# WEIGHTED BIORTHOGONAL SPLINE WAVELETS *

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**Abstract.** We construct weighted wavelets by combining the lifting scheme with a weighted inner product. Furthermore we present lazy wavelets which are used to construct weighted wavelets for periodic $B$-splines. Finally we consider examples for these wavelets and compare them with the "standard" lifted wavelets.

**Key words.** spline wavelets, lazy wavelets, weighted wavelets, multiresolution analysis, lifting scheme

**AMS subject classifications.** 65T60, 65D07

## 1. Introduction.

Wavelets play a prominent role in different areas of mathematics and scientific computing such as approximation theory, signal processing and numerical computation. Typical applications of wavelets include hierarchical visualizations of geometric objects, data compression and numerical simulation.

In particular, spline wavelets have become a powerful mathematical tool for the hierarchical representation of geometric objects, combining the properties of splines and wavelets. Depending on the application, different types of spline wavelets have been constructed. This includes (compactly supported) spline wavelets on a bounded interval e.g. [1, 3, 5, 6, 11], spline wavelets with minimal support e.g. [8] and non-uniform spline wavelets e.g. [9].

In the present paper we construct 1-periodic spline wavelets with the following properties. The constructed wavelets have a compact support such that analysis and synthesis can be done in linear time. Moreover these spline wavelets have improved approximation properties in a certain region of the whole interval, at the expense of the other parts.

The construction consists of two steps. As first step we construct lazy wavelets with a small support. In the second step we modify them by increasing the $L^2$-orthogonality with respect to a weighted inner product. In the wavelet literature, such products have also been considered in [13, 14]. This modification is achieved by using the lifting scheme, a method for modifying existing wavelets, cf. [10, 12, 14]. As the result we obtain *weighted spline wavelets*.

The remainder of this paper is organized as follows. In Section 2 we give a short outline of the general concept of wavelets. Section 3 describes the construction of lazy wavelets for periodic $B$-splines. Section 4 introduces the concept of weighted wavelets, which are wavelets constructed with the help of lifting and a weighted inner product. In Section 5 we present examples of weighted wavelets for periodic $B$-splines and compare these wavelets with the "standard" lifted wavelets.

## 2. Preliminaries.

In this section we give a short outline of the general concept of wavelets. We write $\mathbb{N}$ for the naturals numbers and $\mathbb{N}_0$ for the natural numbers with 0. We follow the notation in [12].

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Let $V^0 \subset V^1 \subset V^2 \subset \ldots \subset L^2([0,1])$ be a nested set of linear function spaces with finite dimension, such that $\overline{\bigcup_{j=0}^{\infty} V^j} \supset L^2([0,1])$. Let $W^0 \subset W^1 \subset W^2 \subset \ldots$ be a sequence of wavelet spaces with finite dimension such that $V^{j+1} = V^j \oplus W^j$ for all $j \in \mathbb{N}_0$. We denote the dimension of $V^j$ by $v(j)$ and the dimension of $W^j$ by $w(j)$. Furthermore let $\phi_0^j, \ldots, \phi_{v(j)-1}^j$ be a basis of $V^j$ and $\psi_0^j, \ldots, \psi_{w(j-1)}^j$ be a basis of $W^j$. The functions $\phi_i^j$ are called *scaling functions* and the functions $\psi_i^j$ are called *wavelets*. For ease of notation we use row vectors of functions as

$$\Phi^j := [\phi_0^j, \ldots, \phi_{v(j)-1}^j] \text{ and } \Psi^j := [\psi_0^j, \ldots, \psi_{w(j)-1}^j]. \tag{2.1}$$

Every function $f^j$ of $V^j$ and $g^j$ of $W^j$ can be written as

$$f^j = \Phi^j \mathbf{c}^j \text{ and } g^j = \Psi^j \mathbf{d}^j \tag{2.2}$$

with the column vectors

$$\mathbf{c}^j := [c_0^j, \ldots, c_{v(j)-1}^j]^T \text{ and } \mathbf{d}^j := [d_0^j, \ldots, d_{w(j)-1}^j]^T, \tag{2.3}$$

respectively. We write $(\mathbf{v}^1, \ldots, \mathbf{v}^k)$ for the concatenation of row (column) vectors $\mathbf{v}^1, \ldots, \mathbf{v}^k$.

Since $V^{j-1}$ and $W^{j-1}$ are subsets of $V^j$, there exist constant matrices $P^j$ and $Q^j$ such that $\Phi^{j-1} = \Phi^j P^j$ and $\Psi^{j-1} = \Phi^j Q^j$. These relations can also be expressed by a single equation, using block matrix notation $[\Phi^{j-1}|\Psi^{j-1}] = \Phi^j [P^j|Q^j]$. This equation is referred to as a *two-scale relation* for scaling functions and wavelets (cf. [12]).

The relation between $\mathbf{c}^j$ and $\mathbf{c}^{j-1}$, $\mathbf{d}^{j-1}$ is expressed by

$$\mathbf{c}^j = [P^j|Q^j][\frac{\mathbf{c}^{j-1}}{\mathbf{d}^{j-1}}], \ \mathbf{c}^{j-1} = A^j \mathbf{c}^j \text{ and } \mathbf{d}^{j-1} = B^j \mathbf{c}^j \tag{2.4}$$

with $[\frac{A^j}{B^j}] = [P^j|Q^j]^{-1}$. Constructing $\mathbf{c}^j$ from $\mathbf{c}^{j-1}$ and $\mathbf{d}^{j-1}$ is called *synthesis*. The process of splitting the coefficients $\mathbf{c}^j$ into coefficients $\mathbf{c}^{j-1}$ and $\mathbf{d}^{j-1}$ is called *analysis*. The matrices $P^j, Q^j$ are called *synthesis filters*, the matrices $A^j, B^j$ are called *analysis filters*.

The different types of wavelets (orthogonal, semiorthogonal and biorthogonal) are distinquished by whether or not scaling functions and wavelets satisfy certain orthogonality relations. For more detail we refer to [12].

Finally we recall the concept of uniform stability for a wavelet construction e.g. [2, 6]. We say that $\{\Phi^j\}_{j\in\mathbb{N}_0}$ ($\{\Psi^j\}_{j\in\mathbb{N}_0}$) is *uniformly stable* if there exist positive constants $M_1, M_2$ ($N_1, N_2$) such that

$$M_1||\mathbf{c}||_{l^2} \leq ||\Phi^j \mathbf{c}||_{L^2} \leq M_2||\mathbf{c}||_{l^2} \quad (N_1||\mathbf{d}||_{l^2} \leq ||\Psi^j \mathbf{d}||_{L^2} \leq N_2||\mathbf{d}||_{l^2}) \tag{2.5}$$

for all sequences $\mathbf{c} \in \mathbb{R}^{v(j)}$ ($\mathbf{d} \in \mathbb{R}^{w(j)}$) and $j \in \mathbb{N}_0$.

A stronger condition of stability is Riesz stability, see e.g. [2, 4].

**3. Constructing lazy wavelets for periodic $B$-splines.** In this section we describe a method for constructing biorthogonal wavelets for periodic $B$-splines. We will obtain biorthogonal wavelets with banded synthesis matrices $P^j, Q^j$ and analysis matrices $A^j, B^j$. Therefore analysis and synthesis can be done in linear time.

$$
\text{m rows}\ \left[\begin{array}{cccc}
\vdots & & & \\
0\ \}\, o & & & \\
v_0 & \vdots & & \\
\vdots & 0\ \}\, s & & \\
\vdots & v_0 & \vdots & \ddots \\
v_l & \vdots & 0\ \}\, s & \\
0 & \vdots & v_0 & \\
\vdots & v_l & \vdots & \ddots \\
 & 0 & \vdots & \\
 & \vdots & v_l & \ddots \\
 & & 0 & \\
 & & \vdots & \ddots
\end{array}\right]
$$

$n$ columns

FIG. 3.1. $\mathrm{PBM}(m, n, o, s, [v_0, \ldots, v_l])$.

