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Difference Equations in Massive Higher Order Calculations

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The calculation of massive 2–loop operator matrix elements, required for the higher order Wilson coefficients for heavy flavor production in deeply inelastic scattering, leads to new types of multiple infinite sums over harmonic sums and related functions, which depend on the Mellin parameter N. We report on the solution of these sums through higher order difference equations using the summation package `Sigma`.

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Contents

1. Introduction

Single scale quantities in renormalizable quantum field theories, such as anomalous dimensions and massless Wilson coefficients, are most simply represented in terms of (finite) harmonic sums. This representation holds at least up to 3–loop order for massless Yang–Mills theories [1]. The corresponding Feynman-parameter integrals are such that nested harmonic sums [2, 3] appear in a natural way, working in Mellin space. They are written either as S– or as Z–sums,

$$\begin{aligned} S_{m_1,\ldots,m_k}(N) &= \sum_{N\geq i_1\geq i_2\ldots\geq i_k>0} \frac{\prod_{l=1}^{k}[\mathrm{sign}(m_l)]^{i_l}}{i_l^{|m_l|}} \\ Z_{m_1,\ldots,m_k}(N) &= \sum_{N\geq i_1> i_2\ldots> i_k>0} \frac{\prod_{l=1}^{k}[\mathrm{sign}(m_l)]^{i_l}}{i_l^{|m_l|}} \ . \end{aligned} \tag{1.1}$$

Indeed a wide class of single scale quantities, including the anomalous dimensions and massless Wilson coefficients for unpolarized and polarized space- and time-like processes to 3–loop order, the Wilson coefficients for the Drell-Yan process and pseudoscalar and scalar Higgs boson production in hadron scattering in the heavy quark mass limit, as well as the soft- and virtual corrections to Bhabha scattering in the on-mass-shell scheme to 2–loop order [4] can be represented in this way. Single scale massive quantities at 2 loops, like the unpolarized and polarized heavy-flavor Wilson coefficients in the region $Q^2 \gg m^2$ – although the respective Feynman parameter-integrals exhibit a more involved structure – belong to this class too [5–8]. Finite harmonic sums obey algebraic, cf. [9], and structural relations [10]. The compact representations being obtained in Mellin space have to be mapped to momentum-fraction space to use the respective quantities in experimental analyzes. The Mellin inversion requires the analytic continuation of the harmonic sums [11] w.r.t. the Mellin index $N \in \mathbf{C}$.

Calculating massive operator matrix elements in Mellin space, which contribute to the heavy-flavor Wilson coefficients in deeply inelastic scattering, [6–8, 12], new types of infinite sums occur if compared to massless calculations. In the latter case, summation algorithms as `summer` [3], `nestedsums` [13] and `Xsummer` [14] may be used to calculate the respective sums. `summer` and `Xsummer` are based on `FORM` [15], while `nestedsums` is based on `GiNaC`. The new sums which emerged in [6–8, 12] can be calculated in different ways. In Ref. [6], we choose analytic methods together with summation. Another way consists in applying general summation algorithms in the field of computer algebra. The beginning was made by Gosper's telescoping algorithm [16] for hypergeometric terms[1]. Concerning practical applications, Zeilberger's extension of Gosper's algorithm to creative telescoping [17, 18] can be considered as the breakthrough in symbolic summation. The recent summation package `Sigma` [19] that can be executed in the computer algebra system `Mathematica` is going to open up completely new possibilities in symbolic summation: Based on Karr's $\Pi\Sigma$-difference fields [20] and further refinements [21], the package contains summation algorithms [22] that allow to attack not only hypergeometric sums, like Gosper's and Zeilberger's algorithms, but sums involving also indefinite nested sums, like, e.g., S–sums or Z–sums (1.1).

In this note we report on using this package for the summation of a series of new sums and comment on different techniques, which may be used as well.

2. The underlying summation principles

In this section, we discuss methods of the summation package `Sigma` that are relevant to discover and prove the identities given below. Similar to hypergeometric summation [18], `Sigma` relies on the following summation techniques.

