$, $\mathrm{sign}(\mathcal{P}_{+d}(h)) = \mathrm{sign}(\mathcal{P}_{-d}(h)) = (1, 2)$.

Example 4 *Consider the curve of equation* $x^5 - y^2 + 2yx^2 - x^4 = 0$. $\mathcal{P}(h) = (h^2, h^4 + h^5)$ *is a place of the curve centered at* $(0, 0)$. $\mathrm{sign}(\mathcal{P}(h)) = (2, 4)$, *hence* $q - 2p = 0$ *and the critical distance is* $\frac{1}{2}$. *Thus, from Theorem 7 we deduce that if* $d \neq \frac{1}{2}$, *the local shape of* $\mathcal{P}(h)$ *is preserved. Now, let analyze the local shape for the critical distance. We apply the formulas in this section to get that* $\mathcal{P}_{+\frac{1}{2}}(h) = (-\frac{5}{4}h^3 + 2h^6 + \cdots, \frac{1}{2} - \frac{3}{2}h^5 + \cdots)$ *and* $\mathcal{P}_{-\frac{1}{2}}(h) = (2h^2 + \frac{5}{4}h^3 + \cdots, -\frac{1}{2} + 2h^4 + \cdots)$. *Hence, the local shape of* $\mathcal{P}(h)$ *and* $\mathcal{P}_{-\frac{1}{2}}(h)$ *is (I), but the local shape of* $\mathcal{P}_{+\frac{1}{2}}(h)$ *is (IV).*

6 Good local behavior of a rational curve.

In the previous section we have studied the problem of determining whether the offsetting process, corresponding to a given distance d, preserves the local shape of a given real place $\mathcal{P}(h)$ of $\mathcal{C}$. Now, we are interested in detecting whether this property holds for all the real places of $\mathcal{C}$, when a real distance is fixed. This motivates the following definition.

Definition 10 *Let* $d_0 \in \mathbb{R}^*$. *We say that* $\mathcal{O}_{d_0}(\mathcal{C})$ *has a* good local behavior *if for every real place* $\mathcal{P}(h)$ *of* $\mathcal{C}$ *the local shapes of* $\mathcal{P}(h)$ *and* $\mathcal{P}_{\pm d_0}(h)$ *are the same.*

Taking into account Theorem 5, if good local behavior holds, one may claim that the offsetting process locally preserves the topology of the curve $\mathcal{C}$. In this section, we address the problem of checking algorithmically whether a given curve has good local behavior for a given distance. We start our analysis assuming that $\mathcal{C}$ is $\mathbb{R}$-normal (see Subsection 2.2.). In addition, in the sequel, $\varphi(t) := (X(t), Y(t))$ is an $\mathbb{R}$-normal proper parametrization of $\mathcal{C}$. In particular, this implies that $\mathcal{C}$ does not have isolated singularities (see Theorem 1), and that every real point of $\mathcal{C}$ is reachable via $\varphi(t)$ by means of real parameter values. In this situation, we provide sufficient conditions to ensure good local behavior, and we also present an algorithm to decide whether any curve $\mathcal{C}$, in the above conditions, has good local behavior, or not. In Remark 4, we show how to proceed when the normality hypothesis does not hold.

The main tool that we use in this section is the notion of *curvature*, which is widely studied in the context of Differential Geometry (see for example (4)). More precisely, if t_0 is a *regular* value of the parameter (*regular* here means that the speed vector $\varphi'(t_0)$ is not zero), then the curvature of the curve at the point $\varphi(t_0)$ can be computed by using the following formula:

$$k(t_0) := \frac{X'(t_0)Y''(t_0) - X''(t_0)Y'(t_0)}{(X'(t_0)^2 + Y'(t_0)^2)^{3/2}}.$$

Observe that if $t_0 \neq t_1$ but $\varphi(t_0) = \varphi(t_1)$ (which happens at a self-intersection of the curve), $k(t_0)$ and $k(t_1)$ are not necessarily equal. Thus, intuitively speaking, curvature has to do not exactly with points of the curve, but with places. Note that if $P_0 = \varphi(t_0) = \varphi(t_1)$ is a self-intersection of the curve, then there are at least two places of the curve centered at P_0, one of them corresponding to t_0 and another one corresponding to t_1. Furthermore, it is well-known (see also (4)) that the curvature is invariant for reparametrizations, except perhaps for the sign.

Curvature has also been used in (5) to characterize some aspects of the local shape of the offset, like the appearance of cusps in the offset. However, in (5) the parametrization $\varphi(t)$ is assumed to be regular, i.e. verifying that the speed vector does not vanish for any real value of the parameter. Here we do not impose that condition to $\varphi(t)$, so there may be points generated by non-regular real values of the parameter. We will refer to these points as *singular points of the parametrization*, and we will refer to the real values of the parameter that generate them as *singular values* of the parameter. Observe that since $\varphi(t)$ is rational, the number of real singular values of the parameter is necessarily finite. Now, for these values of the parameter, curvature is not defined. Nevertheless, in the following subsection we will see that by studying the signature of the places centered at the points generated by singular values of the parameter, the behavior of the curvature in the vicinity of these points can be described. This will allow us to extend the notion of curvature also to singular points of the parametrization, giving rise to the notion of extended curvature, which will be the key to detect good local behavior.

