PARAMETERIZED TELESCOPING PROVES ALGEBRAIC INDEPENDENCE OF SUMS

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Abstract. Usually creative telescoping is used to derive recurrences for sums. In this article we show that the non-existence of a creative telescoping solution, and more generally, of a parameterized telescoping solution, proves algebraic independence of certain types of sums. Combining this fact with summation-theory shows transcendence of whole classes of sums.

1. Introduction

Telescoping [7] and creative telescoping [25, 15] for hypergeometric terms and its variations [14, 10, 12, 4] are standard tools in symbolic summation. All these techniques are covered by the following formulation of the parameterized telescoping problem: **Given** sequences $f_1(k), \dots, f_d(k)$ over a certain field $\mathbb{K}$, **find**, if possible, constants $c_1, \dots, c_d \in \mathbb{K}$ and a sequence $g(k)$ such that

$$g(k+1) - g(k) = c_1 f_1(k) + \cdots + c_d f_d(k). \tag{1}$$

If one succeeds in this task, one gets, with some mild extra conditions, the sum-relation

$$g(n+1) - g(r) = c_1 \sum_{k=r}^{n} f_1(k) + \cdots + c_d \sum_{k=r}^{n} f_d(k) \tag{2}$$

for some $r \in \mathbb{N} = \{0, 1, \dots\}$ big enough. Note that $d = 1$ gives telescoping. Moreover, given a bivariate sequence $f(m,k)$, one can set $f_i(k) := f(m+i-1, k)$ which corresponds to creative telescoping.

Since Karr's summation algorithm [8] and its extensions [23, 21] can solve the parameterized telescoping problem in the difference field setting of $\Pi\Sigma^*$-fields, we get a rather flexible algorithm which is implemented in the package Sigma [19]: the $f_i(k)$ can be arbitrarily nested sums and products; see e.g., [13, 6].

In this article we apply $\Pi\Sigma^*$-field theory [8, 17] to get new theoretical insight: If there is no solution to (1) within a given $\Pi\Sigma^*$-field setting, then the sums in (2) can be represented in a larger $\Pi\Sigma^*$-field by transcendental extensions; see Theorem 3.1. Motivated by this fact, we construct a difference ring monomorphism which links elements from the larger $\Pi\Sigma^*$-field to the sums

$$S_1(n) = \sum_{k=r}^{n} f_1(k), \dots, S_d(n) = \sum_{k=r}^{n} f_d(k) \tag{3}$$

in the ring of sequences over $\mathbb{K}$. In particular, this construction transfers the transcendence properties from the $\Pi\Sigma^*$-world into the sequence domain. In order to accomplish this task, we restrict to generalized d'Alembertian extensions, a huge subclass of $\Pi\Sigma^*$-extensions.

Summarizing, parameterized telescoping in combination with $\Pi\Sigma^*$-fields gives a criterion to check algorithmically the transcendence of sums of type (3); see Theorem 4.1. Combining this criterion with results from summation theory, like [1, 10, 2, 24], shows that whole classes of sequences are transcendental. E.g., the harmonic numbers $\{H_n^{(i)} \mid i \geq 1\}$ with $H_n^{(i)} := \sum_{k=1}^{n} \frac{1}{k^i}$ are transcendental over $\mathbb{Q}(n)$.

The general structure of this article is as follows. In Section 2 we present the basic notions of difference fields, and we introduce $\Pi\Sigma^*$-extensions together with the subclass of generalized d'Alembertian extensions. In Section 3 we show the correspondence of parameterized telescoping and the construction of a certain type of Σ^*-extensions. In Section 4 we construct a difference ring monomorphism that carries over the transcendence properties from

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a given d'Alembertian extension to the ring of sequences. This leads to a transcendence decision criterion of sequences in terms of generalized d'Alembertian extensions. In Section 5 we illustrate our criterion by various examples. Finally, we present the analogous criterion for products in Section 6.

2. Basic notions: $\Pi\Sigma^*$-extensions and generalized d'Alembertian extension

Subsequently we introduce the basic concepts of difference fields that shall pop up later.

A *difference ring*[1] (resp. field) $(\mathbb{A}, \sigma)$ is a ring $\mathbb{A}$ (resp. field) with a ring automorphism (resp. field automorphism) $\sigma : \mathbb{A} \to \mathbb{A}$. The set of constants $\mathrm{const}_\sigma\mathbb{A} = \{k \in \mathbb{A} \,|\, \sigma(k) = k\}$ forms a subring (resp. subfield) of $\mathbb{A}$. In this article we always assume that $\mathrm{const}_\sigma\mathbb{A}$ is a field, which we usually denote by $\mathbb{K}$. We call $\mathrm{const}_\sigma\mathbb{A}$ the *constant field* of $(\mathbb{A}, \sigma)$.

A *difference ring homomorphism* (resp. monomorphism) $\tau : \mathbb{A}_1 \to \mathbb{A}_2$ between two difference rings $(\mathbb{A}_1, \sigma_1)$ and $(\mathbb{A}_2, \sigma_2)$ is a ring homomorphism (resp. monomorphism) with the additional property that $\tau(\sigma_1(f)) = \sigma_2(\tau(f))$ for all $f \in \mathbb{A}_1$.

A difference ring (resp. difference field) $(\mathbb{E}, \sigma)$ is a *difference ring extension* (resp. *difference field extension*) of a difference ring (resp. difference field) $(\mathbb{A}, \sigma')$ if $\mathbb{A}$ is a subring (resp. subfield) of $\mathbb{E}$ and $\sigma'(f) = \sigma(f)$ for all $f \in \mathbb{A}$; since σ and σ' agree on $\mathbb{A}$, we do not distinguish them anymore.

Now we are ready to define $\Pi\Sigma^*$-extensions and generalized d'Alembertian extensions in which we will represent our indefinite nested sums and products.

A difference field extension $(\mathbb{F}(t), \sigma)$ of $(\mathbb{F}, \sigma)$ is called a $\Pi\Sigma^*$-*extension* if both difference fields share the same field of constants, t is transcendental over $\mathbb{F}$, and $\sigma(t) = t + a$ for some $a \in \mathbb{F}^*$ (a sum) or $\sigma(t) = a\,t$ for some $a \in \mathbb{F}^*$ (a product). If $\sigma(t)/t \in \mathbb{F}$ (resp. $\sigma(t) - t \in \mathbb{F}$), we call the extension also a Π-*extension* (resp. Σ^*-*extension*). In short, we say that $(\mathbb{F}(t_1) \dots (t_e), \sigma)$ is a $\Pi\Sigma^*$-extension (resp. Π-extension, Σ^*-extension) of $(\mathbb{F}, \sigma)$ if the extension is given by a tower of $\Pi\Sigma^*$-extensions (resp. Π-extensions, Σ^*-extensions). A $\Pi\Sigma^*$-*field* $(\mathbb{K}(t_1) \dots (t_e), \sigma)$ over $\mathbb{K}$ is a $\Pi\Sigma^*$-extension of $(\mathbb{K}, \sigma)$ with constant field $\mathbb{K}$.

Example 2.1. Consider the difference field $(\mathbb{Q}(m)(k)(b)(h), \sigma)$ with $\sigma(k) = k + 1$, $\sigma(b) = \frac{m-k}{k+1}$, $\sigma(h) = h + \frac{1}{k+1}$, and $\mathrm{const}_\sigma\mathbb{Q}(m)(k)(b)(h) = \mathbb{Q}(m)$. The extensions k, b, and h form $\Pi\Sigma^*$-extensions over the fields below. $(\mathbb{Q}(m)(k)(b)(h), \sigma)$ is a $\Pi\Sigma^*$-field over $\mathbb{Q}(m)$. $\square$

The following theorem tells us how one can check if an extension is a $\Pi\Sigma^*$-extension.

Theorem 2.1 ([8])**.** *Let $(\mathbb{F}(t), \sigma)$ be a difference field extension of $(\mathbb{F}, \sigma)$ with $t \neq 0$ and $\sigma(t) = \alpha\,t + \beta$ where $\alpha \in \mathbb{F}^*$ and $\beta \in \mathbb{F}$. Then:*

(1) *This is a Σ^*-extension iff $\alpha = 1$ and there is no $g \in \mathbb{F}$ with $\sigma(g) - g = \beta$.*
(2) *This is a Π-extension iff $\beta = 0$ and there is no $n \neq 0$ and $g \in \mathbb{F}^*$ with $\sigma(g) = \alpha^n g$.*

The following remarks are in place:

(1) If $(\mathbb{F}, \sigma)$ is a $\Pi\Sigma^*$-field, algorithms are available which make Theorem 2.1 completely constructive; see [8, 23].