Indefinite summation by telescoping: *Given* an indefinite sum $S(a) = \sum_{k=0}^{a} f(k)$, *find* $g(j)$ such that

$$f(j) = g(j+1) - g(j) \tag{2.1}$$

holds within the summation range $0 \leq j \leq a$. Then by telescoping, we get

$$S(a) = g(a+1) - g(0).$$

Example. For the summand

$$f(j) = \frac{(2j+k+N+2)j!k!(j+k+N)!}{(j+k+1)(j+N+1)(j+k+1)!(j+N+1)!(k+N+1)!} + \frac{j!k!(j+k+N)!\left(-S_1(j)+S_1(j+k)+S_1(j+N)-S_1(j+k+N)\right)}{(j+k+1)!(j+N+1)!(k+N+1)!} \tag{2.2}$$

`Sigma` finds

$$g(j) = \frac{(j+k+1)(j+N+1)j!k!(j+k+N)!\Big(S_1(j)-S_1(j+k)-S_1(j+N)+S_1(j+k+N)\Big)}{kN(j+k+1)!(j+N+1)!(k+N+1)!}.$$

[1] $f(k)$ is hypergeometric in k iff $f(k+1)/f(k) = g(k)$ for some fixed rational function $g(k)$.

Therefore summing (2.1) over j produces

$$\begin{aligned}\sum_{j=0}^{a} f(j) = & \frac{(a+1)!(k-1)!(a+k+N+1)!\,(S_1(a)-S_1(a+k)-S_1(a+N)+S_1(a+k+N))}{N(a+k+1)!(a+N+1)!(k+N+1)!} \\ & +\frac{S_1(k)+S_1(N)-S_1(k+N)}{kN(k+N+1)N!} \\ & +\frac{(2a+k+N+2)a!k!(a+k+N)!}{(a+k+1)(a+N+1)(a+k+1)!(a+N+1)!(k+N+1)!}, \end{aligned} \tag{2.3}$$

for $a \in \mathbf{N}$.

More generally, given a finite set of indefinite nested sums and products (in particular, hypergeometric terms and Z–sums/S–sums) and given $f(j)$ as a rational expression in terms of those sums and products, `Sigma` decides algorithmically, whether there exists a rational expression $g(j)$ in terms of those sums and products such that (2.1) holds; for more details see [19].

Usually, one fails to find a solution $g(j)$ for (2.1). If $f(j)$, depends on an extra discrete parameter, say N, Zeilberger's *creative telescoping paradigm* can be applied.

Deriving recurrences by creative telescoping: *Given* an integer $d > 0$ and given a sum

$$S(a,N) := \sum_{j=0}^{a} f(N,j) \tag{2.4}$$

with an extra parameter N, *find* constants $c_0(N),\ldots,c_d(N)$, free of j, and $g(N,j)$ such that the following summand recurrence

$$c_0(N)f(N,j)+\ldots,c_d(N)f(N+d,j) = g(N,j+1)-g(N,j) \tag{2.5}$$

holds for the summation range $0 \le j \le a$. If one succeeds in this task, one gets by telescoping the recurrence relation

$$c_0(N)S(a,N)+\cdots+c_d(N)S(a,N+d) = g(N,a+1)-g(N,0).$$

Summarizing, one arrives at a recurrence relation of the form

$$c_0(N)S(a,N)+\ldots c_d(N)S(a,N+d) = q(a,N) \tag{2.6}$$

for the sum $S(a,N)$ of order d. Note that a can be specialized, e.g., to N, i.e., one obtains a recurrence for the definite sum $\sum_{k=0}^{N} f(N,k)$. In other instances, see below, one can perform the limit $a \to \infty$ which yields a recurrences for $\sum_{k=0}^{\infty} f(N,k)$.

Creative telescoping has been introduced for hypergeometric terms [17]. More generally, in `Sigma` the summand $f(N,j)$ may consist of indefinite nested sums and products, in particular hypergeometric terms and S–sums/Z–sums.

