Applications of filter coefficients and wavelets parametrized by moments

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1 Introduction

Wavelets and their generalizations are used in many areas of mathematics ranging from harmonic analysis over numerical analysis to signal and image processing, see for example Daubechies [9], Mallat [26], and Strang and Nguyen [39]. A function $\psi \in L^2(\mathbb{R})$ is an *orthonormal wavelet* if the family

$$\psi_{jk}(x) = 2^{j/2}\psi(2^j x - k), \quad \text{for } j, k \in \mathbb{Z},$$

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of translated an dilated versions of ψ is an orthonormal basis of the Hilbert space $L^2(\mathbb{R})$. Alfred Haar gave in his dissertation from 1909, published in [16], the first example of an orthonormal wavelet

$$\psi(x) = \begin{cases} 1, & \text{for } 0 \leq x < \frac{1}{2}, \\ -1, & \text{for } \frac{1}{2} \leq x < 1, \\ 0, & \text{otherwise,} \end{cases}$$

which is now known as the *Haar wavelet*. Daubechies introduced in her seminal paper [8] a general method to construct compactly supported wavelets. Her construction is based on *scaling functions*, satisfying a *dilation equation*

$$\phi(x) = \sum_{k=0}^{N} h_k \phi(2x - k) \tag{1.1}$$

given by a linear combination of real *filter coefficients* h_k and dilated and translated versions of the scaling function, see the next section for an outline.

Imposing conditions on the scaling function gives via the dilation equation (1.1) constraints on the filter coefficients. Orthonormality implies quadratic equations and vanishing moments of the associated wavelet and normalization linear constraints. Daubechies wavelets [8] have the maximal number of vanishing moments for a fixed number of filter coefficients, and so there are only finitely many solutions. Parametrizing all possible filter coefficients that correspond to compactly supported orthonormal wavelets has been studied by several authors [18, 25, 30, 35, 38, 43, 44, 46]. All parametrizations express the filter coefficients in terms of trigonometric functions, and there is no natural interpretation of the angular parameters for the resulting scaling function. Furthermore, one has to solve transcendental constraints for the parameters to find wavelets with more than one vanishing moment.

We gave parametrizations that use the first discrete moments of the filter coefficients as parameters and such that the corresponding wavelets have several vanishing moments first in [33] and then simplified in [32]. See Section 3 for the parametrizations of four to eight filter coefficients with one parameter and at least one, two, and three vanishing moments, respectively. To compute these parametrizations we used symbolic computation and for the more involved equations in particular Gröbner bases, which were introduced by Buchberger in [1], see also [2]. Other applications of Gröbner bases to the design of wavelets and filter coefficients are for example discussed in [4, 5, 15, 22, 23, 28, 29, 36].

As a first application of parametrized wavelets, we showed in [33] how they can by used for compression by computing an optimal parameter for a given signal, see also [17]. In this paper, we describe several other applications. In Section 4, we discuss the regularity of the scaling functions and wavelets corresponding to our parametrizations. We construct wavelets that have a higher Hölder exponent than the Daubechies wavelets. Filter design is another possible application of our parametrizations. We deal with the construction of least asymmetric orthonormal wavelets in Section 5. Finally,

we address the existence of rational filter orthogonal filter coefficients in the last section. For example, we show that there are no orthogonal filters with six nonzero filter coefficients and at least two sum rules.
A Maple worksheet with all computations, several MATLAB functions to produce the figures and a GUI to compute with and illustrate parametrized wavelets are available on request from the author.

2 Wavelets and moments

We outline the construction of orthonormal wavelets based on scaling functions and recall the polynomial equations for the filter coefficients, see for example Daubechies [9] or Strang and Nguyen [39].
Orthonormality of the integer translates $\{\phi(x-l)\}_{l\in\mathbb{Z}}$ in $L^2(\mathbb{R})$, that is,

$$\int \phi(x)\phi(x-l)dx = \delta_{0,l}$$

implies, using the dilation equation (1.1), the quadratic equations

$$\sum_{k\in\mathbb{Z}} h_k h_{k-2l} = 2\delta_{0,l}, \quad \text{for } l \in \mathbb{Z}, \tag{2.1}$$

where we set $h_k = 0$ for $k < 0$ and $k > N$. We can assume that $h_0 h_N \neq 0$. Then with Equation (2.1) we see that N must be odd and the number of filter coefficients even.
If the filter coefficients satisfy the necessary conditions for orthogonality (2.1) and the normalization

$$\sum_{k=0}^{N} h_k = 2, \tag{2.2}$$

then there exists a unique solution of the dilation equation (1.1) in $L^2(\mathbb{R})$ with support $[0, N]$ and for which $\int \phi = 1$, see Lawton [20]. For almost all such scaling functions the integer translates $\{\phi(x-l)\}_{l\in\mathbb{Z}}$ are orthogonal, and then

$$\psi(x) = \sum_{k=0}^{N} (-1)^k h_{N-k}\phi(2x-k) \tag{2.3}$$

is an orthonormal wavelet.
Necessary and sufficient conditions for orthonormality were given by Cohen [6] and Lawton [21], see also Daubechies [9, ch. 6.3.]. The only example with four filter coefficients that satisfies the Equations (2.1) and (2.2) and where the integer translates of the corresponding scaling are not orthogonal is $h_0 = h_3 = 1$ and $h_1 = h_2 = 0$ with the scaling function

$$\phi(x) = \begin{cases} 1/3, & \text{for } 0 \le x < 3, \\ 0, & \text{otherwise.} \end{cases} \tag{2.4}$$