(2) We emphasize that we have a first criterion for transcendence in a difference field: if there is no telescoping solution, then we can adjoin the sum as a transcendental extension without extending the constant field. This criterion will be generalized to parameterized telescoping; see Theorem 3.1. For the product case see Theorem 6.1.

Theorem 2.2 states how a solution g of $\sigma(g) - g = f$ and $\sigma(g) = f g$ looks like in certain types of extensions. The first part follows by [8, Sec. 4.1] and the second part follows by [22, Lemma 6.8]. These results are crucial ingredients to prove Theorems 3.1 and 6.1.

Theorem 2.2. *Let $(\mathbb{F}(t_1, \dots, t_d), \sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{F}, \sigma)$ with constant field $\mathbb{K}$ and $\sigma(t_i) = \alpha_i t_i + \beta_i$ where $\alpha_i, \beta_i \in \mathbb{F}$. Let $f \in \mathbb{F}$ and $g \in \mathbb{F}(t_1, \dots, t_d)$.*

[1] All fields and rings are of characteristic 0 and commutative

(1) *If* $\sigma(g)-g=f$, *then* $g=\sum_{i=1}^{d}c_i\,t_i+w$ *with* $w\in\mathbb{F}$, $c_i\in\mathbb{K}$*; if* $\alpha_i\neq 1$, *then* $c_i=0$.
(2) *If* $\sigma(g)=f\,g$, *then* $g=w\prod_{i=1}^{d}t_i^{c_i}$ *with* $w\in\mathbb{F}$ *and* $c_i\in\mathbb{Z}$*; if* $\beta_i\neq 0$, *then* $c_i=0$.

Subsequently, we will restrict to the following type of extensions. A $\Pi\Sigma^*$-extension $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ of $(\mathbb{F},\sigma)$ with $\sigma(t_i)=\alpha_i\,t_i+\beta_i$ is called *generalized d'Alembertian extension*, in short GA-extension, if $\alpha_i\in\mathbb{F}$ and $\beta_i\in\mathbb{F}[t_1,\dots,t_{i-1}]$ for all $1\leq i\leq e$.

Remark 2.1. One can reorder GA-extensions to $\mathbb{F}(p_1)\dots(p_u)(s_1)\dots(s_v)$ with $u,v\geq 0$ where $\frac{\sigma(p_i)}{p_i}\in\mathbb{F}$ for $1\leq i\leq u$ and $\sigma(s_i)-s_i\in\mathbb{F}[p_1,\dots,p_u,s_1,\dots,s_{i-1}]$ for $1\leq i\leq v$. □

It is easy to see that $(\mathbb{F}[t_1,\dots,t_e],\sigma)$ is a difference ring extension of $(\mathbb{F},\sigma)$. Moreover, if $f\in\mathbb{F}[t_1,\dots,t_e]$, then there are no solutions in $\mathbb{F}(t_1,\dots,t_e)\setminus\mathbb{F}[t_1,\dots,t_e]$.

Theorem 2.3. *Let* $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ *be a generalized d'Alembertian extension of* $(\mathbb{F},\sigma)$ *and* $g\in\mathbb{F}(t_1)\dots(t_e)$*. Then* $\sigma(g)-g\in\mathbb{F}[t_1,\dots,t_e]$ *if and only if* $g\in\mathbb{F}[t_1,\dots,t_e]$.

Proof. The direction from left to right is clear by the definition of GA-extensions. We prove the other direction by induction on the number of extensions. For $e=0$ nothing has to be shown. Now suppose that the theorem holds for e extensions and consider the GA-extension $(\mathbb{F}(t_1)\dots(t_{e+1}),\sigma)$ of $(\mathbb{F},\sigma)$. By Remark 2.1 we can bring $\mathbb{F}(t_1)\dots(t_{e+1})$ to a form where all Σ^*-extensions are on top. Write $t:=t_{e+1}$ and let $\sigma(t)=\alpha\,t+\beta$ with $\alpha\in\mathbb{F}^*$ and $\beta\in\mathbb{F}[t_1,\dots,t_e]$. Now suppose that $\sigma(g)-g=f$ where $g\in\mathbb{F}(t_1,\dots,t_e,t)\setminus\mathbb{F}[t_1,\dots,t_e,t]$ and $f\in\mathbb{F}[t_1,\dots,t_e][t]$. Note that $g\in\mathbb{F}(t_1,\dots,t_e)[t]$; see, e.g., [24, Lemma 3.1]. Hence we can write $g=\sum_{i=0}^{d}g_it^i$ with $g_i\in\mathbb{F}(t_1,\dots,t_e)$. If $e=0$, we are done. Otherwise, suppose that $e>0$ and let $j\geq 0$ be maximal such that $g_j\notin\mathbb{F}(t_1,\dots,t_e)\setminus\mathbb{F}[t_1,\dots,t_e]$. Define $g':=\sum_{i=0}^{j}g_it^i\in\mathbb{F}(t_1,\dots,t_e)[t]$ and $f':=f-\big(\sigma(\sum_{i=j+1}^{d}g_it^i)-\sum_{i=j+1}^{d}g_it^i\big)\in\mathbb{F}[t_1,\dots,t_e][t]$. Since $\sigma(g')-g'=f'$, $\deg(f')\leq\deg(g')=j$. By coefficient comparison we have

$$\alpha^j\sigma(g_j)-g_j=\phi\in\mathbb{F}[t_1,\dots,t_e] \tag{4}$$

where ϕ is the jth coefficient in f'. If $\alpha=1$ or $j=0$, we can apply the induction assumption and conclude that $g_j\in\mathbb{F}[t_1,\dots,t_e]$, a contradiction. Otherwise, suppose that $1\neq\alpha$ and $j\geq 1$. Then by the assumption that all Π-extensions come first, it follows that $\sigma(t_i)/t_i\in\mathbb{F}$ for all $1\leq i\leq e$. Reorder $\mathbb{F}(t_1,\dots,t_e)$ such that $g_j\notin\mathbb{F}(t_1,\dots,t_{e-1})[t_e]$. By Bronstein [5, Cor. 3], see also [18, Cor. 1], we get $g_j=\frac{p}{t_e^m}$ for some $m>0$ and $p\in\mathbb{F}(t_1,\dots,t_{e-1})[t_e]^*$ with $t_e\nmid p$. Hence $\alpha^j\sigma(\frac{p}{t_e^m})-\frac{p}{t_e^m}=\frac{\alpha^j\sigma(p)-a^mp}{a^m t_e^m}=\phi$ with $a:=\frac{\sigma(t_e)}{t_e}\in\mathbb{F}^*$. Since $t_e\nmid p$, also $a\,t_e=\sigma(t_e)\nmid\sigma(p)$, and thus $t_e\nmid\sigma(p)$. Since (4) and $m>0$, $\alpha^j\sigma(p)-a^mp=0$, and hence $\sigma(\frac{1}{p\,t_e^m})=\alpha^j\,\frac{1}{p\,t_e^m}$; a contradiction to Theorem 2.1.2 and the fact that $(\mathbb{F}(t_1,\dots,t_e)(t),\sigma)$ is a Π-extension of $(\mathbb{F}(t_1,\dots,t_e),\sigma)$. □

3. Parameterized telescoping, $\Pi\Sigma^*$-extensions and the ring of sequences

We get the following criterion to check transcendence in a given difference field $(\mathbb{F},\sigma)$.

Theorem 3.1. *Let* $(\mathbb{F},\sigma)$ *be a difference field with constant field* $\mathbb{K}$ *and* $(f_1,\dots,f_d)\in\mathbb{F}^d$*. The following statements are equivalent.*
(1) *There do not exist a* $\mathbf{0}\neq(c_1,\dots,c_d)\in\mathbb{K}^d$ *and a* $g\in\mathbb{F}$ *with*

$$\sigma(g)-g=c_1f_1+\dots+c_df_d. \tag{5}$$

(2) *There is a* Σ^**-extension* $(\mathbb{F}(t_1)\dots(t_d),\sigma)$ *of* $(\mathbb{F},\sigma)$ *with* $\sigma(t_i)=t_i+f_i$ *for* $1\leq i\leq d$.