Example. For $d = 1$ and the summand

$$f(N,j) = \frac{S_1(j)+S_1(N)-S_1(j+N)}{jN(j+N+1)N!} \tag{2.7}$$

`Sigma` computes the solution

$$
\begin{aligned}
c_0(N) &= -N(N+1)^2, \quad c_1(N)(N+1)^3(N+2),\\
g(N,j) &= \frac{jS(1,j)+(-N-1)S(1,N)-jS(1,j+N)-2}{(j+N+1)N!}.
\end{aligned}
\tag{2.8}
$$

Hence, summing (2.5) over $1 \le j \le b$ gives

$$
\begin{aligned}
&-NS(N,b)+(1+N)(2+N)S(N+1,b)\\
&\qquad = \frac{(b+1)\left(S_1(b)+S_1(N)-S_1(b+N)\right)}{(N+1)^2(b+N+2)N!}+\frac{b(b+1)}{(N+1)^3(b+N+1)(b+N+2)N!}
\end{aligned}
\tag{2.9}
$$

for the sum $S(N,b)$.

Summarizing, if we succeed in finding a recurrence[2] of type

$$
c_0(N)S(N)+\dots c_d(N)S(N+d)=q(N), \tag{2.10}
$$

then together with the initial values $S(i)$, $0 \le i < d$, we obtain an alternative representation of the corresponding input sum, $\sum_{j=0}^{a} f(N,j)$ (resp. $\sum_{j=0}^{N} f(N,j)$ or $\sum_{j=0}^{\infty} f(N,j)$).

For many applications, such a result is completely satisfactory. However, if one hunts for a closed form, one can continue as follows.

Recurrence solving: *Given* a recurrence (2.10), *find* linearly independent solutions of the homogeneous version of the recurrence, say $h_1(N),\dots,h_r(N)$, and a particular solution of (2.6), say $p(N)$.

Namely, if we manage to compute sufficiently many solutions, we can find constants $k_1,\dots,k_r$ such that

$$
S(i)=p(i)+k_1\,h_1(i)+\dots+k_r\,h_r(i)\ ,
$$

for all $0 \le i < d$. As a consequence,

$$
S(N)=p(N)+k_1\,h_1(N)+\dots+k_r\,h_r(N),
$$

for all $N \ge 0$.

With `Sigma` we can handle the following situation: Given (2.6) where the $c_0(N),\dots,c_d(N)$ and $q(N)$ are given by indefinite nested sums and products, `Sigma` can compute all solutions $S(N)$ in terms of indefinite nested sums and products (the so-called d'Alembertian solution; see [23] and [19, Section 7.2.3]). We emphasize that `Sigma` finds, as a special case, all solutions in terms of hypergeometric expressions and S–sums/Z–sums.

3. Summation through difference equations: single sums

As outlined in Refs. [6, 7], in course of the calculation of massive 2–loop integrals containing as single (general) scale the Mellin parameter N, different types of sums occur. These summands are

[2] For simplicity, we suppress extra parameters, like e.g., a. Moreover, we assume that $c_d(N) \neq 0$ for all $N \in \mathbf{N}$.

typically products of harmonic sums with different arguments, weighted by summation parameters and contain hypergeometric terms, like binomial factors or Beta-function factors

$$B(N,i) := \frac{\Gamma(N)\Gamma(i)}{\Gamma(N+i)}. \tag{3.1}$$

Here i is the summation–index, which runs from one to infinity. In general, these sums can be expressed in terms of nested harmonic sums, [2,3] and ζ–values. Note that sums containing Beta–functions with different arguments, e.g. $B(i,i),\ B(N+i,i)$, usually do not lead to harmonic sums in the final result. Further quite often binomial sums of the type

$$\sum_{i=0}^{N} \binom{N}{i} F(i)\ , \tag{3.2}$$

with $F(i)$ being a broken–rational function of i times a product of harmonic sums, emerge. This type of sums can as well be treated with the `Sigma`–package but are not considered here.

Some of these sums can be performed by the existing packages [3, 13, 14]. However, there exists so far no automatic computer program to calculate sums which contain Beta–function factors of the type (3.1) and single harmonic sums in the summand.