The corresponding scaling function for the Haar wavelet is the box function

$$\phi(x) = \begin{cases} 1, & \text{for } 0 \leq x < 1, \\ 0, & \text{otherwise.} \end{cases}$$

with the filter coefficients $h_0 = h_1 = 1$. In general, there is no closed analytic form for the scaling function, and for computations with scaling functions and wavelets only the filter coefficients are used.
Vanishing moments of the associated wavelet are related to several properties of the scaling function and wavelet. For example, to regularity, the polynomial reproduction and the approximation order of the scaling function, and the decay of the wavelet coefficients for smooth functions, see Strang and Nguyen [39] and the survey [40] by Unser and Blu for details. The condition that the first p moments of the wavelet ψ vanish

$$\int x^l \psi(x)\, dx = 0, \quad \text{for } l = 0, \dots, p-1$$

is using Equation (2.3) equivalent to the *sum rules*

$$\sum_{k=0}^{N} (-1)^k k^l h_k = 0, \quad \text{for } l = 0, \dots, p-1. \tag{2.5}$$

One then says that ψ has p *vanishing moments* or the filter coefficients satisfy p sum rules.
Since we use *discrete moments*

$$m_n = \sum_{k=0}^{N} h_k k^n$$

of the filter coefficients as a parameters, we recall a well-known recursive relation between discrete and *continuous moments*

$$M_n = \int x^n \phi(x)\, dx$$

of the scaling function. Let ϕ be a scaling function satisfying $M_0 = \int \phi = 1$. Then $m_0 = 2$ and

$$M_n = \frac{1}{2^{n+1} - 2} \sum_{i=1}^{n} \binom{n}{i} m_i M_{n-i},$$

$$m_n = \left(2^{n+1} - 2\right) M_n - \sum_{i=1}^{n-1} \binom{n}{i} m_i M_{n-i}, \quad \text{for } n > 0,$$

see for example Strang and Nguyen [39, p. 396]. Using the recursion we obtain for the first moments

$$M_1 = 1/2\, m_1$$

$$M_2 = 1/6\, m_1^2 + 1/6\, m_2$$

$$M_3 = 1/28\, m_1^3 + 1/7\, m_1 m_2 + 1/14\, m_3$$

and

$$m_1 = 2\,M_1$$
$$m_2 = -4\,M_1^2 + 6\,M_2$$
$$m_3 = 12\,M_1^3 - 24\,M_1 M_2 + 14\,M_3.$$

Explicit formulas expressing the discrete moments in terms of the continuous and vice versa are given in [33].

3 Parametrizations

We give the parametrization from [32] of four, six, and eight filter coefficients corresponding respectively to orthonormal wavelets with at least one, two and three vanishing moments. All families depend on the first discrete moment

$$m = m_1 = \sum_{k=0}^{N} h_k k$$

of the filter coefficients. We also discuss some special parameter values, for example, for the Daubechies wavelets.

3.1 Four filter coefficients

We have the following parametrization of filter coefficients with at least one vanishing moments:

$$\begin{aligned} h_0 &= 1/2 - 1/4\,a - 1/4\,w \\ h_1 &= 1/2 - 1/4\,a + 1/4\,w \\ h_2 &= 1/2 + 1/4\,a + 1/4\,w \\ h_3 &= 1/2 + 1/4\,a - 1/4\,w \end{aligned} \tag{3.1}$$

with $w = \sqrt{4 - a^2}$ and $a = m - 3 \in [-2, 2]$.
Note that for $a = -a$ we obtain the flipped filter coefficients. For $a = 0$ we get the filter coefficients $(0, 1, 1, 0)$, whichcorrespond to a translated Haar scaling function and wavelet. The parameter values $a = -2, 2$ give also Haar scaling functions with the filter coefficients $(1, 1, 0, 0)$ and $(0, 0, 1, 1)$. The *Daubechies wavelet* has two vanishing moments, so we have one more sum rule

$$2\,h_0 - h_1 + h_3 = 0.$$

Substituting the parametrized filter coefficients into this equations and solving for a, we get the two solutions $a = -\sqrt{3}, \sqrt{3}$ with the first discrete moments $m = 3-\sqrt{3}, 3+\sqrt{3}$. The first solution gives the famous Daubechies filters [8]

$$1/4\,(1 + \sqrt{3}, 3 + \sqrt{3}, 3 - \sqrt{3}, 1 - \sqrt{3}) \tag{3.2}$$

and the second the flipped version. See Figure 3.1 for plots of scaling functions for various parameter values.

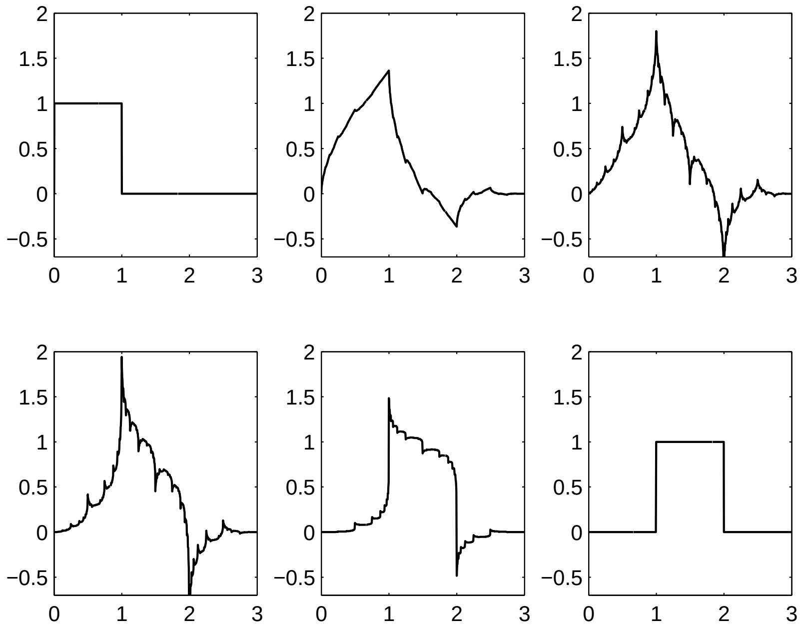

Figure 3.1: Scaling functions for $a = -2, -\sqrt{3}, -1/3\,\sqrt{3} - 2/3$ (first row) and $a = 1/3\,\sqrt{3} - 4/3, -2 + \sqrt{3}, 0$ (second row).