Proof. Suppose that (5) holds for some $\mathbf{0}\neq(c_1,\dots,c_d)\in\mathbb{K}^d$ and $g\in\mathbb{F}$. In addition, assume that there exists a Σ^*-extension $(\mathbb{F}(t_1,\dots,t_d),\sigma)$ of $(\mathbb{F},\sigma)$ with $\sigma(t_i)=t_i+f_i$. Then $\sigma(g)-g=\sum_{i=1}^{d}c_i\big(\sigma(t_i)-t_i\big)=\sigma(\sum_{i=1}^{d}c_i\,t_i)-\sum_{i=1}^{d}c_i\,t_i$, and thus $\sigma(\sum_{i=1}^{d}c_i\,t_i-g)=\sum_{i=1}^{d}c_i\,t_i-g$. Since $\mathrm{const}_\sigma\mathbb{F}(t_1,\dots,t_d)=\mathbb{K}$, there is a $k\in\mathbb{K}$ with $\sum_{i=1}^{d}c_i\,t_i-g+k=0$. Thus there are algebraic relations in the t_i, a contradiction to the definition of $\Pi\Sigma^*$-extensions.

Contrary, let i be maximal such that $(\mathbb{F}(t_1,\dots,t_i),\sigma)$ is a Σ^*-extension of $(\mathbb{F},\sigma)$; suppose that $i<d$. Then there is a $g\in\mathbb{F}(t_1,\dots,t_i)$ with $\sigma(g)-g=f_{i+1}$. By Theorem 2.2.1 there are $c_j\in\mathbb{K}$, $h\in\mathbb{F}$ with $g=h+\sum_{j=1}^{i}c_j\,t_j$. This shows that $\sigma(h)-h=f_{i+1}-\sum_{j=1}^{i}c_j(\sigma(t_j)-t_j)=-c_1\,f_1-\dots-c_i\,f_i+f_{i+1}$. Hence we get a solution for (5) with $\mathbf{0}\neq(c_1,\dots,c_d)\in\mathbb{K}^d$. □

Let $\mathbb{K}$ be a field with characteristic zero. The set of all sequences $\mathbb{K}^{\mathbb{N}}$ with elements $(a_n)_{n=0}^{\infty}=\langle a_0,a_1,a_2,\dots\rangle$, $a_i\in\mathbb{K}$, forms a commutative ring by component-wise addition and multiplication; the field $\mathbb{K}$ can be naturally embedded by identifying $k\in\mathbb{K}$ with the sequence $\boldsymbol{k}:=\langle k,k,k,\dots\rangle$. In order to turn the shift-operation

$$S:\langle a_0,a_1,a_2,\dots\rangle\mapsto\langle a_1,a_2,a_3,\dots\rangle \tag{6}$$

to an automorphism, we follow the construction from [15, Sec. 8.2]: We define an equivalence relation $\sim$ on $\mathbb{K}^{\mathbb{N}}$ with $(a_n)_{n=0}^{\infty}\sim a(b_n)_{n=0}^{\infty}$ if there exists a $\delta\geq 0$ such that $a_k=b_k$ for all $k\geq\delta$. The equivalence classes form a ring which is denoted by $S(\mathbb{K})$; the elements of $S(\mathbb{K})$ will be denoted, as above, by sequence notation. Now it is immediate that $S:S(\mathbb{K})\to S(\mathbb{K})$ with (6) forms a ring automorphism. The difference ring $(S(\mathbb{K}),S)$ is called the *ring of* $\mathbb{K}$*-sequences* or in short the *ring of sequences*.

The main result of our article is that the polynomial ring $\mathbb{F}[t_1,\dots,t_e]$ of a generalized d'Alembertian extension $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ of $(\mathbb{F},\sigma)$ with constant field $\mathbb{K}$ can be embedded in the ring of sequences $S(\mathbb{K})$, provided that $(\mathbb{F},\sigma)$ can be embedded in $S(\mathbb{K})$. More precisely, we will construct a difference ring monomorphism $\tau:\mathbb{F}[t_1,\dots,t_e]\to S(\mathbb{K})$ where the constants $k\in\mathbb{K}$ are mapped to $\boldsymbol{k}=\langle k,k,\dots\rangle$. We will call such a difference ring homomorphism (resp. monomorphism) also a $\mathbb{K}$*-homomorphism* (resp. $\mathbb{K}$*-monomorphism*).

Then the main consequence is that the transcendence properties of d'Alembertian extensions, in particular Theorem 3.1, can be carried over to $S(\mathbb{K})$; see Theorem 4.1.

4. The monomorphism construction

In the following we will construct the $\mathbb{K}$-monomorphism as mentioned in the end of Section 3. Here we use the following lemma which is inspired by [9]; the proof is obvious.

Lemma 4.1. *Let $(\mathbb{A},\sigma)$ be a difference ring with constant field $\mathbb{K}$. If $\tau:\mathbb{A}\to S(\mathbb{K})$ is a $\mathbb{K}$-homomorphism, there is a map* $\mathrm{ev}:\mathbb{A}\times\mathbb{N}\to\mathbb{K}$ *with*

$$\tau(f)=\langle\mathrm{ev}(f,0),\mathrm{ev}(f,1),\dots\rangle \tag{7}$$

for all $f\in\mathbb{A}$ which has the following properties: For all $c\in\mathbb{K}$ there is a $\delta\geq 0$ with

$$\forall i\geq\delta\ \mathrm{ev}(c,i)=c; \tag{8}$$

for all $f,g\in\mathbb{A}$ there is a $\delta\geq 0$ with

$$\forall i\geq\delta:\ \mathrm{ev}(f\,g,i)=\mathrm{ev}(f,i)\,\mathrm{ev}(g,i), \tag{9}$$

$$\forall i\geq\delta:\ \mathrm{ev}(f+g,i)=\mathrm{ev}(f,i)+\mathrm{ev}(g,i); \tag{10}$$

and for all $f\in\mathbb{A}$ and $j\in\mathbb{Z}$ there is a $\delta\geq 0$ with

$$\forall i\geq\delta\ \mathrm{ev}(\sigma^j(f),i)=\mathrm{ev}(f,i+j). \tag{11}$$

Conversely, if we have a map $\mathrm{ev}:\mathbb{A}\times\mathbb{N}\to\mathbb{K}$ *with* (8), (9), (10) *and* (11), *then $\tau:\mathbb{A}\to S(\mathbb{K})$ defined by* (7) *forms a $\mathbb{K}$-homomorphism.*

In order to take into account the constructive aspects, we introduce the following functions.

Definition 4.1. Let $(\mathbb{A},\sigma)$ be a difference ring and $\tau:\mathbb{A}\to S(\mathbb{K})$ be a $\mathbb{K}$-homomorphism defined by (7). τ is called *operation-bounded* by $L:\mathbb{A}\to\mathbb{N}$ if for all $f\in\mathbb{A}$ and $j\in\mathbb{Z}$ with $\delta=\delta(f,j):=L(f)+max(0,-j)$ we have (11) and for all $f,g\in\mathbb{A}$ with $\delta=\delta(f,g):=\max(L(f),L(g))$ we have (9) and (10); such a function is also called *o-function*. τ is called zero-bounded by $Z:\mathbb{F}\to\mathbb{N}$ if for all $f\in\mathbb{F}$ and all $i\geq Z(f)$ we have $\mathrm{ev}(f,i)\neq 0$; such a function is also called *z-function*.

Lemma 4.2. *Let $(\mathbb{A},\sigma)$ be a difference field with constant field $\mathbb{K}$. If $\tau:\mathbb{A}\to S(\mathbb{K})$ is a $\mathbb{K}$-homomorphism with (7), then for all $f\in\mathbb{A}$ we have $\mathrm{ev}(\frac{1}{f},i)=\frac{1}{\mathrm{ev}(f,i)}$ for big enough i. In particular, there is a z-function for τ.*

Proof. $\tau(f^{-1})$ is the inverse of $\tau(f)$, i.e., $\tau(\frac{1}{f})=\frac{1}{\tau(f)}$. Hence, $\mathrm{ev}(\frac{1}{f},k)=\frac{1}{\mathrm{ev}(f,k)}$ for all $k\geq\delta$ for some $\delta\geq 0$. This implies $\mathrm{ev}(f,k)\neq 0$ for all $k\geq\delta$. Hence there is a z-function. □

The following lemma is the crucial tool to design step by step a $\mathbb{K}$-monomorphism for a generalized d'Alembertian extension. This construction we will be used in Theorem 4.1.