**3.1. Periodic $B$-splines.** We consider 1-periodic uniform B-splines of degree $d \in \mathbb{N}_0$. Let $V^j$ be the space spanned by the 1-periodic B-splines that are constructed from the knot sequence

$$(t_0^j, \ldots, t_{2^j(d+1)-1}^j) = \frac{1}{2^j}(0, \frac{1}{d+1}, \frac{2}{d+1}, \ldots, \frac{2^j(d+1)-1}{d+1}). \tag{3.1}$$

Then we have $V^0 \subset V^1 \subset V^2 \subset \ldots$ and $\dim V^j = 2^j(d+1)$.

DEFINITION 1. *Let $m, n, o, s, l \in \mathbb{N}_0$ such that $m = sn$. Let $v = [v_0, v_1, \ldots, v_l] \in \mathbb{R}^{l+1}$. We denote by $\mathrm{PBM}(m, n, o, s, [v_0, \ldots, v_l])$ an $m \times n$ periodic band matrix $[a_{i,j}]_{i=0,\ldots,m-1}^{j=0,\ldots,n-1}$ with offset $o$, shift $s$ and generic column $[v_0, \ldots, v_l]$ such that*

$$
\begin{aligned}
a_{(o+k\cdot s) \bmod m, k} &= v_0 \\
a_{(o+1+k\cdot s) \bmod m, k} &= v_1 \\
&\vdots \\
a_{(o+l+k\cdot s) \bmod m, k} &= v_l
\end{aligned}
$$

*for $k \in \{0, 1, \ldots, n-1\}$, while the remaining matrix entries $a_{i,j}$ vanish, see Figure 3.1.*

EXAMPLE 1.

$$\mathrm{PBM}(6, 3, 1, 2, [1, 2]) = \begin{bmatrix} 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}^T,$$

$$\mathrm{PBM}(6, 3, -1, 2, [1, 2]) = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 \end{bmatrix}^T.$$

**3.2. Lazy spline wavelets of degree 1.** For the case $d = 1$ one gets a biorthogonal wavelet construction by choosing the wavelets $\psi_i^{j-1}$ for $W^{j-1}$ as

$$\psi_i^{j-1} := \phi_{2i}^j \tag{3.2}$$

where $\{\phi_k^j\}_{k=0}^{2^j(d+1)-1}$ is the basis of $V^j$, see [12, 14]. Because the wavelets for $W^{j-1}$ are only a subset of the functions $\phi_i^j$ for $V^j$ and therefore nothing has to be done to compute them, these wavelets have been called *lazy wavelets* by Sweldens [14]. The synthesis filters $P^j, Q^j$ and the analysis filters $A^j, B^j$ are the periodic band matrices

$$P^j = \mathrm{PBM}(2^{j+1}, 2^j, 0, 2, [\frac{1}{2}, 1, \frac{1}{2}]), Q^j = \mathrm{PBM}(2^{j+1}, 2^j, 0, 2, [1]),$$

$$A^j = \mathrm{PBM}(2^{j+1}, 2^j, 1, 2, [1])^T, B^j = \mathrm{PBM}(2^{j+1}, 2^j, -1, 2, [-\frac{1}{2}, 1, -\frac{1}{2}])^T.$$

Now we extend this construction to higher degrees.

**3.3. Lazy spline wavelets of degree $d \geq 1$.** We have 1-periodic uniform $B$-splines of degree $d \in \mathbb{N}_0$. Let $k = d + 1$ the order of these $B$-splines. We use knot insertion for $B$-splines (cf. [3]) to get the refinement matrix

$$P^j = \frac{1}{2^{k-1}} \mathrm{PBM}(2^j k, 2^{j-1} k, 0, 2, [\binom{k}{0}, \binom{k}{1}, \ldots, \binom{k}{k}]).$$

We consider an auxiliary matrix $\bar{P}^j$ which we get from $P^j$ by an index shift. In the first step we construct matrices $\bar{Q}^j, \bar{A}^j, \bar{B}^j$ such that $\bar{A}^j \cdot \bar{P}^j = I$, $\bar{A}^j \cdot \bar{Q}^j = 0$, $\bar{B}^j \cdot \bar{P}^j = 0$ and $\bar{B}^j \cdot \bar{Q}^j = I$. In the second step we get the matrices $Q^j, A^j$ and $B^j$ from $\bar{Q}^j, \bar{A}^j, \bar{B}^j$ by an index shift.

For our construction we will distinguish two cases:

- case: $d$ is odd.
  Let $\bar{P}^j = \mathrm{PBM}(2^j k, 2^j k, -1, 1, [1]) \cdot P^j$,

  $$\bar{P}^j = \frac{1}{2^{k-1}} \mathrm{PBM}(2^j k, 2^{j-1} k, -1, 2, [\binom{k}{0}, \binom{k}{1}, \ldots, \binom{k}{k}]).$$

  We choose

  $$\bar{B}^j = \frac{1}{2^{k-1}} \mathrm{PBM}(2^j k, 2^{j-1} k, 0, 2, [\binom{k}{0}, -\binom{k}{1}, \binom{k}{2}, \ldots, \binom{k}{k}])^T.$$

  $\bar{B}^j$ satisfies $\bar{B}^j \cdot \bar{P}^j = 0$. Now we construct a matrix

  $$\bar{Q}^j = \mathrm{PBM}(2^j k, 2^{j-1} k, 1, 2, [c_0, c_1, \ldots, c_{k-2}]).$$

  Since $\bar{Q}^j$ has to satisfy $\bar{B}^j \cdot \bar{Q}^j = I$, we compute the coefficients of $\bar{Q}^j$ by solving this system of linear equations. This is equivalent to

  $$\begin{bmatrix} \binom{k}{1} & -\binom{k}{2} & \binom{k}{3} & -\binom{k}{4} & \cdots & -\binom{k}{k-2} & \binom{k}{k-1} \\ 0 & -\binom{k}{0} & \binom{k}{1} & -\binom{k}{2} & \cdots & -\binom{k}{k-4} & \binom{k}{k-3} \\ 0 & 0 & 0 & -\binom{k}{0} & \cdots & -\binom{k}{k-6} & \binom{k}{k-5} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\ \binom{k}{5} & -\binom{k}{6} & \binom{k}{7} & -\binom{k}{8} & \cdots & 0 & 0 \\ \binom{k}{3} & -\binom{k}{4} & \binom{k}{5} & -\binom{k}{6} & \cdots & -\binom{k}{k} & 0 \end{bmatrix} \cdot \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{k-3} \\ c_{k-2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \tag{3.3}$$

(We consider the system which we get by multiplying each row of $\bar{B}^j$ with the first column of $\bar{Q}^j$.) This system of linear equations has a $(k-1)\times(k-1)$-coefficient matrix which is regular. Therefore we get a unique solution $c_i$. Furthermore we see that the coefficients $c_i$ do not depend on $j$. Now let $\bar{A}^j$ be the matrix of the following form:

$$\bar{A}^j = \mathrm{PBM}(2^j k, 2^{j-1}k, 0, 2, [-c_{k-2}, c_{k-3}, -c_{k-4}, \dots, c_1, -c_0])^T.$$

Because of this choice of $\bar{A}^j$ it is assured that $\bar{A}^j \cdot \bar{Q}^j = 0$.
In addition to $\bar{A}^j \cdot \bar{Q}^j = 0$, $\bar{B}^j \cdot \bar{Q}^j = I$ and $\bar{B}^j \cdot \bar{P}^j = 0$, the matrix $\bar{A}^j$ satisfies $\bar{A}^j \cdot \bar{P}^j = I$, which can be simplified to equation (3.3).
Finally we choose $Q^j = \mathrm{PBM}(2^j k, 2^j k, -1, 1, [1])\cdot\bar{Q}^j$, $A^j = \bar{A}^j\cdot\mathrm{PBM}(2^j k, 2^j k, -1, 1, [1])$ and $B^j = \bar{B}^j\cdot\mathrm{PBM}(2^j k, 2^j k, 1, 1, [1])$. This choice guarantees that $A^j \cdot P^j = I$, $A^j \cdot Q^j = 0, B^j \cdot P^j = 0$ and $B^j \cdot Q^j = I$.