We have a second parametrization of filter coefficients with at least one vanishing moment:

$$\begin{aligned} h_0 &= 1/2 - 1/4\,a + 1/4\,w \\ h_1 &= 1/2 - 1/4\,a - 1/4\,w \\ h_2 &= 1/2 + 1/4\,a - 1/4\,w \\ h_3 &= 1/2 + 1/4\,a + 1/4\,w \end{aligned} \tag{3.3}$$

with $w = \sqrt{4 - a^2}$ and $a = m - 3 \in [-2, 2]$.
Comparing this solution with the parametrized filter coefficients (3.1), we see that w is replaced by $-w$ and so the two first and the two last filter coefficients are swapped. Note that again for $a = -a$ we obtain the flipped filters.
For $a = 0$ we now get the filter coefficients $(1, 0, 0, 1)$, which give the scaling function (2.4) where the integer translates of the scaling function are not orthogonal. The parameter values $a = -2, 2$ also give Haar scaling functions with the filter coefficients $(1, 1, 0, 0)$ and $(0, 0, 1, 1)$. This parametrization does not contain filter coefficients with

a second vanishing moment. The corresponding scaling functions are, compared to the parametrization (3.1), irregular, see Section 4 for details.

3.2 Six filter coefficients

We have the following parametrization of filter coefficients with with at least two vanishing moments:

$$\begin{aligned} h_0 &= -3/32 - 1/8\,a + 1/32\,a^2 - 1/32\,w \\ h_1 &= 5/32 - 1/8\,a + 1/32\,a^2 + 1/32\,w \\ h_2 &= 15/16 - 1/16\,a^2 + 1/16\,w \\ h_3 &= 15/16 - 1/16\,a^2 - 1/16\,w \\ h_4 &= 5/32 + 1/8\,a + 1/32\,a^2 - 1/32\,w \\ h_5 &= -3/32 + 1/8\,a + 1/32\,a^2 + 1/32\,w \end{aligned} \tag{3.4}$$

with $w = \sqrt{-a^4 + 14\,a^2 + 15}$ and $a = m - 5 \in [-\sqrt{15}, \sqrt{15}]$.
The Daubechies wavelet has one more vanishing moment, that is, it satisfies the sum rule

$$-9\,h_0 + 4\,h_1 - h_2 - h_4 + 4\,h_5 = 0.$$

Substituting the parametrized filter coefficients into this equations and solving for a, we get one real solution $a = -\sqrt{5 + 2\sqrt{10}}$, which gives the filter coefficients

$$1/16\,(1 + \sqrt{10} + w, 5 + \sqrt{10} + 3\,w, 10 - 2\sqrt{10} + 2\,w,$$
$$10 - 2\sqrt{10} - 2\,w, 5 + \sqrt{10} - 3\,w, 1 + \sqrt{10} - w) \tag{3.5}$$

with $w = \sqrt{5 + 2\sqrt{10}}$. The Daubechies filters with four nonzero filter coefficients (3.2) satisfy two sum rules and are therefore contained in this parametrization. Their first discrete moment is $m = 3 - \sqrt{3}$. So here the corresponding parameter $a = -2 - \sqrt{3}$. We get a translated version for $a = -\sqrt{3}$. For $a = -\sqrt{15}$ we obtain

$$1/8\,(3 + \sqrt{15}, 5 + \sqrt{15}, 0, 0, 5 - \sqrt{15}, 3 - \sqrt{15}).$$

The parameter $a = -1$ gives the first coiflet

$$1/16\,(1 - \sqrt{7}, 5 + \sqrt{7}, 14 + 2\sqrt{7}, 14 - 2\sqrt{7}, 1 - \sqrt{7}, -3 + \sqrt{7}),$$

see Daubechies [10] and [9, ch. 8.2.]. For $a = 0$ we get

$$1/32\,(-3 - \sqrt{15}, 5 + \sqrt{15}, 30 + 2\sqrt{15}, 30 - 2\sqrt{15}, 5 - \sqrt{15}, -3 + \sqrt{15}).$$

See Figure 3.2 for plots of scaling functions for various parameter values. The corresponding scaling functions and wavelets for $a > 0$ become increasingly irregular, see Section 4 for details.

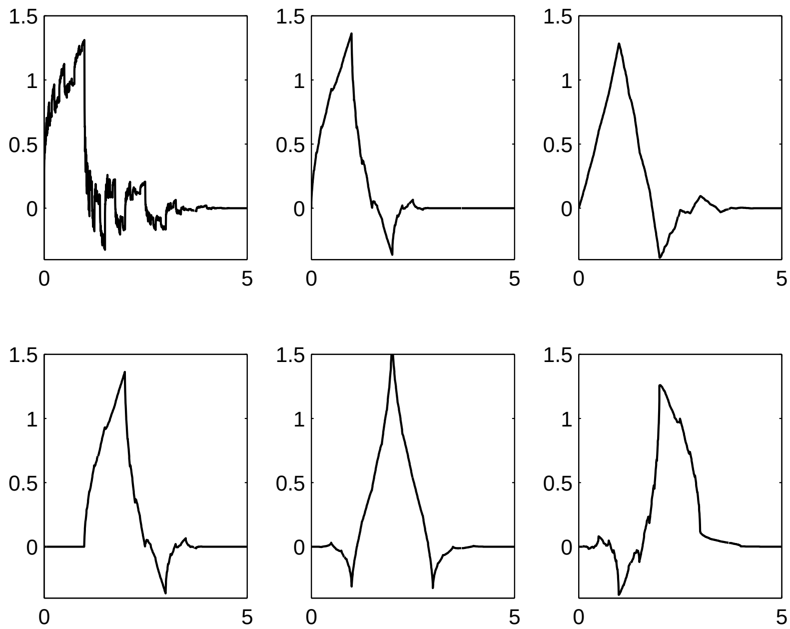

Figure 3.2: Scaling functions for $a = -\sqrt{15}, -2-\sqrt{3}, -\sqrt{5+2\sqrt{10}}$ (first row) and $a = -\sqrt{3}, -1, 0$ (second row).