3.1 The Sigma-approach

As a first example we consider the sum

$$T_1(N) := \sum_{i=1}^{\infty} \frac{B(N,i)}{i+N+2} S_1(i) S_1(N+i)\ . \tag{3.3}$$

We treat the upper bound of the sum as a finite integer, i.e., we consider the truncated version

$$T_1(a,N) := \sum_{i=1}^{a} \frac{B(N,i)}{i+N+2} S_1(i) S_1(N+i),$$

for $a \in \mathbf{N}$. Given this sum as input, we execute `Sigma`'s creative telescoping algorithm and find a recurrence for $T_1(a,N)(=S(a,N))$ of the form (2.6) with order $d=4$. Finally, we perform the limit $a \to \infty$ and we end up at the recurrence

$$\begin{aligned}
&-N(N+1)(N+2)^2\left(4N^5+68N^4+455N^3+1494N^2+2402N+1510\right)T_1(N)\\
&-(N+1)(N+2)(N+3)\left(16N^5+260N^4+1660N^3+5188N^2+7912N+4699\right)T_1(N+1)\\
&+(N+2)(N+4)(2N+5)\left(4N^6+74N^5+542N^4+1978N^3+3680N^2+3103N+767\right)T_1(N+2)\\
&+(N+4)(N+5)\left(16N^6+276N^5+1928N^4+6968N^3+13716N^2+13929N+5707\right)T_1(N+3)\\
&-(N+4)(N+5)^2(N+6)\left(4N^5+48N^4+223N^3+497N^2+527N+211\right)T_1(N+4)\\
&\qquad = P_1(N)+P_2(N)S_1(N)
\end{aligned}$$

where

$$\begin{aligned}
P_1(N) = \Big(&32N^{18}+1232N^{17}+21512N^{16}+223472N^{15}+1514464N^{14}+6806114N^{13}\\
&+18666770N^{12}+15297623N^{11}-116877645N^{10}-641458913N^9-1826931522N^8\\
&-3507205291N^7-4825457477N^6-4839106893N^5-3535231014N^4-1860247616N^3\\
&-684064448N^2-160164480N-17395200\Big)\Big/\left(N^3(N+1)^3(N+2)^3(N+3)^2(N+4)(N+5)\right)
\end{aligned}$$

and

$$P_2(N) = -4\Big((4N^{14}+150N^{13}+2610N^{12}+27717N^{11}+199197N^{10}+1017704N^9+3786588N^8 +10355813N^7+20779613N^6+30225025N^5+31132328N^4+21872237N^3+9912442N^2 +2672360N+362400\Big)/\big(N^2(N+1)^2(N+2)^2(N+3)(N+4)(N+5)\big).$$

In the next step, we apply `Sigma`'s recurrence solver to the computed recurrence and find the four linearly independent solutions

$$h_1(N) = \frac{1}{N+2}, \qquad h_2(N) = \frac{(-1)^N}{N(N+1)(N+2)},$$
$$h_3(N) = \frac{S_1(N)}{N+2}, \qquad h_4(N) = \frac{\big(1+(N+1)S_1(N)\big)(-1)^N}{N(N+1)^2(N+2)},$$

of the homogeneous version of the recurrence plus the particular solution

$$p(N) = \frac{2(-1)^N}{N(N+1)(N+2)}\Bigg[2S_{-2,1}(N)-3S_{-3}(N)-2S_{-2}(N)S_1(N)-\zeta_2 S_1(N) -\zeta_3-\frac{2S_{-2}(N)+\zeta_2}{N+1}\Bigg]-2\frac{S_3(N)-\zeta_3}{N+2}-\frac{S_2(N)-\zeta_2}{N+2}S_1(N) +\frac{2+7N+7N^2+5N^3+N^4}{N^3(N+1)^3(N+2)}S_1(N)+2\frac{2+7N+9N^2+4N^3+N^4}{N^4(N+1)^3(N+2)}$$

of the recurrence itself. Finally, we look for constants $c_1,\dots,c_4$ such that

$$T_1(N) = c_1 h_1(N)+c_2 h_2(N)+c_3 h_3(N)++c_4 h_4(N)+p(N).$$

The calculation of the necessary initial values for $N=0,1,2,3$ does not pose a problem for `Sigma` and we conclude that $c_1=c_2=c_3=c_4=0$. Hence the final result then reads

$$T_1(N) = \frac{2(-1)^N}{N(N+1)(N+2)}\Bigg[2S_{-2,1}(N)-3S_{-3}(N)-2S_{-2}(N)S_1(N)-\zeta_2 S_1(N) -\zeta_3-\frac{2S_{-2}(N)+\zeta_2}{N+1}\Bigg]-2\frac{S_3(N)-\zeta_3}{N+2}-\frac{S_2(N)-\zeta_2}{N+2}S_1(N) +\frac{2+7N+7N^2+5N^3+N^4}{N^3(N+1)^3(N+2)}S_1(N)+2\frac{2+7N+9N^2+4N^3+N^4}{N^4(N+1)^3(N+2)}\ . \tag{3.4}$$