- case: $d$ is even.
  This case works in a similiar way. Let $\bar{P}^j = \mathrm{PBM}(2^j k, 2^j k, -2, 1, [1]) \cdot P^j$. Therefore

$$\bar{P}^j = \frac{1}{2^{k-1}}\mathrm{PBM}(2^j k, 2^{j-1}k, -2, 2, [\binom{k}{0}, \binom{k}{1}, \dots, \binom{k}{k}]).$$

We choose $\bar{B}^j$ as

$$\bar{B}^j = \frac{1}{2^{k-1}}\mathrm{PBM}(2^j k, 2^{j-1}k, 0, 2, [-\binom{k}{0}, \binom{k}{1}, -\binom{k}{2}, \dots, \binom{k}{k}])^T.$$

To construct a matrix

$$\bar{Q}^j = \mathrm{PBM}(2^j k, 2^{j-1}k, 0, 2, [c_0, c_1, \dots, c_{k-2}]),$$

we have to solve $\bar{B}^j \cdot \bar{Q}^j = I$, which is equivalent to solving the following system

$$\begin{bmatrix} -\binom{k}{0} & \binom{k}{1} & -\binom{k}{2} & \cdots & -\binom{k}{k-3} & \binom{k}{k-2} \\ 0 & 0 & -\binom{k}{1} & \cdots & -\binom{k}{k-5} & \binom{k}{k-4} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\ -\binom{k}{5} & \binom{k}{6} & -\binom{k}{7} & \cdots & 0 & 0 \\ -\binom{k}{3} & \binom{k}{4} & -\binom{k}{5} & \cdots & -\binom{k}{k-1} & \binom{k}{k} \end{bmatrix} \cdot \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{k-2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{3.4}$$

(It suffices to consider the system which we get by multiplying each row of $\bar{B}^j$ with the first column of $\bar{Q}^j$.) This system has again a unique solution which is the same for all levels $j$. Now let

$$\bar{A}^j = \mathrm{PBM}(2^j k, 2^{j-1}k, 0, 2, [c_{k-2}, -c_{k-3}, c_{k-4}, \dots, -c_0])^T.$$

Then $\bar{A}^j$ and $\bar{B}^j$ satisfy $\bar{A}^j \cdot \bar{Q}^j = 0$ and $\bar{B}^j \cdot \bar{P}^j = 0$. Furthermore we have chosen $\bar{Q}^j$ in such a way that $\bar{B}^j \cdot \bar{Q}^j = I$. Now it remains to show that $\bar{A}^j \cdot \bar{P}^j = I$ which can be simplified to equation (3.4). Finally we choose $Q^j = \bar{Q}^j$ and $B^j = \bar{B}^j$. In addition let $A^j = \bar{A}^j \cdot \mathrm{PBM}(2^j k, 2^j k, -2, 1, [1])$.

Remark 1. If $d$ is odd, the constructed biorthogonal wavelets are symmetric, but if $d$ is even, they are non-symmetric.

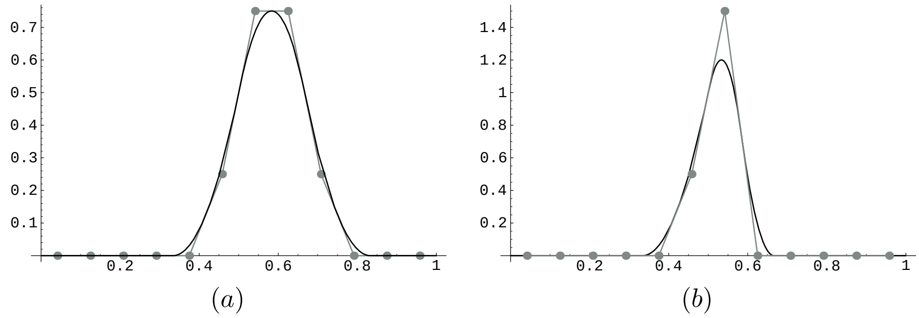



FIG. 3.2. *1-periodic uniform B-splines of degree 2: (a) the scaling functions $\phi_i^j$ and (b) the lazy wavelets $\psi_i^j$ for $j = 1$ with control points and control polygon (grey).*

EXAMPLE 2. Let $d = 2$. Then we get

$$P^j = \frac{1}{4}\mathrm{PBM}(3 \cdot 2^j, 3 \cdot 2^{j-1}, 0, 2, [1, 3, 3, 1]),$$

$$Q^j = \frac{1}{4}\mathrm{PBM}(3 \cdot 2^j, 3 \cdot 2^{j-1}, 0, 2, [2, 6]),$$

$$A^j = \frac{1}{4}\mathrm{PBM}(3 \cdot 2^j, 3 \cdot 2^{j-1}, 2, 2, [6, -2])^T,$$

$$B^j = \frac{1}{4}\mathrm{PBM}(3 \cdot 2^j, 3 \cdot 2^{j-1}, 0, 2, [-1, 3, -3, 1])^T.$$

The wavelets and scaling functions are shown in Figure 3.2.

EXAMPLE 3. For $d = 3$ we have the following biorthogonal wavelet construction given by matrices $P^j, Q^j, A^j$ and $B^j$:

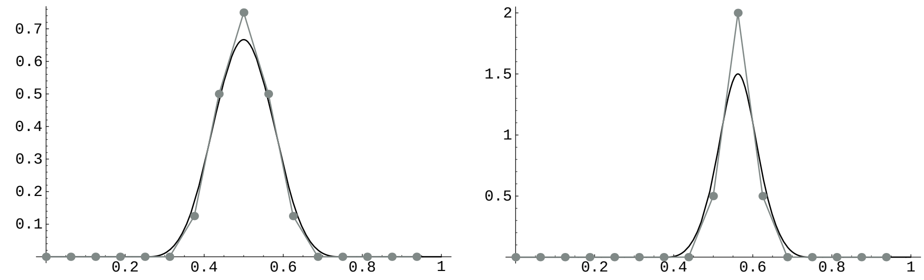
$$P^j = \frac{1}{8}\mathrm{PBM}(2^{j+2}, 2^{j+1}, 0, 2, [1, 4, 6, 4, 1]),$$

$$Q^j = \frac{1}{8}\mathrm{PBM}(2^{j+2}, 2^{j+1}, 0, 2, [4, 16, 4]),$$

$$A^j = \frac{1}{8}\mathrm{PBM}(2^{j+2}, 2^{j+1}, 1, 2, [-4, 16, -4])^T,$$

$$B^j = \frac{1}{8}\mathrm{PBM}(2^{j+2}, 2^{j+1}, -1, 2, [1, -4, 16, -4, 1])^T.$$

The wavelets and scaling functions are shown in Figure 3.3.




$(a)$ $(b)$

FIG. 3.3. 1*-periodic uniform B-splines of degree* 3*:* (*a*) *the scaling functions* $\phi_i^j$ *and* (*b*) *the lazy wavelets* $\psi_i^j$ *for* $j = 1$ *with control points and control polygon* (*grey*)*.*

**3.4. Stability of lazy wavelets.** For the uniform stability of the lazy wavelets we get the following proposition.

PROPOSITION 1. *Let* $d \in \mathbb{N}_0$ *and* $\{\Phi^j\}_{j\in\mathbb{N}_0}$ *be the* 1*-periodic uniform* $B$*-splines of degree* $d$*. Let* $\{\Psi^j\}_{j\in\mathbb{N}_0}$ *be the corresponding lazy wavelets from subsection* 3.3*. Then* $\{\Phi^j \cup \Psi^j\}_{j\in\mathbb{N}_0}$ *is uniformly stable.*

*Proof.* The $B$-splines are uniformly stable. Therefore we have positive constants $M_1, M_2$ such that

$$M_1||\mathbf{c}^j||_{l^2} \le ||\Phi^j\mathbf{c}^j||_{L^2} \le M_2||\mathbf{c}^j||_{l^2} \tag{3.5}$$

for all sequences $\mathbf{c}^j \in \mathbb{R}^{v(j)}$ and $j \in \mathbb{N}_0$.