3.3 Eight filter coefficients

We have the following parametrization of filter coefficients with at least three vanishing moments:

$$
\begin{aligned}
h_0 &= -\frac{1}{512}\,\frac{a^5 - 7\,a^4 - 2\,a^3 + 30\,a^2 - 55\,a - 15 + (1-a)w}{a^2+1}\\
h_1 &= -\frac{1}{512}\,\frac{a^5 - 9\,a^4 + 30\,a^3 + 2\,a^2 - 23\,a + 63 + (1+a)w}{a^2+1}\\
h_2 &= \frac{1}{512}\,\frac{3\,a^5 - 5\,a^4 - 102\,a^3 + 186\,a^2 - 261\,a + 35 + 3(1-a)w}{a^2+1}\\
h_3 &= \frac{1}{512}\,\frac{3\,a^5 - 11\,a^4 - 70\,a^3 + 358\,a^2 - 229\,a + 525 + 3(1+a)w}{a^2+1}\\
h_4 &= -\frac{1}{512}\,\frac{3\,a^5 + 11\,a^4 - 70\,a^3 - 358\,a^2 - 229\,a - 525 + 3(1-a)w}{a^2+1}\\
h_5 &= -\frac{1}{512}\,\frac{3\,a^5 + 5\,a^4 - 102\,a^3 - 186\,a^2 - 261\,a - 35 + 3(1+a)w}{a^2+1}\\
h_6 &= \frac{1}{512}\,\frac{a^5 + 9\,a^4 + 30\,a^3 - 2\,a^2 - 23\,a - 63 + (1-a)w}{a^2+1}\\
h_7 &= \frac{1}{512}\,\frac{a^5 + 7\,a^4 - 2\,a^3 - 30\,a^2 - 55\,a + 15 + (1+a)w}{a^2+1}
\end{aligned}
\tag{3.6}
$$

with

$$w = \sqrt{-a^8 + 36\,a^6 - 182\,a^4 + 1540\,a^2 - 945},$$

$a = m - 7$ and a in the intervals

$$[-\sqrt{\beta}, -\sqrt{\alpha}] \quad \text{or} \quad [\sqrt{\alpha}, \sqrt{\beta}], \tag{3.7}$$

where α denotes the smaller and β the bigger real root of

$$x^4 - 36\,x^3 + 182\,x^2 - 1540\,x + 945,$$

with numerical approximations

$$\sqrt{\alpha} = 0.8113601077\ldots \quad \text{and} \quad \sqrt{\beta} = 5.636256558\ldots$$

The Daubechies wavelet satisfies one more sum rule

$$64\,h_0 - 27\,h_1 + 8\,h_2 - h_3 + h_5 - 8\,h_6 + 27\,h_7 = 0.$$

Substituting the parametrized filter coefficients (3.6) into this equations and solving for a, we get two real solution $a = -\sqrt{\beta}, -\sqrt{\alpha}$, where α denotes the smaller and β the larger real root of

$$x^4 - 28\,x^3 + 126\,x^2 - 1260\,x + 1225$$

or numerically

$$a = -4.989213573\ldots, -1.029063869\ldots$$

The first parameter gives the Daubechies wavelet with extremal phase [9, p. 195] and the second the "least asymmetric" [9, p. 198]. The Daubechies wavelet with six nonzero filter coefficients has the first discrete moment $m = 5 - \sqrt{5 + 2\sqrt{10}}$, so the corresponding parameter value for the parametrization (3.6) is

$$a = -2 - \sqrt{5 + 2\sqrt{10}} = -5.365197664\ldots$$

See Figure 3.3 for plots of scaling functions for various parameter values.

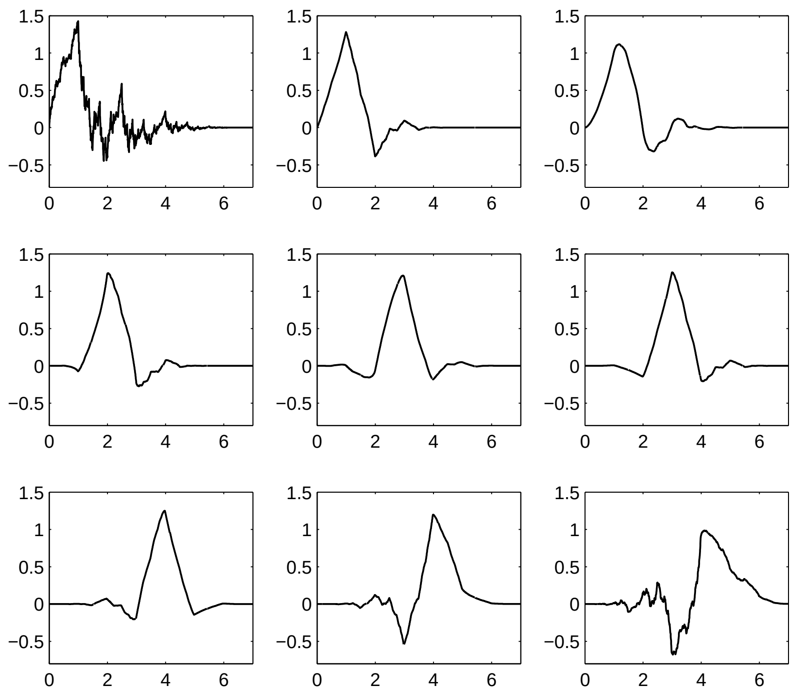

Figure 3.3: Scaling functions for $a = -5.636256559, -5.365197664, -4.989213573$ (first row), $a = -3.009138721, -1.029063869, -.8113601077$ (second row), and $a = .8113601077, 2, 3$ (third row).

4 Regularity of scaling functions and wavelets

In this section, we discuss the regularity or smoothness of the scaling functions and wavelets corresponding to the parametrized filter coefficients from the previous section. The regularity of a function can be measured in different ways, we consider here the Hölder and Sobolev exponent.

