# PARTITION ANALYSIS XII: PLANE PARTITIONS

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Abstract. MacMahon developed Partition Analysis as a calculational and analytic method to produce the generating function for plane partitions. His efforts did not turn out as he had hoped, and he had to spend nearly twenty years finding an alternative treatment. This paper provides a detailed account of our retrieval of MacMahon's original project. One of the key results obtained with Partition Analysis is an extension of a theorem of Gansner which generalizes Stanley's famous trace theorem.

## 1. Introduction

This is the twelfth paper in this series on MacMahon's Partition Analysis. It has been our belief from the beginning that MacMahon's ideas could be best exploited by computer implementation, and that was the genesis of our Partition Analysis project. Our algorithmic version of MacMahon's method has been implemented in the form of the Mathematica package Omega which is freely available via the web; see [20].

In the back of our minds was always MacMahon's melodramatic experience with his own invention. He created Partition Analysis solely to treat the generating functions associated with various classes of plane partitions. This specific project failed, and in this paper, we shall retrieve MacMahon's original project and obtain, using only Partition Analysis, an extension, Theorem 5.6, of a general plane partition theorem originally due to E. R. Gansner [14, Thm. 4.2]. For further remarks on plane partition history, in particular, on how Partition Analysis has led us to a rediscovery and to an alternative proof of Gansner's theorem, we refer the interested reader to [7].

The initial stage of MacMahon's investigations is chronicled by him in [16] where he refines the study of partitions and compositions of multipartite numbers into the theory of plane partitions. On page 658 of [16], MacMahon first states as an unproven assertion that the generating function for plane partitions is, in fact,

$$\prod_{n=1}^{\infty} \frac{1}{(1-q^n)^n}. \tag{1.1}$$

This function has the series expression

$$1 + q + 3q^2 + 6q^3 + 13q^4 + \cdots .$$

*Date*: August 16, 2006.

The first author was partially supported by NSF Grant DMS-0457003.

The second author was partially supported by SFB grant F1305 of the Austrian FWF.

Thus there are 13 two-dimensional or plane partitions of 4, namely:

$$4,\ 3\,1,\ \begin{array}{c}3\\1\end{array},\ 2\,2,\ \begin{array}{c}2\\2\end{array},\ 2\,1\,1,\ \begin{array}{l}2\,1\\1\end{array},\ \begin{array}{c}2\\1\\1\end{array},\ 1\,1\,1\,1,\ \begin{array}{l}1\,1\,1\\1\end{array},\ \begin{array}{l}1\,1\\1\,1\end{array},\ \begin{array}{l}1\,1\\1\\1\end{array},\ \begin{array}{c}1\\1\\1\\1\end{array}.$$

Indeed, MacMahon proceeds directly to a nascent form of what he will later perfect into Partition Analysis, a method inspired by earlier work of Cayley [10] in invariant theory. His object is clear from this early stage: *One should be able to prove general theorems on plane partitions with Partition Analysis.*

Sadly, after two memoirs [16], [17] on developing Partition Analysis for this problem, MacMahon eventually concludes [18, Vol. II, p. 187] with the following comment concerning $\Omega$, the central operator of Partition Analysis, and the generating function for plane partitions with at most $m$ rows and $l$ columns:

"Our knowledge of the $\Omega$ operation is not sufficient to enable us to establish the final form of result. This will be accomplished by the aid of new ideas which will be brought forward in the following chapters."

The "new ideas" did, in fact, allow MacMahon to establish his generating function conjecture. Subsequently, R. P. Stanley [21], [23, Ch. 7], inspired by MacMahon's beginning [18, Vol. II, Art. 495], developed these latter methods into a powerful combinatorial tool, $(P, \omega)$-partitions. Stanley provides an extensive account of his researches in Chapter 7 of [23].

Stanley's treatment of plane partitions also keeps track of the trace of the partition [23, p. 365]. For example, the plane partition

$$\pi = \begin{array}{ccccc} 5 & 5 & 4 & 2 & 2 \\ 5 & 4 & 4 & 1 & \\ 3 & 3 & 2 & & \\ 2 & & & & \end{array} \tag{1.2}$$

of 42 has trace $5 + 4 + 2 = 11$, the sum of the entries on the main diagonal.

Stanley's trace theorem [23, Thm. 7.20.1] (a slightly different version of it is given in [22, Thm. 2.2]) can be extended either by utilizing combinatorial properties of the Burge correspondence, as done by Gansner [14], or by the Partition Analysis treatment that we shall present in the remainder of this paper. Namely, there are lots of diagonals besides the main diagonal. Following Gansner [14], we shall label all these diagonals with integer numbers where the main diagonal is labeled with 0. Starting with label 1, the diagonals above the main diagonal are labeled with positive integers in ascending order. Starting with label $-1$, the diagonals below the main diagonal are labeled with negative integers in descending order.

More formally, one can specify a plane partition $\pi$ as an $r \times c$ matrix $\pi = (a_{i,j})$ of non-negative integers $a_{i,j}$ which are weakly decreasing in rows and columns. (When writing concrete examples of plane partitions, the 0's are often suppressed as e.g. in (1.2).) Then for integer $k$ with $-r+1 \leq k \leq c-1$ the $k$-trace $\mathrm{tr}_k(\pi)$ of $\pi = (a_{i,j})$ is defined as

$$\mathrm{tr}_k(\pi) \div \sum_{\substack{1 \leq i \leq r \\ 1 \leq j \leq c \\ j-i=k}} a_{i,j}. \tag{1.3}$$

For example, for $\pi$ as in (1.2) we have $\mathrm{tr}_0(\pi) = 11$, $\mathrm{tr}_1(\pi) = 9$, $\mathrm{tr}_{-1}(\pi) = 8$, $\mathrm{tr}_2(\pi) = 5$, $\mathrm{tr}_{-2}(\pi) = 3$, etc.

We will also use the standard abbreviation $|\pi|$ for the sum of all elements of $\pi$, i.e.,

$$|\pi| \doteqdot \sum_{\substack{1\le i\le r\\ 1\le j\le c}} a_{i,j} = \sum_{k=-r+1}^{c-1} \mathrm{tr}_k(\pi).$$

We denote by $P_{r,c}$ the set of all plane partitions with $\le r$ rows and $\le c$ columns. In addition, we define $\tau_{r,c}(t_{-r+1},\ldots,t_{-1};t_0,\ldots,t_{c-1};n)$ to be the number of plane partitions $\pi$ in $P_{r,c}$ with $|\pi| = n$ and with trace $\mathrm{tr}_k(\pi) = t_k$ in the $k$th diagonal, $-r < k < c$, and

$$\mathcal{P}_{r,c}(x_{-r+1},\ldots,x_{-1};x_0,\ldots,x_{c-1};q) \doteqdot \sum_{n=0}^{\infty}\sum_{t_{-r+1}=0}^{\infty}\cdots\sum_{t_{c-1}=0}^{\infty} \tau_{r,c}(t_{-r+1},\ldots,t_{-1};t_0,\ldots,t_{c-1};n)q^n \prod_{k=-r+1}^{c-1} x_k^{t_k}.$$

Using only Partition Analysis we shall extend and prove Gansner's theorem [14, Thm.4.2]:

Theorem G. For integers $r, c \ge 1$,

$$\mathcal{P}_{r,c}(x_{-r+1},\ldots,x_{-1};x_0,\ldots,x_{c-1};q) = \prod_{i=1}^{r}\prod_{j=1}^{c} \frac{1}{1 - x_{-i+1}x_{-i+2}\cdots x_{j-1}q^{i+j-1}}.$$

To illustrate, we provide the power series expansion in the case $r = c = 4$ with the relevant plane partitions listed below each term:

$$\begin{array}{ccccc}
1 & +x_0q & +(x_{-1}x_0 & +x_0x_1 & +x_0^2)q^2 \\
\varnothing & 1 & \begin{smallmatrix}1\\1\end{smallmatrix} & 1\,1 & 2
\end{array}$$

$$\begin{array}{cccccc}
+(x_{-2}x_{-1}x_0 & +x_{-1}x_0x_1 & +x_{-1}x_0^2 & +x_0x_1x_2 & +x_0^2x_1 & +x_0^3)q^3 \\
\begin{smallmatrix}1\\1\\1\end{smallmatrix} & \begin{smallmatrix}1&1\\1&\end{smallmatrix} & \begin{smallmatrix}2\\1\end{smallmatrix} & 1\,1\,1 & 2\,1 & 3
\end{array}$$

$$\begin{array}{ccccc}
+(x_{-3}x_{-2}x_{-1}x_0 & +x_{-2}x_{-1}x_0x_1 & +x_{-2}x_{-1}x_0^2 & +x_{-1}^2x_0^2 & +x_{-1}x_0x_1x_2 \\
\begin{smallmatrix}1\\1\\1\\1\end{smallmatrix} & \begin{smallmatrix}1&1\\1&\\1&\end{smallmatrix} & \begin{smallmatrix}2\\1\\1\end{smallmatrix} & \begin{smallmatrix}2\\2\end{smallmatrix} & \begin{smallmatrix}1&1&1\\1&&\end{smallmatrix}
\end{array}$$

$$\begin{array}{ccccccccc}
+2x_{-1}x_0^2x_1 & & +x_{-1}x_0^3 & +x_0x_1x_2x_3 & +x_0^2x_1x_2 & +x_0^2x_1^2 & +x_0^3x_1 & +x_0^4)q^4 & +\cdots \\
\begin{smallmatrix}2&1\\1&\end{smallmatrix} & \begin{smallmatrix}1&1\\1&1\end{smallmatrix} & \begin{smallmatrix}3\\1\end{smallmatrix} & 1\,1\,1\,1 & 2\,1\,1 & 2\,2 & 3\,1 & 4 &
\end{array}$$

We note that Gansner's Theorem G not only generalizes (1.1), but also Stanley's trace theorem [22, Thm. 2.2].

To make this paper as self-contained as possible, we present in Section 2 a brief account on the way how MacMahon's method of Partition Analysis works. In Section 3 we introduce rational function families $Q_{\mathbb{A}}^{\mathbb{X}}$ and $R_{\mathbb{A}}^{\mathbb{X}}$ which together with various symmetry properties will be used throughout the rest of the paper. In

Section 4 we exhibit important connections between $Q_{\mathbb{A}}^{\mathbb{X}}$ and $R_{\mathbb{A}}^{\mathbb{X}}$ which will be used in the proof of our main result, Theorem 5.4. In Section 5 our main theorem, Theorem 5.4, is introduced together with some corollaries. The first consequence is an elegant special case, Theorem 5.5, which turns out to be equivalent to Gansner's Theorem G stated in the Introduction; see Corollary 1. Then Theorem 5.6 is derived as a reformulation of Theorem 5.4 under a special substitution of variables. Finally, using the product representation (5.7) of a certain $Q_{\mathbb{A}}^{\mathbb{X}}$, a simple instance of Theorem 5.6 is given in the form of Corollary 2. It contains as special cases results by Gansner [14] and Bender and Knuth [8]. In Section 6 the Partition Analysis proof of Theorem 5.4 is presented. The paper concludes with a few remarks about what the future of Partition Analysis might be.