6.1 Extended Curvature

In order to properly define the notion of extended curvature, some previous work must be carried out, first. We begin with the following definition:

Definition 11 *Let $\mathcal{P}(h) = (x(h), y(h))$ be a real place of $\mathcal{C}$. Then, the* curvature of $\mathcal{C}$ at $\mathcal{P}(h)$ *is defined as follows:*

$$k_{\mathcal{P}(h)} := \lim_{h \to 0} \left| \frac{x'y'' - x''y'}{(x'^2 + y'^2)^{3/2}} \right|$$

Taking into account that $x(h), y(h) \in \mathbb{R}[[h]]$, because the place is affine and real, and after making calculations with formal power series, one deduces that $(x'y''-x''y')/(x'^2+y'^2)^{3/2}$ can be expressed as $h^p \cdot \sum_{i=0}^{\infty} a_i h^i$, where $p \in \mathbb{Z}$ and $a_i \in \mathbb{R}$; i.e. it is in the quotient field $\mathbb{R}((h))$. Moreover, using that $x(h), y(h)$ are analytic functions in a certain Euclidean neighborhood of 0 (in which case their derivatives are also analytic there), one may see that $\sum_{i=0}^{\infty} a_i h^i$ must be convergent for h sufficiently small. Therefore, $k_{\mathcal{P}(h)} \in \mathbb{R} \cup \{\infty\}$. Furthermore, one may easily see that Definition 11 is independent of the representative of the place, and that $k_{\mathcal{P}(h)}$ stays invariant for translations and orthogonal transformations, so w.l.o.g. it can be assumed that $\mathcal{P}(h)$ has the form of Proposition 3. Now, the following result relates the value of $k_{\mathcal{P}(h)}$ to the signature of $\mathcal{P}(h)$.

Theorem 12 *Let $\mathcal{P}(h) = (\alpha_p h^p, \beta_q h^q + \cdots)$ be a real place of $\mathcal{C}$ with signature (p, q). Then, it holds that:*

$$k_{\mathcal{P}(h)} = \begin{cases} 0 & \text{if } q - 2p > 0 \\ \frac{2|\beta_q|}{\alpha_p^2} & \text{if } q - 2p = 0 \\ \infty & \text{if } q - 2p < 0 \end{cases}$$

Proof. Substituting the derivatives of $X(h) = \alpha_p h^p, Y(h) = \beta_q h^2 + \cdots$ into the expression of $k_{\mathcal{P}(h)}$, and after some calculations, one gets that:

$$k_{\mathcal{P}(h)} = \left| \frac{\alpha_p \beta_q pq \cdot h^{p+q-3} \cdot [(q-p) + \mathcal{O}(h^{p+q-2})]}{\pm \alpha_p^3 p^3 \cdot h^{3(p-1)} \cdot [1 + \mathcal{O}(h^{2(q-p)})]^{3/2}} \right|$$

Since, by the definition of signature, p, q, α_p, β_q and $q - p$ are different from 0, the numerator of $k_{\mathcal{P}(h)}$ has order $p + q - 3$. Also, the order of the denominator is $3(p - 1)$.

Thus, in order to evaluate $k_{\mathcal{P}(h)}$ one has to compare the orders of its numerator and its denominator, respectively. In other words, one has to discuss the sign of $(p+q-3)-3(p-1)$, which is equivalent to discuss the sign of $q-2p$. The rest follows easily. $\square$

Now, we will relate the definition of curvature of a place to the classical notion of curvature. For this purpose, the following definition is also needed:

Definition 13 *Let $\mathcal{P}(h)$ be a real place of $\mathcal{C}$. We say that $t_0 \in \mathbb{R}$ is* the associated parameter (w.r.t. $\varphi(t)$) of $\mathcal{P}(h)$, *if it holds that:*

(i) the center of $\mathcal{P}(h)$ is $\varphi(t_0)$, i.e. $\mathcal{P}(0) = (X(t_0), Y(t_0))$.

(ii) there exists a non-empty Euclidean neighborhood E_0 of t_0 such that $\forall t \in E_0$, there exists $h \in \mathbb{R}$ satisfying that $\varphi(t) = \mathcal{P}(h)$.

Since $\mathcal{C}$ is $\mathbb{R}$-normal, any real place $\mathcal{P}(h)$ has an associated parameter. Moreover, because of condition (ii), the associated parameter of a place is unique, and different places have different associated parameters. Also note that, since we are working with a rational parametrization $\varphi(t)$, for a given parameter value t_0 where $\varphi(t)$ is defined, one can always obtain the place which associated parameter is t_0. For this, one simply has to consider the Laurent expansion of $\varphi(t)$ around t_0.

In addition, since $X(h), Y(h)$ are analytic in a neighborhood of 0, so are their derivatives, and therefore, also using condition (ii), one has that for h sufficiently close to 0, there exists t sufficiently close to t_0 such that $\mathcal{P}(h) = \varphi(t)$, $\mathcal{P}^{(1)}(h) = \varphi'(t)$ and $\mathcal{P}^{(2)}(h) = \varphi''(t)$, simultaneously. Hence, taking into account Definition 11, one has that:

$$k_{\mathcal{P}(h)} = \lim_{t \to t_0} |k(t)|$$

where t_0 is the associated parameter of $\mathcal{P}(h)$. Observe in particular that if t_0 is regular, then $k(t_0)$ is defined and since $\varphi(t)$ is a rational parametrization, $k(t)$ is continuous at $t = t_0$. Hence, whenever t_0 is regular one has that $\lim_{t \to t_0} |k(t)| = |k(t_0)|$. Thus, in that case it holds that $k_{\mathcal{P}(h)} = |k(t_0)|$. In other words, for regular values of the parameter the curvature coincides with the absolute value of the curvature of $\mathcal{C}$ at the corresponding places the parameters are associated with. For singular values of the parameter the curvature is not defined. However, since $k_{\mathcal{P}(h)} = \lim_{t \to t_0} |k(t)|$, with t_0 being associated with $\mathcal{P}(h)$, we have that the curvature at a place centered at a point generated by a singular value t_0, describes the behavior of the classical curvature when t tends to t_0. Hence, by using the notion of curvature at a place one can generalize the notion of curvature to also reach the singular values of the parameter. This leads to the notion of extended curvature, which is made precise in the following definition. Here, we denote by D the following set:

$$D = \{t \in \mathbb{R} | \varphi(t) \text{ is defined}\}$$

Thus, D is the set of real values which are associated with some real place of $\mathcal{C}$.