We first recall the definitions. For $\alpha = n + \beta$, where $n \in \mathbb{N}$ and $0 \leq \beta < 1$, the set $C^\alpha = C^\alpha(\mathbb{R})$ is defined as the set of all functions f that are n times continuously differentiable and such that the nth derivative $f^{(n)}$ is uniformly *Hölder continuous* with exponent β, that is,

$$|f^{(n)}(x+h) - f^{(n)}(x)| \leq C|h|^\beta, \quad \text{for all } x, h \in \mathbb{R},$$

where C is a constant. For $s \geq 0$ the *Sobolev space* $H^s = H^s(\mathbb{R})$ consists of all functions $f \in L^2(\mathbb{R})$ such that $(1 + |\xi|^2)^{s/2}\hat{f}(\xi) \in L^2(\mathbb{R})$, where $\hat{f}$ denotes the fourier transform of f.

To measure the regularity or smoothness of a scaling function ϕ, one is interested in the (optimal) *Sobolev*

$$s_{\max} = \sup\{s : \phi \in H^s\}$$

and *Hölder exponent*

$$\alpha_{\max} = \sup\{\alpha : \phi \in C^\alpha\},$$

respectively. For a scaling function the Hölder exponent satisfies [41]

$$\alpha_{\max} \in [s_{\max} - 1/2, s_{\max}]. \tag{4.1}$$

The regularity of scaling functions is also related to vanishing moments of the corresponding wavelet. Villemoes [41] proved that if $\phi \in H^n$ with $n \in \mathbb{N}$, the filter coefficients satisfy $n+1$ sum rules or equivalently the corresponding wavelet has $n+1$ vanishing moments. So in particular if $\phi \in C^n$, then the filter coefficients satisfy $n+1$ sum rules, see also [9, pp. 153–156].

Eirola [12] and Villemoes [41] showed independently how the optimal Sobolev exponent can be computed from the spectral radius of a matrix depending on the filter coefficients, see also Strang and Nguyen [39] for further details. To find the optimal Hölder exponent is much more involved, see for example [7, 9, 11, 34], but Rioul [34] gave an algorithm to compute good lower bounds for the Hölder exponent. The algorithm produces monotonically increasing lower bounds with an increasing number of iterations, but the storage and computational costs approximately double for each additional iteration.

In Figures 4.1, 4.2 and 4.3 you can see plots of the Sobolev exponent of the corresponding scaling functions and wavelets depending on one parameter. For four filter coefficients the Sobolev exponents range from 0.5 to 1 (parametrization (3.1)) and from 0 to 0.5 (parametrization (3.3)). The maximum 1 is attained for the Daubechies wavelet

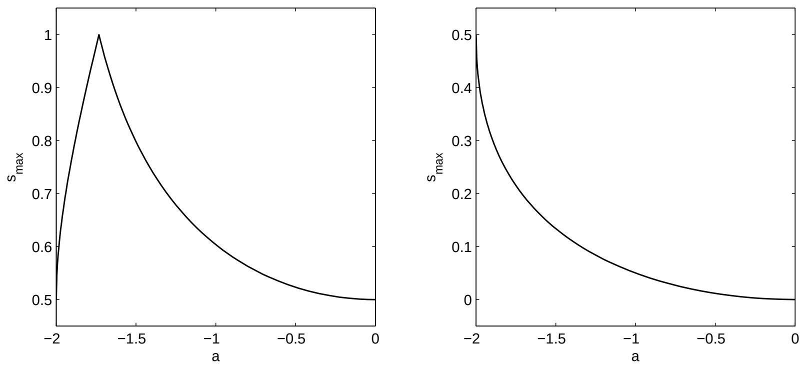

Figure 4.1: Sobolev exponent for scaling functions with four filter coefficients from Equation (3.1) (left) and (3.3) (right).

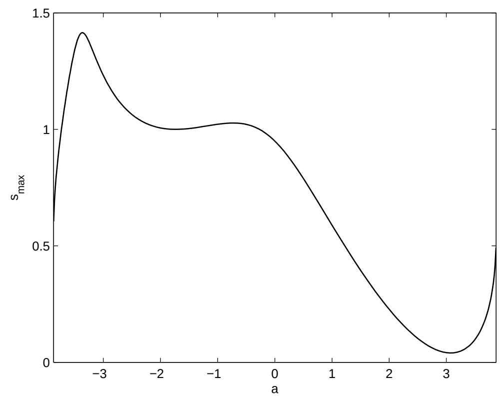

Figure 4.2: Sobolev exponent for scaling functions with six filter coefficients from Equation (3.4).

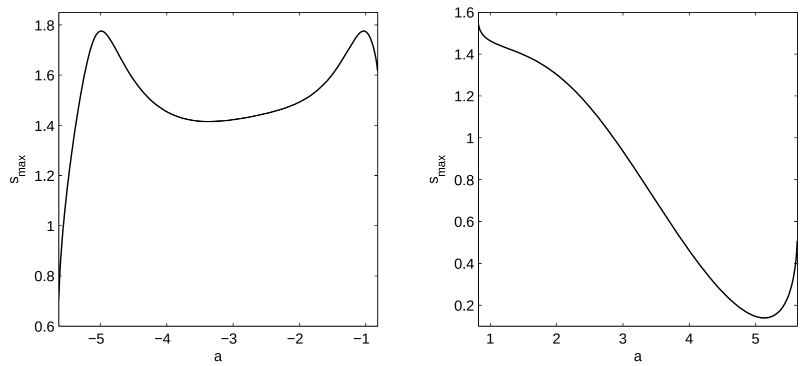

Figure 4.3: Sobolev exponent for scaling functions with eight filter coefficients from Equation (3.6).

