TRUNCATING BINOMIAL SERIES WITH SYMBOLIC SUMMATION

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ABSTRACT. Taking an example from statistics, we show how symbolic summation can be used to find generalizations of binomial identities which involve infinite series. In such generalizations the infinite series are replaced by truncated versions.

1. INTRODUCTION

Consider the identity

$$\sum_{k=m+1}^{\infty} \frac{m}{k(k-1)} \sum_{j=0}^{m} \binom{k}{j} x^{k-j}(1-x)^j = x, \quad m \geq 1,\ 0 \leq x \leq 1, \tag{1}$$

which arose in the theory of records [Blo86, Isr88]. In an attempt to generalize this result Tamás Lengyel [Len06] found the binomial series

$$a_{m,n}(x) = \binom{m}{n-1} \sum_{k=m+1}^{\infty} \binom{k}{n}^{-1} \sum_{j=0}^{m} \binom{k}{j} x^{k-j}(1-x)^j$$

which evaluates for all integers n and m with $0 \leq n-1 \leq m$ to

$$a_{m,n}(x) = \begin{cases} \frac{n}{n-1}\big(1-(1-x)^{n-1}\big), & \text{if } n \geq 2 \text{ and } 0 \leq x \leq 1 \\ -\ln(1-x), & \text{if } n = 1 \text{ and } 0 \leq x < 1. \end{cases} \tag{2}$$

For $n = 2$ one obtains the original identity.

In this note we illustrate how algorithms from symbolic summation can be used to obtain generalizations of (2) where Lengyel's $a_{m,n}(x)$ series is replaced by the truncated version

$$a_{m,n}^{(K)}(x) = \binom{m}{n-1} \sum_{k=m+1}^{K} \binom{k}{n}^{-1} \sum_{j=0}^{m} \binom{k}{j} x^{k-j}(1-x)^j$$

in which $K \geq m+1$.

In Section 2 we present an automatic evaluation of $a_{m,n}(x)$ using Gosper's telescoping and Zeilberger's creative telescoping algorithms, respectively. In Section 3 we observe that for specific values of n creative telescoping can be replaced by plain telescoping. This gives rise to conjecture that identities for the truncated version $a_{m,n}^{(K)}(x)$ exist. In Section 4 we derive such identities which in the limit $K \to \infty$ give (2); see (7) and (8). To obtain the limit for $K \to \infty$, a generating function identity (9) is used. In Section 5 we demonstrate that (9) can be also derived with symbolic summation. In Section 6 some concluding remarks are given.

All computations are done using the package Sigma [Sch04a]; to derive (7) and (8) special Sigma features are applied.

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2. Automatic evaluation of $a_{m,n}(x)$

Before applying computer algebra it is convenient to rewrite $a_{m,n}(x)$ as a series expanded in powers of x. To this end, in $a_{m,n}(x)$ we replace $(1-x)^j$ with $\sum_l(-1)^l\binom{j}{l}x^l$. Then collecting coefficients of x^N $(N=k-j+l)$ turns Lengyel's series into

$$a_{m,n}(x)=\binom{m}{n-1}\sum_{N=1}^{\infty}(-1)^N x^N\sum_{k=m+1}^{\infty}(-1)^k\binom{k}{n}^{-1}A_m(N)$$

where

$$A_m(N)=\sum_{j=0}^{m}(-1)^j\binom{k}{j}\binom{j}{N-k+j}.$$

Now the rest can be done with computer algebra. After loading the Sigma package into the computer algebra system Mathematica

In[1]:= **<< Sigma.m**

Sigma - A summation package by Carsten Schneider © RISC-Linz

we insert the hypergeometric sum `S` $=A_m(N)$. Note that various Sigma functions support the user, like `SigmaSum` for sums, `SigmaProduct` for products, `SigmaPower` for powers, or `SigmaBinomial` for binomials.

