Analysis of dual and dual-primal tearing and interconnecting methods in unbounded domains

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Abstract

Finite element tearing and interconnecting (FETI) methods, boundary element tearing and interconnecting (BETI) methods and the closely related dual-primal methods, FETI-DP and BETI-DP, are special iterative substructuring methods with Lagrange multipliers. For elliptic boundary value problems on bounded domains, the condition number of these methods can be rigorously bounded by $C\,((1+\log(H/h))^2$, where H is the subdomain diameter and h the mesh size. The constant C is independent of H, h and possible jumps in the coefficients of the partial differential equation.

In certain situations, one is interested to avoid artificial boundary conditions but to model the real physical behavior in an exterior domain together with a radiation condition, e. g., in electromagnetic field computations. The present paper gives a detailed analysis on several tearing and interconnecting methods for such unbounded domains. We state appropriate assumptions which result in condition number estimates similar to the one above.

Key words: FETI, BETI, BETI-DP, boundary element method, domain decomposition, iterative substructuring, exterior problems

1 Introduction

Boundary element tearing and interconnecting (BETI) and dual-primal BETI (BETI-DP) methods are robust, parallel domain decomposition methods for solving partial differential equations. The main idea of these methods goes back to the classical finite element tearing and interconnecting (FETI) method which was introduced by Farhat and Roux [34] in 1991, for a more detailed description see also [35, 85, 87]. The FETI methods belong to the class of dual iterative substructuring methods. In contrast to primal iterative substructuring, the finite element subspaces are treated separately on each subdomain including its boundary. The global continuity across subdomain interfaces is enforced by Lagrange multipliers, which leads to a saddle point problem that can be solved iteratively via its dual problem. The dual problem is symmetric positive definite on a subspace. Thus, it can be solved by a preconditioned conjugate gradient (PCG) subspace iteration, with a special preconditioner given by the method. The basic ingredients of FETI methods are just local Dirichlet- and Neumann-solvers on the subdomains, which is probably the main reason that FETI methods have become so popular.

Meanwhile, the classical FETI methods and the more recently developed dual-primal FETI (FETI-DP) methods [30, 31] and balanced domain decomposition by constraints (BDDC) techniques [21, 61] are well established in the field of robust parallel solvers for large-scale finite element equations, see, e. g., [30, 31, 36, 45, 46, 48, 74, 80, 86]. The great success of FETI, FETI-DP and BDDC methods is certainly due to their wide applicability, moderate complexity, scalability and their robustness. The latter properties are not only observed numerically, but they are approved theoretically. As a pioneering work, Mandel and Tezaur [63] gave the first convergence proof for one-level FETI methods with non-redundant Lagrange multipliers for two-dimensional elliptic problems with homogeneous coefficients. They showed that the spectral condition number of the corresponding preconditioned system is bounded by $C\,(1+\log(H/h))^\beta$, with $\beta \leq 3$, where H and h denote the usual scaling of the subdomains and the finite elements, respectively. For a special two-dimensional case, they could show that $\beta \leq 2$. Another breakthrough was the work by Klawonn and Widlund [49] (see also [48]) who introduced and analyzed new one-level FETI methods for three-dimensional elliptic problem with heterogeneous coefficients. They could proof the spectral bound $C\,(1+\log(H/h))^2$, also for redundant Lagrange multipliers (which are usually used in parallel implementations). Furthermore, assuming that the coefficients of the partial differential equation are constant on the subdomains, Klawonn and Widlund showed that the constant C is independent of possible jumps in the coefficients across subdomain interfaces when a special scaling of the preconditioner is applied. For FETI-DP methods, the same bound, $C\,(1+\log(H/h))^2$, was then shown for two-dimensional elliptic problems with homogeneous coefficients by Mandel and Tezaur [64], and for heterogeneous problems in three dimensions in a paper by Klawonn, Widlund and Dryja [50]. Finally, Brenner could prove that the bound $C\,(1+\log(H/h))^2$ is sharp for one-level FETI and FETI-DP in two dimensions [10, 11]. In other words, all these preconditioners are *quasi-optimal*. We note that the FETI methods are closely related to the balancing Neumann-Neumann methods [49]. Indeed, it was shown that the BDDC method gives the same eigenvalues as the FETI-DP method, see, e. g., [9, 62]. For a comprehensive analysis of FETI and FETI-DP methods we refer the reader to the recently published monograph [87] by Toselli and Widlund.

Recently, Langer and Steinbach have introduced boundary element tearing and interconnecting (BETI) methods as boundary element counterpart of the FETI methods [58], the coupled FETI/BETI method [59], see also [56] for some numerical results, and finally, together with Pohoaţă, dual primal BETI (BETI-DP) methods [57]. The BETI methods use boundary element based approximations of local Steklov-Poincaré operators instead of the finite element based Schur complements. The FETI preconditioners can be replaced by preconditioners derived from the corresponding boundary integral operators. Due to spectral arguments, the advantageous properties of FETI methods, such as scalability, robustness, etc., remain valid for BETI methods as well. Furthermore, inexact and data-sparse techniques are available [54].

Coupling boundary element and finite element discretizations, one can benefit from the

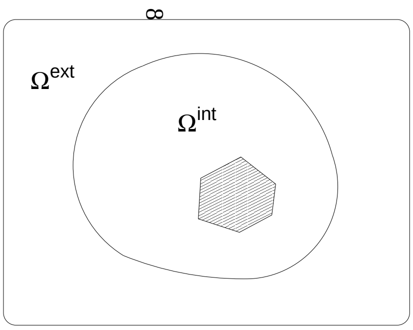

Figure 1: Illustration of the configuration of the bounded domain Ω^{int} and the unbounded domain Ω^{ext}. The shaded region is neither part of Ω^{int} nor Ω^{ext}.

advantages of both discretization techniques. For instance, in electromagnetics, source terms and nonlinearities can be treated more efficiently by the finite element method (FEM) than by the boundary element method (BEM), whereas unbounded exterior domains, moving parts and air regions can efficiently be handled by the BEM [44]. We refer to [18] for the symmetric coupling of finite and boundary elements, and to [16, 38, 42, 43, 53, 82] for using this coupling technique to construct domain decomposition solvers. In the context of coupled FETI/BETI method for nonlinear problems we mention our recent paper [55].

The main focus of the present work is the analysis of coupled FETI/BETI methods for problems including an unbounded exterior domain. In the unbounded case, there are two crucial difficulties. First, the number of the neighboring subdomains of the exterior domain can be arbitrarily large, which is totally in contrast to the usual bounded case. Secondly, we will be faced with different scalings of the local operators, what originates from the fact that the diameter of the exterior boundary can be arbitrarily larger than the diameters of the remaining subdomains. In the following, we restrict ourselves to an elliptic model problem,

$$\begin{aligned} -\nabla \cdot [\alpha(x)\, \nabla u(x)] = f(x) \qquad & \text{for } x \in \Omega^{\text{int}}\,, \\ -\alpha_0\, \Delta u(x) = 0 \qquad & \text{for } x \in \Omega^{\text{ext}}\,, \end{aligned}$$

together with the usual transmission conditions, a suitable radiation condition, and possibly some boundary conditions on the interior boundary. Here, Ω^{int} is bounded, Ω^{ext} is the complement of a bounded domain, and the boundaries of Ω^{int} and Ω^{ext} have a common part, see Figure 1.

One of the key tools of non-overlapping domain decomposition methods is the local *Steklov-Poincaré* operator describing the relation between the Dirichlet and the Neumann data on a local subdomain boundary. In general, these Steklov-Poincaré operators become singular on *floating* subdomains, which have no contribution from the Dirichlet boundary and thus no Dirichlet boundary conditions which make the solution unique, cf. Figure 2, left. There are two main directions to cope with this problem. On one hand, the *one-level* FETI and BETI methods introduce a special projection dealing with the corresponding kernels, resulting in a special kind of coarse space solver. On the other hand, the FETI-DP and BETI-DP methods work with a *primal space*, which plays the role of coarse space handling the global information exchange and ensures that the local operators on the remaining dual space are

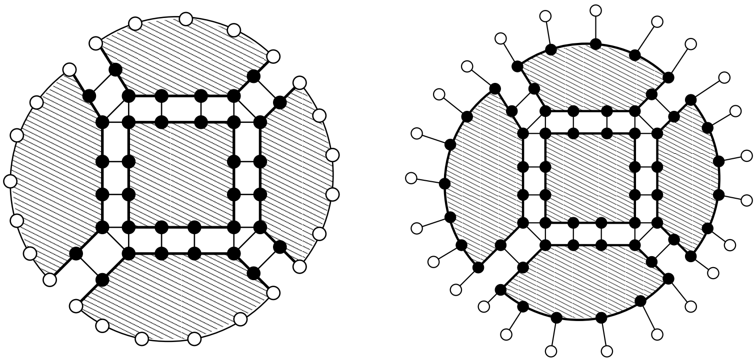

Figure 2: Illustration of three BETI schemes. Left: Standard formulation with one floating subdomain; middle: All-floating formulation; right: BETI-DP formulation with primal vertices. Shaded regions: Subdomains; •–•: Lagrange multipliers; ∘: Dirichlet boundary conditions.

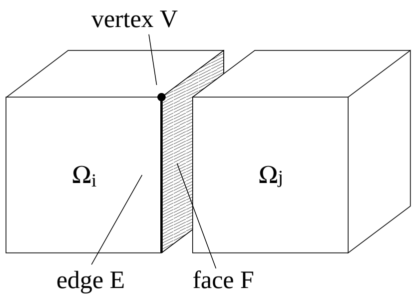

Figure 3: Illustration of a subdomain face, edge and vertex in three dimensions.

regular.

For the construction of the projections occurring in the one-level methods, the knowledge of the local kernels is of utmost importance. Whereas a local kernel corresponding to the Laplace operator can at most be one-dimensional, the operator occurring in linear elastiticy can produce kernels of dimension one up to six, depending on the boundary conditions. In order to cope with this problem, a variant of the one-level method has been introduced independently by Dostál, Horák and Kučera [23] (called *total* FETI) and, for the boundary element case, by Of [70, 71, 72] (called *all-floating* BETI). The idea of these total or all-floating methods is to impose the Dirichlet boundary conditions in terms of additional Lagrange multipliers such that each local operator produces the same kind of kernel, being of the maximal dimension, see also Figure 2, middle.

The main idea of FETI-DP and BETI-DP methods is to introduce *primal* unknowns, which are not "torn" by Lagrange multipliers (cf. Figure 2, right). For the two-dimensional Possion equation, it suffices to use the unknowns at cross points as such primal variables, whereas in three dimensions at least some edge averages have to be added to the primal spaces, cf. [50, 57, 87].

A crucial tool of the analysis of FETI and FETI-DP methods (cf., e. g., [87]) are rather technical Sobolev-type inequalities for finite element functions, in particular estimates for subdomain faces and edges. A *subdomain face* is the open intersection of two adjacent subdomains of a domain decomposition, whereas a *subdomain edge* belongs to more than two subdomains and is part of the boundary of subdomain faces. Finally, the *subdomain vertices* are the endpoints of subdomain edges (cf. Figure 3). The following estimate is of special interest in the FETI analysis. We consider a finite element function v_h on a subdomain face F and the discrete cutoff function θ_F (also defined on F) which vanishes on ∂F and is one on all inner nodes of the face. Then we have that

$$|I^h(\theta_F\, v_h)|^2_{H^{1/2}_{00}(F)} \leq C\,(1+\log(H_F/h))^2 \left\{ |v_h|^2_{H^{1/2}(F)} + \frac{1}{H_F}\|v_h\|^2_{L_2(F)} \right\},$$

for all finite element functions v_h defined on the face F. Here, I^h is the nodal interpolation onto the finite element space, H_F is the face diameter, and h is the mesh size. The precise definition of the Sobolev spaces $H^{1/2}_{00}(F)$ and $H^{1/2}(F)$ as well as the corresponding (semi-)norms will be given later. Many contributions to estimates of this type have been worked out by Bramble, Pasciak and Schatz [6, 7], Bramble and Xu [8], Dryja [24], and in works by Bjørstad and Widlund [4], Dryja, Smith and Widlund [25], and Dryja and Widlund [26, 27]. A comprehensive explanation together with recent results is given by Toselli and Widlund in [87, Chapter 4].

Whenever applying the face estimate in an interior subdomain Ω_i, it is important to note that Ω_i has only a fixed, typically small number of faces and that the face diameter H_F is comparable to the diameter of Ω_i. Hence, we obtain

$$\sum_{F\subset\partial\Omega_i} |v_h|^2_{H^{1/2}(F)} \leq C\Big\{ |v_h|^2_{H^{1/2}(\partial\Omega_i)} + \frac{1}{\operatorname{diam}\Omega_i}\|v_h\|^2_{L_2(\partial\Omega_i)} \Big\}.$$

The scaling factor in front of the L_2-term is essential for the theory of domain decomposition methods, since any constants in estimates should in particular be independent of the subdomain size.

In the case of an unbounded domain, say Ω_0, arbitrary many subdomains can touch its boundary Γ_0, i.e., arbitrary many subdomain faces of the unbounded domain can occur. Here, a first approach might be to use Sobolev interpolation theory. An argument going back to von Petersdorff [89] shows that for all $v \in H^{1/2}(\Gamma_0)$,

$$\sum_{F\subset\Gamma_0} \|v\|^2_{H^{1/2}(F)} \leq C\|v\|^2_{H^{1/2}(\Gamma_0)},$$

where the constant C depends only on the shape of Γ_0. However, we have to read this result for the case that the diameter of Γ_0, which we denote by H_0, is $\mathcal{O}(1)$. By a simple coordinate transformation we obtain the following estimate for arbitrary boundaries Γ_0,

$$\sum_{F\subset\Gamma_0} \left\{ |v|^2_{H^{1/2}(F)} + \frac{1}{H_F}\,\|v\|^2_{L_2(F)} \right\} \leq C\,\frac{H_0}{H_F}\left\{ |v|^2_{H^{1/2}(\Gamma_0)} + \frac{1}{H_0}\,\|v\|^2_{L_2(\Gamma_0)} \right\}.$$

Indeed, the norm induced by the exterior Steklov-Poincaré operator scales exactly like the full $H^{1/2}$-norm inside the brackets on the right hand side. In our analysis we need an estimate similar to above one, but–if possible–robust with respect to the factor H_0/H_F. In this context, there is a strong relation to the theory of iterative substructuring with boundary element methods, see, e. g., the works by Heuer and Stephan [40], Stephan and Tran [84], Ainsworth, McLean and Tran [3], and Ainsworth and Guo [2], only to name a few. There, comparable robust estimates were shown using special coarse spaces.

In the present work, we show how to achieve comparable robustness of the preconditioners of one-level BETI methods under certain restrictions on the geometry and the coefficients, and for BETI-DP methods under the usual assumptions. Of course, the results remain valid for coupled FETI/BETI methods. It will become clear that the coarse space appearing in the dual-primal methods is much more powerful than the one of the one-level methods. We also discuss the computational cost in case of unbounded domains. Here, we will see that in certain situations one-level methods have a better computational complexity than the dual-primal methods.

In the FETI analysis by Toselli, Widlund, Dryja and Klawonn (cf. [49, 50, 87] the face estimates are not related to the usual fractional Sobolev norms (such as the K-interpolation norm or the Sobolev-Slobodeckiĭ norm), but given in terms of the H^1 energy norms of (discrete) harmonic extensions. Indeed, the whole analysis neither needs any trace theorems nor Rellich's embedding theorem, for both of which the concrete size of the constant is usually not at our disposal. In order to keep this concept at least partially alive, we work out spectral properties of the approximated Steklov-Poincaré operators, which are based on the works by Costabel [19], Hsiao, Steinbach and Wendland [42], Steinbach [82], and Steinbach and Wendland [83].

The remainder of this paper is organized as follows. In Section 2 we introduce the notion of boundary integral operators and domain decomposition. Additionally, we derive some results on spectral relations between continuous and approximated Steklov-Poincaré operators. Section 3 contains a detailed formulation of one-level BETI methods for unbounded domains and a full analysis of the corresponding preconditioners. In particular, we discuss both the standard and the all-floating formulation. At the end we give some numerical results. Section 4 is devoted to BETI-DP methods for unbounded domains and the corresponding analysis of the quasi-optimal preconditioner. We give a conclusion in Section 5.

2 Preliminaries

2.1 Basic notation

First of all, we fix some of our basic notations. For a comprehensive introduction to Sobolev spaces we refer to [1, 28], for the finite element method (FEM) to [13, 17], and for the boundary element method (BEM) to [66, 81, 77]

Throughout this paper we denote the dual of a Banach space V by V^* and the duality product between V^* and V by $\langle \cdot, \cdot \rangle$. A linear operator $T : V \to V^*$ is said to be symmetric positive definite (SPD) if $\langle T\, v, w\rangle = \langle T\, w, v\rangle$, $\forall v, w \in V$, and $\langle T\, v, v\rangle > 0$, $\forall v \in V \setminus \{0\}$. The adjoint $T^\top : W \to V^*$ of a linear operator $T : V \to W^*$ is defined by $\langle T^\top w, v\rangle = \langle T\, v, w\rangle$, $\forall v \in V$, $w \in W$.

For a bounded domain $\Omega \subset \mathbb{R}^d$ (with $d = 2$ or 3) with a Lipschitz boundary Γ, let $H^1(\Omega)$,

$H_0^1(\Omega)$ denote the usual Sobolev spaces. The space $H^{1/2}(\Gamma)$ is defined as

$$H^{1/2}(\Gamma) := \left\{ v \in L_2(\Gamma) : \int_\Gamma \int_\Gamma \frac{|u(x) - u(y)|^2}{|x-y|^d} \, dx \, dy < \infty \right\}. \tag{2.1}$$

The spaces $H^{-1}(\Omega)$ and $H^{-1/2}(\Gamma)$ are defined as the duals of $H_0^1(\Omega)$ and $H^{1/2}(\Gamma)$, respectively.

We define the exterior of the bounded domain Ω by

$$\Omega^{\text{ext}} := \mathbb{R}^d \setminus \overline{\Omega}. \tag{2.2}$$

Sometimes we may write Ω^{int} instead of Ω to emphasize that this domain is the interior of its boundary. On Ω^{ext} we define the Sobolev space

$$H^1_{\text{loc}}(\Omega^{\text{ext}}) := \{ u \in \mathcal{D}'(\Omega^{\text{ext}}) : u \in H^1(B_R \cap \Omega^{\text{ext}}) \text{ for all } B_R \supset \Omega \}, \tag{2.3}$$

where B_R is the open ball with radius R and $\mathcal{D}'(\Omega^{\text{ext}})$ the space of distributions on Ω^{ext}, see, e. g., [66, 81].

Furthermore, for an open hypersurface $\Gamma_0 \subset \Gamma$, we denote by $H_{00}^{1/2}(\Gamma_0)$ the space of functions on Γ_0 whose extensions to $\Gamma \setminus \Gamma_0$ by zero are in $H^{1/2}(\Gamma)$. We define the constant functions $\mathbf{1}_\Gamma : \Gamma \to \mathbb{R} : x \mapsto 1$, and $\mathbf{0}_\Gamma : \Gamma \to \mathbb{R} : x \mapsto 0$, and observe that $\mathbf{1}_\Gamma \in H^{1/2}(\Gamma)$. So we can define the orthogonal subspace

$$H_*^{-1/2}(\Gamma) := \{ w \in H^{-1/2}(\Gamma) : \langle w, \mathbf{1}_\Gamma \rangle = 0 \}. \tag{2.4}$$

For a triangulation $\mathcal{T}_h$ of a domain Ω or a boundary Γ, we denote by $V_1^h(\Omega)$ and $V_1^h(\Gamma)$ the spaces of continuous functions that are piecewise linear on the elements of the triangulation. $V_0^h(\Gamma)$ denotes the space of functions that are piecewise constant on the elements. Furthermore, we write Γ_h for the set of nodes of the triangulation $\mathcal{T}_h(\Gamma)$.

In this work, $a \preceq b$ means that some (generic) constant $C > 0$ exists with $a \leq C\, b$. In particular, C will never depend on any mesh size parameter h or (sub)domain diameters H, only on shapes of (sub)domains or elements. Additionally, $a \simeq b$ stands for $a \preceq b$ and $b \preceq a$.

A complete list of notations can be found in the appendix.

2.2 Boundary integral operators

In this section we briefly summarize the notion of the standard boundary integral operators and derive some properties thereof. In particular, we give explicit norm equivalences for approximated Steklov-Poincaré operators.

2.2.1 Basic facts

First, we fix an open, bounded and simply connected domain $\Omega^{\text{int}} \subset \mathbb{R}^d$ ($d = 2, 3$) with a Lipschitz boundary $\Gamma = \partial\Omega^{\text{int}}$. In the following, we work with the exterior domain $\Omega^{\text{ext}} := \mathbb{R}^d \setminus \overline{\Omega}$. Let $\vec{n}$ denote the outward unit normal vector on Γ, i. e., pointing into Ω^{ext}, cf. Figure 4.

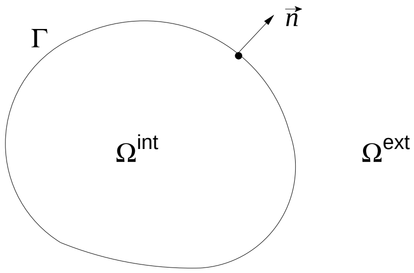

Figure 4: Interior domain Ω^{int}, exterior domain Ω^{ext} and the outward normal vectors $\vec{n}$ on the boundary Γ.

We consider the interior Laplace problem,

$$-\Delta u = 0 \quad \text{in } \Omega^{\text{int}}\,, \tag{2.5}$$

and the Laplace problem on the exterior space,

$$-\Delta u = 0 \quad \text{in } \Omega^{\text{ext}}\,. \tag{2.6}$$

Additionally, we prescribe an appropriate radiation condition for the exterior problem to ensure that the solution u can be represented by a fundamental solution. For a detailed explanation we refer to [20, 66]. In the present work we use the radiation condition

$$u(x) = \mathcal{O}(1) \qquad \frac{\partial u}{\partial \vec{n}}(x) = \begin{cases} \mathcal{O}((|x|\log|x|)^{-1}) & \text{for } d = 2 \\ \mathcal{O}(|x|^{-1}) & \text{for } d = 3 \end{cases} \tag{2.7}$$

as $|x| \to \infty$. Here $\vec{n}$ is understood as the outward unit normal vector on the boundary $\widehat{\Gamma}$ of an arbitrary Lipschitz domain $\widehat{\Omega}$ contained in Ω^{ext}.

It is well known, that for sufficiently smooth boundaries the trace operators

$$\begin{aligned}
\gamma_0^{\text{int}} : H^1(\Omega^{\text{int}}) \to H^{1/2}(\Gamma) : \qquad & (\gamma_0^{\text{int}} u)(x) := \lim_{\tilde{x}\to x} u(\tilde{x}) \quad \text{for } x \in \Gamma, \tilde{x} \in \Omega^{\text{int}}\,, \\
\gamma_0^{\text{ext}} : H^1_{\text{loc}}(\Omega^{\text{ext}}) \to H^{1/2}(\Gamma) : \qquad & (\gamma_0^{\text{ext}} u)(x) := \lim_{\tilde{x}\to x} u(\tilde{x}) \quad \text{for } x \in \Gamma, \tilde{x} \in \Omega^{\text{ext}}\,, \\
\gamma_1^{\text{int}} : H^1(\Omega^{\text{int}}) \to H^{-1/2}(\Gamma) : \qquad & (\gamma_1^{\text{int}} u)(x) := \frac{\partial u}{\partial \vec{n}}(x) \quad \text{for } x \in \Gamma\,, \\
\gamma_1^{\text{ext}} : H^1_{\text{loc}}(\Omega^{\text{ext}}) \to H^{-1/2}(\Gamma) : \qquad & (\gamma_1^{\text{ext}} u)(x) := -\frac{\partial u}{\partial \vec{n}}(x) \quad \text{for } x \in \Gamma
\end{aligned}$$

are bounded linear operators (see, e. g., [66]). In the following, we write $u|_\Gamma$ instead of $\gamma_0^{\text{int}} u$ or $\gamma_0^{\text{ext}} u$ for $u \in H^1(\Omega^{\text{int}})$ or $u \in H^1_{\text{loc}}(\Omega^{\text{ext}})$, respectively.

The fundamental solution of (2.5), (2.6), (2.7) is given by

$$U^*(x, y) = \begin{cases} -\dfrac{1}{2\pi}\log|x-y| & \text{for } d = 2, \\ \dfrac{1}{4\pi}\dfrac{1}{|x-y|} & \text{for } d = 3\,. \end{cases} \tag{2.8}$$

We introduce the following boundary integral operators, the *single layer potential operator* V, the *double layer potential operator* K and its adjoint $K^\top$, and the *hypersingular integral operator* D defined by

$$\begin{aligned}
&V : H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma) : && (V\,t)(x) := \int_\Gamma U^*(x,y)\,t(y)\,ds_y, \\
&K : H^{1/2}(\Gamma) \to H^{1/2}(\Gamma) : && (K\,u)(x) := \int_\Gamma \frac{\partial}{\partial \vec{n}_y} U^*(x,y)\,u(y)\,ds_y, \\
&K^\top : H^{-1/2}(\Gamma) \to H^{-1/2}(\Gamma) : && (K^\top\,t)(x) = \int_\Gamma \frac{\partial}{\partial \vec{n}_x} U^*(x,y)\,t(y)\,ds_y, \\
&D : H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma) : && (D\,u)(x) := -\frac{\partial}{\partial \vec{n}_x} \int_\Gamma \frac{\partial}{\partial \vec{n}_y} U^*(x,y)\,u(y)\,ds_y,
\end{aligned} \tag{2.9}$$

where $x \in \Gamma$.

The following assumption is needed for the ellipticity of the single layer potential operator in two dimensions.

Assumption 2.1. *If $d = 2$, we assume that* $\operatorname{diam}\Omega^{\text{int}} < 1$, *which can always be obtained by a simple scaling.*

Lemma 2.1. *The boundary integral operators defined in (2.9) are linear and bounded operators with the following properties:*

(i) Any weak solution of the interior Laplace problem (2.5) fulfills the Caldéron *system*

$$\begin{pmatrix} \gamma_0^{\text{int}} u \\ \gamma_1^{\text{int}} u \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I - K & V \\ D & \frac{1}{2}I + K^\top \end{pmatrix} \begin{pmatrix} \gamma_0^{\text{int}} u \\ \gamma_1^{\text{int}} u \end{pmatrix}. \tag{2.10}$$

For the exterior problem (2.6) we have

$$\begin{pmatrix} \gamma_0^{\text{ext}} u \\ \gamma_1^{\text{ext}} u \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I + K & V \\ D & \frac{1}{2}I - K^\top \end{pmatrix} \begin{pmatrix} \gamma_0^{\text{int}} u \\ \gamma_1^{\text{ext}} u \end{pmatrix}. \tag{2.11}$$

(ii) The single layer potential V is self-adjoint and (under Assumption 2.1) $H^{-1/2}(\Gamma)$-elliptic, i. e., there exists $c_V > 0$ such that

$$\langle V\,w,\, w\rangle \geq c_V\,\|w\|^2_{H^{-1/2}(\Gamma)},$$

for all $w \in H^{-1/2}(\Gamma)$. Thus, the inverse operator V^{-1} mapping $H^{1/2}(\Gamma)$ to $H^{-1/2}(\Gamma)$ is self-adjoint, elliptic and bounded, and

$$\|v\|_{V^{-1}} := \langle V^{-1}v,\, v\rangle^{1/2}$$

defines a norm on $H^{1/2}(\Gamma)$ equivalent to $\|\cdot\|_{H^{1/2}(\Gamma)}$.

(iii) The single layer potential V is an isomorphism between the subspaces $H_*^{-1/2}(\Gamma)$ and $H_*^{1/2}(\Gamma)$, defined by

$$H_*^{-1/2}(\Gamma) := \{w \in H^{-1/2}(\Gamma) : \langle w,\, \mathbf{1}_\Gamma\rangle = 0\}$$
$$H_*^{1/2}(\Gamma) := \{v \in H^{1/2}(\Gamma) : \langle V^{-1}v,\, \mathbf{1}_\Gamma\rangle = 0\}\,.$$

(iv) The hypersingular integral operator D is self-adjoint, $H_*^{1/2}(\Gamma)$-elliptic, and $H^{1/2}(\Gamma)$-semi-elliptic, i. e., there exists $c_D > 0$ such that

$$\langle D\,v,\, v\rangle \geq c_D\,\|v\|^2_{H^{1/2}(\Gamma)} \quad \textit{for } v \in H_*^{1/2}(\Gamma)\,,$$
$$\langle D\,v,\, v\rangle \geq c_D\,|v|^2_{H^{1/2}(\Gamma)} \quad \textit{for } v \in H^{1/2}(\Gamma)\,,$$

and $\ker D = \operatorname{span}\{\mathbf{1}_\Gamma\}$*.*

(v) The double layer potential operator K provides the contraction properties

$$(1-c_K)\|v\|_{V^{-1}} \leq \|(\tfrac{1}{2}I \pm K)v\|_{V^{-1}} \leq c_K\,\|v\|_{V^{-1}} \quad \textit{for } v \in H_*^{1/2}(\Gamma)\,,$$
$$0 \leq \|(\tfrac{1}{2}I + K)v\|_{V^{-1}} \leq c_K\,\|v\|_{V^{-1}} \quad \textit{for } v \in H^{1/2}(\Gamma)\,,$$
$$(1-c_K)\|v\|_{V^{-1}} \leq \|(\tfrac{1}{2}I - K)v\|_{V^{-1}} \leq \|v\|_{V^{-1}} \quad \textit{for } v \in H^{1/2}(\Gamma)\,,$$

where

$$c_0 := \inf_{v \in H_*^{1/2}(\Gamma)} \frac{\langle D\,v,\, v\rangle}{\langle V^{-1}\,v,\, v\rangle} \in \Big(0,\, \frac{1}{4}\Big)\,,$$
$$c_K := \frac{1}{2} + \sqrt{\frac{1}{4} - c_0} < 1\,.$$

Additionally the following identities hold,

$$\ker(\tfrac{1}{2}I + K) = \operatorname{span}\{\mathbf{1}_\Gamma\}\,, \qquad (\tfrac{1}{2}I + K)\mathbf{1}_\Gamma = \mathbf{0}_\Gamma\,,$$
$$\ker(\tfrac{1}{2}I - K) = \{\mathbf{0}_\Gamma\}\,, \qquad (\tfrac{1}{2}I - K)\mathbf{1}_\Gamma = \mathbf{1}_\Gamma\,.$$

Proof. A proof can be found in [81]. □

Remark 2.1. The constants c_0 and c_K do not depend on the size of the domain Ω^{int} but only on its shape. This can be shown by introducing a simple scaling of the domain and transforming the energy forms induced by D and V. In two dimensions the logarithm in the fundamental solution yields an additive term in the V-form; nevertheless this term vanishes for functions in the space $H_*^{1/2}(\Gamma)$.

2.2.2 Steklov-Poincaré operators

From the first line of the Caldéron system (2.10) and Lemma 2.1, part (ii) we obtain that any solution of the interior Laplace problem (2.5) fulfills the following relation between the Cauchy data,

$$\gamma_1^{\text{int}}u = V^{-1}(\tfrac{1}{2}I + K)\gamma_0^{\text{int}}u \quad \text{on } \Gamma\,.$$

A similar relation can be derived from (2.11) for the exterior problem (2.6), and we arrive at the definition of the *interior and exterior Steklov-Poincaré operators*

$$S^{\mathrm{int}} := V^{-1}\,(\tfrac{1}{2}I + K) \qquad\qquad S^{\mathrm{ext}} := V^{-1}\,(\tfrac{1}{2}I - K) \tag{2.12}$$

which both map from $H^{1/2}(\Gamma)$ to $H^{-1/2}(\Gamma)$ and describe the Dirichlet to Neumann map corresponding to the interior and exterior Laplace problem, respectively. Furthermore, by (2.10), (2.11) we have the identities

$$\begin{aligned} S^{\mathrm{int}} &= D + (\tfrac{1}{2}I + K^{\top})V^{-1}(\tfrac{1}{2}I + K)\,, \\ S^{\mathrm{ext}} &= D + (\tfrac{1}{2}I - K^{\top})V^{-1}(\tfrac{1}{2}I - K)\,, \end{aligned} \tag{2.13}$$

which imply that the S^{int} and S^{ext} are self-adjoint. We define the interior and exterior energy form

$$a_{\Omega^{\mathrm{int}}}(u,\,v) := \int\limits_{\Omega}^{\mathrm{int}} \nabla u(x)\cdot\nabla v(x)\,dx\,, \qquad a_{\Omega^{\mathrm{ext}}}(u,\,v) := \int\limits_{\Omega^{\mathrm{ext}}} \nabla u(x)\cdot\nabla v(x)\,dx\,,$$

for $u,\ v \in H^1(\Omega^{\mathrm{int}})$ and $H^1_{\mathrm{loc}}(\Omega^{\mathrm{ext}})$, respectively. By integration by parts and a density argument (cf. [81]) we see that

$$\begin{aligned} \langle S^{\mathrm{int}}\,v,\,v\rangle &= \min_{\substack{u\in H^1(\Omega^{\mathrm{int}})\\ u|_\Gamma = v}} a_{\Omega}(u,\,u)\,, \\ \langle S^{\mathrm{ext}}\,v,\,v\rangle &= \min_{\substack{u\in H^1_{\mathrm{loc}}(\Omega^{\mathrm{ext}})\\ u|_\Gamma = v}} a_{\Omega^{\mathrm{ext}}}(u,\,u)\,, \end{aligned} \tag{2.14}$$

and from (2.12) that

$$\langle V^{-1}v,\,v\rangle = \langle S^{\mathrm{int}}\,v,\,v\rangle + \langle S^{\mathrm{ext}}\,v,\,v\rangle\,.$$

Lemma 2.2. *(i) The following estimates hold for the Steklov-Poincaré operators:*

$$\begin{aligned} (1-c_K)\langle V^{-1}v,\,v\rangle \le \langle S^{\mathrm{int/ext}}\,v,\,v\rangle &\le c_K\langle V^{-1}v,\,v\rangle && \forall v\in H^{1/2}_*(\Gamma) \\ (1-c_K)\langle V^{-1}v,\,v\rangle \le \langle S^{\mathrm{ext}}\,v,\,v\rangle &\le \langle V^{-1}v,\,v\rangle && \forall v\in H^{1/2}(\Gamma) \\ 0 \le \langle S^{\mathrm{int}}\,v,\,v\rangle &\le c_K\,\langle V^{-1}v,\,v\rangle && \forall v\in H^{1/2}(\Gamma)\,, \end{aligned}$$

with the constraction constant $c_K > 0$, see Lemma 2.1, part (v). In particular, S^{ext} is $H^{1/2}(\Gamma)$-elliptic and S^{int} is $H^{1/2}_*(\Gamma)$-elliptic with $\ker S^{\mathrm{int}} = \mathrm{span}\{\mathbf{1}_\Gamma\}$.