Using more refined algorithms of `Sigma`, see e.g., [24], even a first order difference equation can be obtained

$$(N+2)T_1(N)-(N+3)T_1(N+1)$$
$$= 2\frac{(-1)^N}{N(N+2)}\Bigg(-\frac{3N+4}{(N+1)(N+2)}\Big(\zeta_2+2S_{-2}(N)\Big)-2\zeta_3-2S_{-3}(N)-2\zeta_2 S_1(N)-4S_{1,-2}(N)\Bigg) +\frac{S_2(N)-\zeta_2}{N+1}+\frac{N^6+8N^5+31N^4+66N^3+88N^2+64N+16}{N^3(N+1)^2(N+2)^3}S_1(N) +2\frac{N^5+5N^4+21N^3+38N^2+28N+8}{N^4(N+1)^2(N+2)^2}\ . \tag{3.5}$$

However, in setting up Eq. (3.5), use had to be made of further sums of less complexity, which had to be calculated separately. As above, we can easily solve the recurrence and obtain again the result (3.4).

Here and in the following we applied various algebraic relations between harmonic sums to obtain a simplification of our results, cf. [9].

3.2 Alternative approaches

As a second example we consider the sum

$$T_2(N) := \sum_{i=1}^{\infty} \frac{S_1^2(i+N)}{i^2} , \tag{3.6}$$

which does not contain a Beta–function. In a first attempt, we proceed as with the first example $T_1(N)$. Namely, the naive application of `Sigma` yields a fifth order difference equation, which is clearly too complex for this sum. However, similar to the situation $T_1(N)$, `Sigma` can reduce it to a third order relation which reads

$$\begin{aligned}
&T_2(N)(N+1)^2 - T_2(N+1)(3N^2+10N+9)\\
&+T_2(N+2)(3N^2+14N+17) - T_2(N+3)(N+3)^2\\
&= \frac{6N^5+48N^4+143N^3+186N^2+81N-12}{(N+1)^2(N+2)^3(N+3)^2} - 2\frac{2N^2+7N+7}{(N+1)(N+2)^2(N+3)}S_1(N)\\
&+\frac{-2N^6-24N^5-116N^4-288N^3-386N^2-264N-72}{(N+1)^2(N+2)^3(N+3)^2}\zeta_2 .
\end{aligned} \tag{3.7}$$

Solving this recurrence relation in terms of harmonic sums gives a closed form; see (3.14) below.

Still (3.7) represents a rather involved way to solve the problem. A better way consists in first mapping the numerator $S_1^2(i+N)$ into a linear representation, which can be achieved using Euler's relation

$$S_a^2(N) = 2S_{a,a}(N) - S_{2a}(N), \quad a > 0 . \tag{3.8}$$

This is realized in `summer` by the basis–command for general–type harmonic sums,

$$T_2(N) = \sum_{i=1}^{\infty} \frac{2S_{1,1}(i+N) - S_2(i+N)}{i^2} . \tag{3.9}$$

As outlined in Ref. [3], sums of this type can be evaluated by considering the difference

$$D_2(j) \;=\; T_2(j) - T_2(j-1) = 2\sum_{i=1}^{\infty} \frac{S_1(j+i)}{i} - \sum_{i=1}^{\infty} \frac{1}{i^2(j+i)^2} . \tag{3.10}$$

The solution is then obtained by summing the difference (3.10) to

$$T_2(N) = \sum_{j=1}^{N} D_2(j) + T_2(0) . \tag{3.11}$$

The sums in Eq. (3.10) are now calculable trivially or are of less complexity than the original sum. In the case considered here, only the first sum on the left hand side is not trivial. However, after partial fractioning, one can repeat the same procedure, resulting into another difference equation, which is now easily calculable. Thus using this technique, the solution of Eq. (3.6) can be obtained by summing two first order difference equations or solving a second order one. The above procedure is well known and some of the summation–algorithms of `summer`, [3], are based on it. As a consequence, infinite sums with an arbitrary number of harmonic sums with the same argument can be performed using this package. Note that sums containing harmonic sums with different arguments, see e.g Eq. (3.6, A.2), can in principle be summed automatically using the same approach. However, this feature is not yet built into `summer`.