## 2. Generating Functions and Partition Analysis

In this section we introduce to MacMahon's method of Partition Analysis and present various representations of generating functions in terms of MacMahon's fundamental $\Omega_{\geq}$ operator. The section concludes with a reduction argument, Lemma 2.5, which is used as a key ingredient for the proof of our main result, Theorem 5.4 in Section 6.

**Definition 2.1.** Given an $m \times n$ matrix $X = (x_{i,j})$ we define

$$p_{m,n}(X) \doteqdot$$

$$p_{m,n}\begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ x_{2,1} & \cdots & x_{2,n} \\ \vdots & \ddots & \vdots \\ x_{m,1} & \cdots & x_{m,n} \end{pmatrix} \doteqdot \sum_{(a_{i,j}) \in P_{m,n}} x_{1,1}^{a_{1,1}} \cdots x_{1,n}^{a_{1,n}} \cdots x_{m,1}^{a_{m,1}} \cdots x_{m,n}^{a_{m,n}}$$

where $P_{m,n}$ consists of all $m \times n$ matrices $(a_{i,j})$ over non-negative integers $a_{i,j}$ such that $a_{i,j} \geq a_{i,j+1}$ and $a_{i,j} \geq a_{i+1,j}$.

Hence $p_{m,n}(X)$ is the generating function for all plane partitions with at most $m$ rows and $n$ columns. For instance, it is easily seen that for $m \geq 1$,

$$p_{m,1}(X) = \frac{1}{(1 - x_{1,1})(1 - x_{1,1}x_{2,1}) \cdots (1 - x_{1,1}x_{2,1} \cdots x_{m,1})}; \tag{2.1}$$

and, by symmetry, for $n \geq 1$,

$$p_{1,n}(X) = \frac{1}{(1 - x_{1,1})(1 - x_{1,1}x_{1,2}) \cdots (1 - x_{1,1}x_{1,2} \cdots x_{1,n})}. \tag{2.2}$$

Already for $m = n = 2$ the numerator is different from 1. To find the rational function representation of $p_{2,2}(X)$ we follow MacMahon [18, Vol. II, p. 183] to illustrate his method. To this end we recall the definition of the key ingredient of Partition Analysis, the Omega Operator $\Omega_{\geq}$:

$$\underset{\geq}{\Omega} \sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_k=-\infty}^{\infty} A_{s_1,\ldots,s_k} \lambda_1^{s_1} \cdots \lambda_k^{s_k} \doteqdot \sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_k=-\infty}^{\infty} A_{s_1,\ldots,s_k}$$

where the domain of the $A_{s_1,\ldots,s_k}$ is the field of rational functions over $\mathbb{C}$ in several complex variables and the $\lambda_i$ are restricted to a neighborhood of the circle $|\lambda_i| = 1$.

In addition, the $A_{s_1,\dots,s_k}$ are required to be such that any of the series involved is absolute convergent within the domain of the definition of $A_{s_1,\dots,s_k}$.

It is important to note that throughout the paper the operator $\Omega_{\geq}$ is supposed to act only on the Greek letters $\lambda$ or $\mu$, or on corresponding indexed versions like $\lambda_i$ and $\mu_i$, or $\lambda_{i,j}$ and $\mu_{i,j}$. The parameters uneffected by $\Omega_{\geq}$ will be denoted by letters from the Latin alphabet.

Now the first step to compute the closed form of $p_{2,2}(X)$ is to derive what MacMahon called the "crude form" of the generating function:

$$\begin{aligned} p_{2,2}(X) = p_{2,2}\begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix} &= \sum_{(a_{i,j})\in P_{2,2}} x_{1,1}^{a_{1,1}} x_{1,2}^{a_{1,2}} x_{2,1}^{a_{2,1}} x_{2,2}^{a_{2,2}} \\ &= \underset{\geq}{\Omega} \sum_{a_{i,j}\geq 0} \lambda_{1,1}^{a_{1,1}-a_{1,2}} \lambda_{2,1}^{a_{2,1}-a_{2,2}} \mu_{1,1}^{a_{1,1}-a_{2,1}} \mu_{1,2}^{a_{1,2}-a_{2,2}} x_{1,1}^{a_{1,1}} x_{1,2}^{a_{1,2}} x_{2,1}^{a_{2,1}} x_{2,2}^{a_{2,2}} \\ &= \underset{\geq}{\Omega} \frac{1}{(1-x_{1,1}\lambda_{1,1}\mu_{1,1})\left(1-\frac{x_{2,1}\lambda_{2,1}}{\mu_{1,1}}\right)\left(1-\frac{x_{1,2}\mu_{1,2}}{\lambda_{1,1}}\right)\left(1-\frac{x_{2,2}}{\lambda_{2,1}\mu_{1,2}}\right)}. \end{aligned} \tag{2.3}$$

Note that the "crude form" in the last line has been obtained by geometric series summation.

The next step is to eliminate the $\lambda$ and the $\mu$ variables from the "crude form". To this end MacMahon compiled tables of elimination rules like [18, Vol. II, p. 102],

$$\underset{\geq}{\Omega} \frac{1}{(1-a\lambda)\left(1-\frac{b}{\lambda}\right)} = \frac{1}{(1-a)(1-ab)}. \tag{2.4}$$

MacMahon's fundamental rules are elementary to prove. For instance, by geometric series expansion the left side of (2.4) equals

$$\underset{\geq}{\Omega} \sum_{i,j\geq 0} \lambda^{i-j} a^i b^j = \underset{\geq}{\Omega} \sum_{j,k\geq 0} \lambda^k a^{j+k} b^j,$$

where the summation parameter $i$ has been replaced by $j+k$. Finally, $\Omega_{\geq}$ sets $\lambda$ to 1 which completes the proof of (2.4).

Equipped with the above rule, we are in the position to eliminate the $\lambda$ and the $\mu$ variables from (2.3). Using (2.4) we eliminate successively $\lambda_{1,1}$, $\lambda_{2,1}$, and $\mu_{1,2}$:

$$\begin{aligned} p_{2,2}(X) &= \underset{\geq}{\Omega} \frac{1}{(1-x_{1,1}\mu_{1,1})\left(1-\frac{x_{2,1}\lambda_{2,1}}{\mu_{1,1}}\right)(1-x_{1,1}x_{1,2}\mu_{1,1}\mu_{1,2})\left(1-\frac{x_{2,2}}{\lambda_{2,1}\mu_{1,2}}\right)} \\ &= \underset{\geq}{\Omega} \frac{1}{(1-x_{1,1}\mu_{1,1})\left(1-\frac{x_{2,1}}{\mu_{1,1}}\right)(1-x_{1,1}x_{1,2}\mu_{1,1}\mu_{1,2})\left(1-\frac{x_{2,1}x_{2,2}}{\mu_{1,1}\mu_{1,2}}\right)} \\ &= \underset{\geq}{\Omega} \frac{1}{(1-x_{1,1}\mu_{1,1})\left(1-\frac{x_{2,1}}{\mu_{1,1}}\right)(1-x_{1,1}x_{1,2}\mu_{1,1})(1-x_{1,1}x_{1,2}x_{2,1}x_{2,2})}. \end{aligned}$$

After partial fraction decomposition, (2.4) can be applied again to eliminate $\mu_{1,1}$:

$$\begin{aligned}
&p_{2,2}(X)\\
&\quad= \underset{\geq}{\Omega} \frac{1}{(1-x_{1,1}\mu_{1,1})\left(1-\frac{x_{2,1}}{\mu_{1,1}}\right)(1-x_{1,2})(1-x_{1,1}x_{1,2}x_{2,1}x_{2,2})}\\
&\qquad - \underset{\geq}{\Omega} \frac{x_{1,2}}{(1-x_{1,1}x_{1,2}\mu_{1,1})\left(1-\frac{x_{2,1}}{\mu_{1,1}}\right)(1-x_{1,2})(1-x_{1,1}x_{1,2}x_{2,1}x_{2,2})}\\
&\quad= \frac{1}{(1-x_{1,1})(1-x_{1,1}x_{2,1})(1-x_{1,2})(1-x_{1,1}x_{1,2}x_{2,1}x_{2,2})}\\
&\qquad - \frac{x_{1,2}}{(1-x_{1,1}x_{1,2})(1-x_{1,1}x_{1,2}x_{2,1})(1-x_{1,2})(1-x_{1,1}x_{1,2}x_{2,1}x_{2,2})}\\
&\quad= \frac{1-x_{1,1}^2x_{1,2}x_{2,1}}{(1-x_{1,1})(1-x_{1,1}x_{1,2})(1-x_{1,1}x_{2,1})(1-x_{1,1}x_{1,2}x_{2,1})(1-x_{1,1}x_{1,2}x_{2,1}x_{2,2})}.
\end{aligned}$$

For increasing $m, n \geq 2$ the numerator polynomial of the rational function form of $p_{m,n}(X)$ is getting more and more complicated. To get a better handle on it, we follow MacMahon and transform the general $p_{m,n}(X)$ series into its "crude form". To this end we need again to invoke the elimination rule (2.4) and also the following straightforward generalization for $m \geq 2$,

$$\begin{aligned}
(2.5)\qquad &\underset{\geq}{\Omega}(1-a_1\lambda_1)^{-1}\cdot\prod_{i=2}^{m-1}\left(1-a_i\frac{\lambda_i}{\lambda_{i-1}}\right)^{-1}\cdot\left(1-a_m\frac{1}{\lambda_{m-1}}\right)^{-1}\\
&\quad=\frac{1}{(1-a_1)(1-a_1a_2)\cdots(1-a_1a_2\cdots a_m)},
\end{aligned}$$

which is obtained by successive application of (2.4). In addition, it will be convenient to introduce some short-hand notation.

**Definition 2.2.** For $i, j \geq 1$, we define $X_i^{(j)} \doteqdot x_{1,j}\cdots x_{i,j}$, $X_0^{(j)} \doteqdot 1$; $\Lambda_i^{(j)} \doteqdot \lambda_{1,j}\cdots\lambda_{i,j}$ and $\Lambda_0^{(j)} \doteqdot 1$.

Our first version of a "crude form" of $p_{m,n}(X)$ reads as follows.