(ii) Poincaré's fundamental theorem holds, i. e.,

$$\begin{aligned} \langle S^{\mathrm{ext}}\,v,\,v\rangle &\le \frac{c_K}{1-c_K}\langle S^{\mathrm{int}}\,v,\,v\rangle && \forall v\in H^{1/2}_*(\Gamma)\,, \\ \langle S^{\mathrm{int}}\,v,\,v\rangle &\le \frac{c_K}{1-c_K}\langle S^{\mathrm{ext}}\,v,\,v\rangle && \forall v\in H^{1/2}(\Gamma)\,. \end{aligned}$$

(iii) For $v \in H^{1/2}(\Gamma)$, the decomposition $v = \widetilde{v} + v_0 \mathbf{1}_\Gamma$ with $v_0 \in \mathbb{R}$ and $\widetilde{v} \in H_*^{1/2}(\Gamma)$ (i. e., $\langle V^{-1}\widetilde{v}, v_0\mathbf{1}_\Gamma\rangle = 0$) is unique and orthogonal in the S^{ext} inner product, i. e.,

$$\langle S^{\text{ext}}\,\widetilde{v}, v_0\mathbf{1}_\Gamma\rangle = 0\,.$$

(iv) We have

$$\langle S^{\text{ext}}\,v, v\rangle \simeq \langle S^{\text{int}}\,v, v\rangle + \frac{\langle V^{-1}v, \mathbf{1}_\Gamma\rangle^2}{\langle V^{-1}\mathbf{1}_\Gamma, \mathbf{1}_\Gamma\rangle}\,.$$

Proof. We start with the proof of part (iii): For $v \in H^{1/2}(\Gamma)$, we obtain from $\widetilde{v} = v - v_0\mathbf{1}_\Gamma$ and the orthogonality relation $\langle V^{-1}(v - v_0\mathbf{1}_\Gamma), \mathbf{1}_\Gamma\rangle = 0$ that

$$v_0 = \frac{\langle V^{-1}v, \mathbf{1}_\Gamma\rangle}{\langle V^{-1}\mathbf{1}_\Gamma, \mathbf{1}_\Gamma\rangle}\,,$$

which implies uniqueness. Using that $(\frac{1}{2}I - K)\mathbf{1}_\Gamma = \mathbf{1}_\Gamma$ (Lemma 2.1, part (v)) we obtain

$$\langle S^{\text{ext}}\,v_0\mathbf{1}_\Gamma, \widetilde{v}\rangle = \langle V^{-1}(\tfrac{1}{2}I - K)v_0\mathbf{1}_\Gamma, \widetilde{v}\rangle = \langle V^{-1}v_0\mathbf{1}_\Gamma, \widetilde{v}\rangle = 0\,,$$

which proves the orthogonality (iii).

Part (i): For $v \in H_*^{1/2}(\Gamma)$, the Cauchy-Schwarz inequality and the contraction properties stated in Lemma 2.1, part (v) yield

$$\langle S^{\text{int/ext}}\,v, v\rangle = \langle V^{-1}(\tfrac{1}{2}I \pm K)v, v\rangle \le \|(\tfrac{1}{2}I \pm K)v\|_{V^{-1}}\|v\|_{V^{-1}}^2 \le c_K\|v\|_{V^{-1}}^2\,.$$

On the other hand, for all $v \in H_*^{1/2}(\Gamma)$ we have

$$\begin{aligned}\langle S^{\text{int/ext}}\,v, v\rangle &= \langle V^{-1}(\tfrac{1}{2}I \pm K)v, v\rangle = \langle V^{-1}v, v\rangle - \langle V^{-1}(\tfrac{1}{2}I \mp K)v, v\rangle\\ &\ge \langle V^{-1}v, v\rangle - c_K\|v\|_{V^{-1}}^2 \ge (1 - c_K)\langle V^{-1}v, v\rangle\,.\end{aligned}$$

The inequalities on the full space $H^{1/2}(\Gamma)$ can be derived using the decomposition (iii) and the mapping properties of $(\frac{1}{2}I \pm K)$, see Lemma 2.1, part (v).

Statement (ii) is an immediate consequence of (i), see also [19].

Finally with the decomposition from part (iii) and Poincaré's fundamental theorem (ii) we obtain that

$$\begin{aligned}\langle S^{\text{ext}}\,v, v\rangle = \langle S^{\text{ext}}\,\widetilde{v}, \widetilde{v}\rangle + \langle S^{\text{ext}}\,v_0\mathbf{1}_\Gamma, v_0\mathbf{1}_\Gamma\rangle &\simeq \langle S^{\text{int}}\,\widetilde{v}, \widetilde{v}\rangle + \langle V^{-1}\,v_0\mathbf{1}_\Gamma, v_0\mathbf{1}_\Gamma\rangle\\ &= \langle S^{\text{int}}\,v, v\rangle + (v_0)^2\,\langle V^{-1}\mathbf{1}_\Gamma, \mathbf{1}_\Gamma\rangle\,,\end{aligned}$$

which implies (iv).

□

2.2.3 Approximations of the Steklov-Poincaré operators

From equation (2.13) we see, that for a fixed $v \in H^{1/2}(\Gamma)$,

$$S^{\text{int/ext}}\,v = D\,v + (\tfrac{1}{2}I \pm K^\top)w\,,$$

where $w \in H^{-1/2}(\Gamma)$ solves

$$\langle V\,w,\,\tau\rangle = \langle(\tfrac{1}{2}I \pm K)v,\,\tau\rangle \qquad \forall\tau \in H^{-1/2}(\Gamma)\,. \tag{2.15}$$

In the following, we consider a shape-reglar triangulation $\mathcal{T}_h$ of the polygonal boundary Γ. Recall that $V_0^h(\Gamma)$ is the space of piecewise constant functions on the corresponding elements. According to [82], we define the boundary element approximations

$$S_{\mathrm{BEM}}^{\mathrm{int/ext}}\,v = D\,v + (\tfrac{1}{2}I \pm K^\top)w_h\,, \tag{2.16}$$

where $w_h \in V_0^h(\Gamma)$ solves the projected equation

$$\langle V\,w_h,\,\tau_h\rangle = \langle(\tfrac{1}{2}I \pm K)v,\,\tau_h\rangle \qquad \forall\tau_h \in V_0^h(\Gamma)\,. \tag{2.17}$$

The approximations $S_{\mathrm{BEM}}^{\mathrm{int/ext}}$ are self-adjoint, they fulfill the same ellipticity properties as the original operators $S^{\mathrm{int/ext}}$, and estimates for the error $S^{\mathrm{int/ext}} - S_{\mathrm{BEM}}^{\mathrm{int/ext}}$ in terms of h are available. We mention that $\ker S_{\mathrm{BEM}}^{\mathrm{int}} = \ker S^{\mathrm{int}} = \mathrm{span}\{\mathbf{1}_\Gamma\}$. Moreover, the matrix representations of the restrictions of $S_{\mathrm{BEM}}^{int/ext}$ to $V_1^h(\Gamma) \to V_1^h(\Gamma)^*$ are symmetric and can be expressed by the standard BEM matrices, e. g.,

$$\mathbf{S}_{\mathrm{BEM},h}^{\mathrm{int}} = \mathbf{D}_h + (\tfrac{1}{2}\mathbf{M}_h^\top + \mathbf{K}_h^\top)\mathbf{V}_h^{-1}(\tfrac{1}{2}\mathbf{M}_h + \mathbf{K}_h)\,,$$

where $\mathbf{D}_h$, $\mathbf{K}_h$ and $\mathbf{V}_h$ are the matrix representations of the hypersingular, double layer potential and the single layer potential operator, and $\mathbf{M}_h$ is a mass matrix. For more details, we refer again to [82]. However, in this work we need the equivalence of the related energy forms on the space $V_1^h(\Omega)$, which is presented detail in the subsequent section.

There exists a similar approximation $S_{\mathrm{FEM}}^{\mathrm{int}}$ of the interior operator S^{int} by finite elements, based on a shape-regular triangulation $\mathcal{T}_h$ of the domain Ω^{int}, cf. [82]. Restricted to the space $V_1^h(\Gamma)$, the corresponding energy form reads

$$\langle S_{\mathrm{FEM}}^{\mathrm{int}}\,v_h,\,v_h\rangle = \min_{\substack{u_h \in V_1^h(\Omega^{\mathrm{int}})\\ u_h|_\Gamma = v_h}} a_{\Omega^{\mathrm{int}}}(u_h,\,u_h) \qquad \forall v_h \in V_1^h(\Gamma)\,. \tag{2.18}$$

For the usual nodal FE basis, the matrix representation of $S_{\mathrm{FEM}}^{\mathrm{int}}$ is exactly the Schur complement of the FEM stiffness matrix eliminating the interior nodal unknowns.

2.2.4 Spectral relations of $S^{\mathrm{int/ext}}$, $S_{\mathrm{BEM}}^{\mathrm{int/ext}}$ and $S_{\mathrm{FEM}}^{\mathrm{int}}$

Lemma 2.3. *The following spectral equivalence relations between the exact and approximated Steklov-Poincaré operators hold.*

$$\begin{aligned}
\langle S^{\mathrm{int}}\,v_h,\,v_h\rangle \le \langle S_{\mathrm{FEM}}^{\mathrm{int}}\,v_h,\,v_h\rangle &\le C_T\langle S^{\mathrm{int}}\,v_h,\,v_h\rangle && \forall v_h \in V_1^h(\Gamma)\\
\frac{c_0}{c_K}\langle S^{\mathrm{int}}\,v,\,v\rangle \le \langle S_{\mathrm{BEM}}^{\mathrm{int}}\,v,\,v\rangle &\le \langle S^{\mathrm{int}}\,v,\,v\rangle && \forall v \in H^{1/2}(\Gamma)\\
\langle S_{\mathrm{BEM}}^{\mathrm{ext}}\,v,\,v\rangle &\le \langle S^{\mathrm{ext}}\,v,\,v\rangle && \forall v \in H^{1/2}(\Gamma)\\
\frac{c_0}{c_K}\langle S^{\mathrm{ext}}\,v,\,v\rangle \le \langle S_{\mathrm{BEM}}^{\mathrm{ext}}\,v,\,v\rangle & && \forall v \in H_*^{1/2}(\Gamma)\\
\frac{c_0}{c_K}\langle S^{\mathrm{int}}\,v,\,v\rangle \le \langle S_{\mathrm{BEM}}^{\mathrm{ext}}\,v,\,v\rangle & && \forall v \in H^{1/2}(\Gamma)\,.
\end{aligned} \tag{2.19}$$

Additionally, in three dimensions, or if $\operatorname{diam}\Omega = \mathcal{O}(1)$ *in two dimensions,*

$$\begin{aligned} \langle S^{\mathrm{int}}\, v,\, v\rangle + \frac{1}{\operatorname{diam}\Gamma}\|u\|^2_{L_2(\Gamma)} &\le C_{\mathrm{ext}}\, \langle S^{\mathrm{ext}}_{\mathrm{BEM}}\, v,\, v\rangle \qquad \forall v \in H^{1/2}(\Gamma)\,, \\ \langle S^{\mathrm{ext}}\, v,\, v\rangle &\le C'_{\mathrm{ext}}\, \langle S^{\mathrm{ext}}_{\mathrm{BEM}}\, v,\, v\rangle \qquad \forall v \in H^{1/2}(\Gamma)\,. \end{aligned} \tag{2.20}$$

We point out that the constants c_0*,* c_K*,* C_T*,* C_{ext} *and* C'_{ext} *depend only on the shape of the domain* Ω*.*

Proof. The following proof of the estimates (2.19) can essentially be found in [82]. By (2.14) and (2.18), it becomes clear that

$$\langle S^{\mathrm{int}}\, v_h,\, v_h\rangle \le \langle S^{\mathrm{int}}_{\mathrm{FEM}}\, v_h,\, v_h\rangle \qquad \forall v_h \in V^h_1(\Gamma)\,.$$

The discrete trace theorem **[REF]** states that under appropriate assumptions on the mesh, there exists a constant $C_T > 0$ independent of h such that

$$\langle S^{\mathrm{int}}_{\mathrm{FEM}}\, v_h,\, v_h\rangle \le C_T \cdot \langle S^{\mathrm{int}}\, v_h,\, v_h\rangle \qquad \forall v_h \in V^h_1(\Gamma)\,.$$

For the boundary element approximations $S^{\mathrm{int}}_{\mathrm{BEM}}$, $S^{\mathrm{ext}}_{\mathrm{BEM}}$ of the continuous Steklov-Poincaré operator we can apply Lemma A.1 with $V = H^{1/2}(\Gamma)$, $V_h = V^h_0$, $a(\cdot,\,\cdot) = \langle V\,\cdot,\,\cdot\rangle$ and $f = (\frac{1}{2}I \pm K)v$, and obtain that

$$\langle V\, w_h,\, w_h\rangle \le \langle V\, w,\, w\rangle\,, \tag{2.21}$$

Using the identities

$$\begin{aligned} \langle S^{\mathrm{int/ext}}\, v,\, v\rangle &= \langle D\, v, v\rangle + \langle V\, w,\, w\rangle\,, \\ \langle S^{\mathrm{int/ext}}_{\mathrm{BEM}}\, v,\, v\rangle &= \langle D\, v, v\rangle + \langle V\, w_h,\, w_h\rangle\,, \end{aligned} \tag{2.22}$$

which follow from (2.12), (2.15), (2.16), (2.17) and the adjoint relation between K and $K^\top$, we immediately get

$$\langle S^{\mathrm{int/ext}}_{\mathrm{BEM}}\, v,\, v\rangle \le \langle S^{\mathrm{int/ext}}\, v,\, v\rangle \qquad \forall v \in H^{1/2}(\Gamma)\,.$$

For the opposite direction, we first obtain for $\widetilde{v} \in H^{1/2}_*(\Gamma)$,

$$\begin{aligned} \langle S^{\mathrm{int/ext}}_{\mathrm{BEM}}\, \widetilde{v},\, \widetilde{v}\rangle &= \langle D\, \widetilde{v},\, \widetilde{v}\rangle + \langle V\, w_h, w_h\rangle \\ &\ge \langle D\, \widetilde{v},\, \widetilde{v}\rangle \ge c_0 \langle V^{-1}\widetilde{v},\, \widetilde{v}\rangle \ge \frac{c_0}{c_K}\langle S^{\mathrm{int/ext}}\, \widetilde{v},\, \widetilde{v}\rangle\,, \end{aligned} \tag{2.23}$$

where we have used the ellipticity of V, Lemma 2.1, part (v) and Lemma 2.2, part (i). Secondly, using the V^{-1}-orthogonal splitting $v = v_0 \mathbf{1}_\Gamma + \widetilde{v}$ with $\widetilde{v} \in H^{1/2}_*(\Gamma)$ and $v_0 \in \mathbb{R}$, see Lemma 2.2, part (iii), we get

$$\langle S^{\mathrm{int/ext}}_{\mathrm{BEM}}\, v,\, v\rangle \ge \langle D\, v,\, v\rangle = \langle D\, \widetilde{v},\, \widetilde{v}\rangle \ge \frac{c_0}{c_K}\langle S^{\mathrm{int}}\, \widetilde{v},\, \widetilde{v}\rangle = \frac{c_0}{c_K}\langle S^{\mathrm{int}}\, v,\, v\rangle\,.$$

This finishes the proof of the estimates (2.19).

Now, we prove the two estimates (2.20) of the $S^{\mathrm{ext}}_{\mathrm{BEM}}$ energy form from below by the S^{ext} form and a regularized S^{int} form on the whole space $H^{1/2}(\Gamma)$. In order achieve this goal, we recall once again that

$$\langle S^{\mathrm{ext}}_{\mathrm{BEM}}\, v,\, v\rangle = \langle D\, v,\, v\rangle + \langle V\, w_h,\, w_h\rangle$$

with w_h defined by (2.17). We choose $w_0 \in \mathbb{R}$ by

$$w_0 = \frac{\langle (\frac{1}{2}I - K)v,\, \mathbf{1}_\Gamma\rangle}{\langle V\,\mathbf{1}_\Gamma,\, \mathbf{1}_\Gamma\rangle}\,,$$

which is well-defined due to Assumption 2.1. Consequently, we get the relation

$$\langle V\, w_0 \mathbf{1}_\Gamma,\, \mathbf{1}_\Gamma\rangle = \langle (\tfrac{1}{2}I - K)v,\, \mathbf{1}_\Gamma\rangle$$

which is the projection of equation (2.21) to $\mathrm{span}\{\mathbf{1}_\Gamma\} \subset V_0^h \subset H^{-1/2}(\Gamma)$. Again, by Lemma A.1 (now with $V = V_0^h$, $V_h = \mathrm{span}\{\mathbf{1}_\Gamma\}$ and $a(\cdot,\cdot) = \langle V\cdot,\, \cdot\rangle$) we obtain

$$\langle V\, w_h,\, w_h\rangle \geq \langle V\, w_0\mathbf{1}_\Gamma,\, w_0\mathbf{1}_\Gamma\rangle = \frac{\langle (\frac{1}{2}I - K)v,\, \mathbf{1}_\Gamma\rangle^2}{\langle V\,\mathbf{1}_\Gamma,\, \mathbf{1}_\Gamma\rangle} = |\Psi(v)|^2\,. \tag{2.24}$$

with the linear functional $\Psi : H^{1/2}(\Gamma) \to \mathbb{R}$ defined by

$$\Psi(v) := \frac{\langle (\frac{1}{2}I - K)v,\, \mathbf{1}_\Gamma\rangle}{\sqrt{\langle V\,\mathbf{1}_\Gamma,\, \mathbf{1}_\Gamma\rangle}} \qquad \text{for } v \in H^{1/2}(\Gamma)\,.$$

We observe that Ψ is bounded in the $H^{1/2}$-norm and, most importantly, that the definition of Ψ is independent of the discretization parameter h. Furthermore, $|\Psi(v)|$ defines a semi-norm that becomes a norm on the constant functions, since for some $v_0 \in \mathbb{R}$ with $\Psi(v_0\mathbf{1}_\Gamma) = 0$ we have

$$0 = \langle (\tfrac{1}{2}I - K)v_0\mathbf{1}_\Gamma,\, \mathbf{1}_\Gamma\rangle = \langle v_0\mathbf{1}_\Gamma,\, \mathbf{1}_\Gamma\rangle = v_0 \cdot |\Gamma|\,,$$

thus $v_0 = 0$. On the other hand, we obtain from (2.14) and (2.23) that

$$\langle D\, v,\, v\rangle \geq \frac{c_0}{c_K}\langle S^{\mathrm{int}}\, v,\, v\rangle = \frac{c_0}{c_K} \min_{\substack{u \in H^1(\Omega)\\ u|_\Gamma = v}} |u|^2_{H^1(\Omega)}\,. \tag{2.25}$$

Due to Sobolev's norm theorem (see, e. g., [87, Theorem A.12] or [69]),

$$|u|^2_{H^1(\Omega)} + |\Psi(u|_\Gamma)|^2 \simeq \|u\|^2_{H^1(\Omega)} \simeq |u|^2_{H^1(\Omega)} + \frac{1}{\operatorname{diam}\Gamma}\|u|_\Gamma\|^2_{L_2(\Gamma)}\,,$$

where the scaling $1/\operatorname{diam}\Gamma$ is obtained by dilation from a domain with unit diameter. Combining this result with (2.22), (2.24) and (2.25) we obtain

$$\langle S^{\mathrm{ext}}_{\mathrm{BEM}}\, v,\, v\rangle = \langle D\, v,\, v\rangle + \langle V\, w_h,\, w_h\rangle \geq \frac{c_0}{c_K} \min_{\substack{u \in H^1(\Omega)\\ u|_\Gamma = v}} \Big\{|u|^2_{H^1(\Omega)} + |\Psi(v)|^2\Big\}$$

$$\succeq \min_{\substack{u \in H^1(\Omega)\\ u|_\Gamma = v}} \Big\{|u|^2_{H^1(\Omega)} + \frac{1}{\operatorname{diam}\Gamma}\|v\|^2_{L_2(\Gamma)}\Big\} = \langle S^{\mathrm{int}}\, v,\, v\rangle + \frac{1}{\operatorname{diam}\Gamma}\|v\|^2_{L_2(\Gamma)}\,.$$

In other words, there exists a constant $C_{\text{ext}} > 0$ such that

$$\langle S^{\text{int}}\, v,\, v\rangle + \frac{1}{\operatorname{diam}\Gamma}\|v\|^2_{L_2(\Gamma)} \leq C_{\text{ext}}\,\langle S^{\text{ext}}_{\text{BEM}}\, v,\, v\rangle\,. \tag{2.26}$$

The constant C_{ext} is independent of the discretization parameter h, and by a scaling argument one can show that in three dimensions C_{ext} is independent of the size of Ω, i. e., it depends only on the shape of Ω. In two dimensions, the single layer potential V does not scale linearly due to the logarithmic term in the fundamental solution. Since we have to scale the domain anyway, we can always achieve $\operatorname{diam}\Omega \simeq \mathcal{O}(1)$.

Finally, due to Lemma 2.2, part (iv) we have

$$\langle S^{\text{ext}}\, v,\, v\rangle \simeq \langle S^{\text{int}}\, v,\, v\rangle + \frac{\langle V^{-1}v,\, \mathbf{1}_\Gamma\rangle^2}{\langle V^{-1}\mathbf{1}_\Gamma,\, \mathbf{1}_\Gamma\rangle}\,.$$

Since $\Upsilon(v) := \langle V^{-1}v,\, \mathbf{1}_\Gamma\rangle/\|\mathbf{1}_\Gamma\|_{V^{-1}}$ is a bounded linear functional that reproduces the constant functions, we can apply Sobolev's norm theorem once more and using (2.26) we can conclude that

$$\langle S^{\text{ext}}\, v,\, v\rangle \leq C'_{\text{ext}}\langle S^{\text{ext}}_{\text{BEM}}\, v,\, v\rangle\,, \tag{2.27}$$

where C'_{ext} depends (at least in three dimensions) only on the shape of Ω.

This finishes the proof of Lemma 2.3. □

2.2.5 Newton potentials

The solution of the boundary value problem

$$\begin{aligned} -\Delta u(x) &= f(x) \quad \text{for } x \in \Omega\\ u(x) &= 0 \qquad\ \ \text{for } x \in \Gamma \end{aligned}$$

defines the Newton potential $N : H^{-1}(\Omega) \to H^{-1/2}(\Gamma)$

$$N\,f(x) := -\frac{\partial u}{\partial \vec{n}}\,.$$

For a FEM discretization of Ω, we denote the full stiffness matrix by $\mathbf{K}$ and group the unknowns corresponding to interior unknowns (subscript I) and boundary unknowns (subscript Γ). The discrete FE equation reads

$$\begin{pmatrix} \mathbf{K}_{\Gamma\Gamma} & \mathbf{K}_{\Gamma I}\\ \mathbf{K}_{\Gamma I}^\top & \mathbf{K}_{II} \end{pmatrix}\begin{pmatrix} \mathbf{0}\\ \mathbf{u}_I \end{pmatrix} = \begin{pmatrix} \mathbf{f}_\Gamma + \mathbf{t}\\ \mathbf{f}_I \end{pmatrix},$$

where $\mathbf{t}$ corresponds to the contribution from the Neumann data $\partial u/\partial\vec{n}$. We define the matrix $\mathbf{N}$ by

$$\mathbf{N} := \Big[\, \mathbf{I}_\Gamma \;\Big|\; -\mathbf{K}_{\Gamma I}(\mathbf{K}_{II})^{-1} \Big]\,,$$

where $\mathbf{I}_\Gamma$ is the identity matrix corresponding to the unknowns on the boundary. Thus, for $\mathbf{f} = \begin{bmatrix} \mathbf{f}_\Gamma\\ \mathbf{f}_I \end{bmatrix}$ we have $\mathbf{N}\,\mathbf{f} = \mathbf{f}_\Gamma - \mathbf{K}_{\Gamma I}(\mathbf{K}_{II})^{-1}\mathbf{f}_I = -\mathbf{t}$. The corresponding linear operator

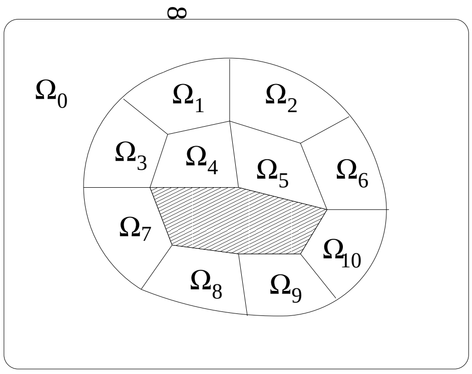

Figure 5: Non-overlapping domain decomposition of Ω into the unbounded subdomain Ω_0 and the bounded subdomains $\Omega_1, \ldots, \Omega_{10}$. The shaded region does not belong to Ω.

$N_h : V_1^h(\Omega)^* \to V_1^h(\Gamma)^*$ is known to be a stable approximation of the exact Newton potential, cf. [82]. A suitable BEM approximation of N can also be found in [82].

The general Dirichlet to Neumann map corresponding to the boundary value problem

$$\begin{aligned} -\Delta u(x) &= f(x) \qquad \text{for } x \in \Omega \\ u(x) &= g(x) \qquad \text{for } x \in \Gamma \end{aligned}$$

is given by

$$\frac{\partial u}{\partial \vec{n}} = S^{\text{int}}\, g - N\, f\,.$$

2.3 Domain decomposition – Definitions and assumptions

We consider an open connected domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, which can be decomposed into finitely many non-overlapping *subdomains* Ω_i, $i \in \mathcal{I}$, where $\mathcal{I} \subset \mathbb{N}_0$ is a finite index set, so that

$$\overline{\Omega} = \bigcup_{i\in\mathcal{I}} \overline{\Omega}_i\,, \qquad\qquad \Omega_i \cap \Omega_j = \emptyset \quad \text{if } i \neq j\,. \tag{2.28}$$

We assume that all the subdomains Ω_i, $i \in \mathcal{I} \setminus \{0\}$ are bounded and simply connected, and that, if the original domain Ω is unbounded, the subdomain Ω_0 is unbounded whereas its complement $\Omega_0^\complement$ is bounded and simply connected too, cf. Figure 5. Furthermore, we assume that the local boundaries $\Gamma_i := \partial\Omega_i$ are polygonal and Lipschitz. The outward unit normal vectors on Γ_i are denoted by $\vec{n}_i$. We emphasize that $\vec{n}_0$ points into $\Omega_0^\complement$ which is the interior of Γ_0. The *subdomain interfaces* Γ_{ij}, regarded as open sets, are defined by $\overline{\Gamma}_{ij} := \Gamma_i \cap \Gamma_j$. If $\partial\Omega$ is not empty, we assume that it is polygonal too, and that it splits in two disjoint parts, a Dirichlet boundary Γ_D, which is regarded as a closed set, and an open Neumann boundary Γ_N, such that

$$\partial\Omega = \Gamma_D \cup \Gamma_N\,.$$

The subdomain diameters are naturally defined by

$$H_i := \operatorname{diam}\Omega_i \qquad \text{for } i \in \mathcal{I}\setminus\{0\}\,,$$
$$H_0 := \operatorname{diam}\Omega_0^{\complement}\,.$$

In the following, we consider a shape-regular, quasi uniform triangulation $\mathcal{T}_h(\Gamma_i)$ of each subdomain boundary Γ_i such that the meshes match on interface boundaries. The minimal mesh size of $\mathcal{T}_h(\Gamma_i)$ is denoted by h_i. For $i \neq 0$ we can always extend the boundary mesh $\mathcal{T}_h(\Gamma_i)$ to a shape-regular quasi-uniform triangulation $\mathcal{T}_h(\Omega_i)$ of the whole subdomain Ω_i.

We define the *interface* Γ_I and the *skeleton* Γ_S by

$$\Gamma_I := \bigcup_{i,j\in\mathcal{I}} \overline{\Gamma}_{ij}\setminus\Gamma_D\,, \qquad \Gamma_S := \bigcup_{i\in\mathcal{I}}\Gamma_i = \Gamma_I \cup \partial\Omega\,.$$

In three dimensions the interface Γ_I is the union of

- *subdomain faces* F, regarded as open sets, which are shared by two subdomains,
- *subdomain edges* E, regarded as open sets, which are shared by more than two subdomains,
- *subdomain vertices* V, which are endpoints of subdomain edges,

cf. [87], see also Figure 3 on page 4. In two dimensions, there are only subdomain edges (which are open sets shared by two subdomains) and their endpoints, the subdomain vertices.

Regarding $\partial\Omega$ as an additional subdomain, we can also split the skeleton Γ_S into subdomain faces, edges and vertices in a straight forward manner. For a fixed $i \in \mathcal{I}$, let $\mathcal{V}_i$, $\mathcal{E}_i$, $\mathcal{F}_i$ denote the sets of subdomain vertices, edges and faces of Γ_S that belong to Γ_i, respectively. We set $\mathcal{V} := \bigcup_{i\in\mathcal{I}}\mathcal{V}_i$, $\mathcal{E} := \bigcup_{i\in\mathcal{I}}\mathcal{E}_i$, and $\mathcal{F} := \bigcup_{i\in\mathcal{I}}\mathcal{F}_i$.

The face and edge diameters are denoted by $H_F := \operatorname{diam} F$ and $H_E := \operatorname{diam} E$, respectively. Whenever we like to emphasize that a face, edge or vertex is shared by at least two subdomains Ω_i, Ω_j, we write F_{ij}, E_{ij} or V_{ij}, and we define $H_{ij} := \operatorname{diam} F_{ij}$ (or $\operatorname{diam} E_{ij}$ in two dimensions).

The following assumption, which basically ensures that the subdomains cannot be very thin, is typical for the theory of iterative substructuring methods, and it is needed for the convergence analysis of BETI and BETI-DP preconditioners, cf. Assumption 4.3 in [87].

Assumption 2.2. 1. *Each subdomain Ω_i, for $i \neq 0$ is the union of shape-regular coarse triangular or tetrahedral elements of a conforming mesh $\mathcal{T}_H$ and the number of coarse elements forming an individual subdomain is uniformly bounded.*

2. *There exist moderately bounded constants $\underline{\sigma}$, $\overline{\sigma} > 0$ such that,*

$$\underline{\sigma}\,H_i \le H_F \le \overline{\sigma}\,H_i \qquad \forall i \in \mathcal{I}\setminus\{0\} \text{ and } \forall F \in \mathcal{F}_i\,,$$

in three dimensions, and

$$\underline{\sigma}\,H_i \le H_E \le \overline{\sigma}\,H_i \qquad \forall i \in \mathcal{I}\setminus\{0\} \text{ and } \forall E \in \mathcal{E}_i\,,$$

in two dimensions.

Furthermore, we define

$$\frac{H}{h} := \max_{\substack{i,j\in\mathcal{I}\\ \Gamma_{ij}\neq\emptyset}} \frac{H_{ij}}{h_i}\,. \tag{2.29}$$

Note, that H_0 is allowed to be arbitrary large in comparison to the other subdomain diameters, and typically $H_0/h_0 \gg H/h$.

Remark 2.2. The first point in Assumption 2.2 is quite restrictive. In particular it is needed for the Sobolev type inequalities relating finite element functions (see Section 3.3.4 of the present paper and in [87, Chapter 4]). However, there are recent works on extending the theory to less regular domains in two dimensions, cf. [22, 47].

2.4 The model problem

Throughout the paper we deal with the following model problem in our domain Ω which is decomposed according to (2.28): Find $u \in H^1_{\text{loc}}(\Omega)$ such that

$$\begin{aligned} -\alpha_i \Delta u(x) &= \bar{f}_i(x) \quad \text{for } x \in \Omega_i\,, i \in \mathcal{I}\,,\\ \alpha_i \frac{\partial u}{\partial \vec{n}_i}(x) + \alpha_j \frac{\partial u}{\partial \vec{n}_j}(x) &= 0 \quad \text{for } x \in \Gamma_{ij},\ i, j \in \mathcal{I} \end{aligned} \tag{2.30}$$

holds in weak sense, where u fulfills additionally the radiation condition (2.7) for $|x| \to \infty$. We assume that $\bar{f}_0 \equiv 0$ if $0 \in \mathcal{I}$. Recall, that Ω_0 is the unbounded exterior of Γ_0 and that the normal vector $\vec{n}_0$ points into the interior of Γ_0, cf. Figure 4. In the case that $\partial\Omega$ is not empty, we assume Dirichlet and/or Neumann conditions on parts of it, i. e., $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N$, and

$$u(x) = g_D(x) \quad \text{for } x \in \Gamma_D\,, \tag{2.31}$$

$$\alpha_i \frac{\partial u}{\partial \vec{n}_i}(x) = g_N(x) \quad \text{for } x \in \Gamma_N \cap \Gamma_i\,, \tag{2.32}$$

for some $g_D \in H^{1/2}(\Gamma_D)$, $g_N \in H^{-1/2}(\Gamma_N)$. We assume that Γ_N is the union of subdomain faces, edges and vertices, i. e., the interface between Γ_D and Γ_N is aligned with the subdomain edges and vertices. We assume further that $0 \in \mathcal{I}$ or $\Gamma_D \neq \emptyset$, such that either the radiation condition or the Dirichlet boundary conditions guarantee the uniqueness of the solution.