A third way to obtain the sum (3.6) consists of using integral representations of harmonic sums, [2]. One may represent the sum in terms of the following integrals

$$\begin{aligned} T_2(N) &= 2\sum_{i=1}^{\infty}\int_0^1 dx \frac{x^{i+N}}{i^2}\left(\frac{\ln(1-x)}{1-x}\right)_+ - \sum_{i=1}^{\infty}\left(\int_0^1 dx\frac{x^{i+N}}{i^2}\frac{\ln(x)}{1-x}+\frac{\zeta_2}{i^2}\right) \\ &= 2\mathbf{M}\left[\left(\frac{\ln(1-x)}{1-x}\right)_+\mathrm{Li}_2(x)\right](N+1) - \left(\mathbf{M}\left[\frac{\ln(x)}{1-x}\mathrm{Li}_2(x)\right](N+1)+\zeta_2^2\right) . \end{aligned} \tag{3.12}$$

Here the Mellin–transform is defined as

$$\mathbf{M}[f(x)](N) := \int_0^1 dx\, x^{N-1}\ f(x) . \tag{3.13}$$

Eq. (3.12) can then be easily calculated since the corresponding Mellin–transforms are well–known, cf. [2]. Either of these three methods above leads to

$$T_2(N) = \frac{17}{10}\zeta_2^2 + 4S_1(N)\zeta_3 + S_1^2(N)\zeta_2 - S_2(N)\zeta_2 - 2S_1(N)S_{2,1}(N) - S_{2,2}(N) . \tag{3.14}$$

As a third example we would like to evaluate the sum

$$T_3(N) = \sum_{i=1}^{\infty}\frac{S_1^2(i+N)S_1(i)}{i} . \tag{3.15}$$

Note that Eq. (3.15) is divergent. In order to treat the divergent pieces, we introduce the symbol σ_1

$$\sigma_1 := \lim_{a\to\infty}\sum_{i=1}^{a}\frac{1}{i} . \tag{3.16}$$

The application of `Sigma` to the sum (3.15) yields a fourth order difference equation

$$\begin{aligned} &(N+1)^2(N+2)T_2(N) - (N+2)\left(4N^2+15N+15\right)T_2(N+1) \\ &+(2N+5)\left(3N^2+15N+20\right)T_2(N+2) - (N+3)\left(4N^2+25N+40\right)T_2(N+3) \\ &+(N+3)(N+4)^2T_2(N+4) \\ = &\ \frac{6N^5+73N^4+329N^3+684N^2+645N+215}{(N+1)^2(N+2)^2(N+3)^2} + \frac{6N^2+19N+9}{(N+1)(N+2)(N+3)}S_1(N) , \end{aligned} \tag{3.17}$$

which can be solved. As in the foregoing example the better way to calculate the sum is to first change $S_1^2(i+N)$ into a linear basis representation, cf. [3],

$$T_3(N) = \sum_{i=1}^{\infty} \frac{2S_{1,1}(i+N) - S_2(i+N)}{i} S_1(i) \ . \tag{3.18}$$

One may now calculate $T_3(N)$ using telescoping for the difference

$$D_3(j) \ = \ T_3(j) - T_3(j-1) = 2\sum_{i=1}^{\infty} \frac{S_1(i+j)S_1(i)}{i(i+j)} - \sum_{i=1}^{\infty} \frac{S_1(i)}{i(i+j)^2} \ , \tag{3.19}$$

with

$$T_3(N) = \sum_{j=1}^{N} D_2(j) + T_3(0) \ . \tag{3.20}$$

One finally obtains

$$\begin{aligned} T_3(N) \ = \ & \frac{\sigma_1^4}{4} + \frac{43}{20}\zeta_2^2 + 5S_1(N)\zeta_3 + \frac{3S_1^2(N) - S_2(N)}{2}\zeta_2 - 2S_1(N)S_{2,1}(N) \\ & + S_1^2(N)S_2(N) + S_1(N)S_3(N) - \frac{S_2^2(N)}{4} + \frac{S_1^4(N)}{4} \ . \end{aligned} \tag{3.21}$$