**Lemma 2.3.** *Given an $m\times n$ matrix $X = (x_{i,j})$. For $m \geq 1$ and $n \geq 2$,*

$$\begin{aligned}
(2.6)\qquad p_{m,n}(X) = &\underset{\geq}{\Omega}\prod_{i=1}^{m}\left(1-X_i^{(1)}\Lambda_i^{(1)}\right)^{-1}\\
&\times\prod_{i=1}^{m}\prod_{j=2}^{n-1}\left(1-X_i^{(j)}\frac{\Lambda_i^{(j)}}{\Lambda_i^{(j-1)}}\right)^{-1}\cdot\prod_{i=1}^{m}\left(1-\frac{X_i^{(n)}}{\Lambda_i^{(n-1)}}\right)^{-1}.
\end{aligned}$$

*Proof.* We code the inequalities $a_{i,j} \geq a_{i,j+1}$ by $\lambda_{i,j}^{a_{i,j}-a_{i,j+1}}$ and $a_{i,j} \geq a_{i+1,j}$ by $\mu_{i,j}^{a_{i,j}-a_{i+1,j}}$. Consequently,

$$
\begin{aligned}
&p_{m,n}(X) \\
&= \underset{\geq}{\Omega} \sum_{a_{i,j} \geq 0} \prod_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} x_{i,j}^{a_{i,j}} \prod_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n-1}} \lambda_{i,j}^{a_{i,j}-a_{i,j+1}} \prod_{\substack{1 \leq i \leq m-1 \\ 1 \leq j \leq n}} \mu_{i,j}^{a_{i,j}-a_{i+1,j}} \\
&= \underset{\geq}{\Omega} (1 - x_{1,1}\ \lambda_{1,1}\ \mu_{1,1})^{-1} \left(1 - x_{m,1} \frac{\lambda_{m,1}}{\mu_{m-1,1}}\right)^{-1} \left(1 - x_{1,n} \frac{\mu_{1,n}}{\lambda_{1,n-1}}\right)^{-1} \\
&\times \left(1 - x_{m,n} \frac{1}{\lambda_{m,n-1}\mu_{m-1,n}}\right)^{-1} \prod_{i=2}^{m-1} \left(1 - x_{i,1} \frac{\lambda_{i,1}\mu_{i,1}}{\mu_{i-1,1}}\right)^{-1} \\
&\times \left(1 - x_{i,n} \frac{\mu_{i,n}}{\lambda_{i,n-1}\mu_{i-1,n}}\right)^{-1} \cdot \prod_{j=2}^{n-1} \left(1 - x_{1,j} \frac{\lambda_{1,j}\mu_{1,j}}{\lambda_{1,j-1}}\right)^{-1} \\
&\times \left(1 - x_{m,j} \frac{\lambda_{m,j}}{\lambda_{m,j-1}\mu_{m-1,j}}\right)^{-1} \cdot \prod_{\substack{2 \leq i \leq m-1 \\ 2 \leq j \leq n-1}} \left(1 - x_{i,j} \frac{\lambda_{i,j}\mu_{i,j}}{\lambda_{i,j-1}\mu_{i-1,j}}\right)^{-1}
\end{aligned}
$$

by geometric series summation. Simple rearrangement of the factors gives

$$
\begin{aligned}
p_{m,n}(X) = \underset{\geq}{\Omega} &(1 - x_{1,1}\ \lambda_{1,1}\ \mu_{1,1})^{-1} \\
&\times \prod_{i=2}^{m-1} \left(1 - x_{i,1}\lambda_{i,1} \frac{\mu_{i,1}}{\mu_{i-1,1}}\right)^{-1} \left(1 - x_{m,1}\lambda_{m,1} \frac{1}{\mu_{m-1,1}}\right)^{-1} \\
&\times \prod_{j=2}^{n-1} \left[ \begin{array}{r} \left(1 - \frac{x_{1,j}\ \lambda_{1,j}}{\lambda_{1,j-1}}\ \mu_{1,j}\right)^{-1} \prod_{i=2}^{m-1} \left(1 - \frac{x_{i,j}\ \lambda_{i,j}}{\lambda_{i,j-1}} \frac{\mu_{i,j}}{\mu_{i-1,j}}\right)^{-1} \\ \times \left(1 - \frac{x_{m,j}\ \lambda_{m,j}}{\lambda_{m,j-1}} \frac{1}{\mu_{m-1,j}}\right)^{-1} \end{array} \right] \\
&\times \left(1 - \frac{x_{1,n}}{\lambda_{1,n-1}}\ \mu_{1,n}\right)^{-1} \prod_{i=2}^{m-1} \left(1 - \frac{x_{i,n}}{\lambda_{i,n-1}} \frac{\mu_{i,n}}{\mu_{i-1,n}}\right)^{-1} \\
&\times \left(1 - \frac{x_{m,n}}{\lambda_{m,n-1}}\ \frac{1}{\mu_{m-1,n}}\right)^{-1}
\end{aligned}
$$

which reduces to the right side of (2.6) after using rule (2.5) $n$ times to eliminate the $\mu_{i,1}$, $\mu_{i,2}$, etc., up to $\mu_{i,n}$. □

The product in (2.6) can be reduced further by eliminating all the $\lambda_{m,j}$ variables. This gives the following second version of a representation in "crude form".

**Lemma 2.4.** *Given an $m \times n$ matrix $X = (x_{i,j})$. For $m \geq 1$ and $n \geq 2$,*

$$
\begin{aligned}
p_{m,n}(X) = (1 - X_m^{(1)} \cdots X_m^{(n)})^{-1} \underset{\geq}{\Omega} \prod_{i=1}^{m-1} \left(1 - X_i^{(1)} \Lambda_i^{(1)}\right)^{-1} \\
\times \prod_{i=1}^{m-1} \prod_{j=2}^{n-1} \left(1 - X_i^{(j)} \frac{\Lambda_i^{(j)}}{\Lambda_i^{(j-1)}}\right)^{-1} \cdot \prod_{i=1}^{m-1} \left(1 - \frac{X_i^{(n)}}{\Lambda_i^{(n-1)}}\right)^{-1} \\
\times \prod_{j=1}^{n-1} \left(1 - X_m^{(1)} \cdots X_m^{(j)} \Lambda_{m-1}^{(j)}\right)^{-1}.
\end{aligned} \tag{2.7}
$$

*Proof.* By Lemma 2.3,

$$
\begin{aligned}
p_{m,n}(X) = \underset{\geq}{\Omega} \prod_{i=1}^{m-1} \left(1 - X_i^{(1)} \Lambda_i^{(1)}\right)^{-1} \prod_{i=1}^{m-1} \prod_{j=2}^{n-1} \left(1 - X_i^{(j)} \frac{\Lambda_i^{(j)}}{\Lambda_i^{(j-1)}}\right)^{-1} \\
\times \prod_{i=1}^{m-1} \left(1 - \frac{X_i^{(n)}}{\Lambda_i^{(n-1)}}\right)^{-1} \cdot \left(1 - X_m^{(1)} \Lambda_{m-1}^{(1)} \lambda_{m,1}\right)^{-1} \\
\times \prod_{j=2}^{n-1} \left(1 - \frac{X_m^{(j)} \Lambda_{m-1}^{(j)}}{\Lambda_{m-1}^{(j-1)}} \frac{\lambda_{m,j}}{\lambda_{m,j-1}}\right)^{-1} \left(1 - \frac{X_m^{(n)}}{\Lambda_{m-1}^{(n-1)}} \frac{1}{\lambda_{m,n-1}}\right)^{-1},
\end{aligned}
$$

and (2.7) follows after applying (2.5) with respect to all the $\lambda_{m,j}$. □

The proof of our main theorem, Theorem 5.4 in Section 6, is based on the following basic reduction lemma which is an immediate consequence of Lemma 2.4.

**Lemma 2.5.** *For $m \geq 1$ and $n \geq 1$,*

$$
\begin{aligned}
p_{m+1,n+1} \begin{pmatrix} x_{1,1} & \cdots & x_{1,n} & z_0 \\ x_{2,1} & \cdots & x_{2,n} & z_1 \\ \vdots & \ddots & \vdots & \vdots \\ x_{m+1,1} & \cdots & x_{m+1,n} & z_m \end{pmatrix} = \left(1 - z_0 \cdots z_m \prod_{\substack{1 \leq i \leq m+1 \\ 1 \leq j \leq n}} x_{i,j}\right)^{-1} \\
\times \underset{\geq}{\Omega} p_{m+1,n} \begin{pmatrix} x_{1,1} & \cdots & x_{1,n-1} & \lambda_0 x_{1,n} \\ x_{2,1} & \cdots & x_{2,n-1} & \lambda_1 x_{2,n} \\ \vdots & \ddots & \vdots & \vdots \\ x_{m,1} & \cdots & x_{m,n-1} & \lambda_{m-1} x_{m,n} \\ x_{m+1,1} & \cdots & x_{m+1,n-1} & x_{m+1,n} \end{pmatrix} \prod_{i=1}^{m} \left(1 - \frac{z_0 \cdots z_{i-1}}{\lambda_0 \cdots \lambda_{i-1}}\right)^{-1}.
\end{aligned} \tag{2.8}
$$

*Proof.* To connect to Lemma 2.4 we introduce the renaming $\lambda_l \to \lambda_{l+1,n}$ of variables. The case $n = 1$ follows from Lemma 2.4 after applying (2.1) to rewrite

$$
p_{m+1,1} \begin{pmatrix} \lambda_{1,1} x_{1,1} \\ \vdots \\ \lambda_{m,1} x_{m,1} \\ x_{m+1,1} \end{pmatrix}
$$

into product form. For $n \geq 2$ it is convenient to define

$$H_{m,n} \doteqdot \prod_{i=1}^{m}\left(1 - X_i^{(1)}\Lambda_i^{(1)}\right)^{-1} \prod_{i=1}^{m}\prod_{j=2}^{n-1}\left(1 - X_i^{(j)}\frac{\Lambda_i^{(j)}}{\Lambda_i^{(j-1)}}\right)^{-1} \times \prod_{j=1}^{n-1}\left(1 - X_{m+1}^{(1)}\cdots X_{m+1}^{(j)}\Lambda_m^{(j)}\right)^{-1}.$$

Applying Lemma 2.4 to the right side of (2.8) gives

$$\left(1 - z_0\cdots z_m \prod_{\substack{1\leq i\leq m+1\\ 1\leq j\leq n}} x_{i,j}\right)^{-1} \underset{\geq}{\Omega} H_{m,n}\left(1 - \Lambda_m^{(n)} \prod_{\substack{1\leq i\leq m+1\\ 1\leq j\leq n}} x_{i,j}\right)^{-1} \times \prod_{i=1}^{m}\left[\left(1 - \frac{z_0\cdots z_{i-1}}{\Lambda_i^{(n)}}\right)^{-1}\left(1 - \frac{\Lambda_i^{(n)}X_i^{(n)}}{\Lambda_i^{(n-1)}}\right)^{-1}\right].$$

Finally it is easy to check that applying Lemma 2.4 to left side of (2.8) gives the same expression. □

## 3. Rational Functions and Symmetry

In this section we introduce rational functions which together with various symmetry properties will be used throughout the rest of the paper.