Lemma 4.3. *Let $(\mathbb{F}(t_1)\dots(t_e)(t),\sigma)$ be a GA-extension of $(\mathbb{F},\sigma)$ with $\mathbb{K}:=\mathrm{const}_\sigma\mathbb{F}$. Let $\tau:\mathbb{F}[t_1]\dots[t_e]\to S(\mathbb{K})$ be a $\mathbb{K}$-homomorphism (resp. $\mathbb{K}$-monomorphism). Then:*

(1) *There is a $\mathbb{K}$-homomorphism (resp. $\mathbb{K}$-monomorphism) $\tau':\mathbb{F}[t_1]\dots[t_e][t]\to S(\mathbb{K})$ with $\tau'(f)=\tau(f)$ for all $f\in\mathbb{F}[t_1,\dots,t_e]$.*
(2) *If there is an o-function for τ, then there is an o-function for τ'.*
(3) *If there is a computable z-function for τ restricted on $\mathbb{F}$ and a computable o-function for τ, then there is a computable o-function for τ'.*

Proof. By Lemma 4.2 there is a z-function $Z:\mathbb{F}\to\mathbb{N}$ for τ restricted on $\mathbb{F}$. Let τ be defined by (7), denote $\mathbb{A}:=\mathbb{F}[t_1,\dots,t_e]$ and suppose that $\sigma(t)=\alpha\,t+\beta$ with $\alpha\in\mathbb{F}^*$ and $\beta\in\mathbb{A}$. First suppose that $\alpha=1$. Then there is a $\delta\geq 0$ with (11) where $f:=\beta$. Let $r:=\delta+1$ and consider the sequence given by

$$\mathrm{ev}(t,k)=\sum_{i=r}^{k}\mathrm{ev}(\beta,i-1)+c\quad\text{for some}^2 c\in\mathbb{K}.\tag{12}$$

Let $j\geq 0$. Then by construction we have for all $k\geq r$:

$$\begin{aligned}\mathrm{ev}(\sigma^j(t),k)&=\mathrm{ev}(t+\sum_{i=0}^{j-1}\sigma^i(\beta),k)=\mathrm{ev}(t,k)+\sum_{i=0}^{j-1}\mathrm{ev}(\sigma^i(\beta),k)+c\\&=\mathrm{ev}(t,k)+\sum_{i=0}^{j-1}\mathrm{ev}(\beta,k+i)+c=\mathrm{ev}(t,k+j).\end{aligned}$$

Similarly, if $j<0$, then $\mathrm{ev}(t,k+j)=\mathrm{ev}(\sigma^j(t),k)$ for all $k\geq r-j$. This proves (11) for $f=t$ and all $j\in\mathbb{Z}$ with $\delta=r+\max(-j,0)$. Now suppose that $\beta=0$. Then there is a $\delta\geq 0$ with (11) where $f:=\alpha$. Let $r:=\max(Z(\alpha),\delta)+1$ and consider the sequence given by

$$\mathrm{ev}(t,k)=c\prod_{i=r}^{k}\mathrm{ev}(\alpha,i-1)\quad\text{for some } c\in\mathbb{K}^*.\tag{13}$$

Analogously, it follows that for all $j\in\mathbb{Z}$ we have $\mathrm{ev}(t,k+j)=\mathrm{ev}(\sigma^j(t),k)$ for all $k\geq r+\max(-j,0)$. Moreover, since $\mathrm{ev}(\alpha,i-1)\neq 0$ for all $i\geq r$, $\mathrm{ev}(t,k)\neq 0$ for all $k\geq r$. Finally, we extend ev from $\mathbb{A}$ to $\mathbb{A}[t]$ by

$$\mathrm{ev}'(\sum_{i=0}^{n}f_it^i,k)=\sum_{i=0}^{n}\mathrm{ev}(f_i,k)\mathrm{ev}(t,k)^i.$$

Moreover, if we choose $\delta\geq r$ big enough (depending on the f_i), we get (11) for $f=\sum_{i=0}^{n}f_it^i$. Similarly, we can find for all $f,g\in\mathbb{A}[t]$ a $\delta\geq 0$ with (9) and (10). Moreover, (8) holds, since ev' restricted on $\mathbb{A}$ equals ev. Summarizing, if we define $\tau':\mathbb{A}[t]\to S(\mathbb{K})$ following (7), τ' forms a $\mathbb{K}$-homomorphism by Lemma 4.1. In particular, if $\beta=0$, then $\tau'(t)\neq\mathbf{0}$.

[2] The constant c can be chosen arbitrarily; this gives extra freedom in the design of τ'.

Suppose that there is in addition an o-function $L : \mathbb{A} \to \mathbb{N}$. Then the r from above can be defined by $r := \max(Z(\alpha), L(\alpha)) + 1$ or $r := L(\beta) + 1$, respectively. Define $L' : \mathbb{A}[t] \to \mathbb{N}$ by

$$L'(f) = \begin{cases} L(f) & \text{if } f \in \mathbb{A}, \\ \max(r, L(f_1), \dots, L(f_n)) & \text{if } f = \sum_{i=0}^{n} f_i t^i \notin \mathbb{A}. \end{cases}$$

Then one can check that L' is an o-function for τ'. E.g., for $f = \sum_{i=0}^{m} f_i t^i$, $g = \sum_{i=0}^{n} g_i t^i$:

$$\begin{aligned} \mathrm{ev}(f\,g, k) &= \mathrm{ev}(\sum_{j=0}^{m+n} t^j \sum_{i=0}^{j} f_i g_{j-i}, k) = \sum_{j=0}^{m+n} \mathrm{ev}(t,k)^j \sum_{i=0}^{j} \mathrm{ev}(f_i g_{j-i}, k) \\ &= \Big(\sum_{i=0}^{m} \mathrm{ev}(f_i, k)\mathrm{ev}(t,k)^i\Big)\Big(\sum_{j=0}^{n} \mathrm{ev}(g_j, k)\mathrm{ev}(t,k)^j\Big) = \mathrm{ev}(f,k)\mathrm{ev}(g,k) \end{aligned}$$

for all $k \geq \max(r, f_0, \dots, f_m, g_0, \dots, g_n) = \max(L(f), L(g))$. In particular, if L and Z are computable, then L' is computable.

Finally, suppose that τ is a $\mathbb{K}$-monomorphism, but the extended $\mathbb{K}$-homomorphism τ' is not injective. Then there is an $f \in \mathbb{A}[t] \setminus \mathbb{A}$ with $\tau'(f) = \mathbf{0}$. Take such an $f = \sum_{i=0}^{n} f_i t^i$ where $\deg(f) = n > 0$ is minimal, and define

$$h := \sigma(f_n)\alpha^n f - f_n \sigma(f) = \sigma(f_n)\alpha^n \sum_{i=0}^{n} f_i t^i - f_n \sum_{i=0}^{n} \sigma(f_i)(\alpha\, t + \beta)^i \in \mathbb{A}[t]. \tag{14}$$

Since $\mathbf{0} = S(\mathbf{0}) = S(\tau'(f)) = \tau'(\sigma(f))$, we have $\tau'(h) = \tau(\sigma(f_n)\alpha^n)\tau'(f) - \tau(f_n)\tau'(\sigma(f)) = \mathbf{0}$. In addition, $\deg(h) < n$ by construction. Moreover, we conclude that $h \notin \mathbb{A}$ as follows. Suppose that $h \in \mathbb{A}$. Since $\tau'(h) = \tau(h)$ and τ is injective, it follows $h = 0$. With (14) we get $\sigma(f)/f \in \mathbb{F}(t_1)\dots(t_e)$ with $f \notin \mathbb{F}(t_1)\dots(t_e)$. If t is a Σ^*-extension, we get a contradiction by Theorem 2.2.1. Otherwise, suppose that t is a Π-extension. Then $f = u\,t$ with $u \in \mathbb{F}^*$ by Theorem 2.2.2. Hence $\mathbf{0} = \tau'(f) = \tau(u)\,\tau'(t)$. Since $\tau(u) \neq \mathbf{0}$ (τ is injective and $u \neq 0$), $\tau'(t)$ has infinitely many zeros, a contradiction to our construction of τ'. Summarizing, $\tau'(h) = 0$ with $0 < \deg(h) < \deg(f)$, a contradiction to the minimality of $\deg(f)$. □