Definition 14 *Let $t_0 \in D$, and let $\mathcal{P}(h)$ be a real place of $\mathcal{C}$ with associated parameter t_0. The* extended curvature *of $\mathcal{C}$ at $\varphi(t_0)$ (we denote it by $\tilde{k}(t_0)$) is defined as:*

$$\tilde{k}(t_0) := k_{\mathcal{P}(h)}.$$

Remark 3 *Note that the extended curvature essentially coincides with the classical notion of curvature for regular values of the parameter. Furthermore, note that, at a singular value of the parameter, the extended curvature may be formally infinite (see Theorem 12). Observe also that, by definition, the extended curvature $\tilde{k}(t)$ is non-negative.*

6.2 Algorithm

The notion of extended curvature stated in the subsection before allows to formulate the following result. We will use this theorem, first, to give a sufficient condition for $\mathcal{C}$ to have good local behavior for a given distance, and second, to device an algorithm for checking good local behavior.

Theorem 15 *Let $\mathcal{P}(h)$ be a real place of $\mathcal{C}$, t_0 its associated parameter, and $d_0 > 0$. The following statements hold:*

(i) d_0 is critical for $\mathcal{P}(h)$ iff $\tilde{k}(t_0) = \frac{1}{d_0}$.

(ii) There are finitely many real places of $\mathcal{C}$ whose signature (p, q) verifies $q - 2p < 0$.

(iii) There are finitely many real places of $\mathcal{C}$ whose local shape is not preserved for d_0.

(iv) If the local shape of $\mathcal{P}(h)$ is not preserved for d_0, then either $\tilde{k}(t_0) = \frac{1}{d_0}$ (i.e. d_0 is critical for $\mathcal{P}(h)$), or $\tilde{k}(t_0) = \infty$ (i.e. $q - 2p < 0$ where $\text{sign}(\mathcal{P}(h)) = (p, q)$*).*

Proof. (i) follows directly from the definition of critical distance (see Definition 6 in Section 5), Definition 14, and Theorem 12. In order to prove (ii), we observe that if the signature (p, q) of a real place $\mathcal{P}(h)$ verifies that $q - 2p < 0$, then $\tilde{k}(t_0)$, where t_0 is associated with $\mathcal{P}(h)$, is infinite (see Theorem 12). This can only happen when t_0 is a singular value of the parameter (note that otherwise $\tilde{k}(t_0) = |k(t_0)|$ and is finite), and since $\varphi(t)$ is a rational parametrization, there are only finitely many singular values of the parameter. Since different places have different associated parameters, (ii) follows.

Now, let us see (iii). From Corollary 8, one has that if $\mathcal{P}(h)$ is a real place of $\mathcal{C}$ whose local shape is not preserved when $d = d_0$, then one of the two following situations arise: (a) $q - 2p = 0$ and d_0 is critical for $\mathcal{P}(h)$; (b) $q - 2p < 0$, with q odd. Now, from statement (i) one may derive that there are just finitely many real places of $\mathcal{C}$ such that d_0 is critical for them (namely, those real places whose associated parameters are the solutions of $\tilde{k}(t) = \frac{1}{d_0}$; note that since $\varphi(t)$ is rational, this equation has finitely many solutions, each one corresponding to a different place). Moreover, from statement (ii) one has that there are finitely many real places of $\mathcal{C}$ in the situation (b). Thus, (iii) holds. Finally, (iv) follows from Corollary 8 and statements (i) and (ii). □

This result provides the following corollary, which gives a sufficient condition for $\mathcal{C}$ to have good local behavior for a distance d_0.

Corollary 16 (Sufficient Condition) *If, for every $t \in \mathbb{R}$ where $\varphi(t)$ is defined, it holds that $\tilde{k}(t) \neq \infty$ and $\tilde{k}(t) \neq \frac{1}{d_0}$, then $\mathcal{C}$ has good local behavior for d_0.*

Proof. It follows from the statement (iv) in Theorem 15. □

In order to algorithmically check the hypotheses in Corollary 16 for a given d_0, on one hand one has to study the existence of real solutions of the equation $|k(t)| = \frac{1}{d_0}$, and on the other hand one has to compute the singular values t_i of the parameter, and $\lim_{t\to t_i} |k(t)|$, to afterwards check whether any of these limits is equal to $\frac{1}{d_0}$, or infinite.

Theorem 15 also gives a clue on how to check good local behavior for the case when some hypothesis in Corollary 16 does not hold. Indeed, by Theorem 15 there are only finitely many real places of $\mathcal{C}$ whose local shape may not be preserved. Therefore, examining this finitely many cases, we can check good local behavior. In fact, let $\mathcal{P}(h)$ be one of these real places, let t_0 be its associated parameter, and let $\text{sign}(\mathcal{P}(h)) = (p, q)$, then one of the following cases hold:

(A) t_0 is regular and $|k(t_0)| = \frac{1}{d_0}$ (note that here $\tilde{k}(t_0) = |k(t_0)|$); so, d_0 is critical for $\mathcal{P}(h)$.

(B) t_0 is singular and $\lim_{t\to t_0} |k(t)| = \frac{1}{d_0}$; thus, d_0 is critical for $\mathcal{P}(h)$ and $\tilde{k}(t_0) = \frac{1}{d_0}$.