**Definition 3.1.** Let $\mathbb{A} = \{A_0, A_1, \ldots, A_m\}$ and $\mathbb{X} = \{X_0, X_1, \ldots, X_n\}$ be two sets of variables. For $A \in \mathbb{A}$ define

$$q_{\mathbb{A}}^{\mathbb{X}}(A) \doteqdot \frac{\prod_{X\in\mathbb{X}}\left(1 - \frac{A}{X}\right)}{\prod_{A'\in\mathbb{A}\setminus\{A\}}\left(1 - \frac{A}{A'}\right)}.$$

For example, for $\mathbb{A}=\{A_0, A_1, A_2\}$ and $\mathbb{X} =\{X_0, X_1\}$,

$$q_{\mathbb{A}}^{\mathbb{X}}(A_0) = \frac{\left(1 - \frac{A_0}{X_0}\right)\left(1 - \frac{A_0}{X_1}\right)}{\left(1 - \frac{A_0}{A_1}\right)\left(1 - \frac{A_0}{A_2}\right)}, q_{\mathbb{A}}^{\mathbb{X}}(A_1) = \frac{\left(1 - \frac{A_1}{X_0}\right)\left(1 - \frac{A_1}{X_1}\right)}{\left(1 - \frac{A_1}{A_0}\right)\left(1 - \frac{A_1}{A_2}\right)}, \text{ etc.}$$

Throughout the paper we use the convention $\prod_{A\in\varnothing} f(A) \doteqdot 1$, hence, for example, $q_{\{A_0\}}^{\varnothing}(A_0) = 1$. Similarly, we will use $\bigcup_{l\in\varnothing} S_l \doteqdot \varnothing$; for example, if $n = 0$ then

$$\{X_0, X_1, \ldots, X_{n-1}\} = \bigcup_{0\leq l\leq n-1}\{X_l\} = \varnothing.$$

Subsequently the rational functions $q_{\mathbb{A}}^{\mathbb{X}}$ will play a crucial role. In particular, they serve as building blocks of two fundamental families of rational functions which we define next.

**Definition 3.2.** Given $\mathbb{A} = \{A_0, A_1, \ldots, A_{n+1}\}$ and $\mathbb{X} = \{X_0, X_1, \ldots, X_n\}$ where $n \geq 0$. We define recursively a rational function $Q_{\mathbb{A}}^{\mathbb{X}}$ in $z_0, z_1, \ldots, z_{n+1}$ and in the variables from $\mathbb{A}$ and $\mathbb{X}$ by

$$Q_{\{A_0\}}^{\varnothing}(z_0) \doteqdot \frac{1}{1 - A_0 z_0}$$

and

$$Q_{\mathbb{A}}^{\mathbb{X}}(z_0, z_1, \ldots, z_{n+1}) \doteqdot \frac{1}{1 - A_0 A_1 \cdots A_{n+1} z_0 z_1 \cdots z_{n+1}} \times \sum_{i=0}^{n+1} q_{\mathbb{A}}^{\mathbb{X}}(A_i) Q_{\mathbb{A}\backslash\{A_i\}}^{\mathbb{X}\backslash\{X_n\}}(z_0, z_1, \ldots, z_n).$$

**Definition 3.3.** Given $\mathbb{A} = \{A_0, A_1, \ldots, A_{n+1}\}$ and $\mathbb{X} = \{X_0, X_1, \ldots, X_n\}$ where $n \geq 0$. We define recursively a rational function $R_{\mathbb{A}}^{\mathbb{X}}$ in $w_0, w_1, \ldots, w_{n+1}$ and $z_0, z_1, \ldots, z_{n+1}$ and in the variables from $\mathbb{A}$ and $\mathbb{X}$ by

$$R_{\{A_0\}}^{\varnothing}(w_0; z_0) \doteqdot \frac{1}{(1 - A_0 w_0)(1 - A_0 w_0 z_0)}$$

and

$$\begin{aligned} &R_{\mathbb{A}}^{\mathbb{X}}(w_0, \ldots, w_{n+1}; z_0, \ldots, z_{n+1}) \\ &\doteqdot \frac{1}{1 - A_0 \cdots A_{n+1} w_0 \cdots w_{n+1} z_0 \cdots z_{n+1}} \\ &\quad \times \sum_{i=0}^{n+1} \frac{q_{\mathbb{A}}^{\mathbb{X}}(A_i)}{1 - A_i w_{n+1}} \Big( R_{\mathbb{A}\backslash\{A_i\}}^{\mathbb{X}\backslash\{X_n\}}(w_0, \ldots, w_n; z_0, \ldots, z_n) \\ &\qquad - A_i w_{n+1} R_{\mathbb{A}\backslash\{A_i\}}^{\mathbb{X}\backslash\{X_n\}}(w_0, \ldots, w_{n-1}, w_n w_{n+1} A_i; z_0, \ldots, z_n) \Big). \end{aligned}$$

Next we state elementary properties of the $Q_{\mathbb{A}}^{\mathbb{X}}$ and $R_{\mathbb{A}}^{\mathbb{X}}$ that will be used later. The first one is immediate from the Definitions 3.1 and 3.2.

**Lemma 3.4.** *Given $\mathbb{A} = \{X_0, X_1, \ldots, X_n\}$ and $\mathbb{X} = \{X_0, X_1, \ldots, X_{n-1}\}$, $n \geq 0$. Then*

$$Q_{\mathbb{A}}^{\mathbb{X}}(z_0, z_1, \ldots, z_n) = \prod_{i=0}^{n} (1 - X_0 \cdots X_i z_0 \cdots z_i)^{-1}.$$

For general $\mathbb{A}$ and $\mathbb{X}$ there is no such factored form but both, $Q_{\mathbb{A}}^{\mathbb{X}}$ and $R_{\mathbb{A}}^{\mathbb{X}}$, have important symmetry properties. For such considerations it is convenient to recall the concept of Lagrange symmetrization.

For a given set $\mathbb{A} = \{A_0, A_1, \ldots, A_n\}$ of variables let $K(\mathbb{A}, z)$ denote the field of rational functions in $z$ and in the $A_i$ with coefficients from a field $K$.

**Definition 3.5.** Given $\mathbb{A} = \{A_0, \ldots, A_n\}$, $n \geq 0$, let $f(A_0, \ldots, A_{n-1}; z)$ be a rational function from $K(\mathbb{A}\backslash\{A_n\}, z)$ that is symmetrical in all the variables from $\mathbb{A}\backslash\{A_n\}$. We denote $f$ in short by $f_{\mathbb{A}\backslash\{A_n\}}(z)$. In order to define the Lagrange symmetrization $L(f)$ of $f$ we proceed as follows. For each $i \in \{0, \ldots, n-1\}$ let $f_{\mathbb{A}\backslash\{A_i\}}(z)$ denote that rational function from $K(\mathbb{A}\backslash\{A_i\}, z)$ which is produced by replacing in $f$ the variable $A_i$ by $A_n$. Obviously each $f_{\mathbb{A}\backslash\{A_i\}}(z)$ is symmetrical in all the variables from $\mathbb{A}\backslash\{A_i\}$, and we define $L(f) \in K(\mathbb{A})$ by

$$L(f) \doteqdot \sum_{i=0}^{n} \frac{f_{\mathbb{A}\backslash\{A_i\}}(A_i)}{\prod_{A' \in \mathbb{A}\backslash\{A_i\}} (A_i - A')}.$$

Definition 3.5 introduces Lagrange symmetrization in a version which was studied extensively by A. Lascoux; see e.g. [15]. Lemma 3.6 states its crucial symmetry property which is easily verified.

**Lemma 3.6.** *$L(f)$ as in Definition 3.5 is symmetrical in all the variables from $\mathbb{A}$.*

Lemma 3.6 implies an important symmetry of our fundamental rational functions $Q_{\mathbb{A}}^{\mathbb{X}}$ and $R_{\mathbb{A}}^{\mathbb{X}}$.

**Lemma 3.7.** *Given $\mathbb{A} = \{A_0, A_1, \ldots, A_n\}$ and $\mathbb{X} = \{X_0, X_1, \ldots, X_{n-1}\}$, $n \geq 0$. The rational functions $Q_{\mathbb{A}}^{\mathbb{X}}$ and $R_{\mathbb{A}}^{\mathbb{X}}$ are symmetrical in all the variables from $\mathbb{A}$.*

*Proof.* The proof is immediate from Lemma 3.6 by induction on $n$. E.g., the symmetry of $Q_{\mathbb{A}}^{\mathbb{X}}$ is implied by the fact that for $n \geq 1$,

$$Q_{\mathbb{A}}^{\mathbb{X}}(z_0, z_1, \ldots, z_n) = (-1)^n \frac{A_0 A_1 \cdots A_n}{1 - A_0 A_1 \cdots A_n z_0 z_1 \cdots z_n} L(f)$$

where for each $i \in \{0, \ldots, n\}$,

$$f_{\mathbb{A}\setminus\{A_i\}}(z) = \frac{1}{z} \prod_{X \in \mathbb{X}} \left(1 - \frac{z}{X}\right) Q_{\mathbb{A}\setminus\{A_i\}}^{\mathbb{X}\setminus\{X_{n-1}\}}(z_0, z_1, \ldots, z_{n-1}).$$

The symmetry of $R_{\mathbb{A}}^{\mathbb{X}}$ is proved analogously. □

Setting $z_0 = 0$ in $Q_{\mathbb{A}}^{\mathbb{X}}$ will become important; see Theorem 5.5 in Section 5.

**Lemma 3.8.** *Given $\mathbb{A} = \{A_0, A_1, \ldots, A_n\}$ and $\mathbb{X} = \{X_0, X_1, \ldots, X_{n-1}\}$, $n \geq 0$. Then*

$$Q_{\mathbb{A}}^{\mathbb{X}}(0, z_1, \ldots, z_n) = 1.$$

*Proof.* We proceed by induction on $n$. According to Definition 3.2, to proceed from $n-1$ to $n$, $n \geq 1$, we need to show that

$$\sum_{i=0}^{n} q_{\mathbb{A}}^{\mathbb{X}}(A_i) = 1. \tag{3.1}$$

In view of Definition 3.5, we rewrite identity (3.1) to

$$\sum_{i=0}^{n} \frac{f_{\mathbb{A}\setminus\{A_i\}}(A_i)}{\prod_{A' \in \mathbb{A}\setminus\{A_i\}}(A_i - A')} = \frac{(-1)^n}{A_0 A_1 \cdots A_n} \tag{3.2}$$

where $f_{\mathbb{A}\setminus\{A_i\}}(z) = \frac{1}{z} \prod_{X \in \mathbb{X}} \left(1 - \frac{z}{X}\right)$. After expanding the product, identity (3.2) is implied by the relations

$$\sum_{i=0}^{n} \frac{A_i^k}{\prod_{A' \in \mathbb{A}\setminus\{A_i\}}(A_i - A')} = 0 \qquad (0 \leq k \leq n-1) \tag{3.3}$$

and

$$\sum_{i=0}^{n} \frac{A_i^{-1}}{\prod_{A' \in \mathbb{A}\setminus\{A_i\}}(A_i - A')} = \frac{(-1)^n}{A_0 A_1 \cdots A_n} \tag{3.4}$$

which are folklore in the context of classical Lagrange interpolation; see e.g. Sect. 1.3 in [19]. □

Writing $\deg_{X_i}$ for the polynomial degree with respect to $X_i$, the following lemma is immediate from Definition 3.3.