Theorem 4.1 (Main result)**.** *Let $(\mathbb{F}(t_1)\dots(t_e), \sigma)$ be a generalized d'Alembertian-extension of $(\mathbb{F}, \sigma)$ with $\mathbb{K} := \mathrm{const}_\sigma\mathbb{F}$, and let $(f_1, \dots, f_d) \in \mathbb{F}[t_1, \dots, t_e]^d$. If we have given a $\mathbb{K}$-monomorphism $\tau' : \mathbb{F}[t_1, \dots, t_e] \to S(\mathbb{K})$ with (7), then the following is equivalent:*

(1) *There is no $g \in \mathbb{F}[t_1, \dots, t_e]$ and $\mathbf{0} \neq (c_1, \dots, c_d) \in \mathbb{K}^d$ with (5).*
(2) *The sequences $(S_1(n))_n, \dots, (S_d(n))_n$ given by*

$$S_1(n) := \sum_{k=r}^{n} \mathrm{ev}(f_1, k), \dots, S_d(n) := \sum_{k=r}^{n} \mathrm{ev}(f_d, k), \tag{15}$$

for some r big enough, are transcendental over $\tau'(\mathbb{F}[t_1, \dots, t_e])$.

If $\tau : \mathbb{F} \to S(\mathbb{K})$ is a $\mathbb{K}$-monomorphism, then a $\mathbb{K}$-monomorphism τ' with $\tau'(f) = \tau(f)$ for all $f \in \mathbb{F}$ exits. If τ has a (computable) o- and z-function, τ' has a (computable) o-function.

Proof. Denote $\mathbb{E} := \mathbb{F}(t_1, \dots, t_e)$, suppose there is a $\mathbb{K}$-monomorphism $\tau' : \mathbb{E} \to S(\mathbb{K})$, and let $(f_1, \dots, f_d) \in \mathbb{F}[t_1, \dots, t_e]^d$. Now assume that the first statement holds. Thus, by Theorems 2.3 and 3.1 there is a Σ^*-extension $(\mathbb{E}(s_1, \dots, s_d), \sigma)$ of $(\mathbb{E}, \sigma)$ with $\sigma(s_i) = s_i + f_i$. Obviously, $(\mathbb{E}(s_1, \dots, s_d), \sigma)$ is a GA-extension of $(\mathbb{F}, \sigma)$. Hence we can apply Lemma 4.3 iteratively and get a $\mathbb{K}$-monomorphism $\tau'' : \mathbb{K}[t_1, \dots, t_e][s_1, \dots, s_d] \to S(\mathbb{K})$. By the construction of the monomorphism, see (12) (we choose $c = 0$), it follows that for each $1 \leq i \leq d$ we have $\mathrm{ev}(s_i, n) = S_i(n)$ for all $n \geq r$ for some $r \geq 0$. Since the s_i are transcendental over $\mathbb{E}$ and τ'' is a $\mathbb{K}$-monomorphism, the second statement follows. Conversely, suppose that the first statement does not hold. Then we get (1) with $g(k) := \mathrm{ev}(g, k)$ and $f_i(k) := \mathrm{ev}(f_i, k)$ with k big enough, say $k \geq r$. Summing this equation over $r \leq k \leq n$ gives a relation of the form (2), i.e., the sums in (3) are algebraic. Thus the second statement does not hold.

Suppose there is a $\mathbb{K}$-monomorphism $\tau : \mathbb{F} \to S(\mathbb{K})$. By iterative application of Lemma 4.3, there is a $\mathbb{K}$-monomorphism $\tau : \mathbb{K}[t_1, \dots, t_e] \to S(\mathbb{K})$ with (7) and $\tau'(f) = \tau(f)$ for all $f \in \mathbb{F}$. The constructive part claimed in the theorem is an immediate consequence of Lemma 4.3. □

The following remarks are in place.

(1) We are mainly interested in the following application: Given (3), start with an underlying $\Pi\Sigma^*$-field $(\mathbb{F}, \sigma)$ and try to construct a d'Alembertian extension $(\mathbb{F}(t_1)\dots(t_e), \sigma)$ of $(\mathbb{F}, \sigma)$ together with a $\mathbb{K}$-monomorphism such that we have $f_i \in \mathbb{F}[t_1, \dots, t_e]$ and $\text{ev}(f_i, k) = f_i(k)$. Here one must choose within the monomorphism construction the initial values c in (12) and (13) accordingly; in the summation package Sigma this is done completely automatically. Then one can check transcendence of the $S_i(n)$ by showing the existence or non-existence of a parameterized telescoping solution in $(\mathbb{F}[t_1, \dots, t_e], \sigma)$. Here summation packages like Sigma or additional insight into the structure of the f_i might help; see Section 5.

(2) In order to apply Theorem 4.1, we need a $\mathbb{K}$-monomorphism for the underlying $\Pi\Sigma^*$-field $(\mathbb{F}, \sigma)$. We give a criterion when this is possible in Theorem 4.2. Applying this result we get, e.g., $\mathbb{K}$-monomorphisms for the rational case, the q-rational case and the mixed case.

Subsequently, we need the following additional notion for a rational function field $\mathbb{F}(t)$: We say that $\frac{p}{q} \in \mathbb{F}(t)$ is in reduced representation, if $p, q \in \mathbb{F}[t]$, $\gcd(p, q) = 1$, and q is monic.

Theorem 4.2. *Let $(\mathbb{F}(t), \sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{F}, \sigma)$ with constant field $\mathbb{K}$ and let $\tau : \mathbb{F}[t] \to S(\mathbb{K})$ be a $\mathbb{K}$-homomorphism (resp. $\mathbb{K}$-monomorphism).*

(1) *There is a z-function for τ iff there is a $\mathbb{K}$-homomorphism (resp. $\mathbb{K}$-monomorphism) $\tau' : \mathbb{F}(t) \to S(\mathbb{K})$.*

(2) *Let Z and L be z- and o-functions for τ. Then there is a $\mathbb{K}$-homomorphism (resp. $\mathbb{K}$-monomorphism) $\tau' : \mathbb{F}(t) \to S(\mathbb{K})$ with a z-function Z' and an o-function L'. If Z and L are computable, then Z' and L' are computable.*

Proof. (1) The direction from right to left follows by Lemma 4.2. Suppose that Z is a z-function for τ. Let $\frac{p}{q} \in \mathbb{F}(t)$ be in reduced representation. Then we extend ev to $\mathbb{F}(t)$ by

$$\text{ev}(\frac{p}{q}, k) = \begin{cases} 0 & \text{if } k < Z(q) \\ \frac{\text{ev}(p,k)}{\text{ev}(q,k)} & \text{if } k \geq Z(q) \end{cases}.$$

The properties (8), (9), (10), (11) can be carried over from $\mathbb{F}[t]$ to $\mathbb{F}(t)$. By Lemma 4.1 we get a $\mathbb{K}$-homomorphism $\tau' : \mathbb{F}(t) \to S(\mathbb{K})$ with (7). Finally, suppose that τ is injective. Take $f = \frac{p}{q}$ in reduced form such that $\mathbf{0} = \tau'(f) = \frac{\tau(p)}{\tau(q)}$. Since $\text{ev}(q, k) \neq 0$ for all $k \geq Z(q)$, $\tau(p) = 0$. As τ is injective, $p = 0$ and thus $f = 0$. This proves that τ' is injective.