(C) t_0 is singular and $\lim_{t\to t_0} |k(t)| = \infty$; thus, $\tilde{k}(t_0) = \infty$ and $q - 2p < 0$.

Hence, one has to check whether the local shape of the places whose associated parameters are the numbers in the situations (A), (B), (C) are preserved, or not. In case (C), from Corollary 8 one has that the local shape of the place is preserved only when q is even, so these places can be checked by examining q. In case (B), the only possibility is to compute the signature of the place of $\mathcal{C}$ associated with the parameter, and the signatures of the places it generates in the offset (see Section 4), in order to compare them with the original ones. Finally, in case (A) one can use the following lemma, which can be easily derived from the results in Section 2.5 of (5) (in fact, it suffices to adapt Definition 2.3 of (5) to our terminology). Here, note that since $\varphi(t)$ is rational, $k(t)$ and and its derivative $k'(t)$ are defined over the same real set, i.e. $k(t_0)$ exists if and only if the derivative $k'(t_0)$ exists.

Lemma 17 *Let $\mathcal{P}(h)$ be a real place of $\mathcal{C}$ with associated parameter t_0. Let t_0 be regular, and d_0 critical for $\mathcal{P}(h)$. If $k'(t_0) \neq 0$, then the local shape of either $\mathcal{P}_{+d_0}(h)$ or $\mathcal{P}_{-d_0}(h)$), is (I) or (III).*

This lemma provides the following corollary:

Corollary 18 *In the preceding situation, the local type of $\mathcal{P}(h)$ is not preserved. Therefore, $\mathcal{C}$ has not good local behavior.*

Proof: Since t_0 is regular, the local type of $\mathcal{P}(h)$ is either (II) or (IV). The rest follows from Lemma 17. □

If the hypotheses of Lemma 17 do not hold, the alternative is to directly compute the signature of the place of $\mathcal{C}$ and the signatures of the places generated by it in the offset in order to compare them, like one does in case (B).

Therefore, we can derive the following algorithm to check good local behavior:

<u>ALGORITHM:</u> `Good Local Behavior`

Given a real, proper, rational, $\mathbb{R}$-normal parametrization $\varphi(t)$ of an algebraic curve $\mathcal{C}$, and a distance $d_0 \in \mathbb{R}$, the algorithm checks whether $\mathcal{O}_{d_0}(\mathcal{C})$ has good local behavior.

(1) Compute the function curvature $k(t)$.

(2) Check the existence of real solutions of the equation $|k(t)| = \frac{1}{d_0}$.

(3) Compute the singular values $t_{r+1}, \ldots, t_s$ of the parameter. For $i = 1, \ldots, s$, compute $\lim_{t \to t_i} |k(t)|$.

(4) If the equation in step (2) has no real solution, and the limits in step (3) are all finite and different from $\frac{1}{d_0}$ or $\varphi(t)$ has no singular value, then Return "$\mathcal{O}_{d_0}(\mathcal{C})$ `has good local behavior`".

(5) Solve the equation $|k(t)| = \frac{1}{d_0}$. Let $t_1, \ldots, t_r$ be the real solutions.

(6) If there exists $i \in \{1, \ldots, r\}$ such that $k'(t_i) \neq 0$ then Return "$\mathcal{O}_{d_0}(\mathcal{C})$ `does not have good local behavior`".

(7) For $i = 1, \ldots, s$ do:

  (7.1) Compute the signature (p_i, q_i) of the place $\mathcal{P}_i(h)$ associated with t_i.

(7.2) If $q_i - 2p_i < 0$ and q_i is odd, then `Return "`$\mathcal{O}_{d_0}(\mathcal{C})$` does not have good local behavior"`. If some of the following situations happen: (a) $q_i - 2p_i > 0$; (b) $q_i - 2p_i < 0$ with q_i even; (c) $q_i - 2p_i = 0$ but d_0 is not critical for $\mathcal{P}_i(h)$, then set $i := i + 1$ and go to Step (7.1).

(7.3) Compute the signatures $(\tilde{p}_{i,\pm d_0}, \tilde{q}_{i,\pm d_0})$ of the places $\mathcal{P}_{i,\pm d_0}(h)$ generated by $\mathcal{P}_i(h)$ in the offset.

(7.4) Compare the local shapes of $\mathcal{P}_i(h)$ and of $\mathcal{P}_{i,\pm d_0}(h)$. If they are different, `Return "`$\mathcal{O}_{d_0}(\mathcal{C})$` does not have good local behavior"`.

(8) `Return "`$\mathcal{O}_{d_0}(\mathcal{C})$` has good local behavior"`.

Remark 4 (The non-normal case)
In the non-normal case, the above algorithm allows to decide whether the local shapes of the real places of $\mathcal{C}$ centered at points generated by real values of the parameter t, are preserved or not. Thus, in order to complete the study, one must also analyze the real places of $\mathcal{C}$ which are generated only by complex values of the parameter. There are only finitely many points of this type (see Proposition 1 in (15)) and can be computed (see algorithm $\mathbb{L}$-degenerations in (15)). Then, in order to finish the analysis, one can compute the real places of $\mathcal{C}$ which are centered at these points, and use the results in Section 5.

The ideas of the section are illustrated by the following examples.