**Lemma 3.9.** *Given* $\mathbb{A} = \{A_0, A_1, \ldots, A_n\}$ *and* $\mathbb{Y} = \{1/X_0, 1/X_1, \ldots, 1/X_{n-1}\}$ *with* $n \geq 0$. *The expression* $R_{\mathbb{A}}^{\mathbb{Y}}(w_0, \ldots, w_n; z_0, \ldots, z_n)$ *can be viewed as a multivariate polynomial in the* $X_i$ *with coefficients being rational functions in the* $A_i$, *where*

$$\deg_{X_i} R_{\mathbb{A}}^{\mathbb{Y}} \leq n - i \qquad (0 \leq i \leq n-1). \tag{3.5}$$

Now we are ready for another fundamental result based on Lagrange symmetrization.

**Lemma 3.10.** *Given* $\mathbb{A} = \{A_0, A_1, \ldots, A_{n+1}\}$ *and* $\mathbb{Y} = \{Y_0, Y_1, \ldots, Y_n\}$, $n \geq 0$. *Then*

$$\sum_{i=0}^{n+1} A_i \frac{q_{\mathbb{A}}^{\mathbb{Y}}(A_i)}{1 - \frac{A_i}{Y_n}} R_{\mathbb{A}\backslash\{A_i\}}^{\mathbb{Y}\backslash\{Y_n\}}(w_0, \ldots, w_{n-1}, w_n A_i; z_0, \ldots, z_n) = 0 \tag{3.6}$$

*for arbitrary variables* $w_i$ *and* $z_j$.

*Proof.* Let $g$ denote the left side of (3.6). First we note that $g$ is a rational function being symmetrical in $A_0, A_1, \ldots, A_{n+1}$. This, in view of Definition 3.5, is implied by the Lemmas 3.6 and 3.7, since $g = (-1)^{n+1} A_0 A_1 \ldots A_{n+1} L(f)$ where

$$f_{\mathbb{A}\backslash\{A_i\}}(z) = \prod_{Y \in \mathbb{Y}\backslash\{Y_n\}} \left(1 - \frac{z}{Y}\right) R_{\mathbb{A}\backslash\{A_i\}}^{\mathbb{Y}\backslash\{Y_n\}}(w_0, \ldots, w_{n-1}, w_n z; z_0, \ldots, z_n).$$

Secondly, we introduce another set $\mathbb{X} = \{X_0, \ldots, X_n\}$ of variables such that after the substitutions $Y_i = 1/X_i$, $0 \leq i \leq n$, $g$ can be viewed as a polynomial $g(X_0, X_1, \ldots, X_n)$ in the $X_i$ with coefficients being rational functions in the $A_i$, $w_j$ and $z_k$. Note that $\deg_{X_n} g \leq 0$ since $1 - A_i/Y_n = 1 - A_i X_n$ is a factor of $q_{\mathbb{A}}^{\mathbb{Y}}(A_i)$. Moreover, from Lemma 3.9 we obtain the degree bounds with respect to the $X_i$ where $0 \leq i \leq n-1$, namely

$$\deg_{X_i} g \leq n - i + 1. \tag{3.7}$$

For $k \in \{0, \ldots, n+1\}$ define

$$g_{\mathbb{A}}^{\mathbb{X}}(k) \doteqdot g\left(\frac{1}{A_0}, \frac{1}{A_1}, \ldots, \frac{1}{A_{n-k}}, X_{n-k+1}, X_{n-k+2}, \ldots, X_n\right).$$

Note that $g_{\mathbb{A}}^{\mathbb{X}}(0)$ is $g$ for the special choice $\mathbb{Y} = \{A_0, A_1, \ldots, A_n\}$. Because of $\deg_{X_n} g \leq 0$ we have that $g_{\mathbb{A}}^{\mathbb{X}}(0) = g_{\mathbb{A}}^{\mathbb{X}}(1)$. As a crucial ingredient of the proof we show that for all $k \in \{1, \ldots, n\}$:

$$g_{\mathbb{A}}^{\mathbb{X}}(k) = 0 \quad \text{implies} \quad g_{\mathbb{A}}^{\mathbb{X}}(k+1) = 0. \tag{3.8}$$

Consequently, to prove (3.6), which is nothing but $g_{\mathbb{A}}^{\mathbb{X}}(n+1) = 0$, it is sufficient to prove $g_{\mathbb{A}}^{\mathbb{X}}(0) = 0$.

First we will prove (3.8). Owing to $g_{\mathbb{A}}^{\mathbb{X}}(k) = 0$ there exists a polynomial $h = h(X_{n-k}, \ldots, X_n)$ such that

$$g_{\mathbb{A}}^{\mathbb{X}}(k+1) = \left(X_{n-k} - \frac{1}{A_{n-k}}\right) h.$$

Suppose that permutations $\sigma$ of $\mathbb{A}$ act in the usual way on rational functions in the $A_l$, i.e.,

$$\sigma r(A_0, A_1, \ldots, A_{n+1}) = r(A_{\sigma(0)}, A_{\sigma(1)}, \ldots, A_{\sigma(n+1)}).$$

For each $i \in \{0, \dots, k+1\}$ let $\sigma_i$ be such a permutation which, in addition, leaves each of the $A_0, A_1, \dots, A_{n-k-1}$ fixed and which exchanges $A_{n-k}$ with $A_{n-k+i}$. Owing to the full symmetry of $g$ with respect to all the elements of $\mathbb{A}$ we have for all such $\sigma_i$,

$$g_{\mathbb{A}}^{\mathbb{X}}(k+1) = \sigma_i g_{\mathbb{A}}^{\mathbb{X}}(k+1) = \left(X_{n-k} - \frac{1}{A_{n-k+i}}\right) \sigma_i h.$$

Hence $g_{\mathbb{A}}^{\mathbb{X}}(k+1)$, viewed as a univariate polynomial in $X_{n-k}$, has at least $k+2$ different roots, namely $1/A_{n-k}, 1/A_{n-k+1}, \dots, 1/A_{n+1}$. But from (3.7) we have that $\deg_{X_{n-k}} g_{\mathbb{A}}^{\mathbb{X}}(k+1) \le k+1$, hence $g_{\mathbb{A}}^{\mathbb{X}}(k+1)$ must be the zero polynomial which proves (3.8).

Finally our proof is completed by showing $g_{\mathbb{A}}^{\mathbb{X}}(0) = 0$ which is equality (3.6) for $\mathbb{Y} = \{A_0, A_1, \dots, A_n\}$. For this special choice, according to the definition of $q_{\mathbb{A}}^{\mathbb{Y}}$, the sum in (3.6) reduces to the two summands for $i = n$ and $i = n+1$. In other words, it remains to prove the identity

$$\begin{aligned} &R_{\mathbb{X}\cup\{A\}}^{\mathbb{X}}(w_0, \dots, w_{n-1}, w_n B; z_0, \dots, z_n) \\ &= R_{\mathbb{X}\cup\{B\}}^{\mathbb{X}}(w_0, \dots, w_{n-1}, w_n A; z_0, \dots, z_n) \end{aligned} \tag{3.9}$$

for arbitrary variables $A$, $B$ and $\mathbb{X} = \{X_0, \dots, X_{n-1}\}$, $n \ge 0$. For $n = 0$, i.e. $\mathbb{X} = \varnothing$, (3.9) is immediate from Definition 3.3. For $n \ge 1$ Definition 3.3 gives

$$\begin{aligned} &R_{\mathbb{X}\cup\{A\}}^{\mathbb{X}}(w_0, \dots, w_n; z_0, \dots, z_n) \\ &= \frac{1}{1 - X_0 \cdots X_{n-1} A w_0 \cdots w_n z_0 \cdots z_n} \\ &\quad \times \frac{1}{1 - A w_n} \Big( R_{\mathbb{X}}^{\mathbb{X}\setminus\{X_{n-1}\}}(w_0, \dots, w_{n-1}; z_0, \dots, z_{n-1}) \\ &\qquad - A w_n R_{\mathbb{X}}^{\mathbb{X}\setminus\{X_{n-1}\}}(w_0, \dots, w_{n-2}, w_{n-1} w_n A; z_0, \dots, z_{n-1}) \Big). \end{aligned}$$

Using this relation to rewrite the left and the right hand side of (3.9) verifies the equality (3.9), and the proof of Lemma 5 is completed. □

## 4. Relations between $Q_{\mathbb{A}}^{\mathbb{X}}$ and $R_{\mathbb{A}}^{\mathbb{X}}$

In this section we exhibit identities, used in the proof of our main result, Theorem 5.4, that relate the rational functions $Q_{\mathbb{A}}^{\mathbb{X}}$ and $R_{\mathbb{A}}^{\mathbb{X}}$.

**Lemma 4.1.** *Given $\mathbb{A} = \{A_0, A_1, \dots, A_n\}$, $\mathbb{Y} = \{Y_0, Y_1, \dots, Y_{n-1}\}$, $n \ge 0$, and additional arbitrary variables $A_{n+1}, w_0, \dots, w_{n+1}$, and $z_0, \dots, z_n$. Then*

$$\begin{aligned} &\underset{\ge}{\Omega} \frac{Q_{\mathbb{A}}^{\mathbb{Y}}(w_0\lambda_0, \dots, w_n\lambda_n)}{(1 - A_0 \cdots A_{n+1} w_0 \cdots w_{n+1} \lambda_0 \cdots \lambda_n) \prod_{k=0}^{n} \left(1 - \frac{z_0 \cdots z_k}{\lambda_0 \cdots \lambda_k}\right)} \\ &= \frac{1}{1 - A_{n+1} w_{n+1}} \big( R_{\mathbb{A}}^{\mathbb{Y}}(w_0, \dots, w_n; z_0, \dots, z_n) - A_{n+1} w_{n+1} \\ &\quad \times R_{\mathbb{A}}^{\mathbb{Y}}(w_0, \dots, w_{n-1}, w_n w_{n+1} A_{n+1}; z_0, \dots, z_n) \big). \end{aligned} \tag{4.1}$$

*Proof.* We proceed by induction on $n$. For $n = 0$ the verification of (4.1) is a simple exercise using the elimination rule

$$\underset{\ge}{\Omega} \frac{1}{(1 - a\lambda)(1 - b\lambda)\left(1 - \frac{c}{\lambda}\right)} = \frac{1 - abc}{(1-a)(1-b)(1-ac)(1-bc)} \tag{4.2}$$

from MacMahon's table [18, Vol. II, pp. 102–103]. For the step from $n-1$ to $n$, $n \geq 1$, we invoke Definition 3.2 together with the partial fraction decomposition

$$\frac{1}{(1-a)(1-ab)} = \frac{1}{1-b}\left(\frac{1}{1-a} - \frac{b}{1-ab}\right).$$