4. A double sum example

Using the Mellin-Barnes integral representation [25], massive two–loop integrals which occur in the calculation of polarized and unpolarized massive operator matrix elements [6–8] result into double infinite series of the kind

$$S(N) = \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} f(N,k,j) \ . \tag{4.1}$$

Here $N \in \mathbf{N}$ denotes the Mellin variable. In the following, we consider an example dealt with in [6,8] where

$$\begin{aligned} f(N,k,j) = & \frac{\Gamma(k+1)}{\Gamma(k+2+N)}\Gamma(\varepsilon)\Gamma(1-\varepsilon)\frac{\Gamma(j+1-2\varepsilon)\Gamma(j+1+\varepsilon)(\Gamma(k+j+1+N)}{\Gamma(j+1-\varepsilon)\Gamma(j+2+N)\Gamma(k+j+2)} \\ & + \frac{\Gamma(k+1)}{\Gamma(k+2+N)}\Gamma(-\varepsilon)\Gamma(1+\varepsilon)\frac{\Gamma(j+1+2\varepsilon)\Gamma(j+1-\varepsilon)\Gamma(k+j+1+\varepsilon+N)}{\Gamma(j+1)\Gamma(j+2+\varepsilon+N)\Gamma(k+j+2+\varepsilon)}, \end{aligned}$$

ε denoting the dimensional regularization parameter of the momentum integrals performed in $D \ = \ 4 \ + \ \varepsilon$ space-time dimensions. $f(N,k,j)$ obeys the expansion

$$f(N,k,j) = f_0(N,k,j) + \varepsilon f_1(N,k,j) + \varepsilon^2 f_1(N,k,j) + \dots \ , \tag{4.2}$$

which is derived using

$$\frac{\Gamma(n+\varepsilon)}{\Gamma(1+\varepsilon)\Gamma(n)} = 1 + \sum_{k=1}^{n-1} \varepsilon^k Z_{\vec{1}_k}(n-1) \ , \tag{4.3}$$

with $\vec{1}_k = \underbrace{\{1,...,1\}}_{k}$, cf. e.g. [2, 26]. Since the summation could not be accomplished with the available packages `summer` [3], `nestedsums` [13], or `Xsummer` [14], this double sum was solved in [6, 8] using suitable integral representations, which allowed the summation. Here special partial differential operators had to be used to map the series expansion efficiently. This method clearly was specialized to the respective cases to be dealt with.

Let us consider the summation of the leading term $f_0(N,k,j)$. The summation for the higher terms in ε can be performed in an analogous way.

We start with the summand $f_0(N,k,j)$ which is given in (2.2), and find the closed form given in (2.3).

Then the limit

$$\sum_{j=0}^{\infty} f_0(N,k,j) = \lim_{a\to\infty}\sum_{j=0}^{a} f_0(N,k,j) = \frac{S_1(k)+S_1(N)-S_1(k+N)}{kN(k+N+1)N!} \tag{4.4}$$

is performed leaving us with an expression in finite harmonic sums which depend on the second summation parameter k. We repeat the foregoing procedure and form a finite sum

$$S(N,b) := \sum_{k=1}^{b} \frac{S_1(k)+S_1(N)-S_1(k+N)}{kN(k+N+1)N!} \Big(= \sum_{k=1}^{b}\sum_{j=0}^{\infty} f_0(N,k,j)\Big), \tag{4.5}$$

$b \in \mathbf{N}$.