In[2]:= **S = SigmaSum[**
SigmaPower[−1, j]SigmaBinomial[k, j]SigmaBinomial[j, N − k + j], {j, 0, m}]

Out[2]= $\sum_{j=0}^{m}(-1)^j\binom{k}{j}\binom{j}{N-k+j}$

Then by telescoping Sigma produces the following closed form for $A_m(x)(=$ `S`$)$:

In[3]:= **SigmaReduce[S]**

Out[3]= $-\frac{(-1)^m(m-k)}{N}\binom{k}{m}\binom{m}{N-k+m}$

This can be also achieved by any implementation of Gosper's algorithm [Gos78], so we omit details here. Hence

$$a_{m,n}(x)=(-1)^m\binom{m}{n-1}\sum_{N=1}^{\infty}\frac{(-1)^N x^N}{N}B_{m,n}(N) \tag{3}$$

where

$$B_{m,n}(N)=\sum_{k=m+1}^{\infty}(-1)^k\binom{k}{n}^{-1}\binom{k}{m}\binom{m}{N-k-m}(k-m).$$

Since telescoping fails to simplify $B_{m,n}(N)$, we continue differently. Namely, for definite sums one can execute the Sigma function

In[4]:= **GenerateRecurrence[**$\sum_{k=m+1}^{\infty}(-1)^k\binom{k}{n}^{-1}\binom{k}{m}\binom{m}{N-k+m}(k-m)$**, N]**

Out[4]= $\{(-1+n-N)\mathrm{SUM}[N]-N\,\mathrm{SUM}[1+N]==0\}$

which is based on creative telescoping [Zei91] and which computes the recurrence Out[4] for $\mathrm{SUM}[N]=B_{m,n}(N)$. This can be also achieved by any implementation of Zeilberger's algorithm, so we omit details here.

The recurrence Out[4] gives $B_{m,n}(N) = (-1)^{N+1}\binom{N-n}{N-1}B_{m,n}(1)$. With the initial value $B_{m,n}(1) = -n(-1)^m\binom{m}{n-1}^{-1}$ we find

$$B_{m,n}(x) = (-1)^{N+m}n\binom{m}{n-1}^{-1}\binom{N-n}{N-1}, \tag{4}$$

i.e.,

$$a_{m,n}(x) = n\sum_{N=1}^{\infty}\frac{x^N}{N}\binom{N-n}{N-1}.$$

Finally, for $n = 1$,

$$a_{m,1}(x) = \sum_{N=1}^{\infty}\frac{x^N}{N} = -\ln(1-x);$$

for $n > 1$, because of $\binom{N-n}{N-1} = (-1)^{N-1}\binom{n-2}{N-1}$,

$$a_{m,n}(x) = -\frac{n}{n-1}\sum_{N=1}^{\infty}(-x)^N\binom{n-1}{N} = -\frac{n}{n-1}(-1+(1-x)^{n-1}).$$

3. A Truncated Variant of the Original Identity (1)

With respect to our starting point, identity (1), the computer algebra approach taken in Section 2 brings along an additional feature. Namely, if $n = 2$, one observes that the $B_{m,2}^{(K)}(N)$ sum telescopes for $N > 1$. Thus one has

$$\begin{aligned} B_{m,2}^{(K)}(N) &= \sum_{k=m+1}^{K}\frac{(-1)^k}{k(k-1)}\binom{k}{m}\binom{m}{N-k+m}(k-m) \\ &= (-1)^K\frac{(K-m)(K-m-N)}{Km(1-N)}\binom{K}{m}\binom{m}{K-N} \end{aligned}$$

for $N > 1$ and arbitrary $K \geq m+1$. Obviously for $N = 1$,

$$B_{m,2}^{(K)}(1) = \frac{(-1)^{m+1}}{m} \quad \text{for } K \geq m+1.$$

Consequently, the truncated version $a_{m,2}^{(K)}(x)$ of $a_{m,2}(x)$ finds the following evaluation for $K \geq m+1$:

$$\begin{aligned} a_{m,2}^{(K)}(x) &= 2(-1)^m m\sum_{N=1}^{K}\frac{(-1)^N x^N}{N}B_{m,2}^{(K)}(N) \\ &= 2x + 2m(-1)^{K+m}\binom{K-1}{m}\sum_{N=2}^{K}\frac{(-1)^N x^N}{N(N-1)}\binom{m-1}{K-N}. \end{aligned} \tag{5}$$

In the limit $K \to \infty$, one retrieves identity (1).