This way the left side of (4.1) can be rewritten as

$$\begin{aligned}
&\frac{1}{1-A_{n+1}w_{n+1}} \\
&\quad\times\left[\sum_{i=0}^{n} q_{\mathbb{A}}^{\mathbb{Y}}(A_i)\,\underset{\geq}{\Omega}\frac{Q_{\mathbb{A}\setminus\{A_i\}}^{\mathbb{Y}\setminus\{Y_{n-1}\}}(w_0\lambda_0,\dots,w_{n-1}\lambda_{n-1})}{(1-A_0\cdots A_n w_0\cdots w_n\lambda_0\cdots\lambda_n)\prod_{k=0}^{n}\left(1-\frac{z_0\cdots z_k}{\lambda_0\cdots\lambda_k}\right)}\right. \\
&\quad - A_{n+1}w_{n+1} \\
&\quad\times\left.\sum_{i=0}^{n} q_{\mathbb{A}}^{\mathbb{Y}}(A_i)\,\underset{\geq}{\Omega}\frac{Q_{\mathbb{A}\setminus\{A_i\}}^{\mathbb{Y}\setminus\{Y_{n-1}\}}(w_0\lambda_0,\dots,w_{n-1}\lambda_{n-1})}{(1-A_0\cdots A_{n+1}w_0\cdots w_{n+1}\lambda_0\cdots\lambda_n)\prod_{k=0}^{n}\left(1-\frac{z_0\cdots z_k}{\lambda_0\cdots\lambda_k}\right)}\right].
\end{aligned}$$

After elimination of $\lambda_n$ by rule (2.4) this expression is equal to

$$\frac{1}{1-A_{n+1}w_{n+1}}\big(h(w_n) - A_{n+1}w_{n+1}h(w_n w_{n+1}A_{n+1})\big) \tag{4.3}$$

where

$$\begin{aligned}
h(w_n) \doteq{}& \frac{1}{1-A_0\cdots A_n w_0\cdots w_n z_0\cdots z_n} \\
&\times\sum_{i=0}^{n} q_{\mathbb{A}}^{\mathbb{Y}}(A_i)\,\underset{\geq}{\Omega}\frac{Q_{\mathbb{A}\setminus\{A_i\}}^{\mathbb{Y}\setminus\{Y_{n-1}\}}(w_0\lambda_0,\dots,w_{n-1}\lambda_{n-1})}{(1-A_0\cdots A_n w_0\cdots w_n\lambda_0\cdots\lambda_{n-1})\prod_{k=0}^{n-1}\left(1-\frac{z_0\cdots z_k}{\lambda_0\cdots\lambda_k}\right)}.
\end{aligned}$$

Applying the induction hypothesis gives

$$\begin{aligned}
h(w_n) ={}& \frac{1}{1-A_0\cdots A_n w_0\cdots w_n z_0\cdots z_n} \\
&\times\sum_{i=0}^{n}\frac{q_{\mathbb{A}}^{\mathbb{Y}}(A_i)}{1-A_iw_n}\Big(R_{\mathbb{A}\setminus\{A_i\}}^{\mathbb{Y}\setminus\{Y_{n-1}\}}(w_0,\dots,w_{n-1};z_0,\dots,z_{n-1}) - A_iw_n \\
&\times R_{\mathbb{A}\setminus\{A_i\}}^{\mathbb{Y}\setminus\{Y_{n-1}\}}(w_0,\dots,w_{n-2},w_{n-1}w_nA_i;z_0,\dots,z_{n-1})\Big) \\
={}& R_{\mathbb{A}}^{\mathbb{Y}}(w_0,\dots,w_n;z_0,\dots,z_n),
\end{aligned}$$

where the last equality is by Definition 3.3. This completes the proof of Lemma 4.1.

□

Finally we show that the $Q_{\mathbb{A}}^{\mathbb{X}}$ can be represented as rational function multiples of certain $R_{\mathbb{B}}^{\mathbb{Y}}$. We state this relation in a form that is convenient for our proof of Theorem 5.4 in Section 6.

**Lemma 4.2.** *For $k \geq 0$ let $\mathbb{X} = \{X_0, X_1, \dots, X_{k-1}\}$ and $\mathbb{B} = \{B_0, B_1, \dots, B_k\}$. For additional variables $x$ and $X_k$ let $\mathbb{Y} = \{X_1/x, X_2/x, \dots, X_k/x\}$ and $\mathbb{A} =$*

$\{xB_0, xB_1, \ldots, xB_k\}$. *Then for arbitrary* $z_0, \ldots, z_k$*:*

$$\prod_{l=0}^{k}\left(1-\frac{xB_l}{X_0}\right) \times R_{\mathbb{B}}^{\mathbb{Y}}\left(\frac{x}{X_0}, \frac{x}{X_1}, \ldots, \frac{x}{X_k}; z_0, \ldots, z_k\right)$$

$$= Q_{\mathbb{A}}^{\mathbb{X}}\left(\frac{z_0}{X_0}, \frac{z_1}{X_1}, \ldots, \frac{z_k}{X_k}\right). \tag{4.4}$$

*Proof.* The case $k = 0$ is immediate from the Definitions 3.2 and 3.3. For the induction step from $k-1$ to $k$, we rewrite the left side of (4.4) by Definition 3.3 which gives

$$\begin{aligned}
&\frac{\prod_{l=0}^{k}\left(1-\frac{xB_l}{X_0}\right)}{1-x^{k+1}\prod_{j=0}^{k}\frac{B_j z_j}{X_j}} \\
&\quad \times \sum_{i=0}^{k} \frac{q_{\mathbb{B}}^{\mathbb{Y}}(B_i)}{1-\frac{xB_i}{X_k}}\Big[R_{\mathbb{B}\backslash\{B_i\}}^{\mathbb{Y}\backslash\{X_k/x\}}\left(\frac{x}{X_0}, \frac{x}{X_1}, \ldots, \frac{x}{X_{k-1}}; z_0, \ldots, z_{k-1}\right) \\
&\quad - \frac{xB_i}{X_k} R_{\mathbb{B}\backslash\{B_i\}}^{\mathbb{Y}\backslash\{X_k/x\}}\left(\frac{x}{X_0}, \frac{x}{X_1}, \ldots, \frac{x}{X_{k-2}}, \frac{x^2 B_i}{X_{k-1}X_k}; z_0, \ldots, z_{k-1}\right)\Big] \\
&\quad = \frac{1}{1-x^{k+1}\prod_{j=0}^{k}\frac{B_j z_j}{X_j}} \sum_{i=0}^{k} \frac{1-\frac{xB_i}{X_0}}{1-\frac{xB_i}{X_k}} q_{\mathbb{B}}^{\mathbb{Y}}(B_i) \frac{\prod_{l=0}^{k}\left(1-\frac{xB_l}{X_0}\right)}{1-\frac{xB_i}{X_0}} \\
&\qquad\qquad \times R_{\mathbb{B}\backslash\{B_i\}}^{\mathbb{Y}\backslash\{X_k/x\}}\left(\frac{x}{X_0}, \frac{x}{X_1}, \ldots, \frac{x}{X_{k-1}}; z_0, \ldots, z_{k-1}\right)
\end{aligned}$$

where the last equality is by Lemma 5. Applying the trivial identity

$$\frac{1-\frac{xB_i}{X_0}}{1-\frac{xB_i}{X_k}} q_{\mathbb{B}}^{\mathbb{Y}}(B_i) = q_{\mathbb{A}}^{\mathbb{X}}(xB_i)$$

and the induction hypothesis, reduces this expression to

$$\begin{aligned}
&\frac{1}{1-x^{k+1}\prod_{j=0}^{k}\frac{B_j z_j}{X_j}} \sum_{i=0}^{k} q_{\mathbb{A}}^{\mathbb{X}}(xB_i) Q_{\mathbb{A}\backslash\{xB_i\}}^{\mathbb{X}\backslash\{X_{k-1}\}}\left(\frac{z_0}{X_0}, \frac{z_1}{X_1}, \ldots, \frac{z_{k-1}}{X_{k-1}}\right) \\
&\qquad = Q_{\mathbb{A}}^{\mathbb{X}}\left(\frac{z_0}{X_0}, \frac{z_1}{X_1}, \ldots, \frac{z_k}{X_k}\right)
\end{aligned}$$

where in the last step Definition 3.2 has been applied. This completes the proof of equality (4.4). □

## 5. The Main Theorem and Corollaries

In this section we state our main theorem, Theorem 5.4, together with some corollaries. The first consequence is an elegant special case, Theorem 5.5, which turns out to be equivalent to Gansner's Theorem G stated in the Introduction; see Corollary 1. Then Theorem 5.6 is derived as a reformulation of Theorem 5.4 under a certain substitution of variables. Finally, using the product representation (5.7) of a specialized $Q_{\mathbb{A}}^{\mathbb{X}}$, a simple instance of Theorem 5.6 is given in the form of Corollary 2. It contains as special cases results by Gansner [14] and Bender and Knuth [8].

First we introduce some convenient definitions.

**Definition 5.1.** Given an $m \times n$ matrix $X = (x_{i,j})$ and a column vector $y = (y_1, y_2, \dots, y_m)^t$, we define

$$X \wedge y \doteqdot \begin{pmatrix} x_{1,1} & \cdots & x_{1,n} & y_1 \\ x_{2,1} & \cdots & x_{2,n} & y_2 \\ \vdots & \ddots & \vdots & \vdots \\ x_{m,1} & \cdots & x_{m,n} & y_m \end{pmatrix}.$$

If $X = \varnothing$, i.e., $X$ is the empty matrix, then $X \wedge y \doteqdot y$.

Of particular importance are matrices of Toeplitz type, i.e., having constant entries along their diagonals.

**Definition 5.2.** For $m, n \geq 1$ let $\boldsymbol{x} = \{x_1, \dots, x_{n+m-1}\}$ be a set of variables. Define an $m \times n$ matrix $T_{m,n}(\boldsymbol{x}) = (t_{i,j})$ by $t_{i,j} \doteqdot x_{n+i-j}$. For $n = 0$ we define $T_{m,0}(\boldsymbol{x}) \doteqdot \varnothing$, the empty matrix.

**Example 5.3.** For $(m,n) = (4,6)$ and $\boldsymbol{x} = \{x_1, \dots, x_9\}$,

$$T_{4,6}(\boldsymbol{x}) = \begin{pmatrix} x_6 & x_5 & x_4 & x_3 & x_2 & x_1 \\ x_7 & x_6 & x_5 & x_4 & x_3 & x_2 \\ x_8 & x_7 & x_6 & x_5 & x_4 & x_3 \\ x_9 & x_8 & x_7 & x_6 & x_5 & x_4 \end{pmatrix}.$$

Now we are ready to state the main theorem of this paper.