We denote the entries of the matrices $P^j$ and $Q^j$ by $p^j_{k,l}$ and $q^j_{k,l}$. Let $m_1^j := \min(\min_{k,l:p^j_{k,l}\neq 0}|p^j_{k,l}|, \min_{k,l:q^j_{k,l}\neq 0}|q^j_{k,l}|)$, $m_2^j := \max(\max_{k,l}|p^j_{k,l}|, \max_{k,l}|q^j_{k,l}|)$ and $n^j := \max(\text{bandwidth of } P^j, \text{bandwidth of } Q^j)$. Due to the structure of $P^j$ and $Q^j$ we get values $m_1^j, m_2^j, n^j \in \mathbb{R}^+$ which are independent of the level $j$. Therefore we use for further computation the constants $m_1 = m_1^j, m_2 = m_2^j$ and $n = n^j$. Now we choose $j \in \mathbb{N}_0$ and a sequence $(\mathbf{c}^j, \mathbf{d}^j) \in \mathbb{R}^{v(j)+w(j)}$ arbitrary but fixed. Then we have

$$\begin{aligned}||\Phi^j\mathbf{c}^j + \Psi^j\mathbf{d}^j||_{L^2} &= ||\Phi^{j+1}P^{j+1}\mathbf{c}^j + \Phi^{j+1}Q^{j+1}\mathbf{d}^j||_{L^2}\\ &\le ||\Phi^{j+1}(P^{j+1}\mathbf{c}^j + Q^{j+1}\mathbf{d}^j)||_{L^2}\\ &\le M_2||P^{j+1}\mathbf{c}^j + Q^{j+1}\mathbf{d}^j||_{l^2}\\ &\le M_2m_2n||(\mathbf{c}^j,\mathbf{d}^j)||_{l^2}.\end{aligned}$$

On the other hand,

$$||\Phi^j\mathbf{c}^j + \Psi^j\mathbf{d}^j||_{L^2} \ge M_1||P^{j+1}\mathbf{c}^j + Q^{j+1}\mathbf{d}^j||_{l^2} \ge M_1m_1||(\mathbf{c}^j,\mathbf{d}^j)||_{l^2}. \tag{3.6}$$

Therefore we get positive constants $S_1, S_2$ with $S_1 = M_1m_1$ and $S_2 = M_2m_2n$ such that

$$S_1||(\mathbf{c}^j,\mathbf{d}^j)||_{l^2} \le ||\Phi^j\mathbf{c}^j + \Psi^j\mathbf{d}^j||_{L^2} \le S_2||(\mathbf{c}^j,\mathbf{d}^j)||_{l^2} \tag{3.7}$$

for all sequences $(\mathbf{c}^j, \mathbf{d}^j) \in \mathbb{R}^{v(j)+w(j)}$ and for $j \in \mathbb{N}_0$. □

Numerical experiments indicate that Riesz stability for the lazy wavelets is not to be expected (see the results in Table 3.1 and Remark 2.4 in [2]).

TABLE 3.1
*Numerical results for $||T^j||$ and $||(T^j)^{-1}||$ for $d = 2, d = 3$. ($T^j$ is the transformation matrix that takes $\mathbf{d}^{(j)} = (\mathbf{c}^0, \mathbf{d}^0, \ldots, \mathbf{d}^{j-1})$ into $\mathbf{c}^j$.)*

| | $d = 2$ | | $d = 3$ | |
|---|---|---|---|---|
| Level $j$ | $\|\|T^j\|\|$ | $\|\|(T^j)^{-1}\|\|$ | $\|\|T^j\|\|$ | $\|\|(T^j)^{-1}\|\|$ |
| 1 | 2.06532 | 2.06532 | 2.61803 | 2.61803 |
| 2 | 3.19532 | 3.2996 | 4.20653 | 5.28273 |
| 3 | 4.71395 | 5.29472 | 6.29703 | 11.204 |
| 4 | 6.80634 | 8.409 | 9.14782 | 23.2115 |
| 5 | 9.72547 | 13.3217 | 13.0304 | 48.5025 |

**4. Constructing weighted wavelets for periodic $B$-splines.** First we recall the lifting scheme, which is a general framework for the construction of wavelets (cf. [10] and [12]). We introduce weighted wavelets which are wavelets constructed with the help of lifting and a weighted inner product. Furthermore we consider the construction of weighted wavelets for 1-periodic $B$-splines.

**4.1. Lifting Scheme.** In this subsection we use the standard inner product namely $\langle f|g\rangle := \int_0^1 f(x) \cdot g(x)dx$. Lifting is a method for constructing biorthogonal wavelets from already existing biorthogonal wavelets defined by matrices $P^j$, $Q^j$, $A^j$ and $B^j$. According to Theorem 8 in [14] we get the synthesis and analysis matrices of the "lifted" wavelets in the following way.

$$[P^j_{\text{lift}}|Q^j_{\text{lift}}] = [P^j|Q^j - P^jS^j] \text{ and}$$
$$[\frac{A^j_{\text{lift}}}{B^j_{\text{lift}}}] = [\frac{A^j + S^jB^j}{B^j}],$$

where $S^j$ is $v(j-1) \times v(j-1)$ matrix.

Depending on the choice of $S^j$ we can construct biorthogonal wavelets with different desirable properties like increased orthogonality, higher vanishing moments etc.

The following example demonstrates a possible application. We assume that we have a biorthogonal wavelet construction defined by matrices $A^j, B^j, P^j$ and $Q^j$. By choosing the coefficients $s^j_{m,n}$ of the matrix $S^j$ we try to make the decomposition $V^j = V^{j-1} \oplus W^{j-1}$ "more orthogonal". In the ideal case we would have

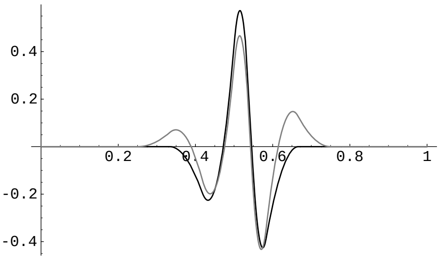
$$\langle \phi_i^{j-1}, \psi_{k,\text{lift}}^{j-1}\rangle = 0 \text{ for all } i, k \tag{4.1}$$

where $\Psi^{j-1}_{\text{lift}} = \Phi^j Q^j_{\text{lift}}$. But in the standard case the system of linear equations (4.1) is over-determined. Instead we find an approximate solution by minimizing the double-sum of the squared errors $\langle \phi_i^{j-1}, \psi_{k,\text{lift}}^{j-1}\rangle^2$,

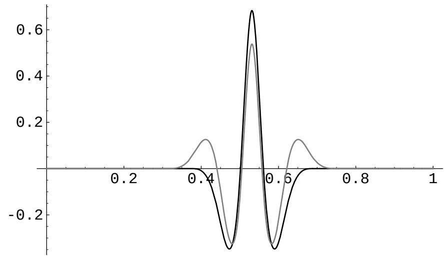
$$\arg\min_{S^j} \sum_{i=0}^{v(j-1)-1} \sum_{k=0}^{w(j-1)-1} \langle \phi_i^{j-1}, \psi_{k,\text{lift}}^{j-1}\rangle^2. \tag{4.2}$$

REMARK 2.
1. Lifting preserves the class of PBM.
2. If $S^j$ is a band matrix then the values of the coefficients along the diagonals are equal. Therefore we can solve the following minimization problem

$$\arg\min_{S^j} \sum_{k=0}^{w(j-1)-1} \langle \phi_i^{j-1}, \psi_{k,\text{lift}}^{j-1}\rangle^2 \tag{4.3}$$




(a)




(b)

FIG. 4.1. *Lifted wavelets* $\psi_i^{j-1}$ *for* 1*-periodic uniform B-splines of (a) degree* 2*; (b) degree* 3*, both for* $j = 3$ *with a band matrix* $S^j$ *with a bandwidth of* 2 *(black) and* 4 *(grey).*

for a fixed but arbitrarily chosen $i \in \{0, \dots, v(j-1)-1\}$ instead of solving (4.2). Without loss of generality we can choose $i = 0$.