2.5 Discrete skeleton formulations

Fixing our triangulations on the subdomains, we define the spaces

$$\begin{aligned} H^{1/2}_D(\Gamma_i) &:= \{u \in H^{1/2}(\Gamma_i) : u|_{\Gamma_i\cap\Gamma_D} = 0\}\,,\\ H^{1/2}_D(\Gamma_S) &:= \{u \in H^{1/2}(\Gamma_S) : u|_{\Gamma_D} = 0\}\,,\\ V^h_{1,D}(\Gamma_i) &:= H^{1/2}_D(\Gamma_i) \cap V^h_1(\Gamma_i)\,,\\ V^h_{1,D}(\Gamma_S) &:= \{u \in H^{1/2}_D(\Gamma_S) : u|_{\Gamma_i} \in V^h_1(\Gamma_i)\}\,. \end{aligned}$$

Let now $\widetilde{g}_D \in H^{1/2}(\Gamma_S)$ be an arbitrary extension of g_D from Γ_D to Γ_S. Then we can write our solution as

$$u = \widetilde{g}_D + u^{(0)} \qquad \text{with } u^{(0)} \in H_D^{1/2}(\Gamma_S)\,.$$

For the sake of simplicity, we assume that g_D, $\widetilde{g}_D$ and g_N are contained in the corresponding discrete spaces. Using the notion of the Steklov-Poincaré operators and Newton potentials (cf. Section 2.2.2), we can rewrite the model problem (2.30)–(2.32) as the following variational problem, cf. [58, 59]: Find $u^{(0)} \in H_D^{1/2}(\Gamma_S)$ such that

$$\sum_{i\in\mathcal{I}} \langle \alpha_i\, S_i\, u^{(0)}|_{\Gamma_i},\, v|_{\Gamma_i}\rangle = \sum_{i\in\mathcal{I}} \langle f_i - \alpha_i\, S_i\, \widetilde{g}_D|_{\Gamma_i},\, v|_{\Gamma_i}\rangle \qquad \forall v \in H_D^{1/2}(\Gamma_S)\,. \tag{2.33}$$

Here,

$$S_i := \begin{cases} S_i^{\text{int}} & \text{for } i \neq 0\,,\\ S_0^{\text{ext}} & \text{for } i = 0 \end{cases}$$

denotes the interior or exterior Steklov-Poincaré operator on Γ_i, and

$$f_i := \begin{cases} N_i \bar{f}_i + g_N|_{\Gamma_i \cap \Gamma_N} \quad \in H^{-1/2}(\Gamma_i) & \text{for } i \neq 0\,,\\ 0 & \text{for } i = 0\,, \end{cases}$$

where N_i denotes the Newton potential on Ω_i.

The Galerkin projection of (2.33) onto the discrete space $V_{1,D}^h$ reads: Find $u_h^{(0)} \in V_{1,D}^h(\Gamma_S)$ such that

$$\sum_{i\in\mathcal{I}} \langle S_{i,h}\, A_i\, u_h^{(0)},\, A_i\, v_h\rangle = \sum_{i\in\mathcal{I}} \langle f_{i,h} - S_{i,h}\, A_i\, \widetilde{g}_D,\, A_i\, v_h\rangle \qquad \forall v_h \in V_{1,D}^h(\Gamma_S)\,. \tag{2.34}$$

Here, A_i denotes the restriction operator from the global space $V_{1,D}^h(\Gamma_S)$ to the local space $V_{1,D}^h(\Gamma_i)$. The operators

$$S_{0,h} := \alpha_0\, S_{i,\text{BEM}}^{\text{ext}}\,, \qquad S_{i,h} := \alpha_i\, S_{i,\text{FEM/BEM}}^{\text{int}}\,, \tag{2.35}$$

involve the approximations of S_i which we have introduced in Section 2.2.3, and the linear forms $f_{i,h}$ denote the corresponding discretizations of the continuous forms f_i. For an individual $i \neq 0$ we can use either the finite element or the boundary element approximation scheme.

Formulation (2.34) is equivalent to the minimization problem

$$\min_{v_h^{(0)} \in V_{1,D}^h(\Gamma_S)} \sum_{i\in\mathcal{I}} \Big[\frac{1}{2}\langle S_{i,h}\, A_i\, v_h^{(0)},\, A_i\, v_h^{(0)}\rangle - \langle f_{i,h} - S_{i,h}\, A_i\, \widetilde{g}_D,\, A_i\, v_h^{(0)}\rangle\Big]\,. \tag{2.36}$$

Using that $u_h = \widetilde{g}_D + u_h^{(0)}$, we obtain by a simple computation the equivalent constrained minimization problem

$$\min_{\substack{v_h \in V_1^h(\Gamma_S)\\ v_h|_{\Gamma_D} = g_D}} \sum_{i\in\mathcal{I}} \Big[\frac{1}{2}\langle S_{i,h}\, A_i\, v_h,\, A_i\, v_h\rangle - \langle f_{i,h},\, A_i\, v_h\rangle\Big]\,. \tag{2.37}$$

The discrete formulations (2.36) and (2.37) are equivalent to a linear system of algebraic equations where the unknowns are the degrees of freedom of the unknown functions $u_h^{(0)} \in V_{1,D}^h(\Gamma_D)$ or $u_h \in V_1^h(\Gamma_S)$. FETI/BETI methods are specific methods to solve these algebraic equations efficiently. In particular, these methods can be implemented on parallel machines in a straight forward manner. The analysis of one-level BETI methods for unbounded domains is presented in Section 3, whereas Section 4 is devoted to BETI-DP methods.

3 One-level BETI methods

In this section, we give the BETI formulation according to [58, 59] and the all-floating BETI formulation according to [71, 72] (which is basically the same as the total FETI formulation in [23]), and extend it to the case of unbounded domains (Section 3.1). Furthermore, we state suitable preconditioners (Section 3.2) and derive appropriate assumptions, under which we can show their quasi-optimality (Section 3.3).

3.1 BETI formulations

3.1.1 Lagrange multipliers

Using the tearing and interconnecting technique, we derive now two different saddle point formulations, one starting from the minimization problem (2.36), to which we refer by the term *standard formulation* (cf. [49, 87]), and the other one starting from problem (2.37), which is called the *all-floating* formulation (cf. [71, 72]), or in the FEM context *total FETI* (cf. [23]).

The standard formulation: Our starting point is the minimization problem (2.36). For simplicity, we assume (only in context of the standard formulation), that $g_D \equiv 0$. Following [58, 59], we introduce separate unknowns $u_{h,i} \in V_{1,D}^h(\Gamma_i)$ for $A_i\, u_h^{(0)}$. Since the triangulation is fixed, we drop the subscript h for a better readability and write u_i instead of $u_{h,i}$. We further introduce Lagrange parameters λ in order to re-enforce the continuity across subdomain interfaces.

For each node $x \in \Gamma_{i,h} \cap \Gamma_{j,h}$ we can introduce a constraint

$$u_i(x) = u_j(x)\,. \tag{3.1}$$

In this work, we consider non-redundant and fully redundant constraints. In the non-redundant case we use the minimal number of necessary constraints, whereas in the fully redundant case, the maximal number of possibile constraints is used, cf. Figure 6.

For the standard formulation of the one-level BETI method, we set

$$W_i := V_{1,D}^h(\Gamma_i)\,, \qquad W := \prod_{i\in\mathcal{I}} W_i\,.$$

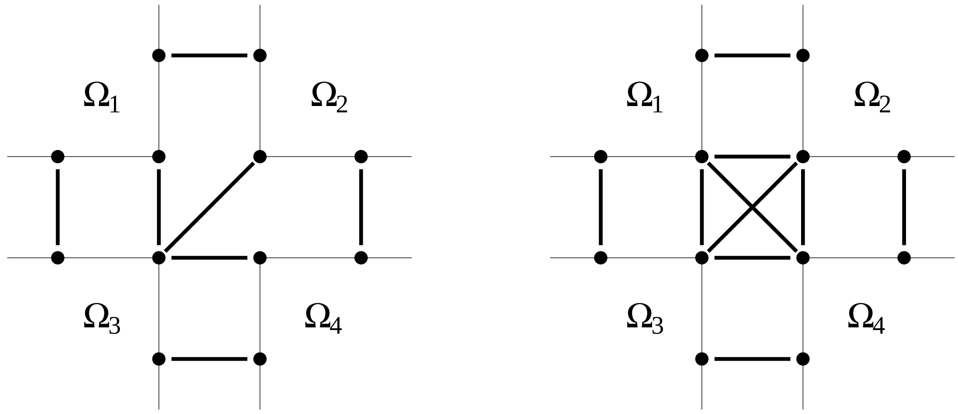

Figure 6: Sketch of non-redundant constraints (left) and fully redundant constraints (right) for a subdomain edge in three dimensions, or a subdomain vertex (cross point) in two dimensions.

Note, that will use a different definition of W_i and W in the next section dealing with the all-floating formulation. We write[1]

$$u = (u_i)_{i\in\mathcal{I}} \in W \,.$$

In the following, we always regard $S_{i,h}$ as operators mapping from W_i to W_i^* and assume that $f_{i,h} \in W_i^*$. Moreover, we define

$$\ker S_{i,h} := \{w \in W_i : S_{i,h} w = 0\} \,,$$

depending on W_i. Throughout the whole section, we work extensively with the product space W of typically discontinuous functions and its subspace $\widehat{W}$ of functions that are continuous across the interface Γ_I, see also [49, 87].

We define the space of Lagrange multipliers by

$$U := \mathbb{R}^M \,, \tag{3.2}$$

where M is the total number of constraints. We identify the dual U^* with $\mathbb{R}^M$ and write $\langle \mu, \lambda\rangle = (\mu, \lambda)_{\ell_2}$ for $\lambda \in U$, $\mu \in U^*$. Furthermore, we introduce a natural basis: The unit vector in U that corresponds to the constraint $u_i(x) = u_j(x)$ for $x \in \Gamma_i \cap \Gamma_j$, is denoted by $\lambda_{x,ij}$. The corresponding unit vector in U^* is denoted by $\mu_{x,ij}$.

We can now define the jump operator $B : W \to U^*$ by

$$\langle B\, w, \lambda_{x,ij}\rangle = w_{\max(i,j)}(x) - w_{\min(i,j)}(x) \,. \tag{3.3}$$

Let $B_i : W_i \to U^*$ defined by

$$\langle B_i\, w_i, \lambda_{x,jk}\rangle = \begin{cases} w_i(x) & \text{for } i = j,\, x \in \Gamma_{i,h} \cap \Gamma_I \,, \\ -w_i(x) & \text{for } i = k,\, x \in \Gamma_{i,h} \cap \Gamma_I \,, \\ 0 & \text{else,} \end{cases} \tag{3.4}$$

[1]Note, that we use the same notation for the continuous solution $u \in H^{1/2}(\Gamma_S)$. However, the difference should always be clear from the context.

where $j > k$. Recall, that $\Gamma_{i,h}$ is the (discrete) set of nodes of the triangulation of Γ_i. By definition, we have $\langle B\,w,\,\lambda\rangle = \sum_{i\in\mathcal{I}}\langle B_i\,w_i,\,\lambda\rangle$ for all $\lambda \in U$. Hence, the constraints (3.1) can be written in the compact form

$$B\,u = 0\,, \tag{3.5}$$

to be read as an equation in U^*. Note, that the operator B can simply be represented by a signed Boolean matrix, i. e., a matrix with entries 0, 1 or −1.

The adjoints $B_i^\top : U \to W_i^*$ and $B^\top : U \to W^*$ are defined by

$$\begin{aligned}\langle B_i^\top\lambda,\,w_i\rangle &= \langle B_i\,w_i,\,\lambda\rangle \qquad \forall\lambda\in U,\, w_i\in W_i\,,\\ \langle B^\top\lambda,\,w\rangle &= \langle B\,w,\,\lambda\rangle \qquad \forall\lambda\in U,\, w\in W\,.\end{aligned}$$

In addition, we define the operator $S : W \to W^*$ by

$$S := \operatorname{diag}\,(S_{i,h})_{i\in\mathcal{I}} \tag{3.6}$$

and the linear form $f \in W^*$ by

$$\langle f,\,w\rangle := \sum_{i\in\mathcal{I}}\langle f_{i,h},\,w_i\rangle \qquad \forall w\in W\,. \tag{3.7}$$

With this notation we can write the minimization problem (2.36) equivalently as the constrained minimization problem

$$\min_{\substack{u\in W\\ Bu=0}} \;\frac{1}{2}\langle S\,u,\,u\rangle - \langle f,\,u\rangle\,.$$

The corresponding saddle point problem reads as follows. Find $(u,\,\lambda) \in W \times U$:

$$\begin{pmatrix} S & B^\top \\ B & 0 \end{pmatrix}\begin{pmatrix} u \\ \lambda \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix}. \tag{3.8}$$

We define the index set of the *floating* subdomains,

$$\mathcal{I}_{\text{float}} := \{i\in\mathcal{I} : \ker S_{i,h} \neq \{0\}\}\,,$$

which is the set of those $i \in \mathcal{I}\setminus\{0\}$ with $\Gamma_i \cap \Gamma_D = \emptyset$ for the standard formulation.

The following assumption is needed later in the convergence analysis, cf. [87].

Assumption 3.1. *For the standard formulation of the one-level BETI method in three dimensions, we assume that in the case that Ω_i touches the Dirichlet boundary Γ_D, the intersection $\Gamma_i \cap \Gamma_D$ consists only of faces and edges, no isolated points.*

The reason for this assumption is that in two dimensions, there is a discrete Poincaré-Friedrichs inequality (cf. Lemma 3.14, page 44) for single-point Dirichlet conditions stating that for an edge $E \in \mathcal{E}_i$,

$$\frac{1}{H_E}\|u\|^2_{L_2(E)} \le C\,(1+\log(H_E/h_i))\,|u|^2_{H^{1/2}(E)}\,,$$

if $u \in V_1^h(E)$ vanishes at one of the endpoints of E. In three dimensions, there is no comparable result for such single-point Dirichlet conditions, at least not with a logarithmic factor; see also [87, Chapter 5, Theorem 5.3 and Remark 5.4].

The all-floating formulation: According to [71, 72], we start with the minimization problem (2.37), and apply the same tearing and interconnecting technique as before, i. e., we introduce separate unknowns $u_i \in V_1^h(\Gamma_i)$ for $A_i\, u_h$ and re-enforce the continuity across interfaces by the same Lagrange multipliers as in the standard formulation. However, for the Dirichlet boundary conditions

$$u_i(x) = g_D(x) \qquad \text{for } x \in \Gamma_{i,h} \cap \Gamma_D\,, \tag{3.9}$$

which are now localized, we introduce additional Lagrange multipliers. The corresponding new unit vectors in the spaces U and U^* are denoted by $\lambda_{x,i}$ and $\mu_{x,i}$, respectively. We extend the jump operators $B_i : W_i \to U^*$ and $B : W \to U^*$ by

$$\left.\begin{aligned} \langle B_i\, w_i,\, \lambda_{x,j}\rangle &= \delta_{ij}\, w_i(x) \\ \langle B\, w,\, \lambda_{x,j}\rangle &= w_j(x) \end{aligned}\right. \qquad \text{for } x \in \Gamma_{S,h} \cap \Gamma_D\,.$$

Together with $b \in U^*$ defined by

$$\begin{aligned} \langle b,\, \lambda_{x,ij}\rangle &= 0 && \text{for } x \in \Gamma_{S,h} \setminus \Gamma_D\,, \\ \langle b,\, \lambda_{x,i}\rangle &= g_D(x) && \text{for } x \in \Gamma_{S,h} \cap \Gamma_D\,, \end{aligned}$$

the modified constraints read

$$B\, u = b\,. \tag{3.10}$$

Since the Dirichlet conditions are already reflected by these constraints, our unknowns u_i are now functions to be found in

$$W_i := V_1^h(\Gamma_i)\,.$$

Since $\ker S_{i,h} = \operatorname{span}\{\mathbf{1}_\Gamma\}$ for all $i \in \mathcal{I} \setminus \{0\}$ and because the exterior Steklov-Poincaré operator $S_{0,h}$ has always a trivial kernel,

$$\mathcal{I}_{\text{float}} = \mathcal{I} \setminus \{0\}\,.$$

which justifies the expression 'all-floating'.

Finally, the corresponding saddle point formulation reads: Find $(u,\, \lambda) \in W \times U$ such that

$$\begin{pmatrix} S & B^\top \\ B & 0 \end{pmatrix} \begin{pmatrix} u \\ \lambda \end{pmatrix} = \begin{pmatrix} f \\ b \end{pmatrix}. \tag{3.11}$$

Note, that for both systems (3.8) and (3.11), $\ker S \cap \ker B = \{0\}$, and thus S is elliptic on $\ker B$. Therefore, the solution $(u,\, \lambda)$ is unique up to adding elements from $\ker B^\top$ to λ.

Remark 3.1. We see that in the all-floating formulation, all the interior operators $S_{i,h}$ have the same kind of kernel. This can be a big advantage in linear elasticity, since in the standard formulation the corresponding local kernels can have dimension from 0 up to 6.

Remark 3.2. We observe that, if Γ_0 has a common part with the Dirichlet boundary, the kernel of $S_{0,h}$ does not change whether we formulate the Dirichlet boundary conditions with Lagrange multipliers, or incorporate them in the space W_0.

The following technical assumption will be essential in Section 3.3 and its reason will become clear in the proof of Lemma 3.12 in Section 3.3.6. Note, that in practice the assumption does not really restrict our method, but is just an implementational detail.

Assumption 3.2. *If $\Gamma_0 \cap \Gamma_D \neq \emptyset$, we assume that the corresponding Dirichlet conditions are **not** formulated by Lagrange multipliers, but incorporated into the space $W_0 := V^h_{1,D}(\Gamma_0)$, also in the all-floating formulation. In accordance with Assumption 3.1 (page 23), we assume that in three dimensions the intersection of Γ_0 and Γ_D does not contain isolated points but is at least in the magnitude of a subdomain edge.*

3.1.2 Dual formulations

In the following, all derivations hold for the standard and the all-floating formulation (the choice of which effects in particular B, U, W_i and $\mathcal{I}_{\text{float}}$), and both the non-redundant and the fully redundant choice of Lagrange multipliers. Any differences will be pointed out explicitly.

We introduce parameterizations $R_i : \mathbb{R} \to \ker S_{i,h} \subset W_i$ of the local kernels and set $Z := \{\xi = (\xi_i)_{i\in\mathcal{I}} \in \mathbb{R}^{|\mathcal{I}|} : \xi_i = 0 \text{ for } i \notin \mathcal{I}_{\text{float}}\}$ and $R := \operatorname{diag}(R_i)_{i\in\mathcal{I}} : Z \to \ker S$. Furthermore, the adjoints $R_i^\top : W_i^* \to \mathbb{R}$ and $R^\top : W^* \to Z$ are defined by

$$\begin{aligned} \langle w, R_i\,\xi_i\rangle &= (R_i^\top w)\xi_i \qquad \forall w \in W_i^*, \quad \forall \xi_i \in \mathbb{R}\,, \\ \langle w, R\,\xi\rangle &= (R^\top w, \xi)_{\ell_2} \quad \forall w \in W^*, \quad \forall \xi \in Z\,. \end{aligned} \tag{3.12}$$

For all $i \in \mathcal{I}_{\text{float}}$, we introduce the regularizations $\widetilde{S}_{i,h} : W_i \to W_i^*$ by

$$\langle \widetilde{S}_{i,h}\, v, w\rangle = \langle S_{i,h}\, v, w\rangle + \beta_i (v, \mathbf{1}_\Gamma)_{L_2(\Gamma_i)} (w, \mathbf{1}_\Gamma)_{L_2(\Gamma_i)} \qquad \text{for } v, w \in W_i\,, \tag{3.13}$$

with $\beta_i > 0$, see also [60]. E. g., we can set $\beta_i := \alpha_i\, H_i^{-1} |\Gamma_i|^{-1}$; the factor $H_i^{-1}|\Gamma_i|^{-1}$ can be obtained by dilation from a domain with unit diameter. For the remaining non-floating subdomains, we simply set $\widetilde{S}_{i,h} := S_{i,h}$, for $i \in \mathcal{I} \setminus \mathcal{I}_{\text{float}}$. We can now define the following pseudo-inverses $S^\dagger_{i,h}$ by

$$S^\dagger_{i,h} := \begin{cases} \big(\widetilde{S}_{i,h}\big)^{-1} & \text{for } i \in \mathcal{I}_{\text{float}}\,, \\ \big(S_{i,h}\big)^{-1} & \text{for } i \notin \mathcal{I}_{\text{float}}\,, \end{cases}$$

and set $\widetilde{S} := \operatorname{diag}(\widetilde{S}_{i,h})_{i\in\mathcal{I}}$ and $S^\dagger := \operatorname{diag}(S^\dagger_{i,h})_{i\in\mathcal{I}}$.

The following lemma summarizes some well-known properties of the Moore-Penrose pseudo inverse in the concrete context of $S^\dagger_{i,h}$.

Lemma 3.1. *For all $i \in \mathcal{I}_{\text{float}}$ and $v \in \operatorname{range} S_{i,h}$,*

$$S_{i,h}\, S^\dagger_{i,h}\, v = v\,,$$

and

$$(S^\dagger_{i,h} v, z)_{L_2(\Gamma_i)} = 0 \qquad \forall z \in \ker S_{i,h}\,.$$

Proof. On one hand, $v \in \operatorname{range} S_{i,h}$ implies $\langle v, \mathbf{1}_\Gamma \rangle = 0$ and thus,

$$\langle S_{i,h}\, S_{i,h}^\dagger\, v, \mathbf{1}_\Gamma \rangle = \langle v, \mathbf{1}_\Gamma \rangle \,.$$

On the other hand, for $w \in W$, with $(w, \mathbf{1}_\Gamma)_{L_2(\Gamma_i)} = 0$, we obtain

$$\langle S_{i,h}\, S_{i,h}^\dagger\, v, w \rangle = \langle \widetilde{S}_{i,h}\, S_{i,h}^\dagger\, v, w \rangle - \beta_i (S_{i,h}^\dagger\, v, \mathbf{1}_\Gamma)_{L_2(\Gamma_i)} (w, \mathbf{1}_\Gamma)_{L_2(\Gamma_i)} = \langle v, w \rangle \,.$$

Combining the last two identities, we can easily derive

$$\langle S_{i,h}\, S_{i,h}^\dagger\, v, w \rangle = \langle v, w \rangle \qquad \forall w \in W \,.$$

Furthermore, we set $u = S_{i,h}^\dagger v$. Thus, $\widetilde{S}_{i,h} u = v$, and we can conclude that

$$\langle S_{i,h}\, u, \mathbf{1}_\Gamma \rangle + \beta_i (u, \mathbf{1}_\Gamma)_{L_2(\Gamma_i)} (\mathbf{1}_\Gamma, \mathbf{1}_\Gamma)_{L_2(\Gamma_i)} = \langle v\,, \mathbf{1}_\Gamma \rangle \,.$$

The first term vanishes since $\mathbf{1}_\Gamma \in \ker S_{i,h}$ and the right hand side is zero because $v \in \operatorname{range} S_{i,h}$. Hence, we obtain that $(u, \mathbf{1}_\Gamma)_{L_2(\Gamma_i)} = 0$ which directly implies above orthogonality.[2] □

We can now express each u_i in terms of the Lagrange multipliers λ by

$$u_i = S_{i,h}^\dagger [f_{i,h} - B_i^\top \lambda] + R_i\, \xi_i \tag{3.14}$$

for some $\xi \in Z$, if the compatibility condition

$$f_{i,h} - B_i^\top \lambda \in \operatorname{range} S_{i,h}$$

is satisfied, which is equivalent to $R_i^\top [f_{i,h} - B_i^\top \lambda] = 0$. We substitute the local solution u_i in the second equation of (3.8) or (3.11) by formula (3.14) and introduce the abbreviations

$$F := B\, S^\dagger B^\top \qquad G := B\, R \qquad d := B\, S^\dagger f \qquad \text{and} \qquad e := R^\top f\,, \tag{3.15}$$

where for the all-floating formulation we have to set

$$d := B\, S^\dagger f - b\,. \tag{3.16}$$

Eventually, we arrive at the dual formulation: Find $(\lambda, \xi) \in U \times Z$ such that

$$\begin{pmatrix} F & -G \\ G^\top & 0 \end{pmatrix} \begin{pmatrix} \lambda \\ \xi \end{pmatrix} = \begin{pmatrix} d \\ e \end{pmatrix} . \tag{3.17}$$

In order to eliminate the kernel correction ξ, we define two special projection operators. For this purpose, we fix an operator $Q : U^* \to U$ (which we will specify later), such that

[2] We could also use other choices of $\widetilde{S}_{i,h}$. For instance, if we regularize by $\beta_i \langle V_i^{-1} v\,, \mathbf{1}_\Gamma \rangle \langle V_i^{-1} w\,, \mathbf{1}_\Gamma \rangle$ (where V_i is the single layer potential operator on Γ_i), we get that $S_{i,h}^\dagger v$ is orthogonal to $\ker S_{i,h}$ in the inner product induced by V_i^{-1} instead of the L_2-product.

$\langle \mu, Q\,\nu \rangle$ defines an inner product on $\operatorname{range} G$. The projections $P : U \to \ker G^\top$ and $P^\top : U^* \to (\operatorname{range} G)^{\perp_Q}$ are now defined by

$$\begin{aligned} P &:= I - Q\,G\,(G^\top Q\,G)^{-1} G^\top , \\ P^\top &:= I - G\,(G^\top Q\,G)^{-1} G^\top Q\,. \end{aligned} \tag{3.18}$$

Here, $\operatorname{range}(P^\top) = (\operatorname{range} G)^{\perp_Q} = \{\mu \in U^* : \langle Q\,\mu, \mu' \rangle \quad \forall \mu' \in \operatorname{range} G\}$.

Note, that $G^\top Q\,G$ is the Galerkin projection of $B^\top Q\,B$ onto $\ker S$. Since $\ker S \cap \ker B = \{0\}$ and Q is SPD on $\operatorname{range} G$, the operator $G^\top Q\,G$ is SPD and so its inverse always exists.

Remark 3.3. Since $G = B\,R$ is a difference operator, $G^\top Q\,G$ is equivalent to the stiffness matrix of a graph Laplace problem, where each floating subdomain Ω_i, $i \in \mathcal{I}_{\text{float}}$ contributes one degree of freedom. The weights or coefficients of this graph Laplace operator are kept in the SPD-operator Q. Thus, an application of P or $P^\top$ implicitly solves a coarse problem with as many unknowns as there are floating subdomains.

Furthermore, according to [49, 87], we define the subspace $V \subset U$ of admissible Lagrange increments and a space $V' \subset U^*$ by

$$\begin{aligned} V &:= \{\lambda \in U : \langle B\,z, \lambda \rangle = 0 \quad \forall z \in \ker S\} = \ker G^\top = \operatorname{range} P\,, \\ V' &:= \{\mu \in U^* : \langle B\,z, Q\,\mu \rangle = 0 \quad \forall z \in \ker S\} = (\operatorname{range} G)^{\perp_Q} = \operatorname{range} P^\top\,. \end{aligned}$$

Additionally, we set

$$\widetilde{V}' := V' \cap \operatorname{range} B\,. \tag{3.19}$$

Remark 3.4. If Q is SPD on V' it is easily shown that the space V' can be identified with the dual of V and $\widetilde{V}'$ can be identified with the dual of the factor space $V_{/\ker B^\top}$.

We introduce the ℓ_2-projector $\widetilde{\Pi} : V' \to \widetilde{V}'$. Then

$$\widetilde{\Pi}\,P^\top = P^\top \widetilde{\Pi}\,, \qquad \widetilde{\Pi}\,F = F, \qquad \widetilde{\Pi}\,G = G, \qquad \widetilde{\Pi}\,d = d\,, \tag{3.20}$$

and it is clear that

$$F\,(\lambda + \ker B^\top) = F\,\lambda\,. \tag{3.21}$$

Multiplying the first equation in (3.17) from the left by $P^\top$ and $(I - P^\top)$, we obtain

$$\begin{aligned} P^\top F\,\lambda &= P^\top d\,, \\ (I - P^\top) F\,\lambda - G\,\xi &= (I - P^\top) d\,. \end{aligned}$$

Note, that due to (3.20) and (3.21) these equations neither change if we multiply them from the left by $\widetilde{\Pi}$ nor if we add an element of $\ker B^\top$ to λ.

Splitting $\lambda = \lambda_0 + \widetilde{\lambda}$ with $\widetilde{\lambda} \in V$, we obtain for the second equation in (3.17) that

$$G^\top \lambda_0 = e\,.$$

We see that we have now decoupled all the equations, and that the solution $(\lambda, \xi) = (\lambda_0 + \widetilde{\lambda}, \xi)$ is given by

$$\lambda_0 = Q\,G\,(G^\top Q\,G)^{-1} e\,, \tag{3.22}$$

$$P^\top F\,\widetilde{\lambda} = P^\top (d - F\,\lambda_0)\,, \tag{3.23}$$

$$\xi = (G^\top Q\,G)^{-1} G^\top Q\,(F\,\lambda - d)\,. \tag{3.24}$$

Hence, if we solve (3.23) for $\widetilde{\lambda}$, we can easily compute λ, ξ, and finally get u from formula (3.14). By $V = \operatorname{range} P$ and $V' = \operatorname{range} P^\top$, we see that on the subspace $V_{/\ker B^\top}$ the operator $P^\top F = P^\top F\,P$ is SPD, and so equation (3.23) can be solved by a preconditioned conjugate gradient (PCG) subspace iteration on $V_{/\ker B^\top}$.

It was proved in [33] that the condition number of the un-preconditioned FETI system grows like $\mathcal{O}(H/h)$, which emphasizes the importance of preconditioners.

3.2 Preconditioners for the one-level method

We are now searching for some preconditioner $M^{-1} : U^* \to U$ for $P^\top F$ on the factor space $V_{/\ker B^\top}$, i. e., we aim for

$$\lambda_{\min} \|\lambda\|_{\ell_2} \le (P\,M^{-1} P^\top F\,\lambda,\, \lambda)_{\ell_2} \le \lambda_{\max} \|\lambda\|_{\ell_2} \qquad \forall \lambda \in V_{/\ker B^\top}$$

with $\lambda_{\max}/\lambda_{\min} \le C$ as small as possible. We informally write

$$\kappa(P\,M^{-1} P^\top F\,P) \le C$$

to be understood in the factor space modulo $\ker B^\top$.

3.2.1 Weighted counting functions

A key tool for the robustness of FETI methods with respect to coefficient jumps are the weighted counting functions, which were first introduced by Sarkis [76], and which play also a crucial role in balancing Neumann-Neumann methods; see [87, Chapter 6] and the references therin.

For a node x of our triangulation, let $\mathbf{N}_x$ denote the set of indices of those subdomains that touch that node. Since this set does not change on edges and faces, we write $\mathbf{N}_E$ and $\mathbf{N}_F$ for the sets of indices of those subdomains that touch the edge E and the face F, respectively. For a fixed number $\gamma \in [1/2, \infty)$, we can now define the weighted counting functions $\delta_i^\dagger \in V_1^h(\Gamma_S)$ by

$$\delta_i^\dagger(x) := \begin{cases} \dfrac{\alpha_i^\gamma}{\sum\limits_{k \in \mathcal{N}_x} \alpha_k^\gamma} & \text{for } x \in \Gamma_{i,h}\,, \\ 0 & \text{for } x \in \Gamma_{S,h} \setminus \Gamma_{i,h}\,. \end{cases} \tag{3.25}$$

These functions provide a the following partition of unity on $\Gamma_{S,h}$:

$$\sum_{i \in \mathcal{I}} \delta_i^\dagger(x) \equiv 1 \qquad \forall x \in \Gamma_{S,h}\,.$$

Lemma 3.2. *The estimate*

$$\alpha_i \left(\delta_j^\dagger(x)\right)^2 \leq \min(\alpha_i,\, \alpha_j) \qquad \forall x \in \Gamma_{i,h} \cap \Gamma_{j,h} \tag{3.26}$$

holds for any choice of the exponent $\gamma \in [1/2, \infty)$.

Proof. We have to prove that

$$\frac{\alpha_i\, \alpha_j^{2\gamma}}{\Big(\sum\limits_{k\in\mathcal{N}_x} \alpha_k^\gamma\Big)^2} \leq \min(\alpha_i,\, \alpha_j)\,,$$

for each $x \in \Gamma_{i,h} \cap \Gamma_{j,h}$. The estimate is trivial if $\alpha_i \leq \alpha_j$. In the other case we note that the function $s \mapsto s^{1-2\gamma}$ is monotonically decreasing for $\gamma \in [1/2, \infty)$, and so we have

$$\frac{\alpha_i\, \alpha_j^{2\gamma}}{\Big(\sum\limits_{k\in\mathcal{N}_x} \alpha_k^\gamma\Big)^2} \leq \frac{\alpha_i^{2\gamma}}{\Big(\sum\limits_{k\in\mathcal{N}_x} \alpha_k^\gamma\Big)^2}\, \alpha_i^{1-2\gamma}\, \alpha_j^{2\gamma} \leq 1 \cdot \alpha_j^{1-2\gamma}\, \alpha_j^{2\gamma} \leq \alpha_j\,.$$

□

3.2.2 Non-redundant Lagrange multipliers

For the case of non-redundant multipliers the following preconditioner was first introduced and analyzed in [49] for the FETI method, and in [58] for the BETI method, respectively. Let us define

$$M^{-1} := (B\, D^{-1} B^\top)^{-1} B\, D^{-1}\, S\, D^{-1} B^\top (B\, D^{-1} B^\top)^{-1} \tag{3.27}$$

with $D := \operatorname{diag}(D_i)_{i\in\mathcal{I}} : W \to W^*$, and $D_i : W_i \to W_i^*$ defined by

$$\begin{aligned} \langle D_i\, \Phi_i \underline{w},\, \Phi_i \underline{w}\rangle &= (\mathbf{D}_i\, \underline{w},\, \underline{w})_{\ell_2}\,, \\ \mathbf{D}_i &:= \operatorname{diag}(\mathbf{d}_{i,x})_{x\in\Gamma_{i,h}}\,, \\ \mathbf{d}_{i,x} &:= \begin{cases} \delta_i^\dagger(x) & x \in \Gamma_{i,h} \setminus \Gamma_D\,, \\ 1 & x \in \Gamma_{i,h} \cap \Gamma_D\,. \end{cases} \end{aligned}$$

Here,

$$\Phi_i : \mathbb{R}^{\dim(W_i)} \to W_i : \underline{w} \mapsto w = \sum_{x_j\in\Gamma_{i,h}} \underline{w}_j \varphi_j$$

denotes the usual FE isomorphism, $\mathcal{N}_x = \{j \in \mathcal{I} : x \in \Gamma_j\}$ is the index set of subdomains touching the node x, and γ is an arbitrary but fixed real exponent with $\gamma \geq 1/2$. Due to their diagonal representation given above, it is clear that D_i^{-1} and D^{-1} always exist. Note, that Γ_D is regarded as a closed set, and that for nodes x lying on subdomain faces included in the Neumann boundary Γ_N also $\mathbf{d}_{i,x} = 1$.

Remark 3.5. In the definition (3.27) of the preconditioner M^{-1}, the local operators $S_{i,h}$ appearing in S may be replaced by any other preconditioners for the local operators $S_{i,h}^\dagger$ which are spectrally equivalent to $S_{i,h}$. There are many preconditioners from the standard finite element literature, cf., e. g., [52, 87]. Another possible choice is the discrete hypersingular operator $D_{i,h}$, see [54, 58, 59].

In the following, we state some lemmas in order to get a certain characterization of our preconditioner at the end of Section 3.2 in Corollary 3.6. This goes along the theory in [87] and is strongly related to the theory of Neumann-Neumann methods.