One observes that $B_{m,n}^{(K)}(x)$ also telescopes for all other specific values of $n \geq 3$. This suggests that a nice truncated generalization like (5) might exist for generic n. To obtain such a formula, one could try to guess a pattern from the evaluations at $n = 2, 3$, etc. However, we prefer to utilize specific features of Sigma which will lead us to an answer in the form of identities (7) and (8).

4. Closed form evaluations of $a_{m,n}^{(K)}(x)$

In contrast to Sections 2 and 3, we will apply Sigma to $a_{m,n}^{(K)}(x)$ in its original form. To this end, it will be convenient to proceed with a slight variation of the truncated generalization of $a_{m,n}^{(K)}(x)$, namely,

$$\alpha_{m,n}^{(K)}(x) := \sum_{k=1}^{K} \frac{x^{m+k}}{\binom{m+k}{n}} D_{m,k}(x)$$

with

$$D_{m,k}(x) = \sum_{j=0}^{m} \frac{(1-x)^j}{x^j} \binom{m+k}{j}.$$

Note that for $K \geq m+1 \geq n \geq 1$,

$$a_{m,n}^{(K)}(x) = \binom{m}{n-1} \alpha_{m,n}^{(K-m)}(x). \tag{6}$$

Also note that we restrict to $x \neq 0$; the case $x = 0$ is trivial.

4.1. **Preprocessing.** A first step to simplify the sum $\alpha_{m,n}^{(K)}(x)$ is to represent the inner sum $D_{m,k}(x)$ as an indefinite sum in k. This means, we try to express $D_{m,k}(x)$ in form of a sum $\sum_{j=0}^{k} f(m,j)$ where the summand $f(m,j)$ is free of k. In order to accomplish this goal, we insert $D_{m,k}(x)$ with

In[5]:= $\mathbf{D = \sum_{j=0}^{m} \frac{(1-x)^j}{x^j} \binom{m+k}{j};}$

and compute a recurrence[1] for SUM$[k] = D_{m,k}(x)$:

In[6]:= **rec = GenerateRecurrence[D, k][[1]]**

Out[6]= $\mathtt{x^m SUM[k] - x^{m+1} SUM[k+1] == (1-x)^{m+1} \binom{k+m}{m}}$

Next, we solve this recurrence with the Sigma function

In[7]:= **recSol = SolveRecurrence[rec, SUM[k]]**

Out[7]= $\{\{0, \prod_{i=1}^{k} \frac{1}{x}\}, \{1, -\left(\prod_{i=1}^{k} \frac{1}{x}\right) \frac{(1-x)^{m+1}}{x^{m+1}} \sum_{i=0}^{k} \frac{i\binom{m+i}{m} x^i}{m+i}\}\}$

This means that $\frac{1}{x^k}$ is a solution of the homogeneous version, and $-\frac{(1-x)^{m+1}}{x^{m+k+1}} \sum_{i=0}^{k} \frac{i\binom{m+i}{m}x^i}{m+i}$ is a particular solution of the recurrence itself. As a consequence we can write

$$D_{m,k}(x) = \frac{c}{x^k} - \frac{(1-x)^{m+1}}{x^{m+k+1}} \sum_{i=0}^{k} \frac{i\binom{m+i}{m}x^i}{m+i}$$

for a constant c which is free of k. Looking at the case $k = 0$ gives

$$\sum_{j=0}^{m} \frac{(1-x)^j}{x^j} \binom{m}{j} = c; \quad \text{i.e., } c = \frac{1}{x^m}.$$

Summarizing, we can represent $\alpha_{m,n}^{(K)}(x) =:$ `A` in the form

[1]As in Out[4], this can be achieved by any implementation of Zeilberger's algorithm.