(2) Let L and Z be o- and z-functions for τ, respectively. Then we extend them to $\mathbb{F}(t)$ by

$$Z'(\frac{p}{q}) = \begin{cases} Z(p) & \text{if } q = 1 \\ \max(Z(p), Z(q)) & \text{if } q \neq 1 \end{cases}, \quad L'(\frac{p}{q}) = \begin{cases} L(p) & \text{if } q = 1 \\ \max(L(p), L(q), Z(q)) & \text{if } q \neq 1 \end{cases} \tag{16}$$

where $\frac{p}{q} \in \mathbb{F}(t)$ is in reduced representation. By construction Z' and L' are z- and o-functions for τ'. If L and Z are computable, then Z' and L' are computable. □

Example 4.1. Let $(\mathbb{K}(n), \sigma)$ be the $\Pi\Sigma^*$-field over $\mathbb{K}$ with $\sigma(n) = n + 1$. Then by our construction we get a $\mathbb{K}$-monomorphism $\tau : \mathbb{K}(n) \to S(\mathbb{K})$ with computable o- and z-functions as follows: Start with the $\mathbb{K}$-monomorphism $\tau : \mathbb{K} \to S(\mathbb{K})$ with $\tau(k) = \boldsymbol{k} = \langle k, k, \dots \rangle$ and take the o-function $L(k) = 0$ and the z-function $Z(k) = 0$ for all $k \in \mathbb{K}$. By Lemma 4.3 we get the $\mathbb{K}$-monomorphism $\tau : \mathbb{K}[n] \to S(\mathbb{K})$ defined by $\text{ev}(p, k) = p(k)$ for all $p \in \mathbb{K}[n]$ and all $k \geq 0$. The resulting o-function is $L(p) = 0$ for all $p \in \mathbb{K}[n]$. Note that the z-function exists since $p(n) \in \mathbb{K}[n]$ can have only finitely many roots. The positive integer roots can be easily computed; see, e.g., [15, page 80]. Hence by Theorem 4.2 we can lift the $\mathbb{K}$-monomorphism from $\mathbb{K}[n]$ to $\mathbb{K}(n)$ together with the o-function L' and z-function Z' given by (16). □

Lemma 4.4. *Let* $(\mathbb{K}(q)(t_1)\dots(t_e)(t),\sigma)$ *be a* $\Pi\Sigma^*$*-field over the rational function field* $\mathbb{K}(q)$ *where* $\sigma(t_i)=\alpha_i t_i+\beta_i$ *with* $\alpha_i,\beta_i\in\mathbb{K}$ *and* $\sigma(t)=q\,t$. *If there is a* $\mathbb{K}$*-monomorphism* $\tau:\mathbb{K}(q)(t_1)\dots(t_e)\to S(\mathbb{K}(q))$ *with (computable) o- and z-functions, there is a* $\mathbb{K}$*-monomorphism* $\tau:\mathbb{K}(q)(t_1)\dots(t_e)(t)\to S(\mathbb{K}(q))$ *with (computable) o- and z-functions.*

Proof. There is a $\mathbb{K}$-monomorphism $\tau':\mathbb{K}(q)(t_1)\dots(t_e)[t]\to S(\mathbb{K}(q))$ with an o-function L' by Lemma 4.3; L' is computable if L is computable. In this construction we can take $\mathrm{ev}(t,k)=q^k$. By [4, Sec. 3.7] there is a Z'-function for $\mathbb{K}(q)(t_1)\dots(t_e)[t]$; it is computable, if Z is computable. By Theorem 4.2 we get a $\mathbb{K}$-monomorphism from $\mathbb{K}(q)(t_1)\dots(t_e)(t)$ to $S(\mathbb{K}(q))$ with o- and z-functions; they are computable, if L',Z' are computable. □

By Example 4.1 and iterative application of Lemma 4.4 based on [4] we get the mixed case.

Corollary 4.1. *Let* $(\mathbb{K}(n)(t_1)\dots(t_e),\sigma)$ *be a* $\Pi\Sigma^*$*-field over the rational function field* $\mathbb{K}:=\mathbb{K}'(q_1)\dots(q_e)$ *where* $\sigma(n)=n+1$ *and* $\sigma(t_i)=q_i t_i$ *for* $1\le i\le e$. *Then there is a* $\mathbb{K}$*-monomorphism* $\tau:\mathbb{K}(n)(t_1)\dots(t_e)\to S(\mathbb{K})$ *with a computable o-function and z-function.*

Note that the use of asymptotic arguments might produce $\mathbb{K}$-monomorphisms for more general $\Pi\Sigma^*$-fields. An open question is, if any $\Pi\Sigma^*$-field over $\mathbb{K}$ can be embedded in $S(\mathbb{K})$.

5. Applications

We illustrate the application of Theorem 4.1 for various classes of sums. Here the summation criterion from [1],[10, Prop. 3.3] and its generalization to $\Pi\Sigma^*$-extensions is substantial.

Theorem 5.1. *(*[24, Cor. 5.1]*) Let* $(\mathbb{F}(t),\sigma)$ *be a* $\Pi\Sigma^*$*-extension of* $(\mathbb{F},\sigma)$ *and* $\frac{p}{q}\in\mathbb{F}(t)$ *be in reduced representation with* $\deg(q)>0$ *and with the property that either* $t\nmid q$ *or* $\frac{\sigma(t)}{t}\notin\mathbb{F}$. *If* $\gcd(\sigma^m(q),q)=1$ *for all* $m>0$, *then there is no* $g\in\mathbb{F}(t)$ *with* $\sigma(g)-g=\frac{p}{q}$.

5.1. **The rational case.** Applying Theorem 4.1 together with Example 4.1 gives the following theorem.

Theorem 5.2. *Let* $f_1(k),\dots,f_d(k)\in\mathbb{K}(k)$. *If there are no* $g(k)\in\mathbb{K}(k)$ *and* $c_1,\dots,c_d\in\mathbb{K}$ *with* (1) *then the sequences* (3)*, for some* r *big enough, are transcendental over* $\mathbb{K}(n)$*, i.e., there is no polynomial* $P(x_1,\dots,x_d)\in\mathbb{K}(n)[x_1,\dots,x_d]$ *with*

$$P(S_1(n),\dots,S_d(n))=0\quad\forall n\ge 0.$$

Corollary 5.1. *Let* $p_1(k),p_2(k),\dots\in\mathbb{K}[k]^*$, $u_1(k),u_2,\dots\in\mathbb{K}[k]^*$ *and* $q\in\mathbb{K}[k]^*$ *with* $\deg(q)>0$ *and* $\gcd(p_i,q)=\gcd(u_i,q)=1$ *for all* $i\ge 1$; *suppose that* $q(k)\neq 0$ *for all* $k\in\mathbb{N}$. *Then the sums* $\sum_{k=1}^n u_1(k)\left(\frac{p_1(k)}{q(k)}\right),\sum_{k=1}^n u_2(k)\left(\frac{p_2(k)}{q(k)}\right)^2,\dots$ *are transcendental over* $\mathbb{K}(n)$.

Proof. Denote $f_i(k):=u_i\left(\frac{p_i}{q}\right)^i$ and suppose there are $g(k)\in\mathbb{K}(k)$ and $c_i\in\mathbb{K}$ with (1) where $d\ge 1$ is minimal. Then it follows that

$$g(k+1)-g(k)=\frac{c_1u_1p_1q^{d-1}+c_2u_2p_2p^{d-2}+\dots+c_du_dp_d}{q^d}=:\frac{v}{q^d}.$$

Since $c_d\neq 0$, $q\nmid c_du_dp_d$. Hence $\gcd(v,q^d)=1$. By Theorem 5.1 it follows that such a $g(k)\in\mathbb{K}(k)$ cannot exist; a contradiction. Hence the corollary follows by Theorem 5.2. □

Example 5.1. Choosing $p_i=u_i=1,q=k$ in Corollary 5.1 proves that the generalized harmonic numbers $H_n,H_n^{(2)},\dots$ are transcendental over $\mathbb{K}(n)$. □

Applying Theorem 4.1 together with Corollary 4.1 accordingly produces the q-versions and the mixed versions of Theorem 5.2 and Corollary 5.1. A typical application is Example 5.2.