Lemma 3.3. *For the non-redundant case, the projection operator $P_D : W \mapsto W$ defined by*

$$P_D := D^{-1} B^\top (B\, D^{-1} B^\top)^{-1} B \tag{3.28}$$

satisfies

$$B\, P_D = B \qquad \text{and} \qquad I - P_D : W \to \widehat{W}\,,$$

i. e., the finite element function $w - P_D\, w$ is continuous across sudomain interfaces for all $w \in W$. Furthermore, $w - P_D\, w$ satisfies the homogeneous Dirichlet boundary conditions, and $P_D\, w$ vanishes at nodes on the Neumann boundary which do not belong to the interface.

Proof. From the definition of P_D we see that $B\, P_D = B$ and $B(I - P_D) = 0$. Therefore, any function $w - P_D\, w$ is continuous across subdomain interfaces. For the all-floating formulation, we obtain already from $B(I - P_D) = 0$ that the function $w - P_D\, w$ satisfies the homogeneous Dirichlet boundary conditions. For the standard formulation, we can observe that $B^\top$ does not contribute to the Dirichlet nodes, and neither does P_D. Hence, $(w - P_D\, w)_i|_{\Gamma_D} = w_i|_{\Gamma_D}$ for all $i \in \mathcal{I}$, and due to our assumption $w_i \in W_i$ already satisfies the homogeneous Dirichlet boundary conditions. For both formulations, $B^\top$ does not contribute to the nodes on the Neumann boundary that do not belong to the interface, and so $P_D\, w$ vanishes there too. □

Lemma 3.4. *With the projection $E_D : W \to \widehat{W}$ defined by*

$$(E_D\, w)_i(x) := \begin{cases} \sum\limits_{j \in \mathcal{N}_x} \delta_j^\dagger(x)\, w_j(x) & \text{for } x \in \Gamma_{i,h} \setminus \Gamma_D\,, \\ 0 & \text{for } x \in \Gamma_{i,h} \cap \Gamma_D\,, \end{cases}$$

the identity

$$P_D = I - E_D$$

holds.

Proof. The proof for the standard formulation can be found in [49, 87]. Note, that on nodes $x \in \Gamma_{i,h} \cap \Gamma_N$ that do not belong to the interface Γ_I, $\mathcal{N}_x$ consists only of one index, namely i. Since $P_D\, w$ gives no contribution to these nodes, the formula $P_D = I - E_D$ remains valid. For the all-floating formulation, we additionally need to consider the nodes on $\Gamma_{i,h} \cap \Gamma_D$. From the previous lemma we know that $(w - P_D\, w)_i|_{\Gamma_D} = 0$. By the definition of E_D also $(E_D\, w)_i|_{\Gamma_D} = 0$. Thus, $I - P_D = E_D$. □

3.2.3 Fully redundant Lagrange multipliers

For the fully redundant case, the following preconditioner is chosen, which was originally proposed for the FETI method in [75] and analyzed in [49],

$$M^{-1} = B_{D^r}\, S\, B_{D^r}^\top \tag{3.29}$$

with $B_{D^r} := \sum_{i\in\mathcal{I}} B_{D^r}^{(i)}$, where the operator $B_{D^r}^{(i)} : W_i^* \to U$ and its adjoint $B_{D_r}^{(i)} : U^* \to W_i$ are defined by

$$[(B_{D^r}^{(i)})^\top \mu_{x,jk}](y) := \begin{cases} \delta_k^\dagger(x) & \text{for } k < j = i,\, x = y, \\ -\delta_j^\dagger(x) & \text{for } i = k < j,\, x = y, \\ 0 & \text{else} \end{cases}$$

for $x \in \Gamma_{i,h} \setminus \Gamma_D$ and $j \in \mathcal{N}_x$, and

$$[(B_{D^r}^{(i)})^\top \mu_{x,j}](y) := \begin{cases} 1 & \text{for } i = j,\, x = y, \\ 0 & \text{else,} \end{cases}$$

for $x \in \Gamma_{i,h} \cap \Gamma_D$. Of course, as in the non-redundant case, the operator S appearing in the definition (3.29) of M^{-1} can be replaced by any spectral equivalent operator, cf. Remark 3.5.

Lemma 3.5. *For the redundant case we have*

$$B_{D^r}^\top B = P_D\,. \tag{3.30}$$

Proof. Using the definition of B (cf. (3.3)) and B_{D^r} and due to the fact that we have fully redundant constraints, we obtain the following formula for the all-floating formulation:

$$(B_{D^r}^{(i)})^\top B\, w = (B_{D^r}^{(i)})^\top \sum_{i\in\mathcal{I}} \Bigg[\sum_{\substack{x\in\Gamma_{i,h}\cap\Gamma_I \\ j\in\mathcal{N}_x}} \big(w_{\max(i,j)}(x) - w_{\min(i,j)}(x)\big)\, \mu_{x,ij} + \\ + \sum_{x\in\Gamma_{i,h}\cap\Gamma_D} w_i(x)\, \mu_{x,i} \Bigg],$$

for all $w \in W$. For the standard formulation, the terms corresponding to $\mu_{x,i}$ must be left out. Hence,

$$\big[(B_{D^r}^{(i)})^\top B\, w\big](x) = \begin{cases} \sum_{j\in\mathcal{N}_x} \delta_j^\dagger(x)\, \big(w_i(x) - w_j(x)\big) & \text{for } x \in \Gamma_{i,h} \setminus \Gamma_D\,, \\ w_i(x) & \text{for } x \in \Gamma_{i,h} \cup \Gamma_D\,. \end{cases}$$

This formula also holds for the standard formulation, since $w_i|_{\Gamma_D} = 0$, and it proves that $B_{D^r}^\top B = I - E_D = P_D$, cf. Lemma 3.4. □

3.2.4 A unified characterization of all BETI formulations

Corollary 3.6. *For all BETI formulations, no matter if non-redundant or fully redundant Lagrange multipliers, standard or all-floating formulation*

$$B^\top M^{-1} B = P_D^\top S\, P_D$$

holds. In other words, the projection $P_D = I - E_D$ characterizes all the preconditioners of the various formulations.

FETI/BETI-PCG-Algorithm

$e = R^\top f$
$\underline{d} = B\, S^\dagger f$
$\boldsymbol{\lambda}^{(0)} = Q\, G(G^\top Q\, G)^{-1}\, e$
$\underline{r}^{(0)} = P^\top(\underline{d} - F\, \boldsymbol{\lambda}^{(0)})$
$\boldsymbol{s}^{(0)} = \boldsymbol{z}^{(0)} = P\, M^{-1}\, \underline{r}^{(0)}$
$\beta_0 = \langle \underline{r}^{(0)}, \boldsymbol{z}^{(0)} \rangle$
$n = 0$
while $(\beta_n \leq \beta_0 \cdot \varepsilon$ and $n < n_{\max})$
{
 $\underline{x}^{(n)} = P^\top F\, \boldsymbol{s}^{(n)}$
 $\alpha_n = \langle \underline{x}^{(n)}, \boldsymbol{s}^{(n)} \rangle$
 $\alpha = \beta_n / \alpha_n$
 $\boldsymbol{\lambda}^{(n+1)} = \boldsymbol{\lambda}^{(n)} + \alpha\, \boldsymbol{s}^{(n)}$
 $\underline{r}^{(n+1)} = \underline{r}^{(n)} - \alpha\, \underline{x}^{(n)}$
 $\boldsymbol{z}^{(n+1)} = P\, M^{-1}\, \underline{r}^{(n+1)}$
 $\beta_{n+1} = \langle \underline{r}^{n+1}, \boldsymbol{z}^{(n+1)} \rangle$
 $\beta = \beta_{n+1} / \beta_n$
 $\boldsymbol{s}^{(n+1)} = \boldsymbol{z}^{(n+1)} + \beta\, \boldsymbol{s}^{(n)}$
 $n = n + 1$
}
$\gamma = (G^\top Q\, G)^{-1} G^\top Q\, (F\, \boldsymbol{\lambda}^{(n)} - \underline{d})$
$u = S^\dagger(f - B^\top \boldsymbol{\lambda}^{(n)}) + R\, \gamma$

Figure 7: The variables in the spaces U and U^* are emphasized in the following way: Bold variables (e. g., $\boldsymbol{s}^{(n)}$) correspond to unknowns in V or $\lambda_0 + V$. The underlined variables (e. g., $\underline{x}^{(n)}$) correspond to unknowns in $\widetilde{V}' = V' \cap \operatorname{range} B$. The key observations is that all operators applied to the bold variables (i. e., F and $B^\top$) are invariant if we add elements from $\ker B^\top$, and any inner product of the form $\langle \underline{x}^{(n)}, \boldsymbol{s}^{(n)} \rangle$ is invariant if we add terms from $\ker B^\top$ to $\boldsymbol{s}^{(n)}$ since $\underline{x}^{(n)} \in \operatorname{range} B$. Furthermore, all terms assigned to the underlined variables, such as $\underline{r}^{(n)}$ really stay in $\widetilde{V}' = V' \cap \operatorname{range} B$.

Figure 7 displays the entire FETI/BETI algorithm, cf. also [58, 59, 87].

Remark 3.6. The present method contains a coarse solver in the projections P, $P^\top$. The name *one-level* method refers to this specific kind of coarse solver, not to the entire algorithm. On the contrary, *two-level* FETI methods were designed for biharmonic and shell problems [29, 32] and, as we have already mentioned in the introduction, in the case of dual-primal methods (FETI-DP, BETI-DP), a completely different coarse solver shows up.

Remark 3.7. The BETI-Algorithm can easily be parallelized. Then the bold variables (see Figure 7) correspond to accumulated vectors, whereas the underlined variables correspond to distributed vectors. The only communication is necessary for the evaluation of the inner products, the preconditioner, the projection and the operations involving the coarse solver $(G^\top Q\, G)^{-1}$.

3.3 Condition number estimates for one-level BETI preconditioners

In this section, we give the condition number estimates for the preconditioners introduced in the last subsection. First, in Section 3.3.1 we state some basic results and specify the operator Q which appears in the projections P and $P^\top$. Secondly, we need to introduce an *extension indicator* in Section 3.3.2 before we state our main result, Theorem 3.1, and related material in Section 3.3.3. The main proof relies first on an estimate concerning the extension indicator which is formulated in Lemma 3.11 and secondly on a special stability result stated in Lemma 3.12. In Section 3.3.4 we elaborate technical tools which we need for the proofs of Lemma 3.11 and Lemma 3.12 which are finally presented in Section 3.3.5 and Section 3.3.6.

3.3.1 Basic results and the operator Q

We define the energy semi-norms

$$|w_i|_{S_{i,h}} := \langle S_{i,h} w_i,\, w_i\rangle^{1/2} \qquad \text{for } w_i \in W_i\,,$$
$$|w|_S := \Big(\sum_{i\in\mathcal{I}} |w|^2_{S_{i,h}}\Big)^2 \qquad \text{for } w \in W\,,$$

and the subspaces

$$(\ker S_{i,h})^\perp := \{w \in W_i : (w,\, z)_{L_2(\Gamma_i)} = 0 \quad \forall z \in \ker S_{i,h}\}$$
$$(\ker S)^\perp := \{w \in W : w_i \in (\ker S_{i,h})^\perp \quad \forall i \in \mathcal{I}\}\,.$$

Note, that $|\cdot|_{S_{i,h}}$ is a norm on $(\ker S_{i,h})^\perp$ and $|\cdot|_S$ is a norm on $(\ker S)^\perp$.

First, we prove that the preconditioner M^{-1} defined in Section 3.2 is SPD on $\widetilde{V}'$.[3]

Lemma 3.7. *If Q is SPD on* range G*, then M^{-1} is SPD on $\widetilde{V}'$, i. e.,*

$$\langle M^{-1}\mu,\, \mu\rangle > 0 \qquad \forall \mu \in \widetilde{V}' \setminus \{0\}\,.$$

[3]In the standard literature, e. g., in [49, 87], the theory is only proved for the diagonal choice of Q. For $Q = M^{-1}$ the definiteness is not shown explicitly. The above lemma shall fix this gap.

Proof. From the definitions (3.27) and (3.29), we see that M^{-1} is symmetric and positive semi-definite, and so M^{-1} is symmetric and positive semi-definite on $\widetilde{V}'$. To show the definiteness we assume that $\langle \mu, M^{-1}\mu\rangle = 0$ for some $\mu \in \widetilde{V}'$. Since $\mu \in \operatorname{range} B$, we can find a $w \in W$ with $B\,w = \mu$. Setting $\widetilde{w} = P_D\,w$ we also have $B\,\widetilde{w} = \mu$, cf. Lemma 3.3, and since P_D is a projection we have $P_D\,\widetilde{w} = \widetilde{w}$. Applying Corollary 3.6 we obtain that

$$0 = \langle \mu, M^{-1}\mu\rangle = \langle B\,\widetilde{w}, M^{-1}B\,\widetilde{w}\rangle = |P_D\,\widetilde{w}|_S^2 = |\widetilde{w}|_S^2\,,$$

We conclude that $\widetilde{w} \in \ker S$. From the definition of $\widetilde{V}'$ (cf. (3.19)) we know that

$$\langle B\,\widetilde{w}, Q\,B\,\widetilde{w}\rangle = \langle B\,\widetilde{w}, Q\,\mu\rangle = 0\,.$$

because $\mu \in \widetilde{V}'$. Finally, since Q is SPD on range G which is the image of $\ker S$ under B, we obtain that $B\,\widetilde{w} = 0$. Therefore, also $\mu = 0$ which proves the definiteness. □

Lemma 3.7 justifies to define an operator $M : V_{/\ker B^\top} \to \widetilde{V}'$ as the inverse of $\widetilde{\Pi}^\top P\,M^{-1} : \widetilde{V}' \to V_{/\ker B^\top}$, see Remark 3.4. Following the proof by Klawonn and Widlund, we show that for some $\lambda_{\min}, \lambda_{\max} > 0$,

$$\lambda_{\min}\langle M\,\lambda, \lambda\rangle \le \langle F\,\lambda, \lambda\rangle \le \lambda_{\max}\langle M\,\lambda, \lambda\rangle \qquad \forall \lambda \in V_{/\ker B^\top}\,. \tag{3.31}$$

Then $\lambda_{\max}/\lambda_{\min}$ is an upper bound for the condition number $\kappa(P\,M^{-1}P^\top F)$ on $V_{/\ker B^\top}$. First, we need several technical lemmas.

Throughout this work, we agree that writing $\sup\limits_{x\in X} \frac{a(x)}{b(x)}$ implicitly excludes those x from X with $a(x) = b(x) = 0$.

Lemma 3.8. *Let Q be SPD on V'. Then the following identities hold:*

$$\langle F\,\lambda, \lambda\rangle = \sup_{w\in W} \frac{\langle B\,w, \lambda\rangle^2}{|w|_S^2} = \sup_{w\in \ker S^\perp} \frac{\langle B\,w, \lambda\rangle^2}{|w|_S^2} \qquad \forall \lambda \in V\,, \tag{3.32}$$

$$\langle M\,\lambda, \lambda\rangle = \sup_{\mu\in\widetilde{V}'} \frac{\langle \mu, \lambda\rangle^2}{\langle \mu, M^{-1}\mu\rangle} \qquad \forall \lambda \in V_{/\ker B^\top}\,, \tag{3.33}$$

$$\langle B\,w, M^{-1}B\,w\rangle = |P_D\,w|_S^2 \qquad \forall w \in W\,. \tag{3.34}$$

Proof. The proof has been presented in [49, 87]. By the definition of $S^\dagger$, $\widetilde{S}$ and Lemma A.2 we get for $\lambda \in V$:

$$\begin{aligned}\langle F\,\lambda, \lambda\rangle = \langle B^\top\lambda, S^\dagger B^\top\lambda\rangle = \langle B^\top\lambda, \widetilde{S}^{-1}B^\top\lambda\rangle &= \sup_{w\in W} \frac{\langle B^\top\lambda, w\rangle^2}{\langle \widetilde{S}\,w, w\rangle}\\ = \sup_{w\in W} \frac{\langle B\,w, \lambda\rangle^2}{\langle S\,w, w\rangle + \sum\limits_{i\in\mathcal{I}_{\text{float}}} \beta_i(w_i\,, \mathbf{1}_\Gamma)_{L_2(\Gamma_i)}^2} &= \sup_{w\in W} \frac{\langle B\,w, \lambda\rangle^2}{|w|_S^2}\,.\end{aligned}$$

In the last step we have used that $\beta_i \ge 0$ and that $(w_i, \mathbf{1}_\Gamma)_{L_2(\Gamma_i)} = 0$ for $w_i \in (\ker S_{i,h})^\perp$. Note, that $\sup_{w\in W}\langle B\,w, \lambda\rangle^2/|w|_S^2$ is well-defined due to our convention on suprema, since

$|w|_S = 0$ implies $\langle B\,w,\,\lambda\rangle = 0$ for $\lambda \in V$. For $w \in W$, $w_0 \in \ker S$ and $\lambda \in V$, one easily shows the shift invariances

$$\langle B\,(w+w_0),\,\lambda\rangle = \langle B\,w,\,\lambda\rangle \qquad \text{and} \qquad |w+w_0|_S = |w|_S\,, \tag{3.35}$$

and altogether, this proves (3.32). The second identity, (3.33), follows from the definition of M and the fact that the dual of $V_{/\ker B^\top}$ is isomorphic to $\widetilde{V}'$, see also the remark on page 27. The third formula (3.34) follows directly from Corollary 3.6. □

Lemma 3.9. *Let Q be SPD on* range G *and symmetric positive semi-definite on U^*. Then, for any $w \in W$, there exists a unique $z_w \in \ker S$ such that $B(w+z_w) \in V'$. Moreover,*

$$z_w = \operatorname*{argmin}_{z \in \ker S} \|B\,(w+z)\|_Q\,, \qquad \textit{and} \qquad \|B\,z_w\|_Q \le \|B\,w\|_Q\,,$$

where $\|\mu\|_Q := \langle \mu,\,Q\,\mu\rangle^{1/2}$. The mapping $w \mapsto z_w$ is linear.

Proof. See [49, 87]. There, z_w is explicitly constructed by

$$z_w = -R\,(G^\top Q\,G)^{-1} G^\top Q\,B\,w\,,$$

from where we see that z_w depends linearly on w. □

Up to now, we have not specified our operator $Q : U^* \to U$ which appears in the projections P, $P^\top$. For the standard case, Klawonn and Widlund (cf. [49, 87]) propose either setting $Q = M^{-1}$ or a special diagonal choice of Q which we generalize for the all-floating case in the following.

The choice $Q = M^{-1}$. In this case, it remains to show that M^{-1} is SPD on range G, such that the projections P, $P^\top$ are well-defined and that the assumptions of Lemma 3.7 and Lemma 3.9 are fulfilled. Note, that with Lemma 3.7, Q is automatically SPD on V', and thus the assumption of Lemma 3.8 is satisfied.

Lemma 3.10. *M^{-1} is symmetric positive definite on* range G*, i. e.,*

$$\langle B\,z,\,M^{-1}B\,z\rangle > 0 \qquad \textit{for } z \in \ker S \setminus \{0\}\,.$$

Proof. We have already seen that M^{-1} is symmetric and positive semi-definite. Assume that $\langle B\,z,\,M^{-1}B\,z\rangle = 0$ holds for some $z \in \ker S$. Due to Corollary 3.6, $|P_D\,z|_S^2 = 0$, and so $P_D\,z = z - E_D\,z \in \ker S$. Consequently, $E_D\,z \in \ker S$ which means that each component of $E_D\,z$ is constant on Γ_i. However, $E_D\,z$ is continuous across subdomain interfaces and satisfies the homogeneous Dirichlet boundary conditions (if present). Additionally, if $0 \in \mathcal{I}$, we know that $z_0 = 0$. Since two arbitrary subdomains can be connected via a path through subdomain faces, an elementary argument shows that $z \equiv 0$. From this we finally conclude that $B\,z = 0$ which shows the definiteness. □

Remark 3.8. The choice $Q = M^{-1}$ is of course expensive because it means for every application of P, $P^\top$ one more application of M^{-1}; additionally the application of $(G^\top Q\,G)^{-1}$ becomes more complicated. The following choice is much cheaper from the computational point of view, although it turns out that a more careful analysis is required.

A diagonal choice of Q. A second possibility is to choose Q diagonal with respect to the basis $\mu_{x,ij}$, $\mu_{x,i}$ which we have introduced in Section 3.1.1. This means, Q is uniquely defined by specifying

$$\begin{aligned} \langle \mu_{x,ij}, Q\,\mu_{x,ij}\rangle &=: Q[\mu_{x,ij}] \qquad \text{for } x \in (\Gamma_{i,h} \cap \Gamma_{j,h}) \setminus \Gamma_D\,, \\ \langle \mu_{x,i}, Q\,\mu_{x,i}\rangle &=: Q[\mu_{x,i}] \qquad \text{for } x \in \Gamma_{i,h} \cap \Gamma_D\,. \end{aligned}$$

In three dimensions, we set for $x \in (\Gamma_{i,h} \cap \Gamma_{j,h}) \cap \Gamma_I$,

$$Q[\mu_{x,ij}] = \begin{cases} \min(\alpha_i, \alpha_j)\big(1 + \log\big(\frac{H_F}{h_i}\big)\big)\frac{h_i^2}{H_F} & \text{for } x \in F \in \mathcal{F}_i \cap \mathcal{F}_j\,, \\ \min(\alpha_i, \alpha_j) h_i & \text{for } x \in E \in \mathcal{E}_i \cap \mathcal{E}_j\,, \\ \min(\alpha_i, \alpha_j) h_i & \text{for } x = V \in \mathcal{V}_i \cap \mathcal{V}_j\,, \end{cases} \tag{3.36}$$

whereas in two dimensions we set

$$Q[\mu_{x,ij}] = \begin{cases} \min(\alpha_i, \alpha_j)\big(1 + \log\big(\frac{H_E}{h_i}\big)\big)\frac{h_i}{H_E} & \text{for } x \in E \in \mathcal{E}_i \cap \mathcal{E}_j\,, \\ \min(\alpha_i, \alpha_j) & \text{for } x = V \in \mathcal{V}_i \cap \mathcal{V}_j\,. \end{cases} \tag{3.37}$$

For the all-floating formulation, we define the additional entries for $x \in \Gamma_{i,h} \cap \Gamma_D$ by

$$Q[\mu_{x,i}] = \begin{cases} \alpha_i\,\big(1 + \log\big(\frac{H_F}{h_i}\big)\big)\frac{h_i^2}{H_F} & \text{for } x \in F \in \mathcal{F}_i\,, \\ \alpha_i\, h_i & \text{for } x \in E \in \mathcal{E}_i\,, \\ \alpha_i\, h_i & \text{for } x = V \in \mathcal{V}_i\,, \end{cases} \tag{3.38}$$

in three dimensions, and

$$Q[\mu_{x,i}] = \begin{cases} \alpha_i\,\big(1 + \log\big(\frac{H_E}{h_i}\big)\big)\frac{h_i}{H_E} & \text{for } x \in E \in \mathcal{E}_i\,, \\ \alpha_i & \text{for } x = V \in \mathcal{V}_i\,. \end{cases} \tag{3.39}$$

in two dimensions. Note, that by Assumption 3.2 (page 25) we have no Lagrange multipliers of the form $\mu_{x,0}$, i. e., no Lagrange multipliers on Γ_0 that enforce Dirichlet boundary conditions, even in the all-floating formulation.

For the case of non-redundant Lagrange multipliers, we need an additional restriction, namely that the constraints are chosen according to the 'fork' distribution depicted in Figure 6 (page 22), where the distinguished node lies on the boundary of the subdomain with the locally largest coefficient, cf. [49, 87].

3.3.2 The extension indicator

The following two definitions and the following lemma are needed to obtain good condition number estimates for the one-level BETI method in the case of certain geometric configurations, in particular if interior Dirichlet boundary conditions are present.

Definition 3.1. *The extension indicator $\gamma_h(\{\Omega_i\}, \Gamma_0, \Gamma_D)$, depending on the domain decomposition $\{\Omega_i\}_{i\in\mathcal{I}}$, on the geometry of the boundary Γ_0, the Dirichlet boundary Γ_D and the discretization (indicated by h), is defined by*

$$\gamma_h(\{\Omega_i\}, \Gamma_0, \Gamma_D) := \sup_{\substack{u_h \in V_1^h(\Gamma_0) \\ u_h|_{\Gamma_0\cap\Gamma_D} = 0}} \frac{|\mathcal{H}^{\text{int}}_{0,D}\, u_h|^2_{H^1(\Omega_{\text{int}})}}{|\mathcal{H}^{\text{int}}\, u_h|^2_{H^1(\Omega_{\text{int}})} + \frac{1}{\operatorname{diam}\Gamma_0}\|u_h\|^2_{L_2(\Gamma_0)}}\,. \tag{3.40}$$

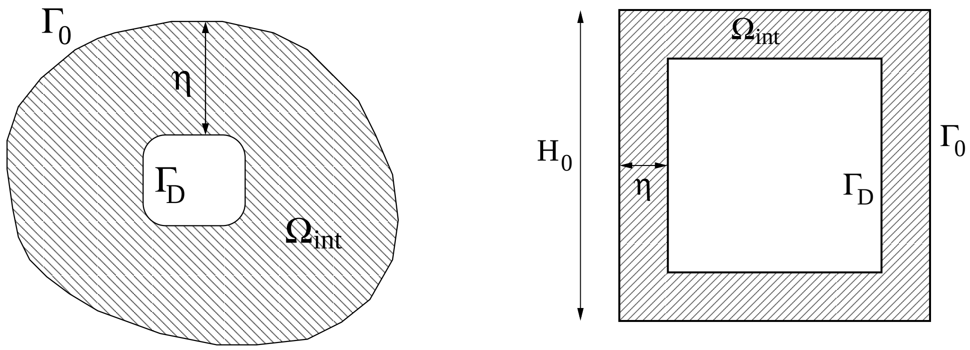

Figure 8: Two ring-shaped configurations of Ω_{int}, Γ_0 and Γ_D. Left: $\eta \simeq H_0$. Right: $\eta \ll H_0$.

Here, $\mathcal{H}^{\text{int}}$ denotes the unrestricted discrete harmonic extension from Γ_0 to the open, bounded domain Ω_{int} defined by

$$\overline{\Omega_{\text{int}}} := \bigcup_{i \in \mathcal{I} \setminus \{0\}} \overline{\Omega_i}\,.$$

I. e., for $u_h \in V_1^h(\Gamma_0)$

$$\mathcal{H}^{\text{int}}\, u_h := \underset{\substack{\widetilde{u}_h \in V_1^h(\Omega_{\text{int}}) \\ \widetilde{u}_h|_{\Gamma_0} = u_h}}{\operatorname{argmin}} |\widetilde{u}_h|^2_{H^1(\Omega_{\text{int}})}\,.$$

The operator $\mathcal{H}^{\text{int}}_{0,D}$ denotes the minimal discrete harmonic extension from Γ_0 to Ω_{int} that satisfies the homogeneous Dirichlet boundary conditions on $\Gamma_D \setminus \Gamma_0$, i. e.,

$$\mathcal{H}^{\text{int}}_{0,D}\, u_h := \underset{\substack{\widetilde{u}_h \in V_1^h(\Omega_{\text{int}}) \\ \widetilde{u}_h|_{\Gamma_0} = u_h}}{\operatorname{argmin}} |\widetilde{u}_h|^2_{H^1(\Omega_{\text{int}})}\,.$$

Definition 3.2. *The shape parameter $\eta > 0$ of Ω_{int} is defined as the largest number such that the boundary layer*

$$\Omega_{\text{int},\eta} := \{x \in \Omega_{\text{int}} : \operatorname{dist}(x, \Gamma_0) < \eta\}\,,$$

can be decomposed into shape-regular patches $\{\omega_j\}_{j\in\mathcal{J}}$ (or covered by them such that $\bigcup_{i\in\mathcal{J}} \omega_j \subset \Omega_{\text{int}}$), where $\operatorname{diam} \omega_j = \mathcal{O}(\eta)$ and $\operatorname{meas}_{d-1}(\partial\omega_j \cap \Gamma_0) = \mathcal{O}(\eta^{d-1})$ for all $j \in \mathcal{J}$, see also Figure 8 and Figure 9.

Lemma 3.11. *Let $w_0 \in V_{1,D}(\Gamma_0)$ be an arbitrary discrete function on the boundary Γ_0 fulfilling the Dirichlet boundary conditions on $\Gamma_0 \cap \Gamma_D$. Then the following estimates hold.*

(i) If $\Gamma_D \setminus \Gamma_0$ is empty we have

$$\gamma_h(\{\Omega_i\}, \Gamma_0, \Gamma_D) = 1$$

(ii) If $\operatorname{dist}(\Gamma_D \setminus \Gamma_0, \Gamma_0) > 0$, we have

$$\gamma_h(\{\Omega_i\}, \Gamma_0, \Gamma_D) \preceq \frac{H_0}{\operatorname{dist}(\Gamma_D \setminus \Gamma_0, \Gamma_0)}\,.$$

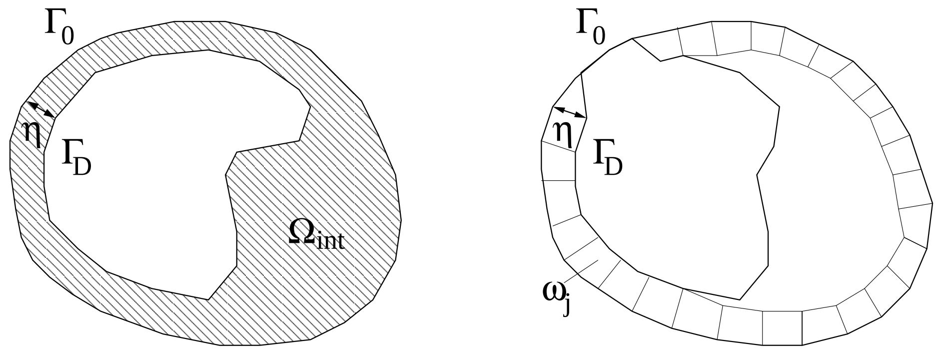

Figure 9: A configuration where Γ_D touches Γ_0. Right: Construction of the patches ω_j according to Definition 3.2.

(iii) Otherwise, if $\Gamma_D \setminus \Gamma_0$ *touches* Γ_0 *(according to Assumption 3.2, page 25),*

$$\gamma_h(\{\Omega_i\}, \Gamma_0, \Gamma_D) \preceq \frac{H_0}{\eta}\Big(1 + \log\Big(\frac{\eta}{h}\Big)\Big)^2$$

holds, with the shape parameter $\eta > 0$ *due to Definition 3.2.*

Proof. We postpone the proof to Section 3.3.4 □

Remark 3.9. In general, estimate (ii) in Lemma 3.11 is sharp. For the two-dimensional ring-shaped domain depicted in Figure 8, right, one can easily show that for $w_0 \equiv 1$,

$$|\mathcal{H}_{0,D}^{\text{int}}\, w_0|^2_{H^1(\Omega_{\text{int}})} \succeq \frac{H_0}{\eta}\,\frac{1}{H_0}\|w_0\|^2_{L_2(\Gamma_0)}\,.$$

The same result can be obtain for the corresponding three-dimensional geometry (a cube with a smaller cube cut out).

Remark 3.10. For certain geometries (in particular since there are no constraints from the Neumann boundary Γ_N on the extension $\mathcal{H}_{0,D}^{\text{int}}\, w_0$), there might exist a larger $\eta > 0$ than the one in Definition 3.2 fulfilling the estimate in Lemma 3.11. On the other hand, we see from Assumption 2.2 that $\eta \leq \min_{F\in\mathcal{F}_0} H_F$.

3.3.3 Main result

Before the main theorem, we state a stability estimate which is crucial in all FETI-type condition number estimates (cf. [87]). For the specific case of the unbounded domain we need a further regularity assumption on Ω_0.

Assumption 3.3. *The domain* Ω_{int} *satisfies an exterior cone condition. More precisely, we can add a layer of auxiliary subdomains around* Ω_{int} *that together with the subdomains* Ω_i, $i \neq 0$ *form a shape-regular coarse triangulation.*

Lemma 3.12. *Assume that* $\alpha_0 \geq \alpha_i$ *for all* $i \in \mathcal{I}$ *and that Assumptions 3.1, 3.2 and 3.3 hold. Furthermore, in two dimensions we require* $\operatorname{diam}\Gamma_0 = \mathcal{O}(1)$. *Additionally, let* $Q = M^{-1}$ *or* *let Q be given by the diagonal choice above. Then, for any $w \in (\ker S)^\perp$ and for the unique z_w given by Lemma 3.9, we have*

$$|P_D(w+z_w)|_S^2 \le C\,\gamma_h(\{\Omega_i\},\Gamma_0,\Gamma_D)\,(1+\log(H/h))^2\,|w|_S^2\,.$$

Note, that H/h is the maximal ratio of H_{ij}/h_i, and not H_i/h_i. The extension indicator $\gamma_h(\{\Omega_i\},\Gamma_0,\Gamma_D)$ can be bounded due to Lemma 3.11.

Additionally, the estimate

$$|P_D(w+z_w)|_S^2 \le C\,\max_{F\in\mathcal{F}_0}\frac{H_0}{H_F}\,(1+\log(H/h))^2\,|w|_S^2\,,$$

holds, even without the assumption $\alpha_0 \ge \alpha_i$.

Proof. We postpone the proof of this lemma to Section 3.3.6. □

Theorem 3.1. *Assume that the assumptions of Lemma 3.12 are fulfilled. Then the BETI preconditioners M^{-1} given in Section 3.2 satisfy the condition number estimates*

$$\kappa(P\,M^{-1}P^\top F\,P) \le C\,\gamma_h(\{\Omega_i\},\Gamma_0,\Gamma_D)\,(1+\log(H/h))^2\,,$$

under the assumption $\alpha_0 \ge \alpha_i$, and

$$\kappa(P\,M^{-1}P^\top F\,P) \le C\,\max_{F\in\mathcal{F}_0}\frac{H_0}{H_F}\,(1+\log(H/h))^2\,,$$

without this assumption. Both estimates are to be understood in the factor space modulo $\ker B^\top$. Note, that importantly H/h is the maximal ratio of the interface diameters H_{ij} and the local mesh size h_i, and not $\max_{i\in\mathcal{I}} H_i/h_i$. The constant C is independent of the mesh parameters h_i, the domain diameters H_i, the interface diameters H_{ij}, the exponent γ and the coefficients α_i, and depends only on the shapes of the subdomains.

Proof. The proof is based on the results stated in Lemma 3.8 and Lemma 3.12 and follows exactly the line of the corresponding proofs in [58, 49, 87]. We give bounds for the eigenvalues in (3.31).