**4.2. Weighted wavelets.** For certain applications of wavelets it is desirable that the ($L^2$-)error of a function $f^j$ with the lower resolution version $f^{j-1}$ is as small as possible in a certain interval or region. On the other hand the size of the error for the other parts may be less important. A possible example in 2D or 3D is that the zero-contour for implicitly defined curves or surfaces is only a small subset of the domain. In this case we want to have wavelets which act "locally" in the region of the zero-contour. We will see that we may construct such wavelets with the help of lifting and by using a weighted inner product. Here we explain the general concept and algorithm of this method for univariate functions. In the case of multivariate functions, the method can be generalized by using a tensor-product construction for the basis functions (cf. [7]).

Let $D^j \subset [0,1]$ be the region of interest and let $w^j : [0,1] \to \mathbb{R}$ such that

$$w^j(x) := \begin{cases} 1 & \text{for } x \in [0,1] \setminus D^j \\ u & \text{for } x \in D^j \end{cases}$$

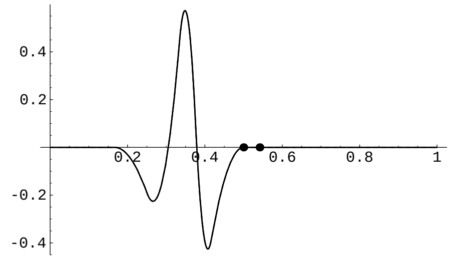
where $u \in \mathbb{R}$ and $u > 1$. For a function $w : [0,1] \to \mathbb{R}$ let $\langle \cdot | \cdot \rangle_w$ be the weighted inner product $\langle f|g \rangle_w := \int_0^1 w(x) \cdot f(x) \cdot g(x) dx$. We generate the matrix $S^j$ by solving the following minimization problem

$$\arg\min_{S^j} \sum_{i=0}^{v(j-1)-1} \sum_{k=0}^{w(j-1)-1} \langle \phi_i^{j-1}, \psi_{k,\text{lift}}^{j-1} \rangle_{w^j}^2 . \tag{4.4}$$

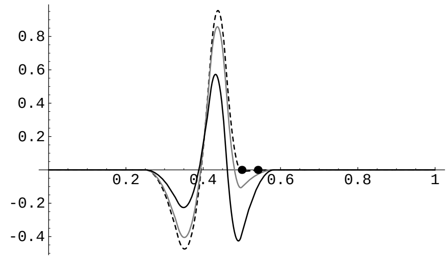
With this construction we try to get wavelets such that the $L^2$-error $|| \ f^j \ - \ f^{j-1} \ ||_{L^2}$ in a certain region is less than by using the "standard" lifted wavelets which are constructed by using the standard inner product by the lifting process. We call these wavelets, which are constructed by using a weighted inner product, *weighted wavelets.*

REMARK 3. Since the values of one column of $S^j$ have an effect on exactly one wavelet $\psi_{k,\text{lift}}^{j-1}$ we can also compute the matrix $S^j$ by solving the following minimization problem

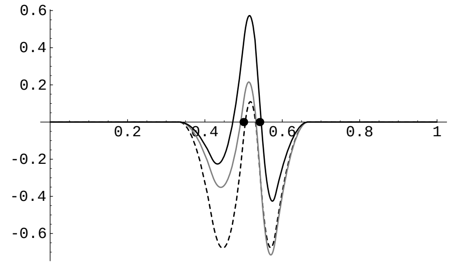
$$\arg\min_{s_{k+1}^j} \sum_{i=0}^{v(j-1)-1} \langle \phi_i^{j-1}, \psi_{k,\text{lift}}^{j-1} \rangle_{w^j}^2 \tag{4.5}$$



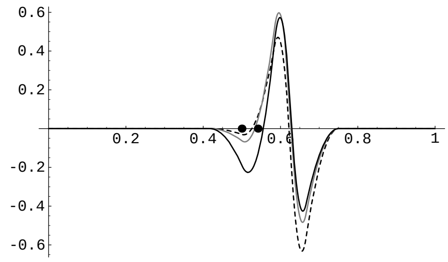

$\psi_3^2(x)$



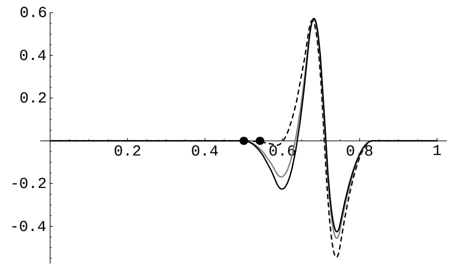

$\psi_4^2(x)$



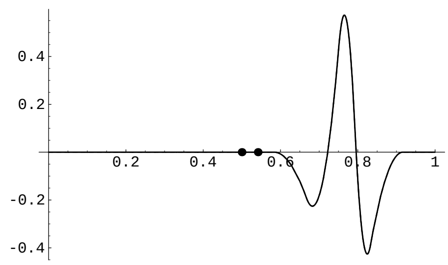

$\psi_5^2(x)$




$\psi_6^2(x)$




$\psi_7^2(x)$




$\psi_8^2(x)$

FIG. 4.2. *Weighted wavelets $\psi_i^{j-1}$ for 1-periodic uniform B-splines of degree 2 for $j = 3$ with $D^j = [\frac{3 \cdot 2^{j-1}}{3 \cdot 2^j}, \frac{3 \cdot 2^{j-1}+1}{3 \cdot 2^j}]$, $u = 1$ (black), $u = 10$ (grey), $u = 100$ (dashed) and a band matrix $S^j$ with a bandwidth of 2. For $u = 10$ and $u = 100$, only $\psi_4^2(x), \psi_5^2(x), \psi_6^2(x)$ and $\psi_7^2(x)$ differ from the "standard" lifted wavelet (u=1). The two dots mark the boundaries of $D^j$.*

for each $k \in \{0, \ldots, w(j-1)-1\}$ where $s_k^j$ is the $k$-th column of $S^j$ instead of solving the larger and more time-consuming problem (4.4). Depending on the choice of the biorthogonal wavelet construction and the region $D^j$ we get a constant number of different minimization problems (4.5). Furthermore if the region $D^j$ does not have an effect on a wavelet then we obtain for this wavelet the "standard" lifted wavelet.

**4.3. Weighted spline wavelets.** In this subsection we explain our construction of weighted wavelets for 1-periodic uniform $B$-splines of degree $d$ and give some properties of these wavelets.

As we have already explained in the last subsection we construct weighted wavelets with the help of lifting and a weighted inner product. In the case of 1-periodic uniform $B$-splines we use the lazy spline wavelets of degree $d$ from subsection 3.3 as starting biorthogonal spline wavelet construction. We choose for the region of interest $D^j$ the union of intervals with the knots $t_i^j$ as endpoints. The best choice for the length of the intervals may differ from application to application. In our case (Example 4 and 5) we have chosen the region of interest as union of intervals with a length of $\frac{2}{(d+1)2^j}$. Furthermore we choose for the weight $u$ a value between 5 and 10. Numerical experiments have shown that this is a reasonable choice for the weight $u$. If $u$ is too high, then analysis may produce additional roots. On the other hand if the weight $u$ is too low, then the effect of the weighted spline wavelets is small. Moreover we choose for $S^j$ a band matrix with a bandwidth $k \in \mathbb{N}$.

Because of these choices only a small number of different weighted spline wavelets has to be considered. All other weighted wavelets can be constructed from them with the help of translation and scaling. A concrete example for weighted spline wavelets is given in subsection 5.1 (Figure 4.2).

The wavelet transform for weighted wavelets differs from the standard wavelet transform in one point. In the analysis process, by splitting $\mathbf{c}^j$ into $\mathbf{c}^{j-1}$ and $\mathbf{d}^{j-1}$ we have also to store the information about the region of interest in level $j$. Otherwise a reconstruction of $\mathbf{c}^j$ from $\mathbf{c}^{j-1}$ and $\mathbf{d}^{j-1}$ is not possible anymore. The implementation of these wavelets can be done in the usual way for wavelets but with a pointer to the region of interest.