Example 5.2. The q-harmonic numbers (and its variations) $\sum_{k=1}^n\frac{q^k}{1-q^k}$, $\sum_{k=1}^n\frac{1}{1-q^k}$, $\sum_{k=1}^n\frac{q^{2k}}{(1-q^k)^2}$, $\sum_{k=1}^n\frac{1}{(1-q^k)^2},\dots$ are all transcendental over $\mathbb{K}(q^k)$. □

Completely analogously to Corollary 5.1 one can show the following corollary

Corollary 5.2. *Let $p_1(k), q_1(k), \dots, p_d(k), q_d(k) \in \mathbb{K}[k]^*$ with $\deg(q_i) > 0$ and $\gcd(p_i, q_i) = 1$. Suppose that $q_i(k) \neq 0$ for all $k \in \mathbb{N}$ and that $\gcd(q_i(k+r), q_j(k)) = 1$ for all $r \in \mathbb{Z}$ and all $1 \leq i < j \leq d$. Then the sums $\sum_{k=1}^{n} \frac{p_1(k)}{q_1(k)}, \dots, \sum_{k=1}^{n} \frac{p_d(k)}{q_d(k)}$ are transcendental over $\mathbb{K}(n)$.*

5.2. **The hypergeometric case.** Suppose that $f(k)$ is a hypergeometric term in k, i.e., there is an $\alpha \in \mathbb{K}(k)$ with $\alpha(r) := \frac{f(r+1)}{f(r)}$ for all r big enough; in short we also write $\alpha(k) := \frac{f(k+1)}{f(k)}$ to define the rational function $\alpha \in \mathbb{K}(k)$.
By [22, Thm. 5.4] $f(k)$ can be represented by a $\Pi\Sigma^*$-field $(\mathbb{K}(k)(t), \sigma)$ over $\mathbb{K}$ with $\sigma(k) = k+1$ and $\sigma(t) = \alpha\, t$ if and only if there is no $r(k) \in \mathbb{K}(k)$ and no root of unity γ with $f(k) = \gamma^k r(k)$. Subsequently, we exclude this special case.

Theorem 5.3. *(Inspired by [15, Sec. 5.6]) Let $f_1(k), \dots, f_d(k)$ be hypergeometric terms with the following properties:* **(1)** *There is a $\Pi\Sigma^*$-field $(\mathbb{K}(k)(t_1)\dots(t_d), \sigma)$ over $\mathbb{K}$ with $\sigma(k) = k+1$ and $\sigma(t_i) = \alpha_i t_i$ with $\alpha_i := \frac{f_i(k+1)}{f_i(k)} \in \mathbb{K}(k)$.* **(2)** *For all $1 \leq i \leq d$, $f_i(k)$ is not Gosper-summable, i.e., there is no $g(k) \in \mathbb{K}(k)$ with $\alpha\, g(k+1) - g(k) = 1$.*
Then the $h_1(n), \dots, h_d(n)$ together with (3), r big enough, are transcendental over $\mathbb{K}(n)$.

Proof. Denote $\mathbb{F} := \mathbb{K}(k)(t_1)\dots(t_d)$. Suppose that there are $\mathbf{0} \neq (c_1, \dots, c_d) \in \mathbb{K}^n$ and $g \in \mathbb{K}(k)[t_1, \dots, t_e]$ with (5) where $f_i := t_i$. By [8, Cor. 2] or [20, Cor. 3], $g = \sum_{i=1}^{d} w_i t_i + u$ with $w_i, u \in \mathbb{K}(n)$. Plugging g into (5) and doing coefficient comparison (the t_i are transcendental!) shows that $\sigma(w_i t_i) - w_i t_i = c_i t_i$ for all $1 \leq i \leq d$. By property (2), $c_i = 0$ for all i; a contradiction. Applying Theorem 4.1 and choosing an appropriate $\mathbb{K}$-monomorphism proves the theorem. □

Example 5.3. The sequences $n!$, $\binom{n}{k}$, $(n+m)!$, $\sum_{k=1}^{n} k!$, $\sum_{k=1}^{n} \binom{m}{k}$ and $\sum_{k=1}^{n}(k+m)!$ are transcendental over $\mathbb{K}(m)(n)$. □

Theorem 5.4. *Let $f(k)$ be a hypergeometric term where $f(k) \neq (-1)^k r(k)$ for some $r(k) \in \mathbb{K}(k)$, and consider the $\Pi\Sigma^*$-field $(\mathbb{K}(k)(t), \sigma)$ over $\mathbb{K}$ with $\sigma(k) = k+1$ and $\sigma(t) = \frac{f(k+1)}{f(k)} t$. Let $r_i(k) \in \mathbb{K}(k)$ for $1 \leq i \leq d$ and set $f_i := r_i t \in \mathbb{K}(k)(t)$. If there are no $c_i \in \mathbb{K}$ and $g \in \mathbb{K}(k)(t)$ with (5), then the following sequences, for r big enough, are transcendental over $\mathbb{K}(n)$:*

$$f(n),\ S_1(n) = \sum_{k=r}^{n} r_1(k) f(k), \dots, S_d(n) = \sum_{k=r}^{b} r_d(k) f(k),$$

i.e., there is no $P(x_0, x_1, \dots, x_d) \in \mathbb{K}(n)[x_0, x_1, \dots, x_d]$ with

$$P(f(n), S_1(n), \dots, S_d(n)) = 0 \quad \forall n \geq 0.$$

Corollary 5.3. *Let $f(\boldsymbol{m}, k)$ be a hypergeometric term in $\boldsymbol{m} = (m_1, \dots, m_u)$ and in k where $f(\boldsymbol{m}, k) \neq (-1)^k r(\boldsymbol{m}, k)$ for some $r(\boldsymbol{m}, k) \in \mathbb{K}(\boldsymbol{m}, k)$. Let $S = \{s_1, \dots, s_d\} \subseteq \mathbb{Z}^u$. Consider the $\Pi\Sigma^*$-field $(\mathbb{K}(\boldsymbol{m})(k)(t), \sigma)$ over $\mathbb{K}(\boldsymbol{m})$ with $\sigma(k) = k+1$ and $\sigma(t) = \frac{f(\boldsymbol{m}, k+1)}{f(\boldsymbol{m}, k)} t$, and define $f_i := \frac{f(\boldsymbol{m}+\boldsymbol{s_i}, k)}{f(\boldsymbol{m}, k)} t \in \mathbb{K}(\boldsymbol{m})(k)(t)$. If there are no $c_i \in \mathbb{K}(\boldsymbol{m})$ and $g \in \mathbb{K}(\boldsymbol{m})(k)(t)$ with (5), then the following sequences, for r big enough, are transcendental over $\mathbb{K}(\boldsymbol{m})(n)$:*

$$S_0(n) = f(\boldsymbol{m}, n),\ S_1(n) = \sum_{k=r}^{n} f(\boldsymbol{m} + \boldsymbol{s_1}, k), \dots, S_d(n) = \sum_{k=r}^{b} f(\boldsymbol{m} + \boldsymbol{s_d}, k).$$

Example 5.4. Consider any hypergeometric sum $S(m) = \sum_{k=0}^{m} f(m, k)$ with $f(m, k)$ hypergeometric in m and k where $f(m, k) \neq (-1)^k r(m, k)$ for some $r(m, k) \in \mathbb{K}(m, k)$. Define $S(m, n) = \sum_{k=r}^{n} f(m, k)$ for some r big enough.
(1) Suppose that Zeilberger's algorithm (or any other algorithm that can handle creative telescoping for hypergeometric terms) fails to compute a recurrence for $S(m)$ of order smaller

or equal than d. Then the sequences $f(m,n), S(m,n), S(m+1,n), \dots, S(m+d,n)$ in n are transcendental over $\mathbb{K}(m)(n)$. E.g., for the Apéry-sum $S(m)=\sum_{k=0}^{m}\binom{m}{k}^2\binom{m+k}{k}$, see [16], one can derive only a recurrences of order 2, but not smaller ones. Hence, the following sequences in n are transcendental over $\mathbb{K}(m)(n)$:

$$\binom{m}{k}^2\binom{m+k}{k},\ \sum_{k=0}^{n}\binom{m}{k}^2\binom{m+k}{k} \text{ and } \sum_{k=0}^{n}\binom{m+1}{k}^2\binom{m+k+1}{k}.$$

(2) In [2] a criterion is given when Zeilberger's algorithm fails to find a creative telescoping solution for a hypergeometric input summand $f(m,k)$. If $f(m,k)$ satisfies this criterion, then all the sequences $h(m,n), S(m,n), S(m+1,n), \dots$ are transcendental over $\mathbb{K}(m)$. A typical example is $f(m,k)=\frac{1}{mk+1}(-1)^k\binom{m+1}{k}\binom{2m-2k-1}{m-1}$; see [2, Exp. 2]. $\square$

Note that the the q-hypergeometric case can be handled completely analogously with our machinery.