Example 5 *Consider the curve $\mathcal{C}$ in Example 1, which can be parametrized by $\varphi(t) = (t^5 + t^6, t^2)$, and the offset curve $\mathcal{O}_{d_0}(\mathcal{C})$ where $d_0 = \frac{1}{3}$. Applying the algorithms in (15), one can see that $\varphi(t)$ is $\mathbb{R}$-normal. Now, the computation of the function curvature $k(t)$ yields:*

$$k(t) = \frac{-6t^2(5 + 8t)}{(25t^6 + 60t^7 + 36t^8 + 4) \cdot \sqrt{t^2(25t^6 + 60t^7 + 36t^8 + 4)}}$$

Then, one checks that the equation $|k(t)| = \frac{1}{d_0} = 3$ has no real solution. Furthermore, one may observe that there is just one singular value of parameter, namely $t = 0$. Since $\lim_{t\to 0} |k(t)| = 0$, which is finite and different from $\frac{1}{d_0} = 3$, one concludes that the hypotheses of Corollary 16 holds. Therefore, the offset $\mathcal{O}_{d_0}(\mathcal{C})$ has good local behavior for $d_0 = \frac{1}{3}$.

Example 6 *Consider the algebraic rational curve parametrized as $\varphi(t) = (t^2, t^4 + t^5)$ (i.e. the curve in Example 4), and let $d_0 = \frac{1}{2}$. One can check that $\varphi(t)$ is $\mathbb{R}$-normal. Now, the function curvature is*

$$k(t) = \frac{2t(8 + 15t)}{(4 + 16t^4 + 40t^5 + 25t^6) \cdot \sqrt{t^2(4 + 16t^4 + 40t^5 + 25t^6)}}$$

The equation $|k(t)| = 2$ has the solution $t_1 = 0.4777915904$. Moreover, the parametrization has one singular value, namely $t = 0$. Since in this case Corollary 16 is not applicable, we go on with the algorithm. Here, we get that $k'(t_1)$ does not vanish. Therefore, we conclude that good local behavior is not present.

Example 7 *(Case of a non-normal parametrization) Let $\mathcal{C}$ be the rational curve parametrized by $(\frac{t}{1+t^5}, \frac{t}{1+t^6})$, and let $d_0 = 1$. First, we observe that $\varphi(t)$ is not $\mathbb{R}$-normal. In fact, $\mathcal{C}$ has two isolated singularities (see Proposition 1). One first checks that in this case the equation $|k(t)| = 1$ has two real solutions, namely $t_1 = -0.499039249630$ and $t_2 = -2.10977984057$. The parametrization has no singular value, but since the hypotheses of Corollary 16 do not hold, more steps of the algorithm need to be executed. Thus, one has to analyze the places whose associated parameters are t_1 and t_2, and for this purpose we check whether the derivatives $k'(t_1), k'(t_2)$ are 0, or not. Since $k'(t_1)$ does not vanish, one concludes that good local behavior is not present for $d_0 = 1$.*

7 Safe intervals

In the previous section, we have addressed the problem of checking whether $\mathcal{C}$ has good local behavior for a particular given distance d_0. Now, we address the problem of characterizing the existence, and actually computing, of an interval I verifying that $\mathcal{O}_d(\mathcal{C})$ has good local behavior for every distance $d \in I$. If this happens, the offset curve corresponding to each distance in the interval, locally shows the same topology as $\mathcal{C}$. We will say that such an interval I is a safe interval.

In this section, we present a systematic way for computing safe intervals. More precisely, here we address the existence of safe intervals of the types $I_\alpha = (0, \alpha)$ or $I_\gamma = (\gamma, \infty)$, with $\alpha, \gamma > 0$, but we do not treat intervals of the form (α, β) where $0 < \alpha < \beta$. In order to study this problem, we assume in the sequel that $\mathcal{C}$ is rational and $\mathbb{R}$-normal, and that $\varphi(t) := (X(t), Y(t))$ is an $\mathbb{R}$-normal proper parametrization of $\mathcal{C}$. The non-normal case will be considered at the end of the section (see Remark 6).

We start with the following result that follows directly from Corollary 8.

Proposition 19 *If there exists a real place $\mathcal{P}(h)$ of $\mathcal{C}$ such that $q - 2p < 0$, with q odd and $\operatorname{sign}(\mathcal{P}(h)) = (p, q)$, then there is no d where $\mathcal{O}_d(\mathcal{C})$ has good local behavior.*

For instance, the curve in Example 3 has a place with signature $(2, 3)$, and therefore it never has good local behavior.

Observe that if $\mathcal{P}(h)$ is such that $q - 2p < 0$, with q odd, then $k_{\mathcal{P}(h)}$ is infinite, and this can only happen when the associated parameter of $\mathcal{P}(h)$ is singular. Since one can always determine the singular values of the parameter, one can always algorithmically check whether the hypothesis in Proposition 19 is fulfilled. If it happens, there does not exist any safe interval. Thus, in the rest of the section we also assume w.l.o.g. that there is no real place $\mathcal{P}(h)$ of $\mathcal{C}$, whose signature (p, q) verifies $q - 2p < 0$, with q odd.

The following result characterizes the existence of a safe interval of the type $I_\alpha = (0, \alpha)$, with $\alpha > 0$. We recall that $D = \{t \in \mathbb{R} | \varphi(t) \text{ exists}\}$.