In[8]:= $\mathbf{A} = \sum_{k=1}^{K} \frac{x - (1-x)^{m+1} \sum_{i=0}^{k} \frac{i\binom{m+i}{m} x^i}{m+i}}{x\binom{k+m}{n}};$

4.2. **The case** $n > 1$**.** Now Sigma is ready to simplify A in one stroke (i.e., no splitting into two sums is needed):

In[9]:= **result = SigmaReduce[A, SimpleSumRepresentation → True]**

Out[9]= $-\frac{K+m-n+1}{(n-1)\binom{K+m}{n}} + \frac{1+m-n}{(n-1)\binom{m}{n}} + \frac{(K+m-n+1)(1-x)^{m+1}}{(n-1)x\binom{K+m}{n}} \sum_{i=0}^{K} \frac{ix^i\binom{i+m}{m}}{i+m} - \frac{(1-x)^{n+1}}{x(n-1)} \sum_{i=1}^{K} \frac{ix^i\binom{i+m}{m}}{\binom{i+m}{n}}$

Remark. Sigma automatically found the additional sum $\sum_{i=1}^{K} ix^i\binom{i+m}{m}\binom{i+m}{n}^{-1}$ in order to simplify $A = \alpha_{m,n}^{(K)}(x)$ to an expression consisting only of single sums; for details of the method see [Sch04b].

Summarizing, using relations like $\binom{i+m-1}{m} = \frac{i}{i+m}\binom{i+m}{m}$ or $i\binom{i+m}{m}\binom{i+m}{n}^{-1} = \binom{m+1}{n}^{-1}\binom{m+i-n}{i-1}$, one obtains that, for $K \geq m+1 \geq n \geq 2$,

$$a_{m,n}^{(K)}(x) = \frac{n}{n-1}\Big[1 - \frac{\binom{m}{n-1}}{\binom{K}{n-1}} + \frac{\binom{m}{n-1}}{\binom{K}{n-1}}(1-x)^{m+1} \sum_{i=0}^{K-m-1} (-1)^i \binom{-m-1}{i} x^i \\ - (1-x)^{m+1} \sum_{i=0}^{K-m-1} (-1)^i \binom{n-m-2}{i} x^i\Big]. \tag{7}$$

In the limit $K \to \infty$, owing to the binomial theorem, we retrieve Lengyel's identity (2).

Concerning the case $n = 2$ we remark that the equivalence of (5) and (7) is based on a non-trivial transformation; e.g., for the elementary special case $x = 1$ it specializes to $(K \geq m+1)$

$$\sum_{N=2}^{K} \frac{(-1)^N}{N(N-1)}\binom{m-1}{K-N} = \frac{(-1)^{K+m+1}}{K}\binom{K-1}{m}^{-1}.$$

Of course, such identities could be proved easily with computer algebra; however, it is instructive to consult background information like the discussion related to [GKP94, (5.41)].

4.3. **The case** $n = 1$**.** After inserting $\alpha_{m,1}^{(K)}(x)$ by

In[10]:= **A1 = A/.n → 1**

Out[10]= $\sum_{k=1}^{K} \frac{x - (1-x)^{m+1} \sum_{i=0}^{k} \frac{i\binom{m+i}{m} x^i}{m+i}}{x(m+k)}$

we get the simplification

In[11]:= **SigmaReduce[A1, SimpleSumRepresentation → True]**

Out[11]= $\Big(1 - (1-x)^{m+1} \sum_{i=0}^{K} x^i \binom{i+m}{m}\Big) \sum_{i=1}^{K} \frac{1}{i+m} + (1-x)^{m+1} \sum_{i=1}^{K} x^i \binom{i+m}{m} \sum_{j=1}^{i} \frac{1}{j+m}$

Remark. Similar as in Out[9] Sigma automatically found the sum expressions $\sum_{i=1}^{K} \frac{1}{i+m}$ and $\sum_{i=1}^{K} x^i\binom{i+m}{m} \sum_{j=1}^{i} \frac{1}{j+m}$ in order to simplify $A1 = \alpha_{m,1}^{(K)}(x)$ in Out[11]; for details see [Sch05].