Lower bound. We show

$$\langle F\,\lambda,\,\lambda\rangle \ge \langle M\,\lambda,\,\lambda\rangle \qquad \forall\lambda\in V_{/\ker B^\top}\,. \tag{3.41}$$

Let us fix $\mu\in\widetilde{V}'$ arbitrary. Since $\operatorname{range} B = U^*$ we can find a $\widetilde{w}\in W$ such that $B\,\widetilde{w}=\mu$. For $\widehat{w} := P_D\,\widetilde{w}$, we obtain $\mu = B\,\widehat{w}$ as well due to Lemma 3.3. Using formulas (3.32) and (3.34) from Lemma 3.8, we conclude that, $\forall\lambda\in V_{/\ker B^\top}$,

$$\langle F\,\lambda,\,\lambda\rangle \overset{(3.32)}{\ge} \frac{\langle B\,\widehat{w},\,\lambda\rangle^2}{|\widehat{w}|_S^2} = \frac{\langle\mu,\,\lambda\rangle^2}{|P_D\,\widetilde{w}|_S^2} \overset{(3.34)}{=} \frac{\langle\mu,\,\lambda\rangle^2}{\langle B\,\widetilde{w},\,M^{-1}B\,\widetilde{w}\rangle} = \frac{\langle\mu,\,\lambda\rangle^2}{\langle\mu,\,M^{-1}\mu\rangle}\,,$$

Since $\mu\in V'$ was arbitrary, we can apply formula (3.33) and finally obtain (3.41).

Upper bound. We show

$$\langle F\,\lambda,\,\lambda\rangle \le C\,(1+\log(H/h))^2\,\langle M\,\lambda,\,\lambda\rangle \qquad \forall\lambda\in V_{/\ker B^\top}\,. \tag{3.42}$$

By Lemma 3.9, for any $w \in (\ker S)^{\perp}$ there exists a unique $z_w \in \ker S$ such that $B(w+z_w) \in \widetilde{V}'$. We fix now an arbitrary $\lambda \in V_{/\ker B^{\top}}$. Apparently, $\langle B\, z_w,\, \lambda\rangle = 0$. Using Lemma 3.8 (formulas (3.32) and (3.33)), Lemma 3.12 and Corollary 3.6 we obtain

$$\begin{aligned}
\langle M\,\lambda,\,\lambda\rangle &= \sup_{\mu\in\widetilde{V}'} \frac{\langle \mu,\,\lambda\rangle^2}{\langle \mu,\, M^{-1}\mu\rangle} \\
&\geq \sup_{w\in(\ker S)^{\perp}} \frac{\langle B\,(w+z_w),\,\lambda\rangle^2}{\langle B(w+z_w),\, M^{-1}B(w+z_w)\rangle} \\
&= \sup_{w\in(\ker S)^{\perp}} \frac{\langle B\,w,\,\lambda\rangle^2}{|P_D(w+z_w)|_S^2} \\
&\geq \frac{1}{C^*\left(1+\log\left(\frac{H}{h}\right)\right)^2} \sup_{w\in(\ker S)^{\perp}} \frac{\langle B\,w,\,\lambda\rangle^2}{|w|_S^2} \\
&= \frac{1}{C^*\left(1+\log\left(\frac{H}{h}\right)\right)^2} \langle F\,\lambda,\,\lambda\rangle\,.
\end{aligned}$$

In the third line we have used and the fact that $\lambda \in V/_{/\ker B^{\top}}$. Moreover, C^* represents the two possible terms in the estimates of P_D, see Lemma 3.12. □

Remark 3.11. The assumption $\alpha_0 \geq \alpha_i$ is, e. g., fulfilled in magnetostatic computations, where the reluctivity coefficient in the exterior space is usually that of free space, $1/\mu_0$, and all other material coefficients equal $1/(\mu_0\mu_r)$ for some $\mu_r \geq 1$.

Two the main ingredients in the proof of Theorem 3.1 are on one hand Lemma 3.11 concerning the extension indicator and on the other hand the estimate stated in Lemma 3.12. Some technical tools for both proofs shall be elaborated in the sequel.

3.3.4 Technical tools

In this section we provide technical tools which we need for the proof of Lemma 3.12. The first class of tools concerning discrete Sobolev type inequalities is an extension of the results summarized in [87, Chapter 4].

Definition 3.3. *For $V \in \mathcal{V}_i$, $E \in \mathcal{E}_i$, $F \in \mathcal{F}_i$, we define θ_V, θ_E, $\theta_F \in V_1^h(\Gamma_i)$ by*

$$\theta_V(x) := \begin{cases} 1 & x = V \\ 0 & \textit{else} \end{cases}$$

$$\theta_E(x) := \begin{cases} 1 & x \in E \\ 0 & \textit{else} \end{cases}$$

$$\theta_F(x) := \begin{cases} 1 & x \in F \\ 0 & \textit{else} \end{cases}$$

where $x \in \Gamma_{i,h}$.

By I^h we denote the nodal interpolator operator mapping the continuous functions on Γ_i onto $V_1^h(\Gamma_i)$. We mention that I^h is stable in the $H^{1/2}$-norm for quadratic functions. For details see, e. g., [87, Chapter 4 and Appendix B].

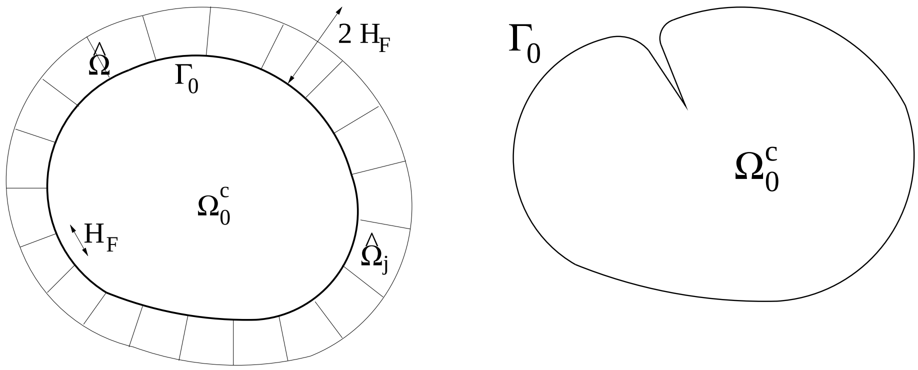

Figure 10: Left: The auxiliary domain $\widehat{\Omega} \subset \Omega_0$ with its non-overlapping decomposition. Right: A configuration of Ω_0 with a small geometric angle.

The functions in Definition 3.3 induce the following partition of unity on Γ_i,

$$\sum_{V\in\mathcal{V}_i} \theta_V(x) + \sum_{E\in\mathcal{E}_i} \theta_E(x) + \sum_{F\in\mathcal{F}_i} \theta_F(x) = 1 \qquad \forall x \in \Gamma_i\,.$$

Hence, for any $u \in V_1^h(\Gamma_i)$, we have

$$u = \sum_{V\in\mathcal{V}_i} I^h(\theta_V\, u) + \sum_{E\in\mathcal{E}_i} I^h(\theta_E\, u) + \sum_{F\in\mathcal{F}_i} I^h(\theta_F\, u)\,. \tag{3.43}$$

Definition 3.4. *In three dimensions, we define for each vertex $V \in \mathcal{V}_i$ its corresponding* vertex patch $\omega_V^{(i)}$*, regarded as an open set, by*

$$\overline{\omega_V^{(i)}} := \bigcup_{F\in\mathcal{F}_i,\, V\in\overline{F}} \overline{F}\,,$$

and for each subdomain edge $E \in \mathcal{E}_i$ its edge patch $\omega_E^{(i)}$*, regarded as an open set, by*

$$\overline{\omega_E^{(i)}} := \bigcup_{F\in\mathcal{F}_i,\, E\subset\overline{F}} \overline{F}\,.$$

For a simplified notation, we define the face patch *of a face as the face itself, i. e., $\omega_F^{(i)} := F$ for $F \in \mathcal{F}_i$. The analogous definitions in two dimensions should be obvious.*

In the following, we introduce special discrete $H^{1/2}$- and $H_{00}^{1/2}$-norms on the subdomain boundaries, faces, and edge and vertex patches. First, we need to construct some auxiliary subdomains in order to cope with the distinguished case of the unbounded domain Ω_0.

We consider a bounded domain $\widehat{\Omega} \subset \Omega_0$ with

$$\{x \in \Omega_0 : \mathrm{dist}\,(x, \Gamma_0) < H_F(x)\} \subset \widehat{\Omega} \subset \{x \in \Omega_0 : \mathrm{dist}\,(x, \Gamma_0) < 2H_F(x)\}\,,$$

where $H_F(x) = \text{diam argmin}_{F\in\mathcal{F}_0} \text{dist}\,(x, F)$ denotes the diameter of the face on Γ_0 that is next to x. The domain $\widehat{\Omega}$ should be chosen in such a way, that it can be decomposed into shape-regular non-overlapping subdomains $\widehat{\Omega}_j$,

$$\widehat{\Omega} = \bigcup_{j\in\widehat{\mathcal{I}}} \widehat{\Omega}_j\,,$$

where the subdomains $\widehat{\Omega}_j$ should satisfy the requirements of Assumption 2.2. Note, that this construction is always possible if Assumption 3.3 holds, whereas it would not be possible in the case of very small angles as depicted in Figure 10, right. Hence, we exclude these cases in our considerations.[4]

Moreover, the subdomain faces, edges and vertices on Γ_0 are again subdomain faces, edges and vertices of the decomposition of $\widehat{\Omega}$. Thus, we can associate to each face $F\in\mathcal{F}_0$ an index j_F such that $\widehat{\Omega}_{j_F}$ touches F; see also Figure 10, left.

We denote by $\widehat{\mathcal{E}}_j$ the subset of edges in $\mathcal{E}_0$ that touch $\widehat{\Omega}_j$, and analogously by $\widehat{\mathcal{V}}_j$ the set of vertices in $\mathcal{V}_0$ touching $\widehat{\Omega}_j$. Finally, we introduce quasi-uniform triangulations on $\widehat{\Omega}_j$ with the mesh size h_0 such that the grids match on Γ_0 and on the interfaces between the domains $\widehat{\Omega}_j$.

Definition 3.5. *Let F be the face shared by Ω_i and Ω_j. For a function $u\in V_1^h(\overline{F})$ vanishing on ∂F, we define the following $H_{00}^{1/2}$-norm,*

$$|u|_{H_{00}^{1/2}(F)} := \begin{cases} \min\left(|u|_{H_{00}^{1/2}(F,\Omega_i)},\, |u|_{H_{00}^{1/2}(F,\Omega_j)}\right) & \text{for } i,\, j\neq 0\,,\\ \min\left(|u|_{H_{00}^{1/2}(F,\widehat{\Omega}_{j_F})},\, |u|_{H_{00}^{1/2}(F,\Omega_j)}\right) & \text{for } i = 0\,,\\ \min\left(|u|_{H_{00}^{1/2}(F,\Omega_i)},\, |u|_{H_{00}^{1/2}(F,\widehat{\Omega}_{j_F})}\right) & \text{for } j = 0\,,\end{cases}$$

where

$$|u|_{H_{00}^{1/2}(F,\mathcal{D})} := \min\left\{|v|_{H^1(\mathcal{D})} : v\in V_1^h(\mathcal{D}),\, v|_F = u,\, v|_{\partial\mathcal{D}\setminus F} = 0\right\}.$$

Apparently, $|u|_{H_{00}^{1/2}(F,\mathcal{D})}$ is only well-defined if $u|_{\partial F} = 0$. For faces on $\partial\Omega$ we define the $H_{00}^{1/2}$-norm as the H^1-norm of the minimal extension to the adjacent subdomain, or auxiliary subdomain. Additionally, we define the corresponding norms on vertex and edge patches,

$$|u|_{H_{00}^{1/2}(\omega_V^{(i)})} := \min\left\{|v|_{H^1(\Omega_i)} : v\in V_1^h(\Omega_i),\, v|_{\omega_V^{(i)}} = u,\, v|_{\Gamma_i\setminus\omega_V^{(i)}} = 0\right\},$$
$$|u|_{H_{00}^{1/2}(\omega_E^{(i)})} := \min\left\{|v|_{H^1(\Omega_i)} : v\in V_1^h(\Omega_i),\, v|_{\omega_E^{(i)}} = u,\, v|_{\Gamma_i\setminus\omega_E^{(i)}} = 0\right\},$$

for $i\neq 0$ and

$$|u|_{H_{00}^{1/2}(\omega_V^{(0)})} := \min\left\{|v|_{H^1(\widehat{\Omega}_V)} : v\in V_1^h(\widehat{\Omega}_V),\, v|_{\omega_V^{(0)}} = u,\, v|_{\partial\Omega_V\setminus\omega_V^{(0)}} = 0\right\},$$
$$|u|_{H_{00}^{1/2}(\omega_E^{(0)})} := \min\left\{|v|_{H^1(\widehat{\Omega}_E)} : v\in V_1^h(\widehat{\Omega}_E),\, v|_{\omega_E^{(0)}} = u,\, v|_{\partial\Omega_E\setminus\omega_E^{(0)}} = 0\right\},$$

[4]In such cases also the constants $c_0^{(0)}$ and $c_K^{(0)}$ would probably reflect this bad behaviour anyway.

with the open, simply connected domains $\widehat{\Omega}_V$, $\widehat{\Omega}_E$ *defined by*

$$\overline{\widehat{\Omega}_V} := \bigcup_{j:V\in\widehat{\mathcal{V}}_j} \overline{\widehat{\Omega}_j}\,, \qquad \overline{\widehat{\Omega}_E} := \bigcup_{j:E\in\widehat{\mathcal{E}}_j} \overline{\widehat{\Omega}_j}\,,$$

Furthermore, we define the full $H^{1/2}$*-norms and* $H^{1/2}$*-semi-norms employing the same definitions, but dropping the zero boundary conditions and (only for the full norms) adding properly scaled* L_2*-terms, e. g., for* i, $j \neq 0$,

$$\|u\|_{H^{1/2}(F)} := \min\Big(\|u\|_{H^{1/2}(F,\Omega_i)}, \|u\|_{H^{1/2}(F,\Omega_j)}\Big)\,,$$

where

$$\|u\|_{H^{1/2}(F,\mathcal{D})} := \min\Big\{\Big(|v|^2_{H^1(\mathcal{D})} + \frac{1}{(\operatorname{diam}\mathcal{D})^2}\|v\|^2_{L_2(\mathcal{D})}\Big)^{1/2} : v \in V_1^h(\mathcal{D}),\, v|_{\overline{F}} = u\Big\}\,.$$

The analogue definitions in two dimensions should be obvious.

Lemma 3.13. *Let the discrete norms* $|\cdot|_{H^{1/2}_{00}(F)}$ *etc. be defined according to Definition 3.5. In three dimensions, assume that* $i \in \mathcal{I}$, $V \in \mathcal{V}_i$, $E \in \mathcal{E}_i$ *and* $F \in \mathcal{F}_i$, *such that* V *is one of the end points of* E, *and* $E \subset \partial F$. *Then there exists a constant* $C > 0$ *depending only on the shapes of those domains* Ω_i *and* $\widehat{\Omega}_j$ *that touch* V, E, *or* F, *such that for all* $u \in V_1^h(\Gamma_i)$,

$$(i) \qquad |I^h(\theta_F\, u)|^2_{H^{1/2}_{00}(F)} \leq C\,(1+\log(H_F/h_i))^2\, \|u\|^2_{H^{1/2}(F)}\,,$$

$$(ii) \qquad |I^h(\theta_{V/E}\, u)|^2_{H^{1/2}_{00}(\omega^{(i)}_{V/E})} \leq C\,(1+\log(H_F/h_i))\, \|u\|^2_{H^{1/2}(F)}\,,$$

$$(iii) \qquad |\theta_F|^2_{H^{1/2}_{00}(F)} \leq C\,(1+\log(H_F/h_i))\, H_F\,,$$

$$(iv) \qquad |\theta_E|^2_{H^{1/2}_{00}(\omega^{(i)}_E)} \leq C\, H_E\,,$$

$$(v) \qquad |\theta_V|^2_{H^{1/2}_{00}(\omega^{(i)}_V)} \leq C\, h_i\,,$$

$$(vi) \qquad \|u\|^2_{L_2(E)} \leq C\,(1+\log(H_F/h_i))\|u\|^2_{H^{1/2}(F)}\,.$$

In (ii), "$\theta_{V/E}$*" means that we can either use* θ_V *or* θ_E. *In two dimensions, the analogous estimates*

$$(vii) \qquad |I^h(\theta_E\, u)|^2_{H^{1/2}_{00}(E)} \leq C\,(1+\log(H_E/h_i))^2\, \|u\|^2_{H^{1/2}(E)}\,,$$

$$(viii) \qquad |I^h(\theta_V\, u)|^2_{H^{1/2}_{00}(\omega^{(i)}_V)} \leq C\,(1+\log(H_E/h_i))\, \|u\|^2_{H^{1/2}(E)},$$

$$(ix) \qquad |\theta_E|^2_{H^{1/2}_{00}(E)} \leq C\,(1+\log(H_E/h_i))\,,$$

$$(x) \qquad |\theta_V|^2_{H^{1/2}_{00}(\omega^{(i)}_V)} \leq C$$

hold. The bounds are known to be sharp (cf., e. g., [12]).

Proof. We give the proof in three dimensions, using the face and edge estimates stated in [87, Chapter 4].

First, we recall the definition of our discrete $H^{1/2}$-norm on F. For simplicity, we assume that F is shared by Ω_i and Ω_j with $i,\, j \neq 0$, and thus $H_F \simeq H_i \simeq H_j$ by Assumption 2.2. Let $\widetilde{u}_i \in V_1^h(\Omega_i)$, $\widetilde{u}_j \in V_1^h(\Omega_j)$ denote the minimizers such that

$$\|u\|^2_{H^{1/2}(F,\Omega_k)} = |\widetilde{u}_k|^2_{H^1(\Omega_k)} + \frac{1}{H_k^2}\|\widetilde{u}_k\|^2_{L_2(\Omega_k)}$$

and $\widetilde{u}_k|_{\overline{F}} = u|_{\overline{F}}$ for $k = i,\, j$.

Face terms – (i) and (iii): By [87, Lemma 4.24], we obtain for $k = i,\, j$

$$|I^h(\theta_F\, u)|^2_{H^{1/2}_{00}(F,\Omega_k)} \preceq (1 + \log(H_F/h_k))^2 \left[|\widetilde{u}_k|^2_{H^1(\Omega_k)} + \frac{1}{H_k^2}\|\widetilde{u}_k\|^2_{L_2(\Omega_k)}\right].$$

By Definition 3.5 we get (i). [87, Lemma 4.25] yields

$$|\theta_F|^2_{H^{1/2}_{00}(F,\Omega_k)} \preceq (1 + \log(H_F/h_k))\, H_F\,,$$

which implies (iii).

Edge terms – (ii), (iv) and (vi): [87, Lemma 4.16] states

$$\|u\|^2_{L_2(E)} \preceq (1 + \log(H_F/h_k)) \left[|\widetilde{u}_k|^2_{H^1(\Omega_k)} + \frac{1}{H_k^2}\|\widetilde{u}_k\|^2_{L_2(\Omega_k)}\right],$$

which implies (vi). Furthermore, [87, Lemma 4.19] gives

$$\begin{aligned} |I^h(\theta_E\, u)|^2_{H^{1/2}_{00}(\omega_E^{(i)},\Omega_i)} &\preceq \|I^h(\theta_E\, u)\|^2_{L_2(E)} \preceq \|u\|^2_{L_2(E)} \\ &\preceq (1 + \log(H_F/h_k)) \left[|\widetilde{u}_k|^2_{H^1(\Omega_k)} + \frac{1}{H_k^2}\|\widetilde{u}_k\|^2_{L_2(\Omega_k)}\right], \end{aligned}$$

for $k = i,\, j$. This gives (ii). From the first line of the last estimate we immediately obtain

$$|\theta_E|^2_{H^{1/2}_{00}(\omega_E^{(i)})} \preceq \|1\|^2_{L_2(E)} \preceq H_E\,,$$

which is (iv).

Vertex terms – (ii) and (v): Using our triangulation, we can estimate $|I^h(\theta_V\, u)|^2_{H^{1/2}_{00}(\omega_V^{(i)},\Omega_i)}$ from above by $|u(V)\, \varphi_V^{(k)}|^2_{H^1(\Omega_k)} \simeq h_k\, u(V)^2$, for $k = i,\, j$, where $\varphi_V^{(k)}$ is the finite element hat function on the triangulation $\mathcal{T}_h(\Omega_k)$ corresponding to the vertex V. This proves (v) by setting $u \equiv 1$. For (ii), apparently, the last term can be bounded by the square of the L_2-norm of u on the edge E, and we can proceed as in the estimates for the edge terms.

The estimates for vertices, edges and faces touching Ω_0, the proof works analogously, except one has to use the auxiliary subdomains $\widehat{\Omega}_j$ in Definition 3.5. The essential point is, that the domains $\widehat{\Omega}_j$ have comparable diameters to those of the faces on Γ_0. □

Lemma 3.14 (discrete Poincaré-Friedrichs inequality)**.** *In three dimensions, for $i \in \mathcal{I}$, we fix a face $F \in \mathcal{F}_i$ and a function $u \in V_1^h(F)$. Assume that for $i \neq 0$, u vanishes on a subdomain edge $E \in \mathcal{E}_i$, and for $i = 0$, that v vanishes on an edge $E \subset \partial F$. Then,*

$$\frac{1}{H_F}\|u\|^2_{L_2(F)} \leq C\,(1 + \log(H_F/h_i))\, |u|^2_{H^{1/2}(F)}\,.$$

In two dimensions, we fix an edge $E \in \mathcal{E}_i$. Then,

$$\frac{1}{H_E}\|u\|^2_{L_2(E)} \leq C\,(1+\log(H_E/h_i))\,|u|^2_{H^{1/2}(E)}\,,$$

if $u \in V_1^h(E)$ vanishes at a vertex $V \in \mathcal{V}_i$ for $i \neq 0$ and $V \in \partial E$ for $i = 0$.

Proof. The three-dimensional result follows immediately from [87, Lemma 4.21]. In two dimensions, we have that $H_E^{-1}\|u\|^2_{L_2(E)} \preceq \|u\|^2_{L_\infty(E)}$ and the desired estimate follows from [87, Lemma 4.15]. □

The next lemma gives an estimate for the difference of mean values of H^1-functions, and will be needed for the coarse-space contributions of the one-level BETI preconditioners.

Lemma 3.15. *Let Ω_i, Ω_j be two neighboring subdomains with comparable diameters $H_i \simeq H_j$ and volumes $|\Omega_i| \simeq |\Omega_j| \simeq H_i^d$, and let $\mathcal{U}$ be an open, simply connected domain with a Lipschitz boundary, such that $\Omega_i \cup \Omega_j \subset \mathcal{U}$, and $\operatorname{diam}\mathcal{U} \simeq H_i$, and $|\mathcal{U}| \simeq |\Omega_i|$. Then, for some $C_Z > 0$ depending only on the shape of $\mathcal{U}$,*

$$\min(H_i,\,H_j)^{d-2}|z_i - z_j|^2 \leq C_Z|u|^2_{H^1(\mathcal{U})} \qquad \forall u \in H^1(\mathcal{U})\,,$$

where

$$z_i = \frac{1}{|\Omega_i|}\int\limits_{\Omega_i} u(x)\,dx\,, \qquad z_j = \frac{1}{|\Omega_j|}\int\limits_{\Omega_j} u(x)\,dx\,.$$

Proof. We use the Bramble-Hilbert lemma [5]. First, we assume that $\operatorname{diam}\mathcal{U} = \mathcal{O}(1)$. We define the linear functional $\phi : H^1(\mathcal{U}) \to \mathbb{R}$ by

$$\phi(u) := \frac{1}{|\Omega_i|}\int\limits_{\Omega_i} u(x)\,dx - \frac{1}{|\Omega_j|}\int\limits_{\Omega_j} u(x)\,dx\,.$$

Obviously, ϕ vanishes for the constant functions in $\mathcal{U}$. Hence, there exists a constant $\widehat{C}_Z$ such that

$$|\phi(u)| \leq \widehat{C}_Z|u|_{H^1(\mathcal{U})}\,.$$

The desired estimate involving the factor $\min(H_i,\,H_j)^{d-2}$ can be obtained by delation from a domain of unit diameter. □

The following lemma states how discrete functions can be split into face terms and be composed from face, edge and vertex terms, with respect to the $S_{i,h}$ energy forms.

Lemma 3.16. *Let the discrete $H^{1/2}$- and $H^{1/2}_{00}$-norm be defined according to Definition 3.5. Then the following stability estimates hold.*

(i) For all $i \in \mathcal{I}$ and $u \in V_1^h(\Gamma_i)$ we have

$$|u|^2_{S_{i,h}} \leq C_1 \sum_{V\in\mathcal{V}_i,\,E\in\mathcal{E}_i,\,F\in\mathcal{F}_i} \alpha_i\,|I^h(\theta_{V/E/F}\,u)|^2_{H^{1/2}_{00}(\omega^{(i)}_{V/E/F})}\,,$$

where in two dimensions, we have to drop the face terms.[5]

[5] The notation $\theta_{V/E/F}$ is a short hand for writing three sums, one over vertices, one over edges and the thrid over faces.

(ii) For all $i \in \mathcal{I}$ and $w \in V_1^h(\Gamma_i)$ we have

$$\sum_{F \in \mathcal{F}_i} \alpha_i \, |w|^2_{H^{1/2}(F)} \leq C_2 \, |w|^2_{S_{i,h}} \,,$$

where in two dimensions the faces must be replaced by edges.

The constants C_1 and C_2 depend only on the shape of Ω_i.

Proof. Part (i): First, we recall, that by Lemma 2.3,

$$|u|^2_{S_{0,h}} \leq \alpha_0 \, |u|^2_{S_0} = \alpha_0 \min_{\substack{v \in H^1_{\text{loc}}(\Omega_0) \\ v|_{\Gamma_0} = u}} |v|^2_{H^1(\Omega_0)} \,,$$

$$|u|^2_{S_{i,h}} \leq \alpha_i \min_{\substack{v \in V_1^h(\Omega_i) \\ v|_{\Gamma_i} = u}} |v|^2_{H^1(\Omega_i)} \,.$$

For all $V \in \mathcal{V}_i$, $E \in \mathcal{E}_i$, $F \in \mathcal{F}_i$, let v_V, v_E, $v_F \in V_1^h(\Omega_i)$ be the minimizers in Definition 3.5 such that

$$|v_{V/E/F}|_{H^1(\Omega_i)} \simeq |I^h(\theta_{V/E/F} \, u)|_{H^{1/2}_{00}(\omega^{(i)}_{V/E/F})} \,.$$

In order to justify above formula we need a "mirror" argument showing that the H^1-norms of the extensions of a function on a face to each of the two adjacent subdomains are equivalent. This can be done with the help of an extension Lemma and the Scott-Zhang interpolation operator [79].

Note, that for $i = 0$, the support of v_F is contained in $\widehat{\Omega}_{j_F}$ and the support of v_V and v_E in $\widehat{\Omega}_V$ and $\widehat{\Omega}_E$, respectively, see also Definition 3.5. We define $\widetilde{v} \in V_1^h(\Omega)$ by

$$\widetilde{v} := \sum_{V \in \mathcal{V}_i} v_V + \sum_{E \in \mathcal{E}_i} v_E + \sum_{F \in \mathcal{F}_i} v_F \,.$$

Clearly, $\widetilde{v}|_{\Gamma_i} = u$, and we can conclude that $|u|^2_{S_{i,h}} \leq \alpha_i |\widetilde{v}|^2_{H^1(\Omega_i)}$. Since the supports of the functions v_V, v_E, v_F have a finite overlap in Ω_i (in particular for $i = 0$), we finally obtain by a coloring argument (cf., e. g., [87, Chapter 2]) that

$$|u|^2_{S_{i,h}} \leq \alpha_i \, C_1 \Big\{ \sum_{V \in \mathcal{V}_i} |v_V|^2_{H^1(\Omega_i)} + \sum_{E \in \mathcal{E}_i} |v_E|^2_{H^1(\Omega_i)} + \sum_{F \in \mathcal{F}_i} |v_F|^2_{H^1(\Omega_i)} \Big\} \,,$$

which proves the desired statement.

Part (ii): First we deal with the case $i \neq 0$. We denote by $\widetilde{w} \in V_1^h(\Omega_i)$ the minimizer with $\widetilde{w}|_{\Gamma_i} = w$ and

$$|\widetilde{w}|^2_{H^1(\Omega_i)} = |w|^2_{S^{\text{int}}_{i,\text{FEM}}} \,. \tag{3.44}$$

Note, that $S_{i,h}$ may either be $\alpha_i \, S^{\text{int}}_{i,\text{FEM}}$ or $\alpha_i \, S^{\text{int}}_{i,\text{BEM}}$ (cf. Section 2.5). In the second case, Lemma 2.3 yields

$$|w|^2_{S^{\text{int}}_{i,\text{FEM}}} \leq C_T^{(i)} \, |w|^2_{S_i} \leq C_T^{(i)} \, \frac{c_K^{(i)}}{c_0^{(i)}} \, |w|^2_{S^{\text{int}}_{i,\text{BEM}}} \,.$$

Furthermore, since the cardinality of $\mathcal{F}_i$ is small for $i \neq 0$, we obtain from the Cauchy-Schwarz inequality and (3.44) that

$$\sum_{F \in \mathcal{F}_i} \alpha_i |w|^2_{H^{1/2}(F)} \preceq \alpha_i \, |\widetilde{w}|^2_{H^1(\Omega_i)} \preceq \alpha_i \, |w|^2_{S^{\text{int}}_{i,\text{FEM}}} .$$

Putting the last two estimates together yields the desired statement.

For $i = 0$, we denote by $\widetilde{w} \in V_1^h(\Omega_0^{\mathrm{c}})$ the minimizer such that

$$|\widetilde{w}|^2_{H^1(\Omega_0^{\mathrm{c}})} = |w|^2_{S^{\text{int}}_{0,\text{FEM}}} .$$

Then we obtain by a finite summation argument that

$$\sum_{F \in \mathcal{F}_0} \alpha_0 |w|^2_{H^{1/2}(F)} \preceq \sum_{\substack{j \in \mathcal{I} \setminus \{0\} \\ \mathcal{F}_0 \cap \mathcal{F}_j \neq \emptyset}} \alpha_0 \, |\widetilde{w}|^2_{H^1(\Omega_j)} \preceq \alpha_0 \, |\widetilde{w}|^2_{H^1(\Omega_0^{\mathrm{c}})} \preceq |w|^2_{S^{\text{int}}_{0,\text{FEM}}} .$$

Finally, we know by Lemma 2.3 that

$$|w|^2_{S^{\text{int}}_{0,\text{FEM}}} \leq C_T^{(0)} \, \frac{c_K^{(0)}}{c_0^{(0)}} \, |w|^2_{S^{\text{ext}}_{0,\text{BEM}}} .$$

□

Next, we focus the missing proof of Lemma 3.11 concerning the extension indicator.

3.3.5 Proof of Lemma 3.11

First, we repeat the statement of Lemma 3.11. Let $w_0 \in V_{1,D}(\Gamma_0)$ be an arbitrary discrete function on the boundary Γ_0 fulfilling the Dirichlet boundary conditions on $\Gamma_0 \cap \Gamma_D$. Then the following estimates hold.

(i) If $\Gamma_D \setminus \Gamma_0$ is empty we have

$$\gamma_h(\{\Omega_i\}, \Gamma_0, \Gamma_D) = 1$$

(ii) If $\operatorname{dist}(\Gamma_D \setminus \Gamma_0, \Gamma_0) > 0$, we have

$$\gamma_h(\{\Omega_i\}, \Gamma_0, \Gamma_D) \preceq \frac{H_0}{\operatorname{dist}(\Gamma_D \setminus \Gamma_0, \Gamma_0)} .$$

(iii) Otherwise, if $\Gamma_D \setminus \Gamma_0$ touches Γ_0 (according to Assumption 3.2, page 25),

$$\gamma_h(\{\Omega_i\}, \Gamma_0, \Gamma_D) \preceq \frac{H_0}{\eta} \Big(1 + \log\Big(\frac{\eta}{h}\Big)\Big)^2$$

holds, with the shape parameter $\eta > 0$ due to Definition 3.2.

The proof borrows several techniques from [87, Chapter 3] (see in particular Lemma 3.10), and recent works by Graham, Lechner and Scheichl [37], and Scheichl and Vainikko [78].

Part (i) is trivial.

Part (ii): First, we observe that for the choice $\eta := \operatorname{dist}(\Gamma_D \setminus \Gamma_0, \Gamma_0) > 0$ we can find a finite covering of the domain $\Omega_{\mathrm{int},\eta}$ by shape-regular patches of diameter η, similar to Definition 3.2. Let $\{\omega_j\}_{j\in\mathcal{J}}$ denote this covering and define $u \in V_1^h(\Omega_{\mathrm{int}})$ by $u = \mathcal{H}^{\mathrm{int}}\, w$.

For the given η, we can find a cutoff function $\chi \in V_1^h(\Omega_{\mathrm{int}})$ such that

(a) $\chi(x) \in [0, 1] \quad \forall x \in \overline{\Omega_{\mathrm{int}}}$,

(b) $\chi|_{\Gamma_0} = 1$ and χ vanishes entirely on $\Omega_{\mathrm{int}} \setminus \Omega_{\mathrm{int},\eta}$, in particular $\chi|_{\Gamma_D\setminus\Gamma_0} = 0$,

(c) $\|\nabla\chi\|_{L_\infty(\Omega_{\mathrm{int}})} \preceq 1/\operatorname{dist}(\Gamma_D \setminus \Gamma_0, \Gamma_0) \simeq 1/\eta$.

These properties can, e. g., be obtained by the following definition of χ: For the finite element nodes $x \in \Omega_{\mathrm{int},h}$ we set

$$\chi(x) := \begin{cases} 1 - \operatorname{dist}(x, \Gamma_0)/\eta & \text{for } \operatorname{dist}(x, \Gamma_0) \le \eta\,, \\ 0 & \text{else,} \end{cases}$$

which uniquely defines χ. Properties (a) and (b) are easy to see. For (c) we observe that for a finite element T,

$$\|\nabla\chi\|_{L_\infty(T)} \preceq \sum_{V\neq V'\in\mathcal{V}_T} \frac{|\chi(V)-\chi(V')|}{|V-V'|}\,,$$

where $\mathcal{V}_T$ is the set of vertices of the element T. It is not hard to see that for above definition of χ,

$$\frac{|\chi(V)-\chi(V')|}{|V-V'|} \le \frac{1}{\eta}\,.$$

Since the number of vertices of an element T is uniformly bounded by a small number we immediately get (c).