Furthermore the computional time for analysis and synthesis is comparable with the "standard" spline wavelet case. If we have precomputed the analysis and synthesis matrices for the "'standard" lifted spline wavelets, then we get the analysis and synthesis matrices for the different weighted spline wavelets by replacing only some of the columns or rows of the precomputed matrices by corresponding precomputed columns or rows.

**4.4. Stability of weighted spline wavelets.** The following proposition shows that the weighted spline wavelets are uniformly stable.

PROPOSITION 2. *Let $d \in \mathbb{N}_0, k \in \mathbb{N}$ and $\{\Phi^j\}_{j\in\mathbb{N}_0}$ be the 1-periodic uniform B-splines of degree $d$. Let $\{\Psi^j\}_{j\in\mathbb{N}_0}$ be a corresponding weighted spline wavelet construction with band matrices $S^j$ with a bandwidth of $k$. Then $\{\Phi^j \cup \Psi^j\}_{j\in\mathbb{N}_0}$ is uniformly stable.*

*Proof.* We denote the entries of the corresponding synthesis matrices $P^j$ and $Q^j$ by $p^j_{k,l}$ and $q^j_{k,l}$. Let $m_1 = \min_j(\min_{k,l:p^j_{k,l}\neq 0} |p^j_{k,l}|, \min_{k,l:q^j_{k,l}\neq 0} |q^j_{k,l}|)$, $m_2 = \max_j(\max_{k,l} |p^j_{k,l}|, \max_{k,l} |q^j_{k,l}|)$ and $n = \max_j(\text{bandwidth of } P^j, \text{ bandwidth of } Q^j)$. Then we can show like in the proof of Proposition 1 that $\{\Phi^j \cup \Psi^j\}_{j\in\mathbb{N}_0}$ is uniformly stable for $S_1 = M_1 m_1$ and $S_2 = M_2 m_2 n$, where $M_1, M_2$ are the constants for the uniform stability of $\{\Phi^j\}_{j\in\mathbb{N}_0}$. □

But again we cannot expect Riesz stability (see results in Table 4.1 and Remark 2.4 in [2]).

**5. Examples.** In this section we give concrete examples for weighted wavelets for 1-periodic uniform $B$-splines. Furthermore we compare the weighted spline wavelets with the "standard" lifted ones.

**5.1. An example.** We consider the biorthogonal wavelet construction for 1-periodic uniform $B$-splines of degree 2 of Example 2 (lazy wavelets of degree 2). Let $D^j$ be the interval $[\frac{3\cdot 2^{j-1}}{3\cdot 2^j}, \frac{3\cdot 2^{j-1}+1}{3\cdot 2^j}]$. We choose $S^j$ as a $3\cdot 2^{j-1} \times 3\cdot 2^{j-1}$-matrix of

TABLE 4.1

*Numerical results for $||T^j||$ and $||(T^j)^{-1}||$ for weighted spline wavelets of degree $d = 2, d = 3$, band matrices $S^j$ with a bandwith of 2, a weight $u = 100$ and weighted intervals $D^j = [\frac{(d+1)\cdot 2^{j-1}}{(d+1)\cdot 2^j}, \frac{(d+1)\cdot 2^{j-1}+1}{(d+1)\cdot 2^j}]$ ($T^j$ is the transformation matrix that takes $\mathbf{d}^{(j)} = (\mathbf{c}^0, \mathbf{d}^0, \ldots, \mathbf{d}^{j-1})$ into $\mathbf{c}^j$.)*

| | $d=2$ | | $d=3$ | |
|---|---|---|---|---|
| Level $j$ | $\|\|T^j\|\|$ | $\|\|(T^j)^{-1}\|\|$ | $\|\|T^j\|\|$ | $\|\|(T^j)^{-1}\|\|$ |
| 1 | 2.22859 | 2.25028 | 2.63551 | 2.69717 |
| 2 | 2.94337 | 3.16852 | 3.23709 | 6.09644 |
| 3 | 4.09352 | 3.79109 | 4.44308 | 12.7002 |
| 4 | 5.76998 | 4.83886 | 6.24833 | 25.0518 |
| 5 | 8.15485 | 6.14869 | 8.8217 | 47.6374 |

the following form:

$$S^j = \begin{bmatrix} s_2 & s_3 & & 0 \\ 0 & s_4 & & 0 \\ 0 & 0 & \ddots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & s_{3\cdot 2^j-1} \\ s_1 & 0 & & s_{3\cdot 2^j} \end{bmatrix}$$

As first we consider the "standard" lifted wavelets. By solving the minimization problem (4.2) we get the following results for $S^j$.

$S^1 = \text{PBM}(3, 3, -1, 1, [0.288, 0.788]), S^j = \text{PBM}(3 \cdot 2^{j-1}, 3 \cdot 2^{j-1}, -1, 1, [0.769, 0.379])$

for $j \geq 2$. The obtained wavelets for $j = 3$ can be seen in Figure 4.1 $(a)$.

Now we consider the construction of weighted wavelets. We get the following results for $S^j$.

- Case $u = 10$

$$S^1 = \begin{bmatrix} 0.442 & 0.393 & 0 \\ 0 & 0.953 & -0.08 \\ 0.451 & 0 & 0.946 \end{bmatrix}.$$

For $j \geq 2$ we have $S^j = \text{PBM}(3 \cdot 2^{j-1}, 3 \cdot 2^{j-1}, -1, 1, [0.769, 0.379])$, except for $s_{3\cdot 2^{j-1}-3} = 0.672$, $s_{3\cdot 2^{j-1}-2} = 0.194$, $s_{3\cdot 2^{j-1}-1} = 0.562$, $s_{3\cdot 2^{j-1}} = 1.227$, $s_{3\cdot 2^{j-1}+1} = 0.123$, $s_{3\cdot 2^{j-1}+2} = 0.873$, $s_{3\cdot 2^{j-1}+3} = 0.284$, $s_{3\cdot 2^{j-1}+4} = 0.823$.

- Case $u = 100$

$$S^1 = \begin{bmatrix} 0.294 & 0.135 & 0 \\ 0 & 1.663 & -0.062 \\ 1.375 & 0 & 0.512 \end{bmatrix}.$$

For $j \geq 2$ we have $S^j = \text{PBM}(3 \cdot 2^{j-1}, 3 \cdot 2^{j-1}, -1, 1, [0.769, 0.379])$, except for $s_{3\cdot 2^{j-1}-3} = 0.778$, $s_{3\cdot 2^{j-1}-2} = 0.02$, $s_{3\cdot 2^{j-1}-1} = 1.018$, $s_{3\cdot 2^{j-1}} = 1.159$, $s_{3\cdot 2^{j-1}+1} = 0.061$, $s_{3\cdot 2^{j-1}+2} = 1.11$, $s_{3\cdot 2^{j-1}+3} = 0.39$, $s_{3\cdot 2^{j-1}+4} = 0.972$.
The obtained weighted wavelets for $j = 3$ can be seen in Figure 4.2.