**Theorem 5.4.** *Let $X_0 = 1$ and $X_k = x_1 \cdots x_k$, $k \geq 1$. For $m, n \geq 0$ let $\mathbb{A} = \{X_n, X_{n+1}, \dots, X_{n+m}\}$, $\mathbb{X} = \{X_0, X_1, \dots, X_{m-1}\}$, and $\boldsymbol{x} = \{x_1, \dots, x_{n+m}\}$. Then for any $z = (z_0, z_1, \dots, z_m)^t$ we have*

$$p_{m+1,n+1}(T_{m+1,n}(\boldsymbol{x}) \wedge z) = p_{m+1,n+1} \begin{pmatrix} x_n & \cdots & x_1 & z_0 \\ x_{n+1} & \cdots & x_2 & z_1 \\ \vdots & \ddots & \vdots & \vdots \\ x_{n+m} & \cdots & x_{m+1} & z_m \end{pmatrix}$$

$$= \prod_{k=0}^{n-1} \frac{1}{\left(1 - \frac{X_n}{X_k}\right)\left(1 - \frac{X_{n+1}}{X_k}\right) \cdots \left(1 - \frac{X_{n+m}}{X_k}\right)} \tag{5.1}$$

$$\times Q_{\mathbb{A}}^{\mathbb{X}}\left(\frac{z_0}{X_0}, \dots, \frac{z_m}{X_m}\right).$$

*Proof.* The proof of Theorem 5.4 is given in Section 6. □

The first corollary of Theorem 5.4 is an elegant special case.

**Theorem 5.5.** *For $m \geq 0$ and $n \geq 1$ let $\boldsymbol{x} = \{x_1, \dots, x_{n+m}\}$. Then*

$$p_{m+1,n}(T_{m+1,n}(\boldsymbol{x})) = p_{m+1,n} \begin{pmatrix} x_n & \cdots & x_2 & x_1 \\ x_{n+1} & \cdots & x_3 & x_2 \\ \vdots & \ddots & \vdots & \vdots \\ x_{n+m} & \cdots & x_{m+2} & x_{m+1} \end{pmatrix}$$

$$= \prod_{k=0}^{n-1} \frac{1}{\left(1 - \frac{X_n}{X_k}\right)\left(1 - \frac{X_{n+1}}{X_k}\right) \cdots \left(1 - \frac{X_{n+m}}{X_k}\right)} \tag{5.2}$$

*where $X_0 = 1$ and $X_k = x_1 \cdots x_k$ $(k \geq 1)$.*

*Proof.* Theorem 5.4 with $z_0 = 0$ gives

$$p_{m+1,n+1}(T_{m+1,n}(\boldsymbol{x}) \wedge z)|_{z_0=0} = p_{m+1,n+1} \begin{pmatrix} x_n & \cdots & x_1 & 0 \\ x_{n+1} & \cdots & x_2 & z_1 \\ \vdots & \ddots & \vdots & \vdots \\ x_{n+m} & \cdots & x_{m+1} & z_m \end{pmatrix}$$

$$= \prod_{k=0}^{n-1} \frac{1}{\left(1 - \frac{X_n}{X_k}\right)\left(1 - \frac{X_{n+1}}{X_k}\right) \cdots \left(1 - \frac{X_{n+m}}{X_k}\right)} \times Q_{\mathbb{A}}^{\mathbb{X}}\left(0, \frac{z_1}{X_1}, \ldots, \frac{z_m}{X_m}\right).$$

By Definition 2.1 it is obvious that

$$p_{m+1,n+1}(T_{m+1,n}(\boldsymbol{x}) \wedge z)|_{z_0=0} = p_{m+1,n}(T_{m+1,n}(\boldsymbol{x})),$$

and the rest of Theorem 5.5 follows immediately from Lemma 3.8. □

**Corollary 1.** *Gansner's Theorem G is equivalent to Theorem 5.5.*

*Proof.* In (5.2) replace $m$ and $n$ by $r-1$ and $c$, respectively. Then replace all the $x_l$ by $qx_{c-l}$ and the statement follows. □

Next we consider a variant of Theorem 5.4 which is obtained by the following substitution of variables in (5.1). First, to relate the result to Theorem G from the Introduction, it will be convenient to replace $m$ and $n$ by $r$ and $c$, respectively. Then, replace all the $x_l$ by $qx_{c-l}$, and all the $z_l$ by $qx_{c-l}z_l$. As a consequence the left side of (5.1) turns into

$$\begin{aligned}
&p_{r+1,c+1}(T_{r+1,c}(\boldsymbol{x}) \wedge z)|_{\substack{x_l \to qx_{c-l} \\ z_l \to qx_{c-l}z_l}} \\
&= \sum_{\pi=(a_{i,j}) \in P_{r+1,c+1}} z_0^{a_{1,c+1}} \cdots z_r^{a_{r+1,c+1}} q^{|\pi|} \prod_{\substack{1 \le i \le r+1 \\ 1 \le j \le c+1}} x_{j-i}^{a_{i,j}} \\
&= \sum_{\pi=(a_{i,j}) \in P_{r+1,c+1}} z_0^{a_{1,c+1}} \cdots z_r^{a_{r+1,c+1}} q^{|\pi|} \prod_{k=-r}^{c} x_k^{\mathrm{tr}_k(\pi)} \\
&= \sum_{n=0}^{\infty} \sum_{t_{-r}=0}^{\infty} \cdots \sum_{t_c=0}^{\infty} \sum_{a_0=0}^{\infty} \cdots \sum_{a_r=0}^{\infty} \tau_{r+1,c+1}(t_{-r}, \ldots, t_{-1}; t_0, \ldots, t_c; a_0, \ldots, a_r; n) \\
&\quad \times q^n \prod_{j=0}^{r} z_j^{a_j} \prod_{k=-r}^{c} x_k^{t_k},
\end{aligned}$$

where

$$\tau_{r+1,c+1}(t_{-r}, \ldots, t_{-1}; t_0, \ldots, t_c; a_0, \ldots, a_r; n) \tag{5.3}$$

denotes the number of plane partitions $\pi = (a_{i,j})$ in $P_{r+1,c+1}$ with $|\pi| = n$ and with trace $\mathrm{tr}_k(\pi) = t_k$ in the $k$th diagonal where $-r \le k \le c$, and with $a_{i,c+1} = a_{i-1}$ where $1 \le i \le r+1$.

Under the same replacements of variables, the right side of (5.1) turns into

$$\prod_{k=0}^{c-1} \frac{1}{\left(1-\frac{X_c}{X_k}\right)\left(1-\frac{X_{c+1}}{X_k}\right)\cdots\left(1-\frac{X_{c+r}}{X_k}\right)} Q_{\mathbb{A}}^{\mathbb{X}}\left(\frac{z_0}{X_0},\dots,\frac{z_r}{X_r}\right)\Bigg|_{\substack{x_l\to q x_{c-l}\\ z_l\to q x_{c-l} z_l}}$$
$$= \prod_{k=0}^{c-1} \frac{1}{(1-q^{c-k}x_0\cdots x_{c-k-1})\cdots(1-q^{c-k+r}x_{-r}\cdots x_{c-k-1})}$$
$$\times Q_{\mathbb{B}}^{\mathbb{Y}}\left(qx_c z_0, z_1, q^{-1}\frac{z_2}{x_{c-1}}, q^{-2}\frac{z_3}{x_{c-2}x_{c-1}},\dots,q^{-r+1}\frac{z_r}{x_{c-r+1}\cdots x_{c-1}}\right)$$
$$= \prod_{i=1}^{r+1}\prod_{j=1}^{c} \frac{1}{1-x_{-i+1}\cdots x_{j-1}q^{i+j-1}} Q_{\mathbb{B}}^{\mathbb{Y}}\left(qx_c z_0, \frac{z_1}{Y_0}, \frac{z_2}{Y_1},\dots,\frac{z_r}{Y_{r-1}}\right)$$

with

$$\mathbb{Y}=\{Y_0,Y_1,\dots,Y_{r-1}\} \quad \text{and} \quad \mathbb{B}=\{Y_c,Y_{c+1},\dots,Y_{c+r}\} \tag{5.4}$$

where

$$Y_k \doteqdot q^k \prod_{j=0}^{k-1} x_{c-k+j}. \tag{5.5}$$

We summarize in the form of a theorem.

**Theorem 5.6.** *For $r,c\ge 0$ let $\tau_{r+1,c+1}(t_{-r},\dots,t_{-1};t_0,\dots,t_c;a_0,\dots,a_r;n)$ be as in (5.3). Then for $\mathbb{Y}$ and $\mathbb{B}$ as in (5.4) and (5.5), respectively, one has that*

$$p_{r+1,c+1}\begin{pmatrix} qx_0 & qx_1 & qx_2 & \cdots & qx_{c-1} & qx_c z_0 \\ qx_{-1} & qx_0 & qx_1 & \cdots & qx_{c-2} & qx_{c-1}z_1 \\ qx_{-2} & qx_{-1} & qx_0 & \cdots & qx_{c-3} & qx_{c-2}z_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ qx_{-r} & qx_{-r+1} & qx_{-r+2} & \cdots & qx_{c-r-1} & qx_{c-r}z_r \end{pmatrix}$$
$$= \sum_{n=0}^{\infty} \sum_{\substack{t_{-r},\dots,t_c\ge 0\\ a_0\ge\dots\ge a_r\ge 0}} \tau_{r+1,c+1}(t_{-r},\dots,t_{-1};t_0,\dots,t_c;a_0,\dots,a_r;n)$$
$$\times q^n \prod_{j=0}^{r} z_j^{a_j} \prod_{k=-r}^{c} x_k^{t_k}$$
$$= \prod_{i=1}^{r+1}\prod_{j=1}^{c} \frac{1}{1-x_{-i+1}\cdots x_{j-1}q^{i+j-1}} Q_{\mathbb{B}}^{\mathbb{Y}}\left(qx_c z_0, \frac{z_1}{Y_0}, \frac{z_2}{Y_1},\dots,\frac{z_r}{Y_{r-1}}\right). \tag{5.6}$$

*Remark.* Our crucial rational functions $Q_{\mathbb{A}}^{\mathbb{X}}$ and $R_{\mathbb{A}}^{\mathbb{X}}$ satisfy many additional relations that are not stated explicitly in this paper but which could be explored further. For example, using the fact that for $X_0=1$, $X_k=x_1\cdots x_k$, $k\ge 1$, and arbitrary $A_i$, $\alpha$, $\beta$, and $\gamma\neq 0$,

$$Q_{\{A_0,\dots,A_k\}}^{\{X_0,\dots,X_{k-1}\}}\left(\frac{\alpha}{X_0},\frac{x_1}{X_1},\dots,\frac{x_{k-1}}{X_{k-1}},\frac{\beta}{\gamma X_{k-1}}\right) = \frac{1-\frac{A_0\cdots A_k}{X_0\cdots X_{k-1}}\alpha}{1-\frac{A_0\cdots A_k}{X_0\cdots X_{k-1}}\frac{\alpha\beta}{\gamma}} \prod_{j=0}^{k}\frac{1}{1-A_j\alpha}, \tag{5.7}$$

one can obtain Gansner's Theorem G differently to Corollary 1. Namely, in (5.6) just set all the $z_j$ to 1. – It should be also noted that the proof of (5.7) is very similar to the proof of Lemma 3.8. Finally we remark that (5.7) can be used to obtain further specializations of Theorem 5.5, for instance, Corollary 2.