since all other filter coefficients satisfy only one sum rule and hence their Sobolev exponent is necessarily less than one. We obtain numerically the maximal Sobolev exponent for respectively six and eight filter coefficients

$$s_{\max} = 1.4150, 1.7757,$$

at the parameter values for the Daubechies wavelets and the minimum is

$$s_{\max} = 0.0399, 0.1393$$

with parameter values

$$a = 3.07768194648051, 5.13160341992728.$$

For more than six filter coefficients it is possible to construct wavelets with a higher Sobolev exponents than the Daubechies wavelets by omitting more than one sum rule, see [24, 27, 42].
In Figures 4.4, 4.5 and 4.6 you can see plots of lower bounds for the Hölder exponent of the corresponding scaling functions and wavelets depending on one parameter, with the bounds from Equation (4.1). We used 24 iteration in the algorithm from [34].
Note that for most and for eight filter coefficients for all parameters the computed lower bound is higher than the lower bound $s_{\max} - 1/2$. The negative lower bound in 4.5 indicates that the corresponding scaling function is not continuous. We obtain numerically the maximal lower bound for the Hölder exponent for respectively four, six and eight filter coefficients

$$\alpha_{24} = 0.5776, 1.1386, 1.6344$$

with parameters

$$a = -1.66260325442517, -3.28211108661493, -4.93905744197576$$

and filter coefficients

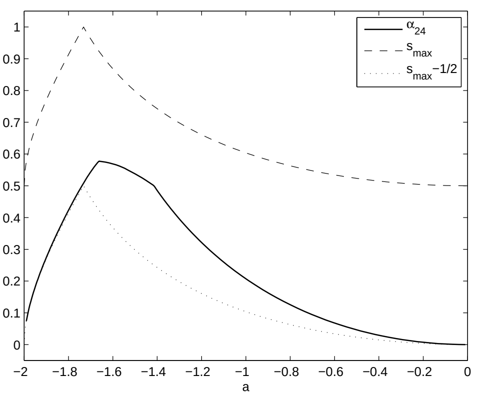

Figure 4.4: Lower bound for Hölder exponent for scaling functions with four filter coefficients from Equation (3.1).

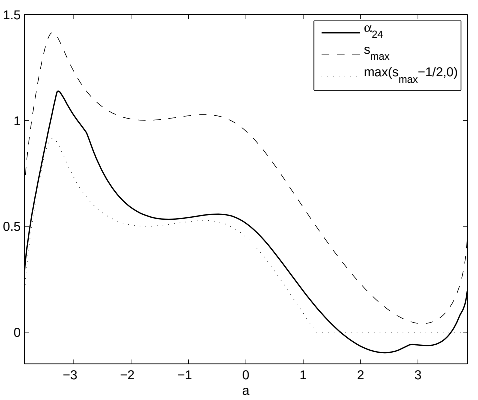

Figure 4.5: Lower bound for Hölder exponent for scaling functions with six filter coefficients from Equation (3.4).

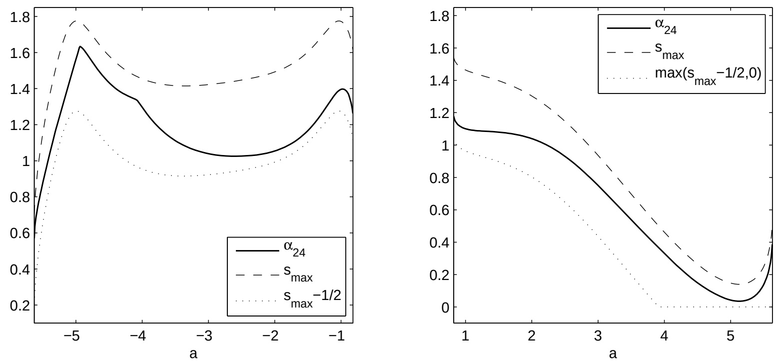

Figure 4.6: Lower bound for Hölder exponent for scaling functions with eight filter coefficients from Equation (3.6).

```
 0.31887001724554, 0.59678079636075,
 0.18112998275446,-0.09678079636075

 0.21634225649014, 0.56180454136425, 0.35257937284659,
-0.08834519690163,-0.06892162933673, 0.02654065553738

 0.15488273436983, 0.49644876596501, 0.45767418856225,
-0.00833281609981,-0.13761439998701, 0.01970151455156,
 0.02505747705493,-0.00781746441676.
```

Daubechies and Lagarias [11] obtained the optimal Hölder exponents for the Daubechies wavelets with a different method (four, six, and eight filter coefficients)

$$\alpha_{\max} = 0.5500, 1.0878, 1.6179,$$

where the last one is for the Daubechies wavelet with extremal phase. So we obtained in all cases wavelets that have a higher Hölder exponent than the Daubechies wavelets.

Daubechies addressed in [10] and [9, p. 242] the question of finding wavelets with more regularity. For four filter coefficients she obtained the rational filter coefficients $(3/5, 6/5, 2/5, -1/5)$, which corresponds to $a = -8/5$ in (3.1), see also Section 6. With the methods from [11] she found that the Hölder exponent of the corresponding scaling function is at least 0.5864. Lang and Heller [19] discussed the general optimization problem of maximizing the Hölder exponent for a fixed number of filter coefficients. They found smoother wavelets than the Daubechies wavelets for more than eight filter coefficients, but the numerical method failed to find the more regular wavelets that we obtained using the explicit parametrization of the filter coefficients.

5 Least asymmetric filters

It is well known [9, p. 252] that if a compactly supported orthonormal wavelet is symmetric or antisymmetric around some axis, then it is the Haar wavelet. Symmetry of the scaling function is in turn equivalent to symmetry of the filter coefficients, see [9, p. 252–253] and [10]. Here we say that the filter coefficients are *symmetric* around $n_0 \in \mathbb{Z}/2$ if

$$h_n = h_{2n_0-n},$$

where we set $h_k = 0$ for $k < 0$ and $k > N$. Symmetric filters are often called *linear phase filters* since the filter coefficients are symmetric around $n_0 \in \mathbb{Z}/2$ if and only if the phase of the *frequency response*