Recall that I^h denotes the nodal interpolator which is continuous in the H^1-norm for quadratic functions (cf. Definition 3.3). We define $\widetilde{u} \in V_1^h(\Omega_{\mathrm{int}})$ by

$$\widetilde{u} := I^h(\chi\, u)$$

and obtain

$$\begin{aligned} |\mathcal{H}^{\mathrm{int}}_{0,D}\, w_0|^2_{H^1(\Omega_{\mathrm{int}})} &\le |\widetilde{u}|^2_{H^1(\Omega_{\mathrm{int}})} = |I^h(\chi\, u)|^2_{H^1(\Omega_{\mathrm{int}})} = |I^h(\chi\, u)|^2_{H^1(\Omega_{\mathrm{int},\eta})} \\ &\preceq \int_{\Omega_{\mathrm{int},\eta}} \big|\nabla\big(\chi(x)\, u(x)\big)\big|^2\, dx \\ &\preceq \int_{\Omega_{\mathrm{int},\eta}} |\nabla\chi(x)|^2\, |u(x)|^2 + |\chi(x)|^2\, |\nabla u(x)|^2\, dx \\ &\le \|\nabla\chi\|^2_{L_\infty(\Omega_{\mathrm{int}})} \|u\|^2_{L_2(\Omega_{\mathrm{int},\eta})} + |u|^2_{H^1(\Omega_{\mathrm{int},\eta})} \\ &\preceq \sum_{j\in\mathcal{J}} \Big\{ \frac{1}{\eta^2}\, \|u\|^2_{L_2(\omega_j)} + |u|^2_{H^1(\omega_j)} \Big\}\,. \end{aligned}$$

By Friedrichs' inequality (see, e. g., [87, Corollary A.15] or [69]), we have that

$$\frac{1}{\eta^2}\,\|u\|^2_{L_2(\omega_j)} \preceq |u|^2_{H^1(\omega_j)} + \frac{1}{\eta}\,\|u\|^2_{L_2(\partial\omega_j\cap\Gamma_0)}\,.$$

Finally, by a finite summation argument and the fact that $H_0 \geq \eta$ we get

$$\begin{aligned}|\mathcal{H}^{\text{int}}_{0,D}\,w_0|^2_{H^1(\Omega_{\text{int}})} &\preceq |u|^2_{\Omega_{\text{int}}} + \frac{1}{\eta}\,\|u\|^2_{L_2(\Gamma_0)}\\ &\preceq \frac{H_0}{\eta}\Big\{|w_0|^2_{H^{1/2}(\Gamma_0)} + \frac{1}{H_0}\|w_0\|^2_{L_2(\Gamma_0)}\Big\}\\ &\preceq \frac{H_0}{\eta}\Big\{\langle S^{\text{int}}_0\,w_0,\,w_0\rangle + \frac{1}{H_0}\|w_0\|^2_{L_2(\Gamma_0)}\Big\}\,,\end{aligned}$$

since $u|_{\Gamma_0} = w_0$.

Part (iii): If $\Gamma_D\backslash\Gamma_0$ touches Γ_0, some of the patches ω_j own a vertex (in two dimensions) or an edge (in three dimensions) that belongs to $\Gamma_D\cap\Gamma_0$ and we cannot use the cutoff function χ anymore. Instead, we use the estimates of Lemma 3.13 which contribute a logarithmic factor. Define for each $j \in \mathcal{J}$

$$\widetilde{u}|_{\omega_j} := \operatorname{argmin}\{v \in V^h_1(\omega_j) : v|_{\Gamma_D} = 0,\ v|_{\partial\omega_{j,h}\backslash\Gamma_D} = u\}\,.$$

Apparently, $\widetilde{u}|_{\Gamma_0} = w_0$, and thus, we obtain

$$\begin{aligned}|\mathcal{H}^{\text{int}}_{0,D}\,w_0|^2_{H^1(\Omega_{\text{int}})} &\preceq \sum_{j\in\mathcal{J}} |\widetilde{u}|^2_{H^1(\omega_j)}\\ &\preceq \sum_{j\in\mathcal{J}} \Big(1+\log\Big(\frac{\eta}{h}\Big)\Big)^2 \Big\{|u|^2_{H^1(\omega_j)} + \frac{1}{\eta}\|u\|^2_{L_2(\partial\omega_j\cap\Gamma_0)}\Big\}\\ &\preceq \Big(1+\log\Big(\frac{\eta}{h}\Big)\Big)^2\,\frac{H_0}{\eta}\Big\{\langle S^{\text{int}}_0\,w_0,\,w_0\rangle + \frac{1}{H_0}\|w_0\|^2_{L_2(\Gamma_0)}\Big\}\,,\end{aligned}$$

where the estimate in the second line can be obtained by Lemma 3.13 (see also the proof of Lemma 3.16) and once again Friedrichs' inequality.

Remark 3.12. First, in Definition 3.2 and in the proof of Lemma 3.11 we have implicitly assumed that the triangulations $\mathcal{T}_h(\Omega_i)$ are conforming with the patches ω_j, which should be in fact coarse elements. This may result in many requirements on the mesh which might be difficult to fulfill. One way out are recent results on substructuring and overlapping methods assuming less regular subdomain boundaries, see [22, 47]. Note further, that Lemma 3.11 is a non-constructive result on extensions. For the construction of explicit extensions we refer to the works by Nepomnyaschikh et. al. [39, 65, 67, 68].

We are now ready to tackle the proof of Lemma 3.12.

3.3.6 Proof of Lemma 3.12

Our goal is to provide the stability estimate

$$|P_D(w+z_w)|^2_S \leq C\,(1+\log(H/h))^2\,|w|^2_S \qquad \forall w \in (\ker S)^\perp\,,$$

with the unique $z_w \in \ker S$ from Lemma 3.9. We give the proof in three dimensions. The two-dimensional case is in fact simpler and can be derived from the following proof in a straight forward manner.

First, recall that $w \in W = \prod_{i\in\mathcal{I}} W_i$. We write $w = \widetilde{w} + \Pi_0 w$ where $\Pi_0\, w$ is the projection that sets all components of w to zero except for its component w_0 corresponding to Γ_0, thus $\widetilde{w}_0 = 0$ and $\widetilde{w}_i = w_i$ for $i \neq 0$. Since the mapping $w \mapsto z_w$ is linear (see Lemma 3.9), we get

$$|P_D(w + z_w)|_S^2 \preceq |P_D(\widetilde{w} + z_{\widetilde{w}})|_S^2 + |P_D(\Pi_0\, w + z_{(\Pi_0\, w)})|_S^2 \,. \tag{3.45}$$

If we have no unbounded domain in our formulation the second term has of course to be dropped. By Lemma 3.3, we have $P_D = I - E_D$. This yields

$$|P_D(w + z_w)|_S^2 \preceq |\widetilde{w}|_S^2 + |E_D\, \widetilde{w}|_S^2 + |P_D\, z_{\widetilde{w}}|_S^2 + |P_D(\Pi_0\, w + z_{(\Pi_0\, w)})|_S^2 \,. \tag{3.46}$$

We treat above terms separately:

Claim 1: $|E_D\, \widetilde{w}|_S^2 \preceq (1 + \log(H/h))^2 |\widetilde{w}|_S^2$, for all $\widetilde{w} \in (\ker S)^\perp$ with $\widetilde{w}_0 = 0$.

Claim 2: $|P_D\, z_{\widetilde{w}}|_S^2 \preceq (1 + \log(H/h))^2 |\widetilde{w}|_S^2$, for all $\widetilde{w} \in (\ker S)^\perp$ with $\widetilde{w}_0 = 0$.

Claim 3: $|P_D(\Pi_0 w + z_{\Pi_0 w})|_S^2 \preceq \gamma_h(\{\Omega_i\}, \Gamma_D, \Gamma_0)\, (1 + \log(H/h))^2\, |w_0|_{S_{0,h}}^2$.

Claim 4: $|P_D(\Pi_0 w + z_{\Pi_0 w})|_S^2 \preceq \max_{F\in\mathcal{F}_0} \frac{H_0}{H_F}\, (1 + \log(H/h))^2 |w_0|_{S_{0,h}}^2$, even if the assumption $\alpha_0 \leq \alpha_i$ does not hold.

Estimate (3.46) and Claim 1–4 imply the desired estimates of Lemma 3.12, i. e.,

$$\begin{aligned} |P_D(w + z_w)|_S^2 &\preceq \gamma_h(\{\Omega_i\}, \Gamma_D, \Gamma_0)\, (1 + \log(H/h))^2\, |w|_S^2 \,, \\ |P_D(w + z_w)|_S^2 &\preceq \max_{F\in\mathcal{F}_0} \frac{H_0}{H_F}\, (1 + \log(H/h))^2 |w|_S^2 \,, \end{aligned}$$

for all $w \in (\ker S)^\perp$. In the following, we give the proofs of Claim 1–4.

Claim 1: We prove

$$|E_D\, \widetilde{w}|_S^2 \preceq (1 + \log(H/h))^2 |\widetilde{w}|_S^2 \qquad \text{for all } \widetilde{w} \in (\ker S)^\perp \text{ with } \widetilde{w}_0 = 0\,.$$

Since $\widetilde{w}_0 = 0$, we are in the situation of standard FETI proof, provided by Klawonn and Widlund [49, 87]. In the following we display the whole proof, extended to the BEM case (cf. Langer and Steinbach [58, 59]) and also to the all-floating formulation.

From the definition of E_D we know in particular that $E_D\, \widetilde{w}$ satisfies the homogeneous Dirichlet boundary conditions. In the following, we use the notation $\mathcal{V}_i^I$, $\mathcal{E}_i^I$ and $\mathcal{F}_i^I$ for the set of vertices, edges and faces on the interface $\Gamma_i \cap \Gamma_I$, respectively. The sub- or superscript I stands for *interface*. Moreover, we denote the sets of vertices, edges and faces on the Dirichlet boundary $\Gamma_i \cap \Gamma_D$ by $\mathcal{V}_i^D$, $\mathcal{E}_i^D$ and $\mathcal{F}_i^D$, respectively, and the set of non-coupling faces on the Neumann boundary by $\mathcal{F}_i^N$. Hence, e. g., $\mathcal{F}_i = \mathcal{F}_i^I \cup \mathcal{F}_i^N \cup \mathcal{F}_i^D$. Finally, we use the same notation also for global sets: The set $\mathcal{V}^I$ denotes the set of all subdomain vertices on the interface, $\mathcal{E}^D$ the set of all Dirichlet edges, and so forth.

Using the partition of unity provided in Definition 3.3, we obtain that for any $i \in \mathcal{I}$,

$$|(E_D\,\widetilde{w})_i|^2_{S_{i,h}} \leq \Big| \sum_{\substack{V\in\mathcal{V}_i^I\\ E\in\mathcal{E}_i^I\\ F\in\mathcal{F}_i^I}} \sum_{j\in\mathcal{N}_{V/E/F}} I^h(\delta_j^\dagger\,\theta_{V/E/F}\,\widetilde{w}_j)\Big|^2_{S_{i,h}} + \Big| \sum_{F\in\mathcal{F}_i^N} I^h(\theta_F\,\widetilde{w}_i)\Big|^2_{S_{i,h}}. \tag{3.47}$$

We apply Lemma 3.16, part (i) and observe that the cardinality of $\mathcal{N}_{V/E/F}$ is small, which justifies the use of the Cauchy-Schwarz inequality. We can obtain

$$|(E_D\,\widetilde{w})_i|^2_{S_{i,h}} \preceq \sum_{\substack{V\in\mathcal{V}_i^I\\ E\in\mathcal{E}_i^I\\ F\in\mathcal{F}_i^I}} \sum_{j\in\mathcal{N}_{V/E/F}} \underbrace{\left(\delta_j^\dagger|_{V/E/F}\right)^2\alpha_i}_{\leq\min(\alpha_i,\alpha_j)} \Big|I^h(\theta_{V/E/F}\,\widetilde{w}_j)\Big|^2_{H^{1/2}_{00}(\omega^{(i)}_{V/E/F})}. \tag{3.48}$$

In the last step, we have used that $\delta_j^\dagger$ is constant on a vertex, edge or face. Thus, we have been able to apply Lemma 3.2. By Lemma 3.13, we can bound the $H^{1/2}_{00}$-norm of $I^h(\theta_{V/E/F}\,\widetilde{w}_j)$ on the patches $\omega^{(i)}_{V/E/F}$ from above in the following way.

- For $j = 0$, we have $\widetilde{w}_j = 0$ and there is nothing to estimate.
- If $j \in \mathcal{I}_{\text{float}}$ we obtain by Lemma 3.13, (i) and (ii) that

$$\begin{aligned}|I^h(\theta_{V/E/F}\,\widetilde{w}_j)|^2_{H^{1/2}_{00}(\omega^{(i)}_{V/E/F})} &\preceq (1+\log(H_F/h_i))^2\,\|\widetilde{w}_j\|^2_{H^{1/2}(F)}\\ &\preceq (1+\log(H_F/h_i))^2\,|\widetilde{w}_j|^2_{H^{1/2}(F)},\end{aligned}$$

  where $F \in \mathcal{F}_j$ and $E \in \mathcal{E}_j$ are chosen such that $E \subset \partial F$, and that V is one of the endpoints of E. In the last step we have used that the fact that $w_j \in (\ker S_j)^\top$ implies $\int_{\Gamma_j} w_j(x)\,ds_x = 0$, and applied Poincaré's inequality on the domain Ω_j, which states that

$$\frac{1}{H_j}\|u\|^2_{L_2(\Omega_j)} \preceq |u|^2_{H^1(\Omega_j)} + \frac{1}{H_j^d}\Big(\int_{\Gamma_j} u(x)\,ds_x\Big)^2 \qquad \forall u\in H^1(\Omega_j),$$

  cf. [87, Lemmas A.13, A.17 and A.18]; see also Definition 3.5 for our definition of the $\|\cdot\|_{H^{1/2}(F)}$-norm.
- If Ω_j is non-floating, we set

$$\overline{w}_j := \frac{1}{|\Gamma_j|}\int_{\Gamma_j}\widetilde{w}_j(x)\,ds_x$$

  (which implies $\int_{\Gamma_j}\widetilde{w}_j(x) - \overline{w}_j\,ds_x = 0$), and Lemma 3.13, (i)–(v) yields

$$\begin{aligned}&|I^h(\theta_{V/E/F}\,\widetilde{w}_j)|^2_{H^{1/2}_{00}(\omega^{(i)}_{V/E/F})}\\ &\preceq |I^h(\theta_{V/E/F}(\widetilde{w}_j-\overline{w}_j))\,|^2_{H^{1/2}_{00}(\omega^{(i)}_{V/E/F})} + |\overline{w}_j|^2\,|I^h(\theta_{V/E/F})|^2_{H^{1/2}_{00}(\omega^{(i)}_{V/E/F})}\\ &\preceq (1+\log(H_F/h_i))^2\,\|\widetilde{w}_j-\overline{w}_j\|^2_{H^{1/2}(F)} + (1+\log(H_F/h_i))\,H_F|\overline{w}_j|^2,\\ &\preceq (1+\log(H_F/h_i))^2\,|\widetilde{w}_j|^2_{H^{1/2}(F)} + (1+\log(H_F/h_i))\,\frac{1}{H_F}\|\widetilde{w}_j\|^2_{L_2(\Gamma_j)}.\end{aligned} \tag{3.49}$$

In the last step we have used Poincaré's inequality and the Cauchy-Schwarz inequality, which shows that

$$|\overline{w}_j|^2 \leq 1/|\Gamma_j| \, \|\widetilde{w}_j\|^2_{L_2(\Gamma_j)} \simeq H_F^{-2} \, \|\widetilde{w}_j\|^2_{L_2(\Gamma_j)} \, .$$

Since Ω_j touches the Dirichlet boundary Γ_D, we can bound the L_2-term on the right hand side from above using the discrete Poincaré-Friedrichs inequality (Lemma 3.14), which contributes another logarithmic factor and leads to the overall bound

$$|I^h(\theta_{V/E/F} \, \widetilde{w}_j)|^2_{H^{1/2}_{00}(\omega^{(i)}_{V/E/F})} \preceq (1 + \log(H/h))^2 \, |\widetilde{w}_j|^2_{H^{1/2}(F)} \, .$$

Knowing that $H_F/h_i \leq H/h$ (cf. formula (2.29)), we have

$$|I^h(\theta_{V/E/F} \, \widetilde{w}_j)|^2_{H^{1/2}_{00}(\omega^{(i)}_{V/E/F})} \preceq (1 + \log(H/h))^2 \, |\widetilde{w}_j|^2_{H^{1/2}(F)} \qquad \forall j \in \mathcal{N}_{V/E/F} \, . \tag{3.50}$$

The second term in (3.47) involving the contributions on the Neumann boundary is treated analogously.

Combining estimates (3.48) and (3.50), and observing that the number of faces of Ω_j is a small for $j \neq 0$, we obtain by Lemma 3.16 part (ii) that

$$\begin{aligned} \sum_{i \in \mathcal{I}} |(E_D \, \widetilde{w})_i|^2_{S_{i,h}} &\preceq (1 + \log(H/h))^2 \sum_{j \in \mathcal{I}\setminus\{0\}} \; \sum_{F \in \mathcal{F}^I_j \cup \mathcal{F}^N_j} \alpha_j \, |\widetilde{w}_j|^2_{H^{1/2}(F)} \\ &\preceq (1 + \log(H/h))^2 \sum_{j \in \mathcal{I}\setminus\{0\}} |\widetilde{w}_j|^2_{S_{j,h}} \\ &\preceq (1 + \log(H/h))^2 \, |\widetilde{w}|^2_S \, , \end{aligned}$$

since $\widetilde{w}_0 = 0$. This finishes the proof of Claim 1.

Claim 2: We prove

$$|P_D \, z_{\widetilde{w}}|^2_S \preceq (1 + \log(H/h))^2 |\widetilde{w}|^2_S \qquad \text{for all } \widetilde{w} \in (\ker S)^\perp \text{ with } \widetilde{w}_0 = 0 \, .$$

Also in this situation, we can basically use the techniques described in [49, 87] taking into account that $\widetilde{w}_0 = 0$, and modify some of the arguments for the all-floating formulation.

For $Q = M^{-1}$, we see from Corollary 3.6 and from Lemma 3.9 that

$$|P_D \, z_{\widetilde{w}}|^2_S = \|B \, z_{\widetilde{w}}\|^2_Q \leq \|B \, \widetilde{w}\|^2_Q = |P_D \, \widetilde{w}|^2_S \preceq |\widetilde{w}|^2_S + |E_D \, \widetilde{w}|^2_S \, .$$

Claim 1 immediately yields

$$|P_D \, z_{\widetilde{w}}|^2_S \preceq (1 + \log(H/h))^2 \, |w|^2_S \, .$$

For the diagonal choice of Q given in Section 3.3.1, we have

$$|P_D\, z_{\widetilde{w}}|_S^2 = \sum_{i\in\mathcal{I}} \Bigg| \sum_{\substack{V\in\mathcal{V}_i^I\\ E\in\mathcal{E}_i^I\\ F\in\mathcal{F}_i^I}} \sum_{j\in\mathcal{N}_{V/E/F}} \frac{\alpha_i^\gamma}{\sum\limits_{k\in\mathcal{N}_{V/E/F}} \alpha_k^\gamma} I^h\big(\theta_{V/E/F}\,(z_{\widetilde{w},i} - z_{\widetilde{w},j})\big) \; + \\ + \sum_{\substack{V\in\mathcal{V}_i^D\\ E\in\mathcal{E}_i^D\\ F\in\mathcal{F}_i^D}} I^h\big(\theta_{V/E/F}\, z_{\widetilde{w},i}\big)\Bigg|_{S_{i,h}}^2 .$$

Subce Ω_0 is non-floating, the 0-component of $z_{\widetilde{w}}$ vanishes. Therefore we can bound above expression by

$$\sum_{i\in\mathcal{I}} \Bigg\{ \sum_{\substack{V\in\mathcal{V}_i^I\\ E\in\mathcal{E}_i^I\\ F\in\mathcal{F}_i^I}} \sum_{j\in\mathcal{N}_{V/E/F}} \min(\alpha_i,\,\alpha_j)\, |I^h\big(\theta_{V/E/F}\,(z_{\widetilde{w},i} - z_{\widetilde{w},j})\big)|^2_{H_{00}^{1/2}(\omega_{V/E/F}^{(i)})} \; + \\ + \sum_{\substack{V\in\mathcal{V}_i^D\\ E\in\mathcal{E}_i^D\\ F\in\mathcal{F}_i^D}} |I^h\big(\theta_{V/E/F}\, z_{\widetilde{w},i}\big)|^2_{H_{00}^{1/2}(\omega_{V/E/F}^{(i)})} \Bigg\},$$

where we have used Lemma 3.16, part (i) and the Cauchy-Schwarz inequality. For the standard formulation, all the Dirichlet terms in the second sum vanish, since $\Gamma_i \cap \Gamma_D \neq \emptyset$ implies that Ω_i is non-floating and thus $z_{\widetilde{w},i} = 0$.

- Using Lemma 3.13, (iii), the contribution from a face $F = F_{ij}$ on the interface can be bounded by
$$C\, \min(\alpha_i,\,\alpha_j)\,\Big(1 + \log\Big(\frac{H_F}{h_i}\Big)\Big)\, H_F |z_{\widetilde{w},i} - z_{\widetilde{w},j}|^2 .$$
The difference $z_{\widetilde{w},i} - z_{\widetilde{w},j}$ is now interpreted as the component of $B\, w_{\widetilde{w}}$ corresponding to $\mu_{x,ij}$ with x being a node on F. Since there are $\mathcal{O}((H_F/h_i)^2)$ such nodes and $Q[\mu_{x,ij}] = \min(\alpha_i,\,\alpha_j)(1 + \log(H_F/h_i))\, h_i^2/H_F$, above term is equivalent to
$$\sum_{x\in F_{ij}\cap\Gamma_{I,h}} Q[\mu_{x,ij}]\, \langle B\, z_{\widetilde{w}},\, \lambda_{x,ij}\rangle^2 .$$

- Lemma 3.13, (iv) yields that contribution from an edge $E = E_{ij}$ on the interface can be bounded by
$$C\, \min(\alpha_i,\,\alpha_j)\, H_E\, |z_{\widetilde{w},i} - z_{\widetilde{w},j}|^2 \qquad (3.51)$$
Assume first, that Ω_i and Ω_j are connected by a Lagrange multiplier. Then the difference can be read again as any component of $B\, z_{\widetilde{w}}$ corresponding to some $\mu_{x,ij}$

with the node x lying on E. Since there are $\mathcal{O}(H_E/h_i)$ such nodes on the edge and $Q[\mu_{x,ij}] = \min(\alpha_i,\, \alpha_j)h_i$, above expression is equivalent to

$$\sum_{x \in E_{ij} \cap \Gamma_{I,h}} Q[\mu_{x,ij}]\, \langle B\, z_{\widetilde{w}},\, \lambda_{x,ij}\rangle^2 .$$

If we have non-redundant Lagrange multipliers, and Ω_i and Ω_j are not directly connected, but both connected to the subdomain Ω_k with the locally largest coefficient, the edge contribution (3.51) is bounded by

$$C \Big\{ \min(\alpha_i,\, \alpha_k)\, H_E\, |z_{\widetilde{w},i} - z_{\widetilde{w},k}|^2 + \min(\alpha_k,\, \alpha_j)\, H_E\, |z_{\widetilde{w},k} - z_{\widetilde{w},j}|^2 \Big\} ,$$

and we can proceed as before, since $E = E_{ik} = E_{kj}$.

- Interface vertex terms are treated similarly to the edge terms.

For the all-floating formulation we additionally have contributions from $\Gamma_i \cap \Gamma_D$ for $i \neq 0$:

- The contribution from a face $F \in \mathcal{F}_i^D$ can be bounded by

  $$C\, \alpha_i\, (1 + \log(H_F/h_i))\, H_F\, |z_{\widetilde{w},i}|^2 .$$

  We observe that $z_{\widetilde{w},i}$ equals each component of $B\, z_{\widetilde{w}}$ corresponding to $\mu_{x,i}$ with the node x lying on the face F. Since there are $\mathcal{O}((H_F/h_i)^2)$ such nodes on the face and $Q[\mu_{x,i}] = \alpha_i(1 + \log(H_F/h_i))\, h_i^2/H_F$, above expression is equivalent to

  $$\sum_{x \in F \cap \Gamma_{i,h}} Q[\mu_{x,i}] \langle B\, z_{\widetilde{w}},\, \lambda_{x,i}\rangle^2 .$$

- Edge and vertex terms on the Dirichlet boundary are treated similarly, and we obtain that they can be bounded by

  $$\sum_{x \in E \cap \Gamma_{i,h}} Q[\mu_{x,i}] \langle B\, z_{\widetilde{w}},\, \lambda_{x,i}\rangle^2 , \qquad Q[\mu_{V,i}] \langle B\, z_{\widetilde{w}},\, \lambda_{V,i}\rangle^2 ,$$

  respectively (see the definition of Q, page 36).

All in all, we have shown that for both formulations, $|P_D\, z_{\widetilde{w}}|_S^2 \preceq \|B\, z_{\widetilde{w}}\|_Q^2$ and due to Lemma 3.9,

$$\|B\, z_{\widetilde{w}}\|_Q^2 \leq \|B\, \widetilde{w}\|_Q^2 .$$

Now, we group the contributions of $\|B\, \widetilde{w}\|_Q^2$ to faces, edges and vertices on subdomain interfaces and on the Dirichlet boundary. We observe the following:

- A subdomain face $F = F_{ij}$ contributes

$$\begin{aligned} &\min(\alpha_i,\, \alpha_j)\, (1 + \log(H_F/h_i))\, \frac{h_i^2}{H_F} \sum_{x \in F \cap \Gamma_{i,h}} \big(\widetilde{w}_i(x) - \widetilde{w}_j(x)\big)^2 \\ &\simeq \min(\alpha_i,\, \alpha_j)\, (1 + \log(H_F/h_i))\, \frac{1}{H_F} \|\widetilde{w}_i - \widetilde{w}_j\|_{L_2(F)}^2 \\ &\leq (1 + \log(H_F/h_i)) \Big\{ \alpha_i \|\widetilde{w}_i\|_{L_2(F)}^2 + \alpha_j \|\widetilde{w}_j\|_{L_2(F)}^2 . \end{aligned}$$

Since $\widetilde{w}_0 = 0$, we can use the same argument as in the proof of Claim 1 (see page 51) and bound the L_2-norms by the $H^{1/2}$-semi-norms: If $j = 0$ there is nothing to estimate. For $j \in \mathcal{I}_{\text{float}}$ we can use the Poincaré inequality, and for non-floating Ω_j we use the discrete Poincaré-Friedrichs inequality (Lemma 3.14). In each case, we obtain the upper bound

$$(1 + \log(H/h))^2 \left\{ \alpha_i |\widetilde{w}_i|^2_{H^{1/2}(F)} + \alpha_j |\widetilde{w}_j|^2_{H^{1/2}(F)} \right\}.$$

- A subdomain edge $E = E_{ij}$ contributes

$$\begin{aligned} \min(\alpha_i, \alpha_j)\, h_i \sum_{x \in E \cap \Gamma_{i,h}} \big(\widetilde{w}_i(x) - \widetilde{w}_j\big)^2 &\simeq \min(\alpha_i, \alpha_j)\, \|\widetilde{w}_i - \widetilde{w}_j\|^2_{L_2(E)} \\ &\preceq \alpha_i\, \|\widetilde{w}_i\|^2_{L_2(E)} + \alpha_j\, \|\widetilde{w}_j\|^2_{L_2(E)} \\ &\preceq (1 + \log(H/h)) \left\{ \alpha_i\, \|\widetilde{w}_i\|^2_{H^{1/2}(F_i)} + \alpha_j\, \|\widetilde{w}_j\|^2_{H^{1/2}(F_j)} \right\}, \end{aligned}$$

  where $F_i \in \mathcal{F}_i$, $F_j \in \mathcal{F}_j$ such that $E \in \partial F_i \cup \partial F_j$. Again, we have used Lemma 3.13, (vi) and the fact that $\widetilde{w}_0 = 0$. With the same argument as for the face contributions we arrive at the upper bound

$$(1 + \log(H/h))^2 \left\{ \alpha_i |\widetilde{w}_i|^2_{H^{1/2}(F_i)} + \alpha_j |\widetilde{w}_j|^2_{H^{1/2}(F_j)} \right\}.$$

- A subdomain vertex $V = V_{ij}$ contributes

$$\min(\alpha_i, \alpha_j)\, h_i \big(\widetilde{w}_i(V) - \widetilde{w}_j(V)\big)^2,$$

  which can be easily be bounded by the sum of the squared L_2-norms of $\widetilde{w}_i$ and $\widetilde{w}_j$ on adjacent edges and further estimated as above.

Additionally, we have to consider the contributions from the Lagrange parameters due to the all-floating formulation. Note, that Assumption 3.2 holds, i. e., there are no such Lagrange parameters acting on Γ_0.

- For $i \neq 0$, a face $F \in \mathcal{F}_i^D$ on the Dirichlet boundary contributes

$$\alpha_i \left(1 + \log\left(\frac{H_F}{h_i}\right)\right) \frac{h_i^2}{H_F} \sum_{x \in F \cap \Gamma_{i,h}} |\widetilde{w}_i(x)|^2 \simeq \alpha_i \left(1 + \log\left(\frac{H_F}{h_i}\right)\right) \frac{1}{H_F} \|\widetilde{w}_i\|^2_{L_2(F)}$$

  and can be treated as above.

- For $i \neq 0$, the edge contribution from $E \in \mathcal{E}_i^D$,

$$\alpha_i\, h_i \sum_{x \in E \cap \Gamma_{i,h}} |\widetilde{w}_i(x)|^2 \preceq \alpha_i\, \|\widetilde{w}_i\|^2_{L_2(E)},$$

  and the vertex contribution from $V \in \mathcal{V}_i^D$,

$$\alpha_i\, h_i\, |\widetilde{w}_i(V)|^2 \preceq \alpha_i \|\widetilde{w}_i\|^2_{L_2(E)},$$

  where $V \in \partial E$, can be estimated as above.

Putting all separate estimates together, we finally obtain by Lemma 3.16, part (ii) that

$$|P_D\, z_{\widetilde{w}}|^2_S \preceq (1 + \log(H/h))^2\, |\widetilde{w}|^2_S,$$

which finishes the proof of Claim 2.

Claim 3 – Case $Q = M^{-1}$: We prove

$$|P_D(\Pi_0\, w + z_{(\Pi_0\, w)})|_S^2 \preceq \gamma_h(\{\Omega_i\}, \Gamma_D, \Gamma_0)\, (1 + \log(H/h))^2\, |w_0|_{S_{0,h}}^2 \,,$$

for any $w \in W$, under the assumption that $Q = M^{-1}$.

Again, Corollary 3.6 and Lemma 3.9 imply

$$\begin{aligned} |P_D\,(\Pi_0\, w + z_{(\Pi_0\, w)})|_S^2 &= \|B\,(\Pi_0\, w + z_{(\Pi_0\, w)})\|_Q^2 \\ &= \min_{z\in\ker S} \|B\,(\Pi_0\, w + z)\|_Q^2 = \min_{z\in\ker S} |P_D(\Pi_0\, w + z)|_S^2 \,. \end{aligned} \tag{3.52}$$

We bound this expression by constructing a special $\widetilde{z} \in \ker S$. By definition, the function $\Pi_0\, w \in W$ has only one non-trivial component, namely w_0. In the following, we use the special extension

$$\mathcal{H}_{0,D}^{\text{int}}\, w_0 = \operatorname{argmin}\left\{|v|_{H^1(\Omega_{\text{int}})} : v \in V_1^h(\Omega_{\text{int}}),\, v|_{\Gamma_0} = w_0,\, v|_{\Gamma_D} = 0\right\}.$$

By Lemma 3.11 and Lemma 2.3, we obtain

$$\begin{aligned} |\mathcal{H}_{0,D}^{\text{int}}\, w_0|_{H^1(\Omega_{\text{int}})}^2 &\leq \gamma_h(\{\Omega_i\}, \Gamma_0, \Gamma_D)\left\{|\mathcal{H}^{\text{int}}\, w_0|_{H^1(\Omega_{\text{int}})}^2 + \frac{1}{H_0}\|w_0\|_{L_2(\Gamma_0)}^2\right\} \\ &\leq \gamma_h(\{\Omega_i\}, \Gamma_0, \Gamma_D)\left\{\langle S_0^{\text{int}}\, w_0,\, w_0\rangle + \frac{1}{H_0}\|w_0\|_{L_2(\Gamma_0)}^2\right\} \\ &\preceq \gamma_h(\{\Omega_i\}, \Gamma_0, \Gamma_D)\, \langle S_{0,\text{BEM}}^{\text{ext}}\, w_0,\, w_0\rangle \,. \end{aligned} \tag{3.53}$$

We are now able to define $\widetilde{z} \in \ker S$ in the following way,

$$\widetilde{z}_i := \begin{cases} \dfrac{1}{|\Omega_i|} \displaystyle\int_{\Omega_i} (\mathcal{H}_{0,D}^{\text{int}}\, w_0)(x)\, dx & \text{for } i \in \mathcal{I}_{\text{float}} \,, \\ 0 & \text{for } i \in \mathcal{I} \setminus \mathcal{I}_{\text{float}} \,. \end{cases} \tag{3.54}$$

Continuing our estimate (3.52) we get

$$|P_D(\Pi_0\, w + z_{(\Pi_0\, w)})|_S^2 \leq |(P_D(\Pi_0\, w + \widetilde{z}))_0|_{S_{0,h}}^2 + \sum_{i\in\mathcal{I}\setminus\{0\}} |(P_D(\Pi_0\, w + \widetilde{z}))_i|_{S_{i,h}}^2 \,. \tag{3.55}$$

We denote the set of vertices, edges and faces on $\Gamma_i \cap \Gamma_D$ by $\mathcal{V}_i^D$, $\mathcal{E}_i^D$ and $\mathcal{F}_i^D$, respectively. Since P_D vanishes on the Neumann boundary (cf. Lemma 3.3) we only need to consider the contributions from the subdomain interfaces and the Dirichlet boundary.