TABLE 5.1
*The average values of* $\gamma^j = \frac{||f^j - f_w^{j-1}||_{L^2(D^j)}}{||f^j - f_{st}^{j-1}||_{L^2(D^j)}}$ *and* $\delta^j = \frac{||f^j - f_w^{j-1}||_{L^2([0,1])}}{||f^j - f_{st}^{j-1}||_{L^2([0,1]))}}$ *of* 1000 *randomly generated functions* $f^j$ *for different levels* $j$ *and for the different values* $u$.

| $u = 10$ | | |
|---|---|---|
| Level $j$ | $\varnothing\ \gamma^j$ | $\varnothing\ \delta^j$ |
| 1 | 0.653719 | 1.11533 |
| 2 | 0.392542 | 1.12419 |
| 3 | 0.389948 | 1.06255 |
| 4 | 0.402308 | 1.03186 |
| 5 | 0.401858 | 1.01657 |

| $u = 100$ | | |
|---|---|---|
| Level $j$ | $\varnothing\ \gamma^j$ | $\varnothing\ \delta^j$ |
| 1 | 0.426994 | 1.61962 |
| 2 | 0.274159 | 1.25497 |
| 3 | 0.275978 | 1.13164 |
| 4 | 0.274846 | 1.07235 |
| 5 | 0.272631 | 1.03637 |

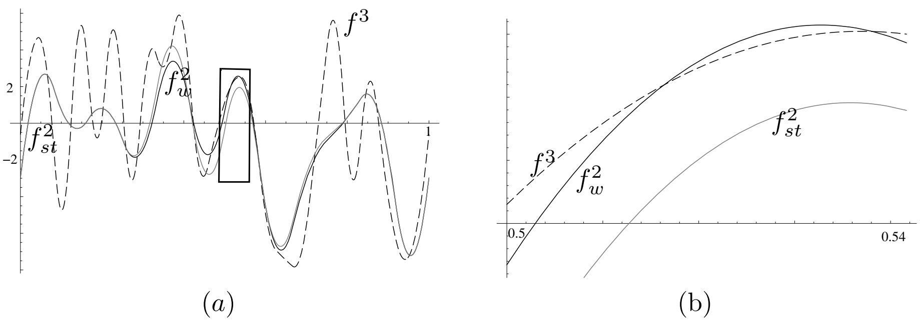



FIG. 5.1. $f_w^2$ *(black) and* $f_{st}^2$ *(grey) for* $f^3$ *(dashed) in* (*a*) *in the whole interval* $[0, 1]$ *and in* (*b*) *in the weighted interval* $D^3 = [\frac{1}{2}, \frac{13}{24}]$.

**5.2. Comparison - $L^2$-error.** Now we compare the $L^2$-error $||f^j - f^{j-1}||$ by using the different wavelet constructions. By $f_{st}^{j-1}$ we denote the resulting function from $f^j$ by applying one analysis step of the "standard" lifted wavelet construction and by $f_w^{j-1}$ we denote the function which we get from $f^j$ by applying one analysis step of the weighted wavelet construction. Table 5.1 compares the ratios of local ($\gamma^j$) and global $L^2$-errors ($\delta^j$) of 1000 randomly generated functions $f^j$. We see that the $L^2$-error $||\ f^j\ -\ f_w^{j-1}\ ||$ on $D^j$ by using weighted wavelets with $u = 10$ and $u = 100$ is only 40% and 27.5% of the $L^2$-error $||\ f^j\ -\ f_{st}^{j-1}\ ||$ on $D^j$, respectively. But on the other hand the $L^2$-error on $[0, 1]$ has increased only a little by using weighted wavelets instead of "standard" lifted wavelets.

Figure 5.1 shows the resulting functions $f_w^2$ and $f_{st}^2$ for a concrete function $f^3$ with coefficients $\mathbf{c}^3 = [5.1, -8.6, 9.1, -4, 8.5, -6.4, 5.4, 2, 7.9, -4.4, -0.4, 2.7, 2.3, -6.4, -7.4, -8.2, -0.2, 8.3, -6.2, 4.7, -3.9, -8.3, -5.4, 4.2]$ in the whole interval $[0, 1]$ and in the weighted interval $D^3 = [\frac{1}{2}, \frac{13}{24}]$.

**5.3. Comparison - Preserving roots.** In this subsection we compare how roots of a spline function are preserved by using the different wavelet constructions.

EXAMPLE 4. Let $f^4$ be a quadratic spline function with the coefficients $\mathbf{c}^4 = [2(16\text{-times}), -2(13\text{-times}), 2(19\text{-times})]$. $f^4$ has two roots $x^4 = 0.3542$ and $y^4 = 0.625$. We compare now how good these roots are preserved by using weighted wavelets and by using "standard" lifted wavelets. For constructing the "standard" lifted wavelets and the weighted wavelets we use the biorthogonal wavelet construction from Example 2 and a band matrix $S^j$ with a bandwidth of 2 for the lifting process. For the weighted wavelets we choose additionally for the weight $u = 10$ and

TABLE 5.2
*Comparison of the roots of $f^j_{st}$ and $f^j_w$ for the original function $f^4$ with two roots $x^4 = 0.3542$ and $y^4 = 0.625$ in Figure 5.2. ( $\sigma^j_{st} = |x^4 - x^j_{st}| + |y^4 - y^j_{st}|$ and $\sigma^j_w = |\ x^4\ -\ x^j_w\ | + |y^4 - y^j_w|$ for $j \in \{1,2,3\}$ )*

| "Standard" lifted wavelets | | | |
|---|---|---|---|
| Level $j$ | $x^j_{st}$ | $y^j_{st}$ | $\sigma^j_{st}$ |
| 3 | 0.35 | 0.6254 | 0.0044 |
| 2 | 0.3449 | 0.6144 | 0.0201 |
| 1 | 0.3359 | 0.6378 | 0.0309 |

| Weighted wavelets | | | |
|---|---|---|---|
| Level $j$ | $x^j_w$ | $y^j_w$ | $\sigma^j_w$ |
| 3 | 0.3517 | 0.6252 | 0.0025 |
| 2 | 0.3529 | 0.617 | 0.0095 |
| 1 | 0.3499 | 0.6248 | 0.0047 |

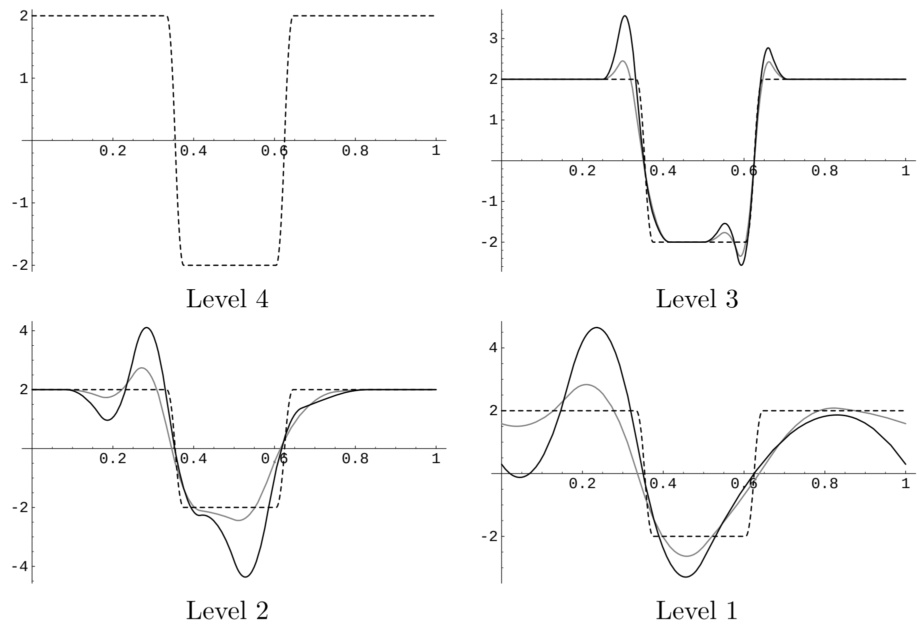



FIG. 5.2. (*Example* 4) *Comparing the resulting functions by using weighted wavelets* (*black*) *and by using "standard" lifted wavelets* (*grey*) *with the original function $f^4$* (*dashed*).

the weighted intervals as follows: $D^4 = [\frac{16}{48}, \frac{18}{48}] \cup [\frac{29}{48}, \frac{31}{48}]$, $D^3 = [\frac{8}{24}, \frac{10}{24}] \cup [\frac{14}{24}, \frac{16}{24}]$ and $D^2 = [\frac{3}{12}, \frac{5}{12}] \cup [\frac{7}{12}, \frac{9}{12}]$. The resulting roots $x^j_{st}, y^j_{st}$ for $f^j_{st}$ and the resulting roots $x^j_w, y^j_w$ for $f^j_w$ for $j \in \{1,2,3\}$ can be seen in Table 5.2. Furthermore we compute in Table 5.2 $\sigma^j_{st} = |x^4 - x^j_{st}| + |y^4 - y^j_{st}|$ and $\sigma^j_w = |x^4 - x^j_w| + |y^4 - y^j_w|$for $j \in \{1,2,3\}$ to compare the both methods. The resulting functions $f^j_{st}$, $f^j_w$ and the original function $f^4$ can be seen in Figure 5.2.