$$h(\xi) = \sum_n h_n e^{in\xi}$$

is a linear function of ξ, that is, if

$$h(\xi) = e^{in_0\xi}|h(\xi)|.$$

So we know that complete symmetry and orthogonality is not possible, and one can only try to find the least asymmetric filter coefficients out of a fixed family. For example, Daubechies discussed in [9] and [10] how to choose the least asymmetric out of the finitely many wavelets with a maximal number of vanishing moments. Another possibility is to omit some vanishing moments and use the additional degrees of freedom to find filters with partial symmetry. Several authors [14, 22, 37] discussed the use of Gröbner bases to find orthogonal filter coefficients with partial symmetry where several pairs of filters are equal. Zhao and Swamy [45] designed least asymmetric orthogonal wavelets with several vanishing moments via numerical optimization.
An immediate application of our parametrized filter coefficients is to find symbolically the least asymmetric filter coefficients using some criteria to measure symmetry. In the following, we discuss some examples, where we minimize the sum of squares error as in [45].
We want to find six filter coefficients satisfying two sum rules such that they are almost symmetric around 2, so that

$$h_0 \approx h_4, h_1 \approx h_3, h_6 \approx 0.$$

Using Maple, we find the minimum of the sum of squares error is attained at $a = \alpha$, where α denotes the largest negative real root of

$$25\,x^{10} - 30\,x^9 - 702\,x^8 + 652\,x^7 + 5866\,x^6 - 3256\,x^5 - 13140\,x^4 - 1036\,x^3 + 5797\,x^2 - 2730\,x - 5190$$

or numerically

$$a = -1.102986298\ldots$$

The filter coefficients are:

```
-0.090589559870111, 0.504872307867382, 1.20692569433612,
 0.516001958861136,-0.116336134466010,-0.02087426672852.
```

See Figure 5.1 for the corresponding scaling function, which has a Sobolev exponent $s_{\max} = 1.0180$ and a lower bound for the Hölder exponent $\alpha_{24} = 0.5370$.

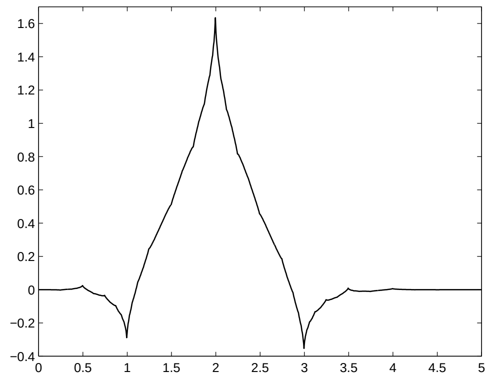

Figure 5.1: Least asymmetric (around 2) scaling function with six filter coefficients and two sum rules.

Now we consider eight filter coefficients. First we want to find filter coefficients that are almost symmetric around 3, so that

$$h_0 \approx h_6, h_1 \approx h_5, h_2 \approx h_4, h_7 \approx 0.$$

The minimum of the sum of squares error is attained at $a = \alpha$, where α denotes the largest negative real root of

$$11025\,x^{24} - 21000\,x^{23} - 901900\,x^{22} + 1407480\,x^{21} + 25484946\,x^{20} - 23935800\,x^{19} - 280989500\,x^{18}$$
$$-149785464\,x^{17} + 837190927\,x^{16} + 6460372400\,x^{15} + 4612440168\,x^{14} - 53422512976\,x^{13}$$
$$-69302308420\,x^{12} + 344858640016\,x^{11} - 84085760856\,x^{10} - 294800719088\,x^{9} + 2435452393919\,x^{8}$$
$$-1913025285928\,x^{7} - 18887356576348\,x^{6} + 10024351195096\,x^{5} + 51733811048402\,x^{4}$$
$$-17259269191640\,x^{3} - 57876449779820\,x^{2} + 8466676099560\,x + 21625605062145$$

or numerically

$$a = -.8395579286\ldots$$

The filter coefficients are:

```
-0.073484394510424,-0.071424517120364, 0.556147092523951,
 1.154912201440016, 0.568048480655853,-0.135661369346454,
-0.050711178669381, 0.052173685026802.
```

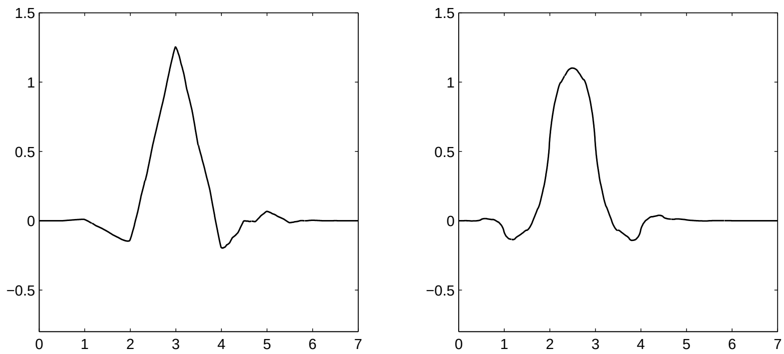

Figure 5.2: Least asymmetric (around 3 left and 2.5 right) scaling function with eight filter coefficients and three sum rules.

See Figure 5.2 (left) for the corresponding scaling function, which has a Sobolev exponent $s_{\max} = 1.6569$ and a lower bound for the Hölder exponent $\alpha_{24} = 1.3080$. Finally, we want to design filters that are almost symmetric around 2.5, so that

$$h_0 \approx h_5, h_1 \approx h_4, h_2 \approx h_3, h_6 \approx 0, h_7 \approx 0.$$

This is related to the example considered in [14, 22], where the authors constructed using Gröbner bases eight orthogonal filters with two sum rules such that $h_0 = h_5$, $h_1 = h_4$ and $h_2 = h_3$. The minimum of the sum of squares error is attained at $a = \alpha$, where α denotes the second largest negative real root of

$$\begin{aligned}&2025\,x^{24}-9000\,x^{23}-168020\,x^{22}+823000\,x^{21}+4733434\,x^{20}-27869720\,x^{19}-46538164\,x^{18}\\&+437384872\,x^{17}-40684609\,x^{16}-3591330192\,x^{15}+3105046936\,x^{14}+20835868016\,x^{13}\\&-35438686580\,x^{12}-64147246896\,x^{11}+233849168056\,x^{10}-48135550128\,x^{9}-894126414729\,x^{8}\\&+1033511750456\,x^{7}+2682874758716\,x^{6}-4634966862792\,x^{5}-4762513155302\,x^{4}\\&+10857513198280\,x^{3}+182957235580\,x^{2}-6268723929720\,x+2258107786305\end{aligned}$$

or numerically

$$a = -1.927469761\ldots$$

The filter coefficients are:

```
-0.114678365799638, 0.127976021526492,  0.977783792709255,
 0.990754350911186, 0.120334952341046, -0.133569326041206,
 0.016559620749336, 0.014838953603528.
```

See Figure 5.2 (right) for the corresponding scaling function, which has a Sobolev exponent $s_{\max} = 1.5026$ and a lower bound for the Hölder exponent $\alpha_{24} = 1.0633$.