The first summand in (3.55) is treated as follows: By the same technique as in the proofs of Claim 1 and Claim 2, and using Lemma 3.2, we can conclude that

$$\begin{aligned} &|(P_D(\Pi_0\, w + \widetilde{z}))_0|_{S_{0,h}}^2 \\ &\quad\preceq \Bigg| \sum_{\substack{V\in\mathcal{V}_0^I \\ E\in\mathcal{E}_0^I \\ F\in\mathcal{F}_0^I}} \sum_{j\in\mathcal{N}_{V/E/F}} I^h\big(\delta_j^\dagger\, \theta_{V/E/F}\,(w_0 - \widetilde{z}_j)\big) + \sum_{\substack{V\in\mathcal{V}_0^D \\ E\in\mathcal{E}_0^D \\ F\in\mathcal{F}_0^D}} I^h(\theta_{V/E/F}\, w_0)\Bigg|_{S_{0,h}}^2 \\ &\quad\preceq \sum_{\substack{V\in\mathcal{V}_0^I \\ E\in\mathcal{E}_0^I \\ F\in\mathcal{F}_0^I}} \sum_{j\in\mathcal{N}_{V/E/F}} \underbrace{\alpha_0\,\big(\delta_j^\dagger\big)^2}_{\leq\alpha_0}\, \big|I^h\big(\theta_{V/E/F}\,(w_0 - \widetilde{z}_j)\big)\big|_{H_{00}^{1/2}(\omega_{V/E/F}^{(0)})}^2 \,. \end{aligned}$$

We have used that w_0 vanishes on $\Gamma_0 \cap \Gamma_D$ due to Assumption 3.2, see page 25. For $j \in \mathcal{I}_{\text{float}}$, we obtain immediately

$$\left|I^h\big(\theta_{V/E/F}\,(w_0 - \widetilde{z}_j)\big)\right|^2_{H^{1/2}_{00}(\omega^{(0)}_{V/E/F})} \preceq (1 + \log(H_{0,j}/h_0))^2\, |\mathcal{H}^{\text{int}}_{0,D}\, w_0|^2_{H^1(\Omega_j)}\,.$$

If Ω_j is non-floating, we have $\widetilde{z}_j = 0$. With the mean value

$$\widehat{z}_j := \frac{1}{|\Omega_j|}\int_{\Omega_j} (\mathcal{H}^{\text{int}}_{0,D}\, w_0)(x)\, dx\,,$$

we can use the same trick as in estimate (3.49) on page 51 and obtain that

$$\begin{aligned}
&\left|I^h\big(\theta_{V/E/F}\,(w_0 - \widetilde{z}_j)\big)\right|^2_{H^{1/2}_{00}(\omega^{(0)}_{V/E/F})}\\
&\quad\preceq (1 + \log(H_{0,j}/h_0))^2\, |\mathcal{H}^{\text{int}}_{0,D}\, w_0 - \widehat{z}_j|^2_{H^1(\Omega_j)} + (1 + \log(H_{0,j}/h_0))\, H_{0,j}\, |\widehat{z}_j|^2\\
&\quad\preceq (1 + \log(H_{0,j}/h_0))^2\, |\mathcal{H}^{\text{int}}_{0,D}\, w_0|^2_{H^1(\Omega_j)}\,,
\end{aligned}$$

where in the last step, we have used the Cauchy-Schwarz inequality, the discrete Poincaré-Friedrichs inequality, and the fact that $\mathcal{H}^{\text{int}}_{0,D}\, w_0$ satisfies the homogeneous Dirichlet boundary conditions on $\Gamma_0 \cap \Gamma_D$. Summarizing, we have shown

$$\begin{aligned}
|(P_D(\Pi_0\, w + \widetilde{z}))_0|^2_{S_{0,h}} &\preceq \sum_{F_{0,j}\in\mathcal{F}_0} (1 + \log(H_{0,j}/h_0))^2\, \alpha_0\, |\mathcal{H}^{\text{int}}_{0,D}\, w_0|^2_{H^1(\Omega_j)}\\
&\preceq (1 + \log(H/h))^2\, \alpha_0\, |\mathcal{H}^{\text{int}}_{0,D}\, w_0|^2_{H^1(\Omega_{\text{int}})}\,.
\end{aligned} \tag{3.56}$$

Now we treat the remaining summands on the right hand side of formula (3.55): For $i \in \mathcal{I} \setminus \{0\}$, we obtain by the same technique described above that

$$\begin{aligned}
|(P_D(\Pi_0\, w + \widetilde{z}))_i|^2_{S_{i,h}} \preceq& \sum_{\substack{V\in\mathcal{V}^D_i\\ E\in\mathcal{E}^D_i\\ F\in\mathcal{F}^D_i}} \alpha_i \left|I^h\big(\theta_{V/E/F}\,\widetilde{z}_i\big)\right|^2_{H^{1/2}_{00}(\omega^{(i)}_{V/E/F})} + \\
&+ \sum_{\substack{V\in\mathcal{V}^I_i\\ E\in\mathcal{E}^I_i\\ F\in\mathcal{F}^I_i}} \sum_{j\in\mathcal{N}_{V/E/F}} \min(\alpha_i, \alpha_j) \left|I^h\big(\theta_{V/E/F}\,(\widetilde{z}_i - (\Pi_0\, w + \widetilde{z})_j)\big)\right|^2_{H^{1/2}_{00}(\omega^{(i)}_{V/E/F})}\,.
\end{aligned} \tag{3.57}$$

Using Lemma 3.13, the contributions on the Dirichlet boundary of non-floating subdomains can be estimated from above by

$$\begin{aligned}
\left|I^h\big(\theta_F\,\widetilde{z}_i\big)\right|^2_{H^{1/2}_{00}(F)} &\preceq (1 + \log(H/h))\, H_i\, |\widetilde{z}_i|^2\,,\\
\left|I^h\big(\theta_{V/E}\,\widetilde{z}_i\big)\right|^2_{H^{1/2}_{00}(\omega^{(i)}_{V/E})} &\preceq H_i |\widetilde{z}_i|^2\,.
\end{aligned}$$

Due to the definition of $\widetilde{z}_i$, we obtain for $i \in \mathcal{I}_{\text{float}}$,

$$\begin{aligned}
H_i |\widetilde{z}_i|^2 &\le H_i \left(\frac{1}{|\Omega_i|}\int_{\Omega_i} (\mathcal{H}^{\text{int}}_{0,D}\, w_0)(x)\, dx\right)^2\\
&\preceq \frac{1}{H_i^2} \|\mathcal{H}^{\text{int}}_{0,D}\, w_0\|^2_{L_2(\Omega_i)} \preceq (1 + \log(H/h))\, |\mathcal{H}^{\text{int}}_{0,D}\, w_0|^2_{H^1(\Omega_i)}\,,
\end{aligned}$$

where we have used the Cauchy Schwarz inequality and the fact that the function $\mathcal{H}^{\text{int}}_{0,D}\, w_0$ satisfies the Dirichlet boundary conditions on $\Gamma_i \cap \Gamma_D \neq \emptyset$. If $\Gamma_i \cap \Gamma_D$ is just a subdomain edge, we still can apply a discrete Poincaré-Friedrichs inequality (cf. Lemma 3.14), which contributes the logarithmic term in the estimate above. Eventually, by a finite summation argument and the assumption that $\alpha_i \leq \alpha_0$, we have

$$\sum_{\substack{V\in\mathcal{V}_i^D\\ E\in\mathcal{E}_i^D\\ F\in\mathcal{F}_i^D}} \alpha_i \left|I^h\big(\theta_{V/E/F}\, \widetilde{z}_i\big)\right|^2_{H^{1/2}_{00}(\omega^{(i)}_{V/E/F})} \preceq (1+\log(H/h))^2\, \alpha_0\, |\mathcal{H}^{\text{int}}_{0,D}\, w_0|^2_{H^1(\Omega)}\,.$$

Next, we treate the interface terms in estimate (3.57). For $j = 0$, the contributions on the subdomain interfaces read

$$\begin{aligned} &\min(\alpha_i, \alpha_0)\left|I^h\big(\theta_{V/E/F}\,(\widetilde{z}_i - (\Pi_0\, w - \widetilde{z})_0)\big)\right|^2_{H^{1/2}_{00}(\omega^{(i)}_{V/E/F})}\\ &\quad\leq \alpha_0 \left|I^h\big(\theta_{V/E/F}\,(\widetilde{z}_i - w_0)\big)\right|^2_{H^{1/2}_{00}(\omega^{(i)}_{V/E/F})} \end{aligned}$$

where F is the face shared by Ω_0 and Ω_i. These terms have already been treated in the estimates for the first summand in (3.55). For $j \neq 0$, we have

$$\widetilde{z}_i - (\Pi_0\, w + \widetilde{z})_j = \widetilde{z}_i - \widetilde{z}_j\,.$$

Lemma 3.13 yields

$$\left|I^h\big(\theta_{V/E/F}\,(\widetilde{z}_i - \widetilde{z}_j)\big)\right|^2_{H^{1/2}_{00}(\omega^{(i)}_{V/E/F})} \preceq (1+\log(H/h))\, H_F\, |\widetilde{z}_i - \widetilde{z}_j|^2\,. \tag{3.58}$$

If both i and j are indices of non-floating subdomains, we can employ Lemma 3.15 and bound the term $H_F\, |\widetilde{z}_i - \widetilde{z}_j|^2$ by a constant times the H^1-norm of $\mathcal{H}^{\text{int}}_{0,D}\, w_0$ on a domain $\mathcal{U}_{ij}$ which satisfies $\Omega_i \cap \Omega_i \subset \mathcal{U}_{ij} \subset \Omega_{\text{int}}$. If, e. g., $i \in \mathcal{I}_{\text{float}}$ and $j \notin \mathcal{I}_{\text{float}}$ (thus $\widetilde{z}_j = 0$), we can insert $\widehat{z}_j := 1/|\Omega_j| \int_{\Omega_j} (\mathcal{H}^{\text{int}}_{0,D}\, w_0)(x)\, dx$ and obtain, by the same arguments and Lemma 3.14, that

$$H_F\, |\widetilde{z}_i|^2 \preceq H_F\, |\widetilde{z}_i - \widehat{z}_j|^2 + H_F\, |\widehat{z}_j|^2 \preceq (1+\log(H/h))\, |\mathcal{H}^{\text{int}}_{0,D}\, w_0|^2_{H^1(\mathcal{U}_{ij})}$$

holds. Since we have only a finite overlap of the domains $\mathcal{U}_{ij}$, we finally obtain the following estimate for the remaining summands in formula (3.55),

$$\sum_{i\in\mathcal{I}\setminus\{0\}} |(P_D(\Pi_0\, w + \widetilde{z}))_i|^2_{S_{i,h}} \preceq (1+\log(H/h))^2\, \alpha_0\, |\mathcal{H}^{\text{int}}_{0,D}\, w_0|^2_{H^1(\Omega_{\text{int}})}\,. \tag{3.59}$$

Combining the estimate (3.53) on page 56 with (3.56) and (3.59), we obtain

$$|P_D(\Pi_0\, w + z_{(\Pi_0\, w)})|^2_S \preceq \gamma_h(\{\Omega_i\}, \Gamma_D, \Gamma_0)\, (1+\log(H/h))^2\, \alpha_0\, \langle S^{\text{ext}}_{0,\text{BEM}}\, w_0, w_0\rangle\,,$$

which is the desired estimate of Claim 3 for the case that $Q = M^{-1}$.

Claim 3 – diagonal Q: We prove

$$|P_D(\Pi_0\, w + z_{(\Pi_0\, w)})|_S^2 \preceq \gamma_h(\{\Omega_i\}, \Gamma_D, \Gamma_0)\,(1+\log(H/h))^2\,|w_0|_{S_{0,h}}^2\,,$$

for any $w \in W$, under the assumption that Q is chosen diagonal according to Section 3.3.1.

First, we can use the coarse space element $\widetilde{z}_i \in \ker S$ defined in equation (3.54) and obtain by a simple triangle inequality that

$$|P_D(\Pi_0\, w + z_{(\Pi_0\, w)})|_S^2 \le 2\left\{|P_D(\Pi_0\, w + \widetilde{z})|_S^2 + |P_D(\widetilde{z} - z_{(\Pi_0\, w)})|_S^2\right\}.$$

The first term can be bounded using the proof of Claim 3 for the case $Q = M^{-1}$. For the second term, we observe that $\widetilde{z} - z_{(\Pi_0\, w)}$, as an element in $\ker S$, is constant on the floating subdomains. Using the same arguments as in the proofs of Claim 2, and Lemma 3.9, we get

$$\begin{aligned}
|P_D(\widetilde{z} - z_{(\Pi_0\, w)})|_S^2 &\preceq \|B\,(\widetilde{z} - z_{(\Pi_0\, w)})\|_Q^2 \\
&\preceq \|B\,(\Pi_0\, w + \widetilde{z})\|_Q^2 + \|B\,(\Pi_0\, w + z_{(\Pi_0\, w)})\|_Q^2 \\
&= \|B\,(\Pi_0\, w + \widetilde{z})\|_Q^2 + \min_{z \in \ker S} \|B\,(\Pi_0\, w + z)\|_Q^2 \\
&\preceq \|B\,(\Pi_0\, w + \widetilde{z})\|_Q^2
\end{aligned}$$

The last term splits into contributions corresponding to subdomain vertices, edges and faces on Γ_I and Γ_D. We distinguish the following cases:

- The contribution from a face F shared by Ω_0 and Ω_i is treated as follows,

$$\begin{aligned}
\sum_{x \in F \cap \Gamma_{i,h}} &\min(\alpha_0, \alpha_i)\,(1+\log(H_F/h_0))\,\frac{h_0^2}{H_F}\,|\widetilde{z}_i - w_0(x)|^2 \\
&\preceq \alpha_0\,(1+\log(H/h))\,\frac{1}{H_F}\,\|\widetilde{z}_i - w_0\|_{L_2(F)}^2\,.
\end{aligned}$$

  If $i \in \mathcal{I}_{\text{float}}$, the Poincaré inequality yields the overall bound

$$\alpha_0\,(1+\log(H/h))\,|\mathcal{H}_{0,D}^{\text{int}}\, w_0|_{H^1(\Omega_i)}^2\,,$$

  whereas for $i \notin \mathcal{I}_{\text{float}}$, $\widetilde{z}_i = 0$. Thus, Lemma 3.14 yields the bound

$$\alpha_0\,(1+\log(H/h))^2\,|\mathcal{H}_{0,D}^{\text{int}}\, w_0|_{H^1(\Omega_i)}^2\,.$$

- Next, we treat the contribution from a face F shared by Ω_i and Ω_j, with $i,\, j \neq 0$:

$$\begin{aligned}
\sum_{x \in F \cap \Gamma_{i,h}} &\min(\alpha_i, \alpha_j)\,(1+\log(H_{ij}/h_i))\,\frac{h_i^2}{H_F}\,|\widetilde{z}_i - \widetilde{z}_j|^2 \\
&\preceq \alpha_0\,(1+\log(H_F/h_i))\,H_F\,|\widetilde{z}_i - \widetilde{z}_j|^2 \\
&\preceq \alpha_0\,(1+\log(H/h))^2\,|\mathcal{H}_{0,D}^{\text{int}}\, w_0|_{H^1(\mathcal{U}_{ij})}^2\,,
\end{aligned}$$

  where we have used the fact that the face contains $\mathcal{O}((H_{ij}/h_i)^2)$ nodes, and the same argumentation as in the paragraph after equation (3.58), page 58.

- An edge E shared by Ω_0 and Ω_i contributes

$$\sum_{x \in E \cap \Gamma_{i,h}} \min(\alpha_0, \alpha_i)\, h_0\, |\widetilde{z}_i - w_0(x)|^2 \preceq \alpha_0\, \|\widetilde{z}_i - w_0\|^2_{L_2(E)}$$
$$\preceq \alpha_0\, (1 + \log(H/h))^2\, |\mathcal{H}^{\rm int}_{0,D}\, w_0|^2_{H^1(\Omega_i)}\,.$$

- An edge E shared by Ω_i and Ω_j, with $i,\, j \neq 0$ contributes $\alpha_0\, H_E\, |\widetilde{z}_i - \widetilde{z}_j|^2$, which can be treated analogously as before.

- A vertex $V \in \Gamma_{0,i}$ contributes

$$\alpha_0\, h_i\, |\widetilde{z}_i - w_0(V)|^2 \preceq \alpha_0\, \|\widetilde{z}_i - w_0\|^2_{L_2(E)}$$

  and can be treated analogously to the edge case.

- The contribution from a vertex $V \in \Gamma_{ij}$ with $i,\, j \neq 0$ reads

$$\alpha_0\, h_i\, |\widetilde{z}_i - \widetilde{z}_j|^2 \leq \alpha_0\, H_E\, |\widetilde{z}_i - \widetilde{z}_j|^2$$

  which is again analogue to the edge case.

- In the all-floating formulation, faces F, edges E and vertices V on $\Gamma_j \cap \Gamma_D$ with $j \neq 0$ contribute

$$\alpha_j\, (1 + \log(H_F/h_j))\, H_F |\widetilde{z}_j|^2\,,$$
$$\alpha_j\, H_E |\widetilde{z}_j|^2\,,$$
$$\alpha_j\, h_i |\widetilde{z}_j|^2\,,$$

  respectively, see also Assumption 3.2. Using the same arguments as before, each of the three terms can be bounded by

$$\alpha_j\, (1 + \log(H/h))^2\, |\mathcal{H}^{\rm int}_{0,D}\, w_0|^2_{H^1(\Omega_j)}\,.$$

Putting the separate estimates together, we obtain by a finite summation argument that

$$|P_D(\widetilde{z} - z_{(\Pi_0\, w)})|^2_S \preceq \|B\, (\Pi_0\, w + \widetilde{z})\|^2_Q \preceq (1 + \log(H/h))^2\, \alpha_0\, |\mathcal{H}^{\rm int}_{0,D}\, w_0|^2_{H^1(\Omega_{\rm int})}\,.$$

From this point on, we can proceed exactly as in the proof of Claim 3 for the case $Q = M^{-1}$, and get the desired result of Claim 3 for the diagonal choice of Q.

Claim 4: We prove

$$|P_D(\Pi_0 w + z_{\Pi_0 w})|^2_S \preceq \max_{F \in \mathcal{F}_0} \frac{H_0}{H_F}\, (1 + \log(H/h))^2 |w_0|^2_{S_{0,h}}\,,$$

for any $w \in W$, even if the assumption $\alpha_0 \leq \alpha_i$ does not hold. In the case of $Q = M^{-1}$ we know from equation (3.52) that

$$|P_D(\Pi_0 w + z_{\Pi_0 w})|^2_S = \min_{z \in \ker S} |P_D(\Pi_0 w + z)|^2_S\,.$$

Again, we choose a special such z, namely $\widetilde{z} \equiv 0$. Following the line of the proof of Claim 3, we obtain

$$
\begin{aligned}
|P_D(\Pi_0 w + z_{\Pi_0 w})|_S^2| &\leq |P_D\, \Pi_0 w|_S^2 \\
&\preceq \sum_{F\in\mathcal{F}_0} (1+\log(H/h))^2\, \alpha_0 \left\{ |w_0|^2_{H^{1/2}(F)} + \frac{1}{H_F}\, \|w_0\|^2_{L_2(F)} \right\} \\
&\preceq (1+\log(H/h))^2 \max_{F\in\mathcal{F}_0} \frac{H_0}{H_F}\, \alpha_0 \left\{ |w_0|^2_{S_0^{\mathrm{int}}} + \frac{1}{H_0} \|w_0\|^2_{L_2(\Gamma_0)} \right\} \\
&\preceq (1+\log(H/h))^2 \max_{F\in\mathcal{F}_0} \frac{H_0}{H_F}\, |w_0|^2_{S_{0,h}}\,,
\end{aligned}
\tag{3.60}
$$

see also the introduction, page 5. A careful inspection reveals that we do not need the assumption $\alpha_0 \geq \alpha_i$.

For the diagonal choice of Q, the triangle inequality yields

$$
|P_D(\Pi_0 w + z_{\Pi_0 w})|_S^2| \leq |P_D\, \Pi_0 w|_S^2 + |P_D\, z_{\Pi_0 w}|_S^2\,.
$$

We have already estimated the first term in (3.60). For the second term, we follow the line of the proof of Claim 2. Here, Assumption 3.2 (page 25) becomes again important. Finally, we obtain

$$
|P_D\, z_{\Pi_0 w}|_S^2 \preceq \sum_{F\in\mathcal{F}_0} (1+\log(H/h))^2\, \alpha_0 \left\{ |w_0|^2_{H^{1/2}(F)} + \frac{1}{H_F} \|w_0\|^2_{L_2(F)} \right\},
$$

which can be absorbed again in estimate (3.60).

This finishes the proof of Claim 4.

Remark 3.13. In Claim 4, we have not made use of the extension indicator. Hence, we can drop the additional assumptions which we need for Lemma 3.11.

Remark 3.14. The estimate of Claim 4 gets worse if the ratio $\max_{F\in\mathcal{F}_0} H_0/H_F$ increases. Slightly modifying the proof of Claim 3, one can show the sub-optimal bound

$$
C\,(1+\log(H/h))^2 \max_{i\in\mathcal{I}} \frac{\alpha_i}{\alpha_0}
$$

of the total condition number, also without using the extension indicator.

Since we have proved Claim 1–4 this finishes also the proof of Lemma 3.12, which means we have completed our argumentation to show the main result of this section, i. e., Theorem 3.1.

We see, that the analysis of one-level BETI methods for unbounded domains is quite involved, and the conditions on the geometry and the coefficients which lead to the optimal bounds seem rather artificial. The situation is completely different for dual-primal BETI methods which we discuss in the Section 4.

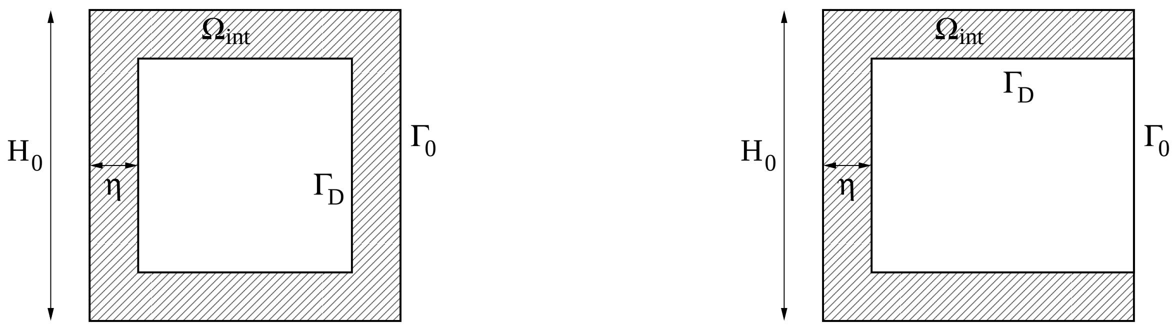

Figure 11: Sketch of the geometries for Table 2 (left) and Table 3 (right).

3.4 Numerical results

In this subsection, we show some results on one-level BETI methods for two-dimensional unbounded domains. Here, we restrict ourselves to the standard formulation with redundant Lagrange multipliers, and the cheap, diagonal choice of the operator Q, see Section 3.3.1 In particular we study the three cases that appear in Lemma 3.11. In Table 1–3, we display the number of PCG iterations for various ratios H/h and H_0/H_F or H_0/η.

$\frac{H_0}{H_i}$	$\frac{H}{h} = 2$	4	8	16	32	64
3	7	8	10	11	12	13
6	9	9	11	13	15	17
12	9	10	12	14	16	18
24	9	10	12	15	17	20
48	9	11	13	15	18	–

Table 1: Number of PCG iterations of the one-level BETI method for a unit square without any interior boundary conditions. Moving right means h-refinement, moving down means increasing the number of subdomains.

In most of the examples the estimates that we worked out are not or only partially reflected. Actually, the condition numbers are much better than expected. This may have two reasons:

- The estimates could be not sharp. Note, that there could, e. g., be a more clever way of choosing $\widetilde{z}$, see the proofs of Claim 3 and 4.
- The examples we computed are still in the preasymptotic range.

$\frac{H_0}{\eta} = 6$

$\frac{H_0}{H_i}$	$\frac{H}{h} = 2$	4	8	16	32	64
6	9	11	13	14	16	19
12	11	12	14	17	21	24
24	11	13	15	18	21	25
48	12	14	16	19	23	–
96	13	15	17	20	–	–

$\frac{H_0}{\eta} = 24$

$\frac{H_0}{H_i}$	$\frac{H}{h} = 2$	4	8	16	32
24	12	14	16	18	22
48	14	16	21	23	27
96	15	19	22	25	–
192	17	21	24	–	–

$\frac{H_0}{\eta} = 96$

$\frac{H_0}{H_i}$	$\frac{H}{h} = 2$	4	8	16
96	17	22	28	32
192	24	29	34	–
384	27	30	–	–

Table 2: Number of PCG iterations of the one-level BETI method for a unit square with Dirichlet conditions on a "hole" (cf. Fig. 11, left) with distance η from Γ_0. Moving right means h-refinement, moving down means increasing the number of subdomains.

	$\frac{H_0}{H_i}$	$\frac{H}{h} = 2$	4	8	16	32	64
$\frac{H_0}{\eta} = 6$	6	11	13	14	16	17	20
	12	12	14	17	19	22	25
	24	13	15	18	20	24	27
	48	13	16	19	21	24	–
	96	13	16	19	22	–	–

	$\frac{H_0}{H_i}$	$\frac{H}{h} = 2$	4	8	16	32	64
$\frac{H_0}{\eta} = 24$	6	14	16	19	22	25	27
	12	16	20	24	27	31	–
	24	17	21	26	29	–	–

Table 3: Number of PCG iterations of the one-level BETI method for a unit square with Dirichlet conditions on a square outline touching Γ_0, see Fig. 11, right. Moving right means h-refinement, moving down means increasing the number of subdomains.

4 BETI-DP methods

Dual-primal FETI (FETI-DP) methods were first introduced by Farhat, Lesoinne, LeTallec, Pierson and Rixen [30] and analyzed in two dimensions by Mandel and Tezaur [64]. Algorithms for the three-dimensional case were contributed by Farhat, Lesoinne and Pierson [31], see also Pierson's dissertation [73]. Finally, a rigorous analysis was given by Klawonn, Widlund and Dryja [50]. See also the works by Brenner et al. [11, 12, 14] for sharp estimates. A comprehensive description of the theory for FETI-DP methods can again be found in the monograph by Toselli and Widlund [87].

The dual-primal BETI (BETI-DP) methods were first introduced by Langer, Pohoaţă and Steinbach [57]. In this section, we give a full analysis on BETI-DP methods for the case of unbounded domains.

4.1 BETI-DP formulations

For the sake of simplicity, we treat only the case of fully redundant Lagrange multipliers, and without loss of generality we assume homogeneous Dirichlet boundary conditions, i. e., $g_D \equiv 0$.

As in Section 3, we start from the minimization problem

$$\min_{u_h \in V_{1,D}^h(\Gamma_S)} \sum_{i\in\mathcal{I}} \left[\frac{1}{2} \langle S_{i,h}\, A_i\, u_h,\, A_i\, u_h \rangle - \langle f_{i,h},\, A_i\, u_h \rangle \right]. \tag{4.1}$$

We recall the definition of $W := \Pi_{i\in\mathcal{I}} W_i$ and $W_i := V_{1,D}^h(\Gamma_i)$, and consider the Steklov-Poincaré operators $S_{i,h} : W_i \to W_i^*$ and $S = \operatorname{diag}(S_{i,h})_{i\in\mathcal{I}} : W \to W^*$.

As in all dual-primal methods, we work with subspaces $\widetilde{W} \subset W$ for which sufficiently many constraints are enforced such that the block operator S is SPD on $\widetilde{W}$. Such spaces are constructed as follows. We choose a *primal space* $\widehat{W}_\Pi \subset V_{1,D}^h(\Gamma_S)$ and a *dual subspace* $\widetilde{W}_\Delta \subset \widetilde{W}$ such that

$$\widetilde{W} = \widehat{W}_\Pi \oplus \widetilde{W}_\Delta\,.$$

Note, that for simplicity we identify continuous functions from $V_1^h(\Gamma_S)$ with the corresponding ones in the product space W. We denote the i-th component of the product space $\widetilde{W}_\Delta$ by $\widetilde{W}_{\Delta,i}$. According to [50] and [57] we display three choices of the space $\widetilde{W}$:

Algorithm A. *The primal subspace* $\widehat{W}_\Pi$ *is spanned by the nodal vertex basis functions* $\theta_V \in V_{1,D}^h(\Gamma_S)$, *where* $V \in \mathcal{V}^I$, *i. e.,* V *runs over all the subdomain vertices on the interface. The local subspace* $\widetilde{W}_{\Delta,i}$ *is defined as the subspace of* W_i *with its elements vanishing on the subdomain vertices, i. e.,*

$$\widetilde{W}_{\Delta,i} := \{ w_i \in W_i : w_i(V) = 0 \ \forall V \in \mathcal{V}_i^I \}\,.$$

Thus, $\widetilde{W}$ *is the subspace of* W *of functions being continuous at the subdomain vertices.*

Algorithm B. *The primal subspace* $\widehat{W}_\Pi$ *is spanned by the nodal vertex basis functions* $\theta_V \in V_{1,D}^h(\Gamma_S)$, $V \in \mathcal{V}^I$, *and the cutoff functions* θ_E, $\theta_F \in V_{1,D}^h(\Gamma_S)$, *where* $E \in \mathcal{E}^I$, $F \in \mathcal{F}^I$.

The local subspace $\widetilde{W}_{\Delta,i}$ is defined as the subspace of W_i where the values at the subdomain vertices vanish, together with the averages

$$\overline{w_i}^E := \frac{1}{|E|}\int_E w_i(x)\,ds_x\,, \qquad \overline{w_i}^F := \frac{1}{|F|}\int_F w_i(x)\,ds_x\,,$$

i. e.,

$$\widetilde{W}_{\Delta,i} := \{w_i \in W_i : w_i(V) = 0\ \forall V \in \mathcal{V}_i^I\,,\ \overline{w_i}^E = 0\ \forall E \in \mathcal{E}_i^I\,,\ \overline{w_i}^F = 0\ \forall F \in \mathcal{F}_i^I\}\,.$$

Thus, $\widetilde{W}$ is the subspace of W of functions being continuous at the subdomain vertices and with continuous edge and face averages.

Algorithm C. *The primal subspace $\widehat{W}_\Pi$ is spanned by the nodal vertex basis functions $\theta_V \in V_{1,D}^h(\Gamma_S)$, $V \in \mathcal{V}^I$, and the cutoff functions $\theta_E \in V_{1,D}^h(\Gamma_S)$, where $E \in \mathcal{E}^I$. The local subspace $\widetilde{W}_{\Delta,i}$ is defined as the subspace of W_i where the values at the subdomain vertices vanish, together with the edge averages, i. e.,*

$$\widetilde{W}_{\Delta,i} := \{w_i \in W_i : w_i(V) = 0\ \forall V \in \mathcal{V}_i^I\,,\ \overline{w_i}^E = 0\ \forall E \in \mathcal{E}_i^I\}\,.$$

Thus, $\widetilde{W}$ is the subspace of W of functions being continuous at the subdomain vertices and with continuous edge averages.

There are even more choices possible, see, e. g., [50, Algorithm D]. Note, that Algorithm A works only well in two dimensions, see also [30, 64] for some numerical experiments on FETI-DP. Its poor behavior in three dimensions relates to the fact that there is no more a discrete Poincaré-Friedrichs inequality (cf. Lemma 3.14) for functions vanishing at single vertices, at least not with a logarithmic factor.

We now formulate the BETI-DP algorithms. Depending on the choice of the space $\widetilde{W}_\Delta$, we define the Schur complement $\widetilde{S} : \widetilde{W}_\Delta \to (\widetilde{W}_\Delta)^*$ by

$$\widetilde{S} := S_\Delta - S_{\Delta\Pi}\, S_\Pi^{-1} S_{\Pi\Delta}\,,$$

where the block operators $S_\Delta : (\widetilde{W}_\Delta \to (\widetilde{W}_\Delta)^*$, $S_{\Pi\Delta} : \widehat{W}_\Pi \to (\widetilde{W}_\Delta)^*$, $S_{\Delta\Pi} : \widetilde{W}_\Delta \to (\widehat{W}_\Pi)^*$ and $S_\Pi : \widehat{W}_\Pi \to (\widehat{W}_\Pi)^*$ satisfy the relations

$$\begin{aligned} \langle S_\Delta\, v_\Delta, w_\Delta\rangle &= \langle S\, v_\Delta, w_\Delta\rangle && \forall v_\Delta, w_\Delta \in \widetilde{W}_\Delta\,,\\ \langle S_{\Delta\Pi}\, v_\Delta, w_\Pi\rangle = \langle S_{\Pi\Delta}\, w_\Pi, v_\Delta\rangle &= \langle S\, v_\Delta, w_\Pi\rangle && \forall v_\Delta \in \widetilde{W}_\Delta, \forall w_\Pi \in \widehat{W}_\Pi\,,\\ \langle S_\Pi\, v_\Pi, w_\Pi\rangle &= \langle S\, v_\Pi, w_\Pi\rangle && \forall v_\Pi, w_\Pi \in \widehat{W}_\Pi\,. \end{aligned}$$

Note, that S_Π is SPD and $\widetilde{S}$ fulfill the minimizing property

$$\langle \widetilde{S}\, w_\Delta, w_\Delta\rangle = \min_{w_\Pi \in \widehat{W}_\Pi} \langle S(w_\Delta + w_\Pi), w_\Delta + w_\Pi\rangle$$

Furthermore, we define the reduced right hand side

$$\widetilde{f} := f_\Delta - S_{\Delta\Pi}\, S_\Pi^{-1} f_\Pi \in (\widetilde{W}_\Delta)^*\,,$$

where f_Δ, f_Π are the projections of f onto the corresponding subspaces. The continuity constraints on the interface are incorporated using fully redundant Lagrange multipliers, but–in contrast to one-level methods–only for the degrees of freedom in $\widetilde{W}_\Delta$. E. g., for Algorithm A there are no Lagrange multipliers corresponding to subdomain vertices. We denote the space of these Lagrange multipliers by U_Δ and define the corresponding jump operator $B_\Delta : \widetilde{W}_\Delta \to U_\Delta^*$ analogously to Section 3. With these definitions, we arrive at the following minimization problem, which is equivalent to (4.1),

$$\min_{\substack{u_\Delta \in \widetilde{W}_\Delta \\ B_\Delta u_\Delta = 0}} \sum_{i \in \mathcal{I}} \left[\frac{1}{2} \langle \widetilde{S}\, u_\Delta, u_\Delta \rangle - \langle \widetilde{f}, u_\Delta \rangle \right]. \tag{4.2}$$

Suppose we have the solution u_Δ. Then the overall solution $u \in \widetilde{W}$ is given by $u = u_\Delta + u_\Pi$ with $u_\Pi = S_\Pi^{-1}(f_\Pi - S_{\Pi\Delta} u_\Delta)$. Above minimization problem is equivalent to the following saddle point problem: Find $(u_\Delta, \lambda) \in \widetilde{W}_\Delta \times U_\Delta$ such that

$$\begin{pmatrix} \widetilde{S} & B_\Delta^\top \\ B_\Delta & 0 \end{pmatrix} \begin{pmatrix} u_\Delta \\ \lambda \end{pmatrix} = \begin{pmatrix} \widetilde{f} \\ 0 \end{pmatrix},$$

where the Lagrange parameters λ are unique up to $\ker B_\Delta^\top$. As an important observation $\widetilde{S}$ is SPD on $\widetilde{W}_\Delta$, and so the inverse $\widetilde{S}^{-1}$ exists. We define the BETI-DP operator $F : U_\Delta \to U_\Delta^*$ and the dual right hand side $d \in U_\Delta^*$ by

$$F := B_\Delta \widetilde{S}^{-1} B_\Delta^\top \qquad \text{and} \qquad d := B_\Delta \widetilde{S}^{-1} \widetilde{f}.$$

Then above saddle point problem reduces to find $\lambda \in U_\Delta$ such that

$$F\,\lambda = d\,.$$

Similar to Section 3.1.2 we see that F is SPD (and invariant) on the factor space $V := (U_\Delta)_{/\ker B_\Delta^\top}$, and maps to $V' := U_\Delta^* \cap \operatorname{range} B_\Delta$. Thus λ can be obtained by a PCG iteration on V. By the same arguments as on page 32 we do not have to care about any contributions from $\ker B_\Delta^\top$ in the Lagrange multipliers during the PCG algorithm. As a main difference to the one-level method discussed in Section 3, no projection P appears, instead we need the coarse solve S_Π^{-1} in every PCG step.