Using $\sum_{i=1}^{K}\frac{1}{i+m}=H_{K+m}-H_m$ where $H_m=1+\frac{1}{2}+\cdots+\frac{1}{m}$ are the harmonic numbers, we can write $a_{m,1}^{(K)}(x)$ in the form ($K\geq m+1$ and $m\geq 0$)

$$a_{m,1}^{(K)}(x)=H_{K+m}-H_m-(1-x)^{m+1}\Big[H_{k+m}\sum_{i=0}^{K}\binom{i+m}{m}x^i-\sum_{i=0}^{K}\binom{i+m}{m}H_{m+i}x^i\Big]. \quad (8)$$

In the limit $K\to\infty$ we retrieve (2), owing to the binomial theorem and the fact [GKP94, (7.43)] that

$$\sum_{i=0}^{\infty}\binom{i+m}{m}H_{m+i}x^i=-\frac{1}{(1-x)^{m+1}}\big(\ln(1-x)-H_m\big). \quad (9)$$

5. Generating Functions with Sigma

Identity (9) can be also derived with computer algebra. A standard method would be to utilize holonomic closure properties by using the packages [SZ94] or [Mal96]. However, in the given context, we find it instructive to show that (9) can be also obtained with Sigma, namely as follows. We input the sum

In[12]:= $\mathbf{E}=\sum_{\mathbf{i=0}}^{\infty}\binom{\mathbf{i+m}}{\mathbf{m}}\mathbf{H_{m+i}x^i};$

and compute a recurrence relation satisfied by it:

In[13]:= **GenerateRecurrence[E, m]**

Out[13]= $-(m+2)(x-1)^2\mathrm{SUM}[m+2]-(2m+3)(x-1)\mathrm{SUM}[m+1]+(-m-1)\mathrm{SUM}[m]==0$

Next, we solve the recurrence with the Sigma function

In[14]:= **recSol = SolveRecurrence[rec[[1]], SUM[m]]**

Out[14]= $\{\{0,\prod_{i=1}^{m}\frac{1}{1-x}\},\{1,\prod_{i=1}^{m}\frac{1}{1-x}\sum_{i=1}^{m}\frac{1}{i}\}\}$

Denoting the left hand side of (9) by $E_m(x)$ this means

$$E_m(x)=\frac{c_0}{(1-x)^m}+c_1\frac{H_m}{(1-x)^m}$$

for constants c_1 and c_2 which are free of m. Finally, we determine the constants c_1 and c_2. The two initial conditions at $m=0$ and $m=1$ lead to

$$c_0=\sum_{i=0}^{\infty}H_ix^i(=E_0(x))$$

and

$$\frac{c_0}{(1-x)}+\frac{c_1}{(1-x)}=\sum_{i=0}^{\infty}(i+1)H_{i+1}x^i(=E_1(x)).$$

Finally, we succeed in expressing a truncated version of $E_1(x)$ by a truncated version of $E_0(x)$; namely by

In[15]:= **SigmaReduce**$[\sum_{\mathbf{i=0}}^{\mathbf{K}}\mathbf{(i+1)H_{i+1}x^i},$ **Tower** $\rightarrow\{\sum_{\mathbf{i=0}}^{\mathbf{K}}\mathbf{H_ix^i}\}]$

Out[15]= $\dfrac{(x-2)x^{K+1}+(K+1)(x-1)H_Kx^{K+1}+1+(1-x)\sum_{i=0}^{K}H_ix^i}{(x-1)^2}$

Hence, with $K \to \infty$ and the assumption that $0 \leq x < 1$, we get

$$E_1(x) = \frac{1}{(1-x)^2} + \frac{1}{1-x} \sum_{i=0}^{\infty} H_i x^i.$$

Thus $c_1 = \frac{1}{1-x}$, and therefore

$$E_m(x) = \frac{\sum_{i=0}^{\infty} H_i x^i}{(1-x)^m} + \frac{H_m}{(1-x)^{m+1}}.$$

Invoking the knowledge $\sum_{i=0}^{\infty} H_i x^i = -\ln(1-x)/(1-x)$, we arrive at (9).

6. Conclusion

With identity (2) as an illustrative example, we showed how symbolic summation methods can be used to find generalizations of binomial identities where infinite series are replaced by truncated versions. In the given context the Sigma package has led us to find the relations (5), (7) and (8) for terminating variants of Lengyel's infinite series $a_{m,n}(x)$.

From a combinatorial point of view it might be interesting to explore whether such truncated variants do also have statistical interpretations, e.g., within the theory of records like identity (1).

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