5.3. **Nested sums.** All what has been said in Example 5.4.1 can be carried over to sequences in terms of generalized d'Alembertian extensions. E.g., in [13] we derived for the sum $S(m):=\sum_{k=0}^{m}(1+5\,\mathrm{H}_k(m-2k)\binom{m}{k}^5)$ a recurrence of order 4 with creative telescoping, but failed to find a recurrence of smaller order. Hence Theorem 4.1 tells us that the sequences

$$(\binom{m}{n})_{n\geq 0},\ \big(H_n\big)_{n\geq 0},\ \big(S(m,n)\big)_{n\geq 0},\dots,\ \big(S(m+3,n)\big)_{n\geq 0} \tag{17}$$

with

$$S(m,n):=\sum_{k=0}^{n}f(m,k)=\sum_{k=0}^{n}(1-5\,k\,\mathrm{H}_k+5\,(m-k)j\,\mathrm{H}_k)\binom{m}{k}^5$$

are transcendental over $\mathbb{K}(m)(n)$. More precisely, Sigma works as follows: it constructs the $\Pi\Sigma^*$-field $(\mathbb{F},\sigma)$ with $\mathbb{F}:=\mathbb{Q}(m)(k)(b)(h)$ from Example 2.1 and designs the $\mathbb{Q}(m)$-monomorphism with $\mathrm{ev}(k,j)=j$, $\mathrm{ev}(b,j)=\prod_{i=1}^{j}\frac{m+1-i}{i}=\binom{m}{j}$, $\mathrm{ev}(h,j)=\sum_{i=1}^{j}\frac{1}{k}=H_j$. Moreover, Sigma takes $f_1=b^5(1+5h(m-2k))$, $f_2=\frac{b^5(m+1)^5(5h(-2k+m+1)+1)}{(-k+m+1)^5}$, $f_3=\frac{b^5(m+1)^5(m+2)^5(5h(-2k+m+2)+1)}{(-k+m+1)^5(-k+m+2)^5}$, and $f_4=\frac{b^5(m+1)^5(m+2)^5(m+3)^5(5h(-2k+m+3)+1)}{(-k+m+1)^5(-k+m+2)^5(-k+m+3)^5}$. This is motivated by the fact $\binom{m+1}{k}=\frac{m+1}{m+1-k}\binom{m}{k}$ which shows that $\mathrm{ev}(f_i,k)=f(m+i-1,k)$. Finally, Sigma proves algorithmically that there are no $g\in\mathbb{F}$ and $c_i\in\mathbb{Q}(m)$ with (5). Hence the transcendence of (17) follows by Theorem 4.1.

We emphasize that the sum $S(m)=S(m,m)$ has completely different properties: it satisfies a recurrence of order two. More precisely, as shown in [13] we get

$$\sum_{j=0}^{m}(1+5\mathrm{H}_j(m-2k))\binom{m}{j}^5=(-1)^m\sum_{j=0}^{m}\binom{m}{j}^2\binom{m+j}{j}.$$

6. A transcendence criterion for products

The product version of Theorem 3.1 is Theorem 6.1.

Theorem 6.1. *Let $(\mathbb{F},\sigma)$ be a difference field with constant field $\mathbb{K}$ and $(f_1,\dots,f_d)\in(\mathbb{F}^*)^n$. The following statements are equivalent.*

(1) *There do not exist $\mathbf{0}\neq(c_1,\dots,c_d)\in\mathbb{Z}^d$ and $g\in\mathbb{F}^*$ with*

$$\frac{\sigma(g)}{g}=f_1^{c_1}\dots f_d^{c_d}. \tag{18}$$

(2) *There is a Π-extension $(\mathbb{F}(t_1,\dots,t_d),\sigma)$ of $(\mathbb{F},\sigma)$ with $\sigma(t_i)=f_it_i$.*

Proof. Suppose that $(\mathbb{F}(t_1,\dots,t_d),\sigma)$ is a Π-extension of $(\mathbb{F},\sigma)$. Moreover, assume that there is a $g\in\mathbb{F}^*$ and $\mathbf{0}\neq(c_1,\dots,c_d)\in\mathbb{Z}^d$ with (18). Let j be maximal with $c_j\neq 0$. Then with $w=gt_1^{-c_1}\dots t_{j-1}^{-c_{j-1}}\in\mathbb{F}(t_1,\dots,t_{j-1})^*$ we get $\sigma(w)=f_j^{c_j}w$, a contradiction to Theorem 2.1.2. Conversely, let j be maximal such that $(\mathbb{F}(t_1,\dots,t_j),\sigma)$ is a Π-extension of $(\mathbb{F},\sigma)$; suppose that $j<d$. By Theorem 2.1.2 there is a $g\in\mathbb{F}(t_1,\dots,t_{j-1})^*$ and $c\in\mathbb{Z}$ with $\sigma(g)=f_j^c g$. By Theorem 2.2.2 it follows that $g=u\,t_1^{c_1}\dots t_{j-1}^{c_{j-1}}$ with $c_i\in\mathbb{Z}$ and $u\in\mathbb{F}$; clearly $u\neq 0$. Thus, $\frac{\sigma(u)}{u}=f_1^{-c_1}\dots f_{j-1}^{-c_{j-1}}f_j^c$ which proves the theorem. □

Note that the existence of a solution of (18) can be checked by Karr's algorithm [8] if $(\mathbb{F},\sigma)$ is a $\Pi\Sigma^*$-field over $\mathbb{K}$. The following theorem is immediate.

Theorem 6.2. *Let $(\mathbb{F},\sigma)$ be a difference field with constant field $\mathbb{K}$, let $\tau:\mathbb{F}\to S(\mathbb{K})$ be a $\mathbb{K}$-monomorphism, and let $(f_1,\dots,f_d)\in(\mathbb{F}^*)^d$. Then the following statements are equivalent:*
(1) *There is no $g\in\mathbb{F}^*$ and $\mathbf{0}\neq(c_1,\dots,c_d)\in\mathbb{Z}^d$ with* (18).
(2) *The sequences $(S_1(n))_n,\dots,(S_d(n))_n$ given by*

$$S_1(n):=\prod_{k=r}^{n}\operatorname{ev}(f_1,k),\dots,S_d(n):=\prod_{k=r}^{n}\operatorname{ev}(f_d,k),$$

for some r big enough, are transcendental over $\tau(\mathbb{F})$.

Example 6.1. The sequences $2^n,3^n,5^n,7^n,\dots$ over the prime numbers are all transcendent over $\mathbb{K}(n)$; compare [8, Exp. 7]. □

The following lemma is a direct consequence of [22, Thm. 4.14]; for the rational case see also [3].

Lemma 6.1. *Let $(\mathbb{F}(t),\sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$. Let $p_1,\dots,p_d,q_1,\dots,q_d\in\mathbb{F}[t]^*$ where $\gcd(\sigma^l(p_i),q_j)=1$ for all $l\in\mathbb{Z}$ and $1\leq i<j\leq d$; set $f_i:=\frac{p_i}{q_i}$. Then there is no $g\in\mathbb{F}(t)^*$ and $(c_1,\dots,c_d)\in\mathbb{Z}^d$ with* (18).

Example 6.2. Let $p_1(k),q_1(k)\dots,p_d(k),q_d(k)\in\mathbb{K}[k]$ with $\gcd(p_i(k+l),q_j(k))=1$ for all i,j and $l\in\mathbb{Z}$. Then the sequences $\prod_{k=r}^{n}\frac{p_1(k)}{q_1(k)},\dots,\prod_{k=r}^{n}\frac{p_d(k)}{q_d(k)}$, for some r big enough, are transcendental over $\mathbb{K}(n)$: □

7. Conclusion

We showed that creative telescoping and, more generally, parameterized telescoping can be applied to obtain a criterion to check algebraic independence of nested sum expressions. Here the summation package Sigma can be used to check transcendence with the computer.

Moreover, using results from summation theory one can show that whole classes of sums are transcendental. Obviously, refinements of summation theory should give also stronger tools to prove or disprove transcendence of sum expressions. E.g., Peter Paule's results [11] enable one to predict the existence of contiguous relations. Using these results might help to refine, e.g., Theorem 5.3.

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