Theorem 20 *There exists a safe interval of the type $I_\alpha := (0, \alpha)$, with $\alpha > 0$, iff $\tilde{k}(t)$ is upper bounded in D. Moreover, if $\forall t \in D$ $\tilde{k}(t) < a$, then $(0, \frac{1}{a})$ is safe.*

Proof: Let us see the left–right implication. First, we assume that there exists $\alpha > 0$ such that $I_\alpha = (0, \alpha)$ is safe, and that $\tilde{k}(t)$ is not upper bounded in D, in which case neither is $|k(t)|$. Since $\varphi(t)$ is rational, then $k(t)$ is continuous at any regular value of the parameter, and so is $|k(t)|$. Moreover, since there are finitely many singular values of the parameter, and since $\mathbb{R} \setminus D$ is also finite, one has that there are only finitely many values of t, which we write as $t_1, \ldots, t_m$, where $|k(t)|$ is not continuous. Moreover, there are also finitely many values of t, denoted as $t_{m+1}, \ldots, t_n$, where $|k(t)|$ is continuous and $k'(t) = 0$; note that $k(t)$ is not constant, since circles and lines have been excluded. Now, since $|k(t)|$ is not upper bounded, by continuity there exists $\beta > 0$ sufficiently big satisfying that: (i) $\beta > \frac{1}{\alpha}$; (ii) there exists $\bar{t} \notin \{t_1, \ldots, t_n\}$, such that $|k(\bar{t})| = \frac{1}{\beta}$; (iii) $k'(\bar{t}) \neq 0$. In this case, by Theorem 15, Lemma 17 and Corollary 18, we have that for $d_0 = \frac{1}{\beta}$, the offset $\mathcal{O}_{d_0}(\mathcal{C})$ has not good local behavior. However, this is a contradiction because $d_0 \in (0, \alpha)$ and $(0, \alpha)$ is a safe interval.

Now, let us see the right–left implication. Let a be an upper bound of $\tilde{k}(t)$ in D, and let us see that $(0, \frac{1}{a})$ is a safe interval. Indeed, let $d_0 \in (0, \frac{1}{a})$. Since $\tilde{k}(t)$ is bounded, then $\tilde{k}(t)$ is not infinite for any $t \in D$. Moreover, since $a < \frac{1}{d_0}$ and $\tilde{k}(t) < a$, one has that $\tilde{k}(t) < \frac{1}{d_0}$. Thus, by Theorem 15, statement (i), d_0 is not critical for any real place of $\mathcal{C}$. Therefore, by Corollary 16, one has that the offset $\mathcal{O}_{d_0}(\mathcal{C})$ has good local behavior, and hence I_α is safe. □

Remark 5 *Observe that if $\tilde{k}(t)$ is bounded, then $\tilde{k}(t)$ is not infinite at any point, so there cannot be any real place of $\mathcal{C}$ whose signature (p, q) verifies that $q - 2p < 0$ (see Theorem 12). Thus, in that case the hypothesis of Proposition 19 needs not to be checked.*

Now, in the following theorem, we also characterize the existence of safe intervals of the type (γ, ∞), with $\gamma > 0$.

Theorem 21 *There exists a safe interval of the type $I_\gamma := (\gamma, \infty)$, with $\gamma > 0$, iff there exists $b > 0$ such that $\forall t \in D$, $b < \tilde{k}(t)$. Moreover, if $\forall t \in D$, $\tilde{k}(t) > b > 0$, the interval $(\frac{1}{b}, \infty)$ is safe.*

Proof: Let us see the left–right implication. We assume that there exists a safe interval $I_\gamma = (\gamma, \infty)$ with $\gamma > 0$, and that there does not exist any $b > 0$ such that $b < \tilde{k}(t)$. Reasoning as in the proof of the previous theorem, one gets that there exists

a regular value $\bar{t} \in D$ verifying that $k'(\bar{t}) \neq 0$ and such that $\tilde{k}(\bar{t})$ is sufficiently close to 0, so that the distance $d_0 := 1/\tilde{k}(\bar{t}) \in I_\gamma$. Hence, by Theorem 15, statement (i), d_0 is critical for the real place $\bar{\mathcal{P}}(h)$ associated with $\bar{t}$. Thus, as in the preceding theorem, by Lemma 17 and Corollary 18, we deduce that $\mathcal{C}$ has not good local behavior for the distance d_0, which is impossible because $d_0 \in I_\gamma$ and it is a safe interval.
Now, let us see the right–left implication. Let $b > 0$ verify that $b < \tilde{k}(t)$, and let $\gamma = 1/b$, $I_\gamma = (\gamma, \infty)$. Now, we consider a distance $d_0 \in I_\gamma$. Hence, $\tilde{k}(t) > b > \frac{1}{d_0}$. Therefore d_0 cannot be critical for any real place of $\mathcal{C}$ (see Theorem 15, statement (i)). Thus, by statement (iv) of Theorem 15, the only real places of $\mathcal{C}$ whose local shape may not be preserved verify $q - 2p < 0$, where (p, q) is its signature. However, we have assumed along the section that there is no real place of $\mathcal{C}$ with signature (p, q) where $q - 2p < 0$ and q is odd. Thus, if $q - 2p < 0$ for a particular place, then q must be even, and in that case by Corollary 8 the local shape of the place is also preserved. □

The proofs of these two theorems provide ways of computing safe intervals. More precisely, in order to check the existence of safe intervals of the type $(0, \alpha)$, one has to check whether $\tilde{k}(t)$ is upper bounded. Thus, for this type of intervals, one may proceed as follows:

(i) Compute $\lim_{t \to \pm\infty} |k(t)|$.

(ii) For every real singular value t_0 compute $\lim_{t \to t_0} |k(t)|$.

(iii) Compute the value of $|k(t)|$ at each real root of $k'(t)$.

Then, it is straightforward that $\tilde{k}(t)$ is upper bounded in D iff the limits in (i) and (ii) are finite. Furthermore, in the affirmative case, an upper bound a of $\tilde{k}(t)$ can be computed as the maximum of the quantities in (i), (ii), (iii).

On the other hand, from Theorem 21 we get that in order to analyze the existence of safe intervals of the type (γ, ∞), one needs to analyze the existence of strictly positive lower bounds of $\tilde{k}(t)$. Now, $\tilde{k}(t)$ has a positive lower bound if and only if the following statements hold: (a) the equation $k(t) = 0$ has no real solution; (b) the minimum ξ of the quantities in (i) and (ii) (see above) is not 0. Furthermore, if these conditions hold we can take the positive lower bound b as the minimum of ξ and the value of $|k(t)|$ at each real root of $k'(t)$.