Remark 4.1. Implementation details will be treated in a forthcoming paper. However, it is clear that for assembling of the coarse matrix, one has to solve a local problem for each primal unknown. This means in particular that in the case of uniform interior subdomain in two-dimensions, one has to solve $\mathcal{O}(H_0/H_E)$ local problems on the exterior domain, where the ratio H_0/H_E can grow arbitrarily large.

4.2 BETI-DP preconditioners and convergence analysis

Analogous to Section 3.2.3 we introduce a scaled jump operator $B_{D,\Delta} := \sum_{i \in \mathcal{I}} B_{D,\Delta}^{(i)} : (\widetilde{W}_\Delta)^* \to U_\Delta$ with $B_{D,\Delta}^{(i)} : (\widetilde{W}_{\Delta,i})^* \to U_\Delta$ defined by its adjoint

$$[(B_{D,\Delta}^{(i)})^\top \mu_{x,jk}](y) := \begin{cases} \delta_k^\dagger & \text{for } k < j = i,\ x = y, \\ -\delta_k^\dagger & \text{for } i = k < j,\ x = y, \\ 0 & \text{else,} \end{cases}$$

for the relevant $x, y \in \Gamma_{I,h}$, with the weighted counting functions $\delta_k^\dagger$ from Section 3.2.1. According to [87, 50] the preconditioner $M^{-1} : V' \to V$ is defined by

$$M^{-1} = B_{D,\Delta}\, S_\Delta\, B_{D,\Delta}^\top$$

We observe, that $S_\Delta = \operatorname{diag}(S_{\Delta,i})_{i\in\mathcal{I}}$ with $S_{\Delta,i} : \widetilde{W}_{\Delta,i} \to (\widetilde{W}_{\Delta,i})^*$. Consequently, the application of S_Δ is purely local and therefore the application of M^{-1} is easily parallelized.

In the sequel we elaborate a characterization of above preconditioner in terms of the projection

$$P_\Delta := B_{D,\Delta}^\top B_\Delta\,,$$

so that $B_\Delta^\top M^{-1} B_\Delta = P_\Delta^\top S_\Delta P_\Delta$, similar to Section 3.2.

Lemma 4.1. *Let the spaces $\widehat{W}_\Pi$ and $\widetilde{W}_\Delta$ be defined according to Algorithm A, B or C. Then, for all $w_\Delta \in \widetilde{W}_\Delta$,*

$$(P_\Delta w_\Delta)_i(x) = \begin{cases} \sum\limits_{j\in\mathcal{N}_x} \delta_j^\dagger(x)\left(w_{\Delta,i}(x) - w_{\Delta,j}(x)\right) & \text{for } x \in \Gamma_{i,h}\setminus\Gamma_D\,,\\ w_{\Delta,i}(x) = 0 & \text{for } x \in \Gamma_{i,h}\cap\Gamma_D\,. \end{cases}$$

In particular, for a face $F \in \mathcal{F}_i^N$, we have $(P_\Delta\, w_\Delta)|_F = 0$. Furthermore, $P_\Delta w_\Delta \in \widetilde{W}_\Delta$ and $B_\Delta P_\Delta = B_\Delta$.

Proof. The proof is analogous to the proof of Lemma 3.5, see also [50]. The only difference is that there are no Lagrange multipliers corresponding to subdomain vertices, however, the values of w_Δ vanish there anyway. □

We can now state a stability estimate similar to Lemma 3.12.

Lemma 4.2. *Let $\widetilde{W}_\Delta$ be defined according to Algorithm B or Algorithm C. Then, for all $w_\Delta \in \widetilde{W}_\Delta$, we have*

$$|P_\Delta w_\Delta|_{S_\Delta}^2 \le C\,(1+\log(H/h))^2\,|w_\Delta|_{\widetilde{S}}^2\,,$$

where $C > 0$ is independent of h_i, H_i, α_i and γ.

Proof. Essentially the following proof can be found in [50], only some of the arguments need to be adapted to the unbounded domain Ω_0. For the sake of completeness, we display the whole proof for the three-dimensional case.

For an arbitrary $w_\Delta \in \widetilde{W}_\Delta$ we determine the minimizing function $w_\Pi \in \widehat{W}_\Pi$ such that for $w = w_\Delta + w_\Pi \in \widetilde{W}$, we have $|w_\Delta|_{\widetilde{S}} = |w|_S$. According to [50], since w_Π is continuous across the subdomain interfaces and vanishes on Γ_D, we obtain by Lemma 4.1 that $P_\Delta w = P_\Delta w_\Delta$. Moreover, recall that $P_\Delta w \in \widetilde{W}_\Delta$, and that S_Δ is identical to S on the subspace $\widetilde{W}_\Delta$. With these considerations it suffices to show that

$$|P_\Delta w|_S^2 \le C\,(1+\log(H/h))^2\,|w|_S^2\,. \tag{4.3}$$

First, Lemma 3.16, part (i) yields

$$
\begin{aligned}
|(P_\Delta w)_i|^2_{S_{i,h}} &= \Big| \sum_{\substack{E\in\mathcal{E}_i^I\\ F\in\mathcal{F}_i^I}} \sum_{j\in\mathcal{N}_{E/F}} I^h\big(\delta_j^\dagger\,\theta_{E/F}(w_i - w_j)\big)\Big|^2_{S_{i,h}} \\
&\preceq \sum_{\substack{E\in\mathcal{E}_i^I\\ F\in\mathcal{F}_i^I}} \sum_{j\in\mathcal{N}_{E/F}} \underbrace{\alpha_i\,\big(\delta_j^\dagger\big)^2}_{\le \min(\alpha_i,\alpha_j)} \Big|I^h\big(\theta_{E/F}(w_i - w_j)\big)\Big|^2_{H^{1/2}_{00}(\omega^{(i)}_{E/F})}.
\end{aligned}
\tag{4.4}
$$

Note, that in (4.4) neither contributions on the Dirichlet boundary nor on subdomain vertices appear, because the function w vanishes there due to the definition of the space $\widetilde{W}$, for both Algorithms B and C.

Face contributions from a face $F = F_{ij}$ can be estimated in the following way,

$$
\begin{aligned}
&\min(\alpha_i,\,\alpha_j)\Big|I^h\big(\theta_F(w_i - w_j)\big)\Big|^2_{H^{1/2}_{00}(F)} \\
&= \min(\alpha_i,\,\alpha_j)\Big|I^h\big[\theta_F\big((w_i - \overline{w_i}^F) - (w_j - \overline{w_j}^F) + (\overline{w_i}^F - \overline{w_j}^F)\big)\big]\Big|^2_{H^{1/2}_{00}(F)} \\
&\preceq (1+\log(H/h))^2\,\min(\alpha_i,\,\alpha_j)\Big\{|w_i - w_j|^2_{H^{1/2}(F)} + \\
&\quad + \frac{1}{H_{ij}}\big\|(w_i - \overline{w_i}^F) - (w_j - \overline{w_j}^F)\big\|^2_{L_2(F)} + |\overline{w_i}^F - \overline{w_j}^F|^2\,\|\theta_F\|^2_{H^{1/2}_{00}(F)}\Big\}.
\end{aligned}
\tag{4.5}
$$

For Algorithm B, $\overline{w_i}^F = \overline{w_j}^F$ by the definition of the space $\widetilde{W}$, and we can estimate the last expression from above by

$$
C\,(1+\log(H/h))^2\left\{\alpha_i|w_i|^2_{H^{1/2}(F)} + \alpha_j|w_j|^2_{H^{1/2}(F)}\right\}.
$$

using the Poincaré inequality. For Algorithm C, Lemma 3.13, (iii) yields

$$
\|\theta_F\|^2_{H^{1/2}_{00}(F)} \preceq (1+\log(H_F/h_i))\,H_F\,. \tag{4.6}
$$

We choose an edge $E\in\mathcal{E}^I$ with $E\subset\partial F$ and obtain

$$
\begin{aligned}
|\overline{w_i}^F - \overline{w_j}^F|^2 &\preceq |\overline{w_i}^F - \overline{w_i}^E|^2 + |\overline{w_j}^F - \overline{w_j}^E|^2 \\
&= \Big|\overline{(w_i - \overline{w_i}^F)}^E\Big|^2 + \Big|\overline{(w_j - \overline{w_j}^F)}^E\Big|^2
\end{aligned}
$$

because $\overline{w_i}^E = \overline{w_j}^E$ due to the definition of $\widetilde{W}$. Using the definition of the edge average, the Cauchy-Schwarz inequality and Lemma 3.13, (vi), we obtain

$$
\begin{aligned}
\Big|\overline{(w_i - \overline{w_i}^F)}^E\Big|^2 &\preceq \frac{1}{H_E}\|w_i - \overline{w_i}^F\|^2_{L_2(E)} \\
&\preceq \frac{1}{H_E}(1+\log(H_F/h_i))\left\{|w_i|^2_{H^{1/2}(F)} + \frac{1}{H_F}\|w_i - \overline{w_i}^F\|^2_{L_2(F)}\right\}.
\end{aligned}
$$

Since $H_E \simeq H_F$, the factor $1/H_E$ cancels with H_F in (4.6), and using again the Poincaré inequality we are finally able to bound the face contribution (4.5) from above by

$$
C\,(1+\log(H/h))^2\left\{\alpha_i|w_i|^2_{H^{1/2}(F)} + \alpha_j|w_j|^2_{H^{1/2}(F)}\right\}.
$$

Edge contributions of (4.4) from an edge $E = E_{ij} \in \mathcal{E}^I$ can be estimated by

$$\begin{aligned}
\min(\alpha_i, \alpha_j)&\big\|I^h\big(\theta_E(w_i - w_j)\big)\big\|^2_{H^{1/2}_{00}(\omega_E^{(i)})} \\
&\preceq \min(\alpha_i, \alpha_j)\|w_i - w_j\|^2_{L_2(E)} \\
&= \min(\alpha_i, \alpha_j)\|(w_i - \overline{w_i}^E) - (w_j - \overline{w_j}^E)\|^2_{L_2(E)} \\
&= \min(\alpha_i, \alpha_j)\Big\|\Big[(w_i - \overline{w_i}^{F_i}) - \overline{(w_i - \overline{w_i}^{F_i})}^E\Big] + \Big[(w_j - \overline{w_j}^{F_j}) - \overline{(w_j - \overline{w_j}^{F_j})}^E\Big]\Big\|^2_{L_2(E)},
\end{aligned}$$

where the faces $F_i \in \mathcal{F}_i$ and $F_j \in \mathcal{F}_j$ are chosen such that $E \in \partial F_i \cap \partial F_j$. In above deduction, we have used Lemma 3.13, (vi) and the fact that for both Algorithms B and C, $\overline{w_i}^E = \overline{w_j}^E$. By the Cauchy-Schwarz inequality, we see that

$$\Big\|\overline{(w_i - \overline{w_i}^{F_i})}^E\Big\|^2_{L_2(E)} \preceq \|w_i - \overline{w_i}^{F_i}\|^2_{L_2(E)}\,.$$

Thus, we obtain

$$\begin{aligned}
\min(\alpha_i, \alpha_j)&\big\|I^h\big(\theta_E(w_i - w_j)\big)\big\|^2_{H^{1/2}_{00}(\omega_E^{(i)})} \\
&\preceq \alpha_i\|w_i - \overline{w_i}^{F_i}\|^2_{L_2(E)} + \alpha_j\|w_j - \overline{w_j}^{F_j}\|^2_{L_2(E)} \\
&\preceq (1 + \log(H_{ij}/h_i))\Big\{\alpha_i\Big[|w_i|^2_{H^{1/2}(F_i)} + \frac{1}{H_{F_i}}\|w_i - \overline{w_i}^{F_i}\|^2_{L_2(F_i)}\Big] + \\
&\qquad\qquad + \alpha_j\Big[|w_j|^2_{H^{1/2}(F_j)} + \frac{1}{H_{F_j}}\|w_j - \overline{w_j}^{F_j}\|^2_{L_2(F_j)}\Big]\Big\} \\
&\preceq (1 + \log(H/h))\Big\{\alpha_i|w_i|^2_{H^{1/2}(F_i)} + \alpha_j|w_j|^2_{H^{1/2}(F_j)}\Big\},
\end{aligned}$$

Eventually, by a finite summation argument, we obtain

$$|P_\Delta w|^2_S \preceq (1 + \log(H/h))^2 \sum_{i\in\mathcal{I}}\sum_{F\in\mathcal{F}_i} \alpha_i|w_i|^2_{H^{1/2}(F)}\,.$$

For $i \neq 0$, we can use Lemma 3.16, part (iii) and bound $\sum_{F\in\mathcal{F}_i}\alpha_i|w_i|^2_{H^{1/2}(F)}$ from above by $C\,|w_i|^2_{S_{i,h}}$. For $i = 0$, we denote the discrete harmonic extension of w_0 from Γ_0 to Ω_0^c by $\mathcal{H}^{\text{int}}\,w_0$. Obviously, for $F = F_{0,j}$, $|w_0|^2_{H^{1/2}(F)} \preceq |\mathcal{H}^{\text{int}}\,w_0|^2_{H^1(\Omega_j)}$. Thus, with a finite summation argument, we have

$$\sum_{F\in\mathcal{F}_i}\alpha_0|w_0|^2_{H^{1/2}(F)} \preceq \alpha_0|\mathcal{H}^{\text{int}}\,w_0|^2_{H^1(\Omega_0^c)} = \alpha_0\,\langle S^{\text{int}}_{0,\text{FEM}}w_0,\, w_0\rangle \preceq |w_0|^2_{S_{0,h}}\,,$$

where the last step follows from Lemma 2.3. Finally, we arrive at the desired estimate (4.3), which concludes the proof of Lemma 4.2. □

Theorem 4.1. *Let the spaces $\widehat{W}_\Pi$ and $\widetilde{W}_\Delta$ be defined according to Algorithm B or C. Then the BETI-DP preconditioner fulfills the following condition number estimate*

$$\kappa(M^{-1}F) \leq C\,(1 + \log(H/h))^2\,,$$

to be understood in the factor space modulo $\ker B_\Delta^\top$*. The constant* C *is independent from* H_i*,* h_i*,* α_i *and* γ*, and we point out that*

$$H/h = \max_{i\in\mathcal{I}} \max_{F\in\mathcal{F}_i^I} H_F/h_i\,.$$

Proof. For completeness, we display the proof which can be found in [50]. The only essential point is the estimate stated in Lemma 4.2. Since S_Δ is SPD on $\widetilde{W}_\Delta$, we can conclude that M^{-1} is SPD on V'. Thus its inverse $M : V \to V'$ exists. We show

$$\langle M\,\lambda,\,\lambda\rangle \le \langle F\,\lambda,\,\lambda\rangle \le C\,(1+\log(H/h))^2\,\langle M\,\lambda,\,\lambda\rangle \qquad \forall\lambda\in V\,. \tag{4.7}$$

Lower bound. Similar to the formula in the one-level method (cf. Lemma 3.8) the following identity holds (see [50]),

$$\langle F\,\lambda,\,\lambda\rangle = \sup_{v_\Delta\in\widetilde{W}_\Delta} \frac{\langle\lambda,\,B_\Delta\,v_\Delta\rangle^2}{|v_\Delta|^2_{\widetilde{S}}}\,.$$

For an arbitrary $\mu\in V'$ there exists a $w_\Delta\in\widetilde{W}_\Delta$ with $\mu = B_\Delta\,w_\Delta$. Since $B_\Delta P_\Delta = B_\Delta$, $\mathrm{range}\,(P_\Delta)\subset\widetilde{W}_\Delta$ (cf. Lemma 4.1) and $|u_\Delta|_{\widetilde{S}} \le |u_\Delta|_{S_\Delta}$ for all $u_\Delta\in\widetilde{W}_\Delta$, we can conclude that

$$\langle F\,\lambda,\,\lambda\rangle \ge \frac{\langle\lambda,\,B_\Delta P_\Delta\,w_\Delta\rangle^2}{|P_\Delta w_\Delta|^2_{\widetilde{S}}} \ge \frac{\langle\lambda,\,B_\Delta\,w_\Delta\rangle^2}{|P_\Delta w_\Delta|^2_{S_\Delta}} = \frac{\langle\lambda,\,\mu\rangle^2}{\langle M^{-1}\mu,\,\mu\rangle}$$

For the choice $\mu = M\lambda$ we arrive at the lower bound in (4.7).

Upper bound. Using Lemma 4.1 we have that for all $\lambda\in V$,

$$\begin{aligned}
\langle F\,\lambda,\,\lambda\rangle &= \sup_{v_\Delta\in\widetilde{W}_\Delta} \frac{\langle\lambda,\,B_\Delta\,v_\Delta\rangle^2}{|v_\Delta|^2_{\widetilde{S}}} \\
&\preceq (1+\log(H/h))^2 \sup_{v_\Delta\in\widetilde{W}_\Delta} \frac{\langle\lambda,\,B_\Delta\,v_\Delta\rangle^2}{|P_\Delta v_\Delta|^2_{\widetilde{S}}} \\
&\preceq (1+\log(H/h))^2 \sup_{v_\Delta\in\widetilde{W}_\Delta} \frac{\langle\lambda,\,B_\Delta\,v_\Delta\rangle^2}{\langle M^{-1}B_\Delta\,w_\Delta,\,B_\Delta\,w_\Delta\rangle} \\
&\preceq (1+\log(H/h))^2 \sup_{\mu\in V'} \frac{\langle\lambda,\,\mu\rangle^2}{\langle M^{-1}\mu,\,\mu\rangle}\,.
\end{aligned}$$

By Lemma A.2 we arrive at the upper bound in (4.7). □

Remark 4.2.

1. Algorithm A gives the same condition number estimate in two dimensions, also in presence of an unbounded domain. The proof of such an estimate requires no new ideas and is therefore skipped.

2. Algorithm D in [50, 87] can be proved for the case of unbounded domains with the same ideas as stated therein.

3. We see that in contrast to the one-level methods, the analysis for dual-primal methods with unbounded domains is much simpler, because the L_2-terms on the faces of the exterior domain Ω_0 can be eliminated at once. Moreover, we do not need any restrictions on the coefficients or on the boundary conditions. One could say that the coarse

spaces W_Π of the dual-primal methods are more powerful than the coarse spaces Z of the one-level methods.

4. It was shown in [11] that for bounded domains, the estimate in Theorem 4.1 is sharp in two dimensions.

5 Conclusion

In this work we have given a detailed analysis of one-level BETI methods and BETI-DP methods for Poisson problems in two- and three-dimensional unbounded domains. In particular, we have discussed several Lagrange multiplier formulation for one-level BETI methods, such as non-redundant vs. redundant, and standard vs. all-floating formulation.

Our analysis of one-level methods for unbounded domains is essentially based on a relatively tricky way of using the coarse space. With no assumptions on the coefficients the sub-optimal bound

$$C \max_{i\neq 0} \frac{H_0}{H_i}\Big(1+\log\Big(\frac{H_i}{h_i}\Big)\Big)^2 . \tag{5.1}$$

for the condition number of the preconditioned BETI-system can be shown. Here, H_0 is the diameter of the complement of the unbounded subdomain Ω_0. This bound can be improved, depending on the geometry of the problem, in particular on the location of interior Dirichlet boundaries. However, for our proving technique, we need the additional assumption that the coefficient α_0 on Ω_0, which is the unbounded exterior domain, is the largest one, i. e., all interior coefficients α_i must be smaller or equal to α_0. This is exactly the case in magnetostatic problems, where α_i is the magnetic reluctivity which attains its largest value in air or vacuum. The main tool in this case is an estimate which bounds the energy norm of a restricted harmonic extension against a non-restricted one. Therefore, we have introduced the *extension indicator*, $\gamma_h(\{\Omega_i\}, \Gamma_0, \Gamma_D)$, which is defined the maximal ratio of energy norms of these extensions. For several specific geometrical configurations we can derive bounds for the indicator $\gamma_h(\{\Omega_i, \Gamma_0, \Gamma_D)$. In the the best case, where interior Dirichlet boundaries are either not present or well-separated from Γ_0, the quasi-optimal estimate of the condition number of the preconditioned BETI-system holds, i. e.,

$$C \max_{i\neq 0} \Big(1+\log\Big(\frac{H_i}{h_i}\Big)\Big)^2 . \tag{5.2}$$

Depending on the location of interior Dirichlet boundaries, bounds in between of (5.1) and (5.2) can explicitly be derived.

Our numerical examples show that one-level BETI methods behave even better than one would expect due to the theoretical bounds.

From the parallel scalability point of view, however, the application of the operators $S_{0,h}$ and $S_{0,h}^{-1}$ is the bottleneck of the computation, since it usually disturbs the load balancing in a parallel scheme. The algorithm can be accelerated by

- introducing a sub-group or processors that all together are responsible for the application of $S_{0,h}$ and $S_{0,h}^{-1}$,

- using an inexact BETI scheme (cf. [54, 45, 51]) with similar sub-parallel preconditioners.

For parallel computations with data-sparse BEM matrices see, e. g., [15]. Furthermore, several Schwarz-type preconditioners for boundary integral operators (which are parallelizable) can, e. g., be found in [2, 41, 88].

For the BETI-DP method, we have proved the quasi-optimal bound

$$C \max_{i \neq 0} \left(1 + \log\left(\frac{H_i}{h_i}\right)\right)^2 \tag{5.3}$$

without any assumptions on the coefficients and just standard assumptions on the domain decomposition. This is possible due to the larger coarse space of BETI-DP methods, which is fairly different to the one of one-level methods. However, in BETI-DP methods the assembly of the coarse grid matrix is more costly, since there might be a large number of primal unknonws on the boundary Γ_0, and for each of these unknowns one has to solve a local problem on Ω_0 to get the corresponding coarse matrix entry; see also Remark 4.1. For the rest of the algorithm, one can apply the same strategy as above to speed up the local operations on Ω_0.

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A Auxiliary results

The following lemma is a well-known energy result on Galerkin projections

Lemma A.1. *Let $V_h \subset V$ be Hilbert spaces and let $a : V \times V \to \mathbb{R}$ be a bounded, symmetric positive definite bilinear form. For some $f \in V^*$ we define $u \in V$ and $u_h \in V_h$ by*

$$\begin{aligned} a(u, v) &= \langle f, v\rangle \qquad \forall v \in V\,, \\ a(u_h, v_h) &= \langle f, v_h\rangle \qquad \forall v_h \in V_h\,. \end{aligned}$$

Then

$$a(u_h, u_h) \leq a(u, u)\,.$$

Proof. We define the energy functional

$$J(v) := \tfrac{1}{2} a(v, v) - \langle f, v\rangle\,.$$

One can easily show that

$$u = \operatorname*{argmin}_{v\in V} J(v)\,, \qquad\qquad J(u) = -\tfrac{1}{2}a(u,\,u)$$
$$u_h = \operatorname*{argmin}_{v_h\in V_h} J(v_h)\,, \qquad\qquad J(u_h) = -\tfrac{1}{2}a(u_h,\,u_h)\,.$$

Since $V_h \subset V$, we have $J(u) \le J(u_h)$ which implies that $a(u_h,\,u_h) \le a(u,\,u)$. □

The next lemma is a well-known result on self-adjoint operators

Lemma A.2. *Let V be a separable Hilbert space and $T : V \to V^*$ be self-adjoint and positive definite, in particular $T^{-1} : V^* \to V$ exists. Then*

$$\langle w,\, T^{-1}\, w\rangle = \sup_{v\in V\setminus\{0\}} \frac{\langle w,\, v\rangle}{\langle T\, v,\, v\rangle} \qquad \forall w \in V^*\,.$$

Proof. For simplicity, we give the proof for the finite dimensional case. In the other case, we need to use the spectral theorem and work with an orthonormal basis. We identify V and V^* with the space $\mathbb{R}^{n\times n}$ and T with a matrix A, where we use the Euklidean scalar product $(\cdot,\,\cdot)$ and the induced norm $\|\cdot\|$. First, as a well known fact, there exists an SPD matrix $A^{1/2}$ with its inverse $A^{-1/2}$ such that $A = A^{1/2}A^{1/2}$ and $A^{-1} = A^{-1/2}A^{-1/2}$. Secondly, by the Cauchy-Schwarz inequality one easily sees that

$$\|x\| = \sup_{y\in\mathbb{R}^n\setminus\{0\}} \frac{(x,\,y)}{\|y\|} \qquad \forall x \in \mathbb{R}^n\,.$$

Thus,

$$\begin{aligned}(w,\, A^{-1}\, w) = (A^{-1/2}\, w,\, A^{-1/2}\, w) &= \sup_{u\in\mathbb{R}^n\setminus\{0\}} \frac{(A^{-1/2}\, w,\, u)^2}{\|u\|^2}\\ &= \sup_{u\in\mathbb{R}^n\setminus\{0\}} \frac{(w,\, A^{-1/2}\, u)^2}{(A\, A^{-1/2}\, u,\, A^{-1/2}\, u)} = \sup_{v\in\mathbb{R}^n\setminus\{0\}} \frac{(w,\, v)^2}{(A\, v,\, v)}\,.\end{aligned}$$

In the last step we have used that $A^{-1/2}$ is bijective and substituted v for $A^{-1/2}u$. □

B List of notations

Vectors

$\lvert x\rvert$	absolute value of $x \in \mathbb{R}$
$\lvert v\rvert$	Euclidean norm of the vector v in $\mathbb{R}^d$
$v \cdot w$	Euclidean inner product in $\mathbb{R}^d$

Spaces

V^*	dual space of a Banach space V
$\langle\cdot,\,\cdot\rangle$	duality product between a dual V^* and a Banach space V
$L_2(\Omega)$	Hilbert space of Lebesgue-measurable functions v on the domain or mannifold Ω where $\int_\Omega \lvert v\rvert^2 dx$ is bounded
$H^1(\Omega)$	Sobolev space of functions in $L_2(\Omega)$ whose weak derivative is in $L_2(\Omega)$
$H_0^1(\Omega)$	Sobolev space of functions in $H^1(\Omega)$ with vanishing trace on $\partial\Omega$
$H_{\mathrm{loc}}^1(\Omega^{\mathrm{c}})$	Sobolev space of distributions $u \in \mathcal{D}'(\Omega^{\mathrm{c}})$ such that $u \in H^1(B_R \cap \Omega^{\mathrm{c}})$ for all open balls $B_R \supset \Omega$
$H^{1/2}(\Gamma)$	trace space of $H^1(\Omega)$ functions where $\Gamma = \partial\Omega$
$H_D^{1/2}(\Gamma)$	subspace of $H^{1/2}(\Gamma)$ with homogeneous Dirichlet conditions on $\Gamma_D \subset \Gamma$
$H^{-1/2}(\Gamma)$	dual of $H^{1/2}(\Gamma)$
$H_{00}^{1/2}(\widetilde{\Gamma})$	space of functions on $\widetilde{\Gamma} \subset \Gamma)$ whose extension by zero is in $H^{1/2}(\Gamma)$
$H_*^{1/2}(\Gamma)$	$= \{v \in H^{1/2}(\Gamma) : \langle V^{-1}\,v,\,\mathbf{1}_\Gamma\rangle = 0\}$, where V is the (elliptic) single layer potential
$H_*^{-1/2}(\Gamma)$	$= \{w \in H^{-1/2}(\Gamma) : \langle w,\,\mathbf{1}_\Gamma\rangle = 0\}$
U	space of Lagrange multipliers, see pg. 22
$V,\,V'$	spaces of admissible Lagrange increments, see pg. 27 and pg. 66
$\widetilde{V}'$	$= V' \cap \operatorname{range} B$
$V_1^h(\Omega)$	discrete space of piecewise linear functions on a triangulation $\mathcal{T}_h$ of Ω
$V_{1,\,D}^h(\Omega)$	subspace of $V_1^h(\Omega)$ with homogeneous Dirichlet boundary conditions on $\Gamma_D \subset \partial\Omega$
$V_1^h(\Gamma)$	discrete space of piecewise linear functions on a triangulation $\mathcal{T}_h$ of the mannifold Γ
$V_{1,\,D}^h(\Gamma)$	subspace of $V_1^h(\Gamma)$ with homogeneous Dirichlet boundary conditions on $\Gamma_D \subset \Gamma$
$V_0^h(\Gamma)$	discrete space of piecewise constant functions on a triangulation $\mathcal{T}_h$ of the mannifold Γ
W	product space $\prod\limits_{i\in\mathcal{I}} W_i$, see pg. 21
W_i	discrete space of piecewise linear functions on Γ_i with or without homogeneous Dirichlet boundary condition, see pages 21 and 24

$\widetilde{W}$	subspace of W of partially continuous functions; $\widetilde{W} = \widehat{W}_\Pi \oplus \widetilde{W}_\Delta$, see Section 4.1
$\widehat{W}_\Pi$	primal space in dual-primal formulation, see Section 4.1
$\widehat{W}_\Delta$	dual space in dual-primal formulation, see Section 4.1

Operators

B	FETI/BETI jump operator, see pg. 22
B_{D^r}	scaled FETI/BETI jump operator for redundant Lagrange multipliers, see pg. 30
$B_{D,\Delta}$	scaled FETI/BETI-DP jump operator for redundant Lagrange multipliers, see pg. 66
B_i	FETI/BETI jump operator on subdomain Ω_i, see pg. 22
B_Δ	FETI/BETI-DP jump operator, see pg. 66
D_i	hypersingular integral operator on Γ_i, see pg. 9; or: FETI/BETI scaling operator, see pg. 29
D	block-diagonal FETI/BETI scaling operator, see pg. 29
E_D	FETI/BETI averaging operator, $E_D = I - P_D$, see pg. 30
F	FETI/BETI operator, $F = B^\top S^\dagger B$, see pg. 26
G	FETI/BETI: $G = B\,R$, see pg. 26
$\mathcal{H}^{\text{int}}$	minimal discrete harmonic extension of a piecewise linear function w from the boundary Γ_0 to the domain Ω_{int}, see Definition 3.1, pg. 36; or to the domain Ω_0^{c}, see pg. 69
$\mathcal{H}^{\text{int}}_{0,D}$	minimal discrete harmonic extension of a piecewise linear function w from the boundary Γ_0 to the domain Ω_{int}, which meets the homogeneous Dirichlet conditions on Γ_D, see Definition 3.1, pg. 36
K, K_i	double layer potential operator on domain Ω or subdomain Ω_i
M^{-1}	FETI/BETI preconditioner, see pages 29 and 30
P	FETI/BETI projection operator, see pg. 27 $P = I - Q\,G\,(G^\top Q\,G)^{-1} G^\top$
P_D	FETI/BETI projection operator, see pg. 30 for the non-redundant and pg. 31 for the redundant case
Q	FETI/BETI scaling operator, see pages 26 and 35
R	FETI/BETI operator generating the kernel $kerS$, see pg. 25

S	$= \operatorname{diag}(S_{i,h})_{i\in\mathcal{I}}$, block Steklov-Poincaré operator, see pg. 23
S^{int}, S^{ext}	interior and exterior Steklov-Poincaré operator, see pg. 11
S_i	Steklov-Poincaré operator on subdomain Ω_i, see pg. 20
$S_{i,h}$	approximated coefficient-weighted Steklov-Poincaré operator on subdomain Ω_i, see pg. 20, $S_{i,h} := \alpha_i S_{i,\mathrm{FEM/BEM}}^{\mathrm{int/ext}}$
$\widetilde{S}$	Schur complement in dual primal methods, see pg. 65
S_Δ, S_Π, $S_{\Pi\Delta}$	dual-primal related restrictions of S, see pg. 65
V, V_i	single layer potential operator on domain Ω or on subdomain Ω_i

Constants and scalars

c_0	$= \inf\limits_{v\in H_*^{1/2}(\Gamma)} \frac{\langle D\,v, v\rangle}{\langle V^{-1}\,v, v\rangle}$, see pg. 10
c_K, $c_K^{(i)}$	contraction constant of a domain Ω or subdomain Ω_i, see pg. 10
$\gamma_h(\{\Omega_i\}, \Gamma_0, \Gamma_D)$	the extension indicator, see Definition 3.1, pg. 36
H_0	diameter of the (bounded) complement Ω_0^c of the unbounded subdomain Ω_0
H_i	diameter of the subdomain Ω_i for $i \neq 0$
h_i	minimal mesh size of the triangulation $\mathcal{T}_h$ of Γ_i or Ω_i
H/h	short hand for the maximum of the local ratio of the largest face diameter and minimal mesh size on one subdomain, see pg. 19

Domains and mannifolds

Γ	generically a boundary or a mannifold
Γ_D	Dirichlet boundary
Γ_i	boundary of the subdomain Ω_i
Γ_I	interface of the domain decomposition, see pg. 18
Γ_I	skeleton of the domain decomposition, see pg. 18
Γ_N	Neumann boundary
E	subdomain edge of the domain decomposition
E_{ij}	subdomain edge between domains Ω_i and Ω_j

F	subdomain face of the domain decomposition
F_{ij}	subdomain face between the domains Ω_i and Ω_j
Ω	generically a domain
Ω_0	0-th subdomain, which is an unbounded domain
Ω_i	i-th subdomain
V	subdomain vertex of the domain decomposition
V_{ij}	subdomain vertex (crosspoint) between the subdomains Ω_i and Ω_j

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