6 Rational filter coefficients

In this section, we address the existence of rational orthogonal filter coefficients. We know from Section 2 that filter coefficients are determined by quadratic equations for orthonormality (2.1) and linear equations for normalization (2.2) and vanishing moments (2.5). Note that all these equations have integer coefficients, and we want to find a rational solution. This leads to "Hilbert's 10th Problem over $\mathbb{Q}$", which asks if there exists an algorithm for deciding the existence of rational points for a system of polynomial equations with integer coefficients. The answer is not known, and despite centuries of effort, even for curves it is an open problem although many results and computational methods are known, see for example Poonen [31] for an introduction and further references. Using our parametrizations, we can reduce the question of rational filter coefficients to finding rational points on curves and give some answers.
The case of four filter coefficients is not difficult. Daubechies [8] already gave a rational parametrization of all orthogonal filter coefficients

$$h_0 = \frac{t\,(t-1)}{t^2+1}, \quad h_1 = \frac{1-t}{t^2+1}, \quad h_2 = \frac{t+1}{t^2+1}, \quad h_3 = \frac{t\,(t+1)}{t^2+1},$$

with $t \in \mathbb{R}$. Note that for $t = -t$ we obtain the flipped filter coefficients. The interval $-1 \le t \le 1$ corresponds to the filter coefficients from (3.1) and $t \le -1, 1 \le t$ to (3.3), except for $(1,0,0,1)$, which are approached for $t \to \infty$ and $t \to -\infty$. The Daubechies wavelet corresponds to $t = -1/\sqrt{3}$. Computing the continued fraction expansion of $-1/\sqrt{3}$, we obtain the periodic expansion

$$-\frac{1}{\sqrt{3}} = [-1;\, 2,\, \overline{2,\, 1}\,]$$

with the first convergents

$$-1,\ -1/2,\ -3/5,\ -4/7,\ -\frac{11}{19},\ -\frac{15}{26},\ -\frac{41}{71},\ -\frac{56}{97},\ -\frac{153}{265},\ -\frac{209}{362}.$$

Taking for example $t = -209/362$, we get a good rational approximation

$$1/174725\,(119339, 206702, 55386, -31977)$$

for the Daubechies filters. Surprisingly, we obtain the filter coefficients corresponding to the most regular scaling function found by Daubechies for the second convergent $t = -1/2$, see Section 4.
In the parametrization (3.4) for six filter coefficients there appears only the square root

$$w = \sqrt{-a^4 + 14\,a^2 + 15}.$$

So the question of the existence of rational filters reduces to finding a rational point $(a,b) \in \mathbb{Q}^2$ on the (hyperelliptic) algebraic curve defined by the equation

$$y^2 = -x^4 + 14\,x^2 + 15 = -(x^2+1)(x^2-15). \tag{6.1}$$

Proposition 6.1 *There are no rational points on the curve defined by Equation* (6.1)*.*

Proof. Substituting $x = X/Z$ and $y = Y/Z^2$ in (6.1) and multiplying by Z^4, we obtain

$$Y^2 = -(X^2 + Z^2)(X^2 - 15\,Z^2)$$

and we equivalently would have to find integers a, b, c with a and c coprime satisfying this equation. Suppose that we had integers a, b, c satisfying

$$b^2 = -(a^2 + c^2)(a^2 - 15\,c^2). \tag{6.2}$$

Then

$$b^2 \equiv (a^2 + c^2)^2 \pmod 2$$

and hence

$$b \equiv (a + c) \pmod 2.$$

This implies that either

$$a \equiv 1,\, c \equiv 0 \pmod 2 \quad \text{or} \quad a \equiv 0,\, c \equiv 1 \pmod 2$$

or, since a and c are coprime,

$$a \equiv c \equiv 1 \pmod 2.$$

In the first case, we then get

$$(a^2 + c^2)^2 \equiv 1 \pmod 4.$$

But then by Equation (6.2)

$$b^2 \equiv -1 \equiv 3 \pmod 4,$$

which is not possible since the only quadratic residues modulo 4, that is, the integers d for which

$$x^2 \equiv d \pmod 4$$

has a solution, are

$$d \equiv 0, 1 \pmod 4.$$

In the second case, we get

$$(a^2 + c^2)^2 \equiv 4 \pmod{16}.$$

But then by Equation (6.2)

$$b^2 \equiv -4 \equiv 12 \pmod{16},$$

which is not possible since the only quadratic residues modulo 16 are

$$d \equiv 0, 1, 4, 9 \pmod{16},$$

and the proposition is proved. □

Corollary 6.2 *There are no rational orthogonal filters with six nonzero filter coefficients and at least two sum rules.*

In the parametrization (3.6) for eight filter coefficients, we have the square root

$$w = \sqrt{-a^8 + 36\,a^6 - 182\,a^4 + 1540\,a^2 - 945}.$$

So we would have to find a rational point on the algebraic curve defined by the equation

$$y^2 = x^8 - 36\,x^6 + 182\,x^4 - 1540\,x^2 + 945.$$

This is a nonsingular curve with genus 3. Hence by Falting's Theorem [13] it has only finitely many rational points, and so there are at most finitely many rational orthogonal filters with eight nonzero filter coefficients and at least three sum rules.

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