All these considerations lead to the following algorithm, that allows to determine whether safe intervals of the types $(0, \alpha)$ and (γ, ∞) exist, and which computes them in the affirmative case. This algorithm works under the hypotheses of rationality and $\mathbb{R}$-normality of $\mathcal{C}$.
ALGORITHM: `Safe-Interval`

Given a real, proper, rational, $\mathbb{R}$-normal parametrization $\varphi(t)$ of an algebraic curve $\mathcal{C}$, the algorithm checks whether there exist safe intervals of the types analyzed in Theorems 20 and 21. Moreover, in the affirmative case it `computes` them.

(1) Compute the function curvature $k(t)$.

(2) Check whether $\tilde{k}(t)$ is upper bounded; for instance as shown above. If it is, then go to (3). Otherwise, compute the singular values, and the signature of the corresponding real places, and check whether any of them verify $q-2p<0$, with q odd. If this happens, then return `Safe intervals do not exist`; else print `Safe interval of the type` $(0,\alpha)$ `does not exist`, and go to (4).

(3) Compute an upper bound a of $\tilde{k}(t)$, for instance as shown above, and return `Safe interval:` $(0, \frac{1}{a})$.

(4) Check the existence of a strictly positive lower bound b of $\tilde{k}(t)$; for instance, as shown above. If it does not exist, then return `Safe interval of the type` (γ,∞)`, does not exist`. Else return `Safe interval:` $(\frac{1}{b},\infty)$.

Example 8 *Consider the rational curve parametrized by* $(\frac{t}{1+t^5}, \frac{t}{1+t^6})$ *(See Example 7). In this case,* $\lim_{t\to\pm\infty}|k(t)|=\infty$ *and* $k(0)=0$*. Therefore,* $\tilde{k}(t)$ *is neither upper bounded nor lower bounded with a positive bound. Hence, safe intervals of the type* $(0,\alpha)$ *or of the type* (γ,∞) *do not exist.*

Remark 6 (The non-normal case)
In the non-normal case, the left–right implications of both Theorem 20 and Theorem 21 still hold. However, the right–left implications of these theorems do not necessarily hold, because we still need to check whether the local shapes of the real places $\mathcal{P}_1(h),\ldots,\mathcal{P}_m(h)$ *of* $\mathcal{C}$ *(that are centered at the non-isolated points that are not generated, via* $\varphi(t)$*, by real values of the parameter) are preserved. Let* (p_i,q_i) *be the signature of* $\mathcal{P}_i(h)$*. In this situation, the following facts show how to proceed:*

- *If any* $\mathcal{P}_j(h)$*, where* $j\in\{1,\ldots,m\}$*, verifies that* $q_j-2p_j<0$ *with* q_j *odd, then there does not exist any safe interval of any type (see Proposition 19).*
- *If* $\tilde{k}(t)$ *is not upper bounded, because of the left–right implication of Theorem 20, there does not exist any safe interval of the type* $(0,\alpha)$*.*
- *If* $\tilde{k}(t)$ *is upper bounded by* $a\in\mathbb{R}$*, we have that:*
  - *If all the places* $\mathcal{P}_j(h)$*,* $j\in\{1,\ldots,m\}$*, verify that* $q_j-2p_j>0$ *or* $q_j-2p_j<0$ *with* q_j *even, then* $I=(0,\frac{1}{a})$ *is a safe interval (see Corollary 8).*

- *If among $\mathcal{P}_1(h), \ldots, \mathcal{P}_m(h)$ there are r real places where $q_j - 2p_j = 0$, their corresponding critical distances being $d_1, \ldots, d_r$, then $I = (0, \alpha)$, where $\alpha = \min\{\frac{1}{a}, d_1, \ldots, d_r\}$, is a safe interval (see Corollary 8).*

- *If $\tilde{k}(t)$ has no positive lower bound, because of the left–right implication of Theorem 21, there does not exist any safe interval of the type (γ, ∞).*

- *If there exists $b > 0$ such that $\tilde{k}(t) > b$, then we have that:*
  - *If all the places $\mathcal{P}_j(h)$, $j \in \{1, \ldots, m\}$, verify that $q_j - 2p_j > 0$ or $q_j - 2p_j < 0$ with q_j even, then $I = (\frac{1}{b}, \infty)$ is a safe interval.*
  - *If there are r real places where $q_j - 2p_j = 0$, with corresponding critical distances $d_1, \ldots, d_r$, then $I = (\gamma, \infty)$, where $\gamma = \max\{\frac{1}{b}, d_1, \ldots, d_r\}$, is a safe interval.*

7.1 Safe Intervals for Conics

We finish this section applying the previous results to the case of irreducible conics. We omit the trivial case of circles where for every real distance different from the radius the offset behaves properly.