We see that in this example the weighted wavelets preserve the roots of the original function $f^4$ better than the "standard" lifted wavelets. Furthermore we can see that the distance $\sigma^j_w$ is only 50% or less of the distance $\sigma^j_{st}$.

EXAMPLE 5. Let $f^4$ be a quadratic spline function given in Figure 5.3. $f^4$ has two roots $x^4 = 0.25$ and $y^4 = 0.5509$. We want to do the same comparison like in Example 4. For this we use the same notations and the same "standard" lifted wavelets. For constructing the weighted wavelets we choose again for the weight $u = 10$ but in difference to Example 4 the weighted intervals in the following way: $D^4 = [\frac{11}{48}, \frac{13}{48}] \cup [\frac{25}{48}, \frac{27}{48}]$, $D^3 = [\frac{5}{24}, \frac{7}{24}] \cup [\frac{12}{24}, \frac{14}{24}]$ and $D^2 = [\frac{2}{12}, \frac{4}{12}] \cup [\frac{6}{12}, \frac{8}{12}]$. The

TABLE 5.3
*Comparison of the roots of* $f^j_{st}$ *and* $f^j_w$ *for the original function* $f^4$ *with two roots* $x^4 = 0.25$ *and* $y^4 = 0.5509$ *in Figure* 5.3. ( $\sigma^j_{st} = |x^4 - x^j_{st}| + |y^4 - y^j_{st}|$ *and* $\sigma^j_w = |\ x^4\ -\ x^j_w\ | + |y^4 - y^j_w|$ *for* $j \in \{1, 2, 3\}$ )

| "Standard" lifted wavelets | | | |
|---|---|---|---|
| Level $j$ | $x^j_{st}$ | $y^j_{st}$ | $\sigma^j_{st}$ |
| 3 | 0.2485 | 0.552 | 0.0026 |
| 2 | 0.2475 | 0.5694 | 0.021 |
| 1 | 0.2155 | 0.5109 | 0.0745 |

| Weighted wavelets | | | |
|---|---|---|---|
| Level $j$ | $x^j_w$ | $y^j_w$ | $\sigma^j_w$ |
| 3 | 0.2495 | 0.5515 | 0.0011 |
| 2 | 0.2497 | 0.5652 | 0.0146 |
| 1 | 0.2418 | 0.541 | 0.0181 |

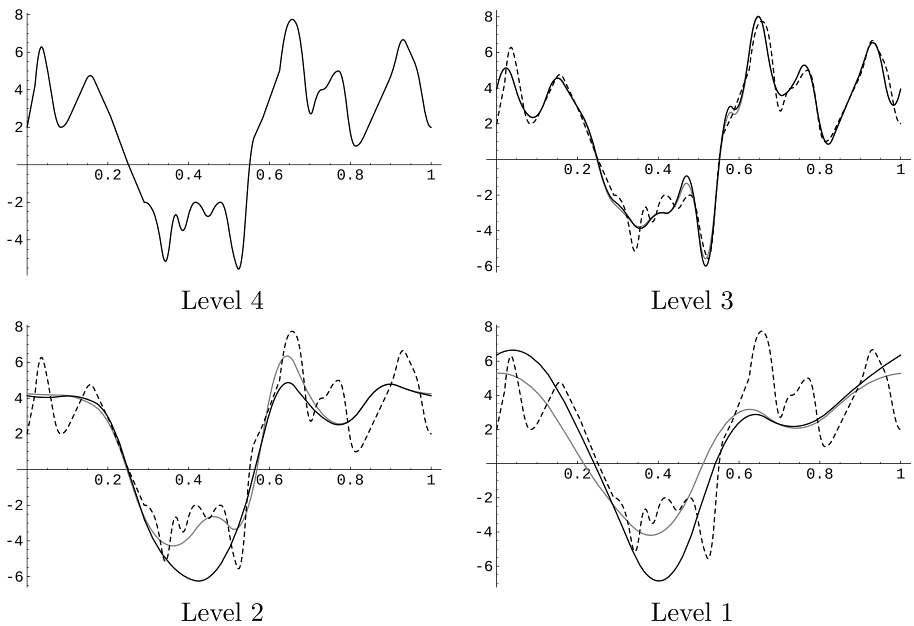



FIG. 5.3. (*Example* 5) *Comparison weighted wavelets* (*black*) *and "standard" lifted wavelets* (*grey*) *with the original function* $f^4$ (*dashed*).

results can be found in Table 5.3 and in Figure 5.3.

We see again that the weighted wavelets preserve the roots of $f^4$ better than the "standard" lifted wavelets.

For further examples (e.g. the preservation of implicitly defined algebraic spline curves) we refer to [7].

**References.**

[1] Kai Bittner. Biorthogonal spline wavelets on the interval. In *Wavelets and splines: Athens 2005*, Mod. Methods Math., pages 93–104. Nashboro Press, Brentwood, TN, 2006.

[2] J. M. Carnicer, W. Dahmen, and J. M. Peña. Local decomposition of refinable spaces and wavelets. *Appl. Comput. Harmon. Anal.*, 3(2):127–153, 1996.

[3] Charles K. Chui and Ewald Quak. Wavelets on a bounded interval. In *Numerical methods in approximation theory, Vol. 9 (Oberwolfach, 1991)*, Internat. Ser. Numer. Math., pages 53–75. Birkhäuser, Basel, 1992.

[4] Albert Cohen. *Numerical analysis of wavelet methods*, volume 32 of *Studies in Mathematics and its Applications*. North-Holland, Amsterdam, 2003.

[5] Albert Cohen, Ingrid Daubechies, and Pierre Vial. Wavelets on the interval and fast wavelet transforms. *Appl. Comput. Harmon. Anal.*, 1(1):54–81, 1993.

[6] Wolfgang Dahmen, Angela Kunoth, and Karsten Urban. Biorthogonal spline wavelets on the interval—stability and moment conditions. *Appl. Comput. Harmon. Anal.*, 6(2):132–196, 1999.

[7] Mario Kapl and Bert Jüttler. Multiresolution analysis for implicitly defined algebraic spline curves with weighted wavelets. *SFB-report.* Will be posted as SFB-report at *www.sfb013.uni-linz.ac.at*.

[8] T. Lyche and K. Mørken. Spline-wavelets of minimal support. In *Numerical methods in approximation theory, Vol. 9 (Oberwolfach, 1991)*, volume 105 of *Internat. Ser. Numer. Math.*, pages 177–194. Birkhäuser, Basel, 1992.

[9] T. Lyche, K. Mørken, and E. Quak. Theory and algorithms for nonuniform spline wavelets. In *Multivariate approximation and applications*, pages 152–187. Cambridge Univ. Press, Cambridge, 2001.

[10] Jürgen Prestin and Ewald Quak. Periodic and spline multiresolution analysis and the lifting scheme. In *Advances in multiresolution for geometric modelling*, Math. Vis., pages 369–390. Springer, Berlin, 2005.

[11] Ewald Quak and Norman Weyrich. Wavelets on the interval. In *Approximation theory, wavelets and applications (Maratea, 1994)*, volume 454 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 247–283. Kluwer Acad. Publ., Dordrecht, 1995.

[12] Eric J. Stollnitz, Tony D. DeRose, and David H. Salesin. *Wavelets For Computer Graphics: Theory and Application*. Morgan Kaufmann Publishers, Inc, first edition, 1996.

[13] W. Sweldens. Compactly supported wavelets which are biorthogonal with respect to a weighted inner product. avalaible at *http://netlib.bell-labs.com/who/wim/*, *Proceedings of the 14th Imacs World Congress.*

[14] W. Sweldens. The lifting scheme: A construction of second generation wavelets. *SIAM J. Math. Anal.*, 29(2):511–546, 1997.