**Corollary 2.** *Let $\tau(t;\alpha,\beta,\gamma;n)$ denote the number of plane partitions $\pi = (a_{i,j})$ in $P_{r,c}$ with $|\pi| = n$ , with 0-trace* $\mathrm{tr}_0(\pi) = t$, *and with $a_{r,1} = \alpha$, $a_{1,c} = \beta$, and $a_{r,c} = \gamma$, then*

$$\sum_{n\geq 0}\sum_{t\geq 0}\sum_{\alpha,\beta,\gamma\geq 0}\tau(t;\alpha,\beta,\gamma;n)q^n x_0^t x_1^\alpha x_2^\beta x_3^\gamma = \prod_{i=1}^{r-1}\prod_{j=1}^{c-1}\frac{1}{1-x_0q^{i+j-1}} \tag{5.8}$$

$$\times\prod_{i=1}^{r-1}\frac{1}{1-x_0x_2q^{c+i-1}}\prod_{j=1}^{c-1}\frac{1}{1-x_0x_1q^{r+j-1}}\frac{1}{1-x_0x_1x_2q^{r+c-1}}\frac{1-x_0^rx_1x_2q^{rc}}{1-x_0^rx_1x_2x_3q^{rc}}.$$

*Note that setting $x_1 = x_2 = x_3 = 1$ gives Stanley's trace theorem* [23, Thm. 7.20.1]*; setting $x_0 = x_3 = 1$ gives Thm. 4.5 in Gansner* [14]*; setting $x_0 = x_2 = x_3 = 1$ gives Cor. 4.6 in* [14] *which is equivalent to a theorem of Bender and Knuth* [8].

## 6. Proof of the Main Theorem

In this section we present the proof of our main result, Theorem 5.4.

The case $m = 0$ is immediate from (2.2). Thus we shall proceed by induction on $n$ assuming that $m \geq 1$. The case $n = 0$ of Theorem 5.4 is settled by Lemma 3.4 together with (2.1).

For the induction step from $n-1$ to $n$, $n \geq 1$, we first apply Lemma 2.5 to rewrite the left side of (5.1). In the resulting $p_{m+1,n}$ expression we rename the $x$-variables by $y_i = x_{i+1}$ for all $i \geq 0$. In view of this renaming of variables, it is convenient to introduce the sets $\mathbb{Y}=\{Y_0,Y_1,\dots,Y_{m-1}\}$ and $\mathbb{B} = \{B_0,B_1,\dots,B_m\}$ where $Y_i = X_{i+1}/x_1$ and $B_i = Y_{n-1+i}$ for $i \geq 0$. Note that $Y_0 = 1$ and $Y_i = y_1\cdots y_i$ for $i \geq 1$.

Now invoking the induction hypothesis on the $p_{m+1,n}$ expression gives

$$p_{m+1,n+1}(T_{m+1,n}(\boldsymbol{x})\wedge z) = \left(1-\prod_{k=0}^{m}\frac{X_{n+k}}{X_k}z_k\right)^{-1}$$

$$\times\prod_{k=0}^{n-2}\frac{1}{(1-\frac{Y_{n-1}}{Y_k})(1-\frac{Y_n}{Y_k})\cdots(1-\frac{Y_{n+m-1}}{Y_k})}\times\underset{\geq}{\Omega}\frac{Q_{\mathbb{B}}^{\mathbb{Y}}(\lambda_0w_0,\dots,\lambda_{m-1}w_{m-1},w_m)}{\prod_{k=0}^{m-1}\left(1-\frac{z_0\cdots z_k}{\lambda_0\cdots\lambda_k}\right)}$$

where $w_i = y_i/Y_i$ for $0\leqslant i\leqslant m$. We abbreviate the $\underset{\geq}{\Omega}$ expression by $H_m$; i.e.,

$$H_m \doteqdot \underset{\geq}{\Omega}\frac{Q_{\mathbb{B}}^{\mathbb{Y}}(\lambda_0w_0,\dots,\lambda_{m-1}w_{m-1},w_m)}{\prod_{k=0}^{m-1}\left(1-\frac{z_0\cdots z_k}{\lambda_0\cdots\lambda_k}\right)}.$$

Applying Definition 3.2 to $H_m$ we obtain that

$$H_m = \sum_{i=0}^{m}q_{\mathbb{B}}^{\mathbb{Y}}(B_i)\underset{\geq}{\Omega}\frac{Q_{\mathbb{B}\setminus\{B_i\}}^{\mathbb{Y}\setminus\{Y_{m-1}\}}(\lambda_0w_0,\dots,\lambda_{m-1}w_{m-1})}{(1-B_0\cdots B_mw_0\cdots w_m\lambda_0\cdots\lambda_{m-1})\prod_{k=0}^{m-1}\left(1-\frac{z_0\cdots z_k}{\lambda_0\cdots\lambda_k}\right)}.$$

Applying Lemma 6 to each of the summands (i.e., $B_i$ is playing the same role as $A_{n+1}$ in Lemma 6) results in

$$\begin{aligned} H_m &= \sum_{i=0}^{m} \frac{q_{\mathbb{B}}^{\mathbb{Y}}(B_i)}{1-B_i w_m}\big(R_{\mathbb{B}\setminus\{B_i\}}^{\mathbb{Y}\setminus\{Y_{m-1}\}}(w_0,\dots,w_{m-1};z_0,\dots,z_{m-1}) \\ &\quad - B_i w_m \cdot R_{\mathbb{B}\setminus\{B_i\}}^{\mathbb{Y}\setminus\{Y_{m-1}\}}(w_0,\dots,w_{m-2},w_{m-1}w_m B_i;z_0,\dots,z_{m-1})\big) \\ &= \sum_{i=0}^{m} \frac{q_{\mathbb{B}}^{\mathbb{Y}}(B_i)}{1-B_i w_m} R_{\mathbb{B}\setminus\{B_i\}}^{\mathbb{Y}\setminus\{Y_{m-1}\}}(w_0,\dots,w_{m-1};z_0,\dots,z_{m-1}); \end{aligned}$$

the second equality is by Lemma 3.10 noting that $w_m = y_m/Y_m = 1/Y_{m-1}$.

Summarizing, in view of $\frac{Y_{n-1+i}}{Y_k} = \frac{X_{n+i}}{X_{k+1}}$ and $X_0 = 1$, we have derived that

$$\begin{aligned} & p_{m+1,n+1}(T_{m+1,n}(\boldsymbol{x}) \wedge z) \\ &= \prod_{k=0}^{n-1} \frac{1}{\left(1-\frac{X_n}{X_k}\right)\left(1-\frac{X_{n+1}}{X_k}\right)\cdots\left(1-\frac{X_{n+m}}{X_k}\right)} \cdot \frac{\prod_{j=0}^{m}(1-X_{n+j})}{1-\prod_{i=0}^{m} X_{n+i}\frac{z_i}{X_i}} \\ &\quad \times \sum_{i=0}^{m} \frac{q_{\mathbb{B}}^{\mathbb{Y}}(B_i)}{1-B_i w_m} R_{\mathbb{B}\setminus\{B_i\}}^{\mathbb{Y}\setminus\{Y_{m-1}\}}(w_0,\dots,w_{m-1};z_0,\dots,z_{m-1}). \end{aligned}$$

This means, to complete the proof of (5.1) it remains to show that

$$\begin{aligned} & \prod_{j=0}^{m}(1-X_{n+j}) \cdot \sum_{i=0}^{m} \frac{q_{\mathbb{B}}^{\mathbb{Y}}(B_i)}{1-B_i w_m} R_{\mathbb{B}\setminus\{B_i\}}^{\mathbb{Y}\setminus\{Y_{m-1}\}}(w_0,\dots,w_{m-1};z_0,\dots,z_{m-1}) \\ &= \sum_{i=0}^{m} q_{\mathbb{A}}^{\mathbb{X}}(X_{n+i}) Q_{\mathbb{A}\setminus\{X_{n+i}\}}^{\mathbb{X}\setminus\{X_{m-1}\}}\left(\frac{z_0}{X_0},\dots,\frac{z_{m-1}}{X_{m-1}}\right), \end{aligned} \tag{6.1}$$

since by Definition 3.2 the right side of (6.1) turns into

$$\left(1-\prod_{i=0}^{m} X_{n+i}\frac{z_i}{X_i}\right) Q_{\mathbb{A}}^{\mathbb{X}}\left(\frac{z_0}{X_0},\dots,\frac{z_m}{X_m}\right).$$

To prove identity (6.1), in view of $X_0 = 1$, $X_{n+i} = x_1 B_i$, $Y_i = X_{i+1}/x_1$, and $w_i = x_1/X_i$, we apply Lemma 4.2 to each of the summands of its left side which then is transformed into

$$\sum_{i=0}^{m} \frac{1-x_1 B_i}{1-B_i w_m} q_{\mathbb{B}}^{\mathbb{Y}}(B_i) Q_{\mathbb{A}\setminus\{X_{n+i}\}}^{\mathbb{X}\setminus\{X_{m-1}\}}\left(\frac{z_0}{X_0},\dots,\frac{z_{m-1}}{X_{m-1}}\right).$$

But this, because of

$$\begin{aligned} q_{\mathbb{A}}^{\mathbb{X}}(X_{n+i}) &= \frac{\prod_{k=0}^{m-1}\left(1-\frac{X_{n+i}}{X_k}\right)}{\prod_{\substack{0\le k\le m\\ k\ne i}}\left(1-\frac{X_{n+i}}{X_{n+k}}\right)} = (1-x_1 B_i)\frac{\prod_{k=1}^{m-1}\left(1-\frac{x_1 B_i}{x_1 Y_{k-1}}\right)}{\prod_{\substack{0\le k\le m\\ k\ne i}}\left(1-\frac{x_1 B_i}{x_1 B_k}\right)} \\ &= \frac{1-x_1 B_i}{1-\frac{B_i}{Y_{m-1}}} q_{\mathbb{B}}^{\mathbb{Y}}(B_i) = \frac{1-x_1 B_i}{1-B_i w_m} q_{\mathbb{B}}^{\mathbb{Y}}(B_i), \end{aligned}$$

turns out to be equal to the right side of (6.1). This completes the proof of identity (5.1) and thus of Theorem 5.4.

## 7. Conclusion

This paper marks a milestone in our study of the implications of Partition Analysis. Many others have become interested in its use; see e.g. [1], [11], [12], and [13]. Also G. Xin [24] has introduced a variation to the algorithm. We hope in the future to examine more recondite types of compositions and partitions along the lines of [4], [5], and [6].

We live in an exciting era when the growing power of computers and extensive use and improvement of computer algebra algorithms promises seemingly boundless vistas for the Omega package and its offsprings to explore.

**Acknowledgement.** We thank Christian Krattenthaler for pointing out the fact that Partition Analysis has led us to a rediscovery of Gansner's theorem.

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