Proposition 22 *For a curve defined by a real polynomial $y - p(x)$ there always exists a safe interval of the type $(0, \alpha)$, but there exists no safe interval of the type (γ, ∞).*

Proof: Let $\mathcal{C}$ be the curve defined by $y - p(x)$. $\varphi(t) := (t, p(t))$ is an $\mathbb{R}$-normal proper parametrization of $\mathcal{C}$. Then $k(t) = p''(t)/\sqrt{(1 + p'(t)^2)^3}$. Now, the result follows observing that $\lim_{t\to\pm\infty} |k(t)| = 0$, and $\varphi(t)$ has no real singular value. $\square$

Corollary 23 (The parabola) *For the parabola $y = ax^2 + bx + c$, where $a \neq 0$, the interval $(0, \frac{1}{2a})$ is safe, and no safe interval of the type (γ, ∞) exists.*

Proof: By Proposition 22 no safe interval of the unbounded type exists. Moreover, let $\varphi(t)$ be as in the proof of Proposition 22. Then $\lim_{t\to\pm\infty} |k(t)| = 0$, $\varphi(t)$ has no real singular value, $t_0 := -b/(2a)$ is the only root of $k'(t)$ and $k(t_0) = 2a$. Thus, $1/(2a)$ is an upper bound of $\tilde{k}(t)$. So, $(0, 1/(2a))$ is safe. $\square$

Proposition 24 (The ellipse) *For the ellipse $\frac{x^2}{a} + \frac{y^2}{b} = 1$, where $a, b > 0$ and $a \neq b$, the intervals $(0, \min\{b/\sqrt{a}, a/\sqrt{b}\})$ and $(\max\{b/\sqrt{a}, a/\sqrt{b}\}, \infty)$ are safe.*

Proof: For simplicity in the computations, we apply w.l.o.g. a translation of vector $(0, -\sqrt{b})$, and we work with the transformed ellipse $\mathcal{C}$ defined by $y^2 a + 2a\sqrt{b}y + bx^2 = 0$.

$\mathcal{C}$ can be parametrized as $\varphi(t) := (2\sqrt{a}t/(t^2+1), -2\sqrt{b}/(t^2+1))$, which is not $\mathbb{R}$–normal, but the only unreachable real point of $\mathcal{C}$ is $(0,0)$. The function curvature is:

$$k(t) = \frac{\sqrt{a}\sqrt{b}(t^2+1)^3}{(at^4 - 2at^2 + a + 4bt^2)^{\frac{3}{2}}}.$$

We check whether $\tilde{k}(t)$ is upper bounded. One has that $\lim_{t\to\pm\infty} |k(t)| = \sqrt{b}/a$, $\varphi(t)$ has no real singular value, and the real zeros of $k'(t)$ are $0, 1, -1$, that provide $|k(0)| = \sqrt{b}/a$ and $|k(\pm 1)| = \sqrt{a}/b$. Thus $\tilde{k}(t)$ is upper bounded by $\max\{\sqrt{b}/a, \sqrt{a}/b\}$. Now, accordingly to the above process, we compute the real place centered at the origin. One gets $\mathcal{P}(h) = (2\sqrt{a}h - 2\sqrt{a}h^3 + \cdots, -2\sqrt{b}h^2 + 2\sqrt{b}h^4 + \cdots)$. Thus $\mathrm{sign}(\mathcal{P}(h)) = (1,2)$ and $d_0 := a/\sqrt{b}$ is critical. Therefore, if $\alpha := \min\{1/\max\{\sqrt{b}/a, \sqrt{a}/b\}, a/\sqrt{b}\} = \min\{b/\sqrt{a}, a/\sqrt{b}\}$, then $(0,\alpha)$ is safe. For the other interval, we observe that $k(t) = 0$ has no real solution. Thus, a positive lower bound of $\tilde{k}(t)$ is $\min\{\sqrt{b}/a, \sqrt{a}/b\}$. Now if $\gamma := \max\{1/\min\{\sqrt{b}/a, \sqrt{a}/b\}, a/\sqrt{b}\} = \max\{b/\sqrt{a}, a/\sqrt{b}\}$, then (γ, ∞) is safe. $\square$

Proposition 25 (The Hyperbola) *For the hyperbola* $\frac{x^2}{a} - \frac{y^2}{b} = 1$*, where* $a, b > 0$*, the interval* $(0, b/\sqrt{a})$ *is safe, and no safe interval of the type* (γ, ∞) *exists.*

Proof: For simplicity in the computations, we apply w.l.o.g. a translation of vector $(\sqrt{a}, 0)$, and we work with the transformed hyperbola $\mathcal{C}$ defined by $-x^2ba + 2a\sqrt{a}bx + a^2y^2$. $\mathcal{C}$ can be parametrized as $\varphi(t) := (2a\sqrt{a}/(-bt^2+a), -2b\sqrt{a}t/(-bt^2+a))$, which is not $\mathbb{R}$–normal, but the only unreachable real point of $\mathcal{C}$ is $(0,0)$. The function curvature is:

$$k(t) = \frac{-\sqrt{a}(bt^2 - a)^3}{(b(4t^2a^2 + a^2 + 2at^2b + t^4b^2)^{\frac{3}{2}})}.$$

We check whether $\tilde{k}(t)$ is upper bounded. One has that $\lim_{t\to\pm\infty} |k(t)| = \sqrt{a}/b$, $\varphi(t)$ has no real singular value, and the real zeros of $k'(t)$ are $0, \pm\sqrt{a/b}$, that provide $|k(0)| = \sqrt{a}/b$ and $|k(\pm\sqrt{a/b})| = 0$. Thus $\tilde{k}(t)$ is upper bounded by $\sqrt{a}/b$. Now, accordingly to the above process, we compute the place centered at the origin, getting $\mathcal{P}(h) = (2\sqrt{a}\, b/a\, h^2 + 2b^2\sqrt{a}/a^2\, h^4 + \cdots, -2\sqrt{a}\, b/a\, h - b^2\sqrt{a}/a^2\, h^3 + \cdots)$. Thus $\mathrm{sign}(\mathcal{P}(h)) = (1,2)$ and $d_0 := b/\sqrt{a}$ is critical. Therefore, $(0, b/\sqrt{a})$ is safe. For the other interval, we observe that $k(t) = 0$ has real solutions. Thus, safe intervals of the type (γ, ∞) does not exist. $\square$

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