First, we execute `Sigma`'s creative telescoping algorithm and compute the recurrence relation (2.9). Again, by limit considerations, it follows that the sum

$$S'(N) := \lim_{b\to\infty} S(N,b) = \sum_{k=1}^{\infty}\sum_{j=0}^{\infty} f_0(N,k,j) \tag{4.6}$$

satisfies the recurrence relation

$$-NS'(N)+(1+N)(2+N)S'(N+1) = \frac{(N+1)S_1(N)+1}{(N+1)^3 N!}.$$

Next, we use `Sigma`'s recurrence solver and compute the general solution

$$\frac{1}{N(N+1)N!}c + \frac{S_1(N)^2+S_2(N)}{2N(N+1)N!}$$

for a constant c. Checking the initial value $S'(1)$ shows that

$$S'(N) = \frac{S_1(N)^2+S_2(N)}{2N(N+1)!}.$$

Hence,

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f_0(N,k,j) = \frac{S_1(N)^2+S_2(N)}{2N(N+1)!} + \sum_{j=0}^{\infty} f_0(N,0,j).$$

Using again `Sigma` we find

$$\begin{aligned}\sum_{j=0}^{a} f_0(N,0,j) &= \sum_{j=0}^{a} \frac{2j+N+2}{(j+1)^2(j+N+1)^2(N+1)!}\\ &= \frac{2a+N+2}{(a+1)^2(a+N+1)^2(N+1)!}\\ &\quad+\frac{S_2(a)}{N(N+1)!}+\frac{S_2(N)}{N(N+1)!}-\frac{S_2(a+N)}{N(N+1)!}. \end{aligned} \tag{4.7}$$

In the limit $a \to \infty$ this later expression simplifies to

$$\frac{S_2(N)}{N(N+1)!}\ .$$

Finally, we find

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f_0(N,k,j) = \frac{S_1(N)^2+3S_2(N)}{2N(N+1)!}.$$

Completely analogously, we compute, for instance, the linear term in the series expansion (4.2):

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty} f_1(N,k,j) = \frac{-S_1(N)^3-3S_2(N)S_1(N)-8S_3(N)}{6N(N+1)!}.$$

A. Appendix

Further examples of sums calculable by `Sigma` and the order of their respective recurrence relation are given below. Note that the inhomogeneous part of the recurrence relations are given in terms of single harmonic sums up to depth 2, ζ–values and polynomials in N only. Below we list a series of examples for infinite sums, which emerged calculating the linear terms in ε for the massive operator matrix elements [12]. The sums were calculated solving higher order difference equations with `Sigma`.

A.1 Sums resulting in fourth order difference equations

$$\begin{aligned}\sum_{i=1}^{\infty}\frac{B(N,i)}{i}S_1(i+N)^2 &= \frac{17}{10}\zeta_2^2+4S_1(N)\zeta_3+2S_{-2}(N)\zeta_2+S_1^2(N)\zeta_2-3S_4(N)\\ &\quad+2S_{-2}^2(N)-4S_1(N)S_3(N)-S_1^2(N)S_2(N)+\frac{S_1^2(N)}{N^2}+2\frac{S_1(N)}{N^3}\\ &\quad+\frac{2}{N^4}-(-1)^N\frac{2S_{-2}(N)+\zeta_2}{N^2}\ , \end{aligned}\tag{A.1}$$

$$\begin{aligned}\sum_{i=1}^{\infty}\frac{S_1(i)S_2(i+N)}{i} &= \frac{\sigma_1^2}{2}\zeta_2-\frac{1}{5}\zeta_2^2-S_1(N)\zeta_3-\frac{S_1^2(N)-3S_2(N)}{2}\zeta_2-2S_{3,1}(N)\\ &\quad+\frac{1}{2}S_4(N)+\frac{S_1^2(N)S_2(N)}{2}+S_1(N)S_3(N)\ , \end{aligned}\tag{A.2}$$

$$\begin{aligned}\sum_{i=1}^{\infty}\frac{S_1(i+N)S_2(i+N)}{i} &= \frac{\sigma_1^2}{2}\zeta_2-\frac{1}{5}\zeta_2^2-S_1(N)\zeta_3+\frac{1}{2}[S_2(N)-S_1^2(N)]\zeta_2+S_{2,2}(N)\\ &\quad+S_1^2(N)S_2(N)+S_1(N)S_3(N)\ . \end{aligned}\tag{A.3}$$

A.2 Sums resulting in third order difference equations

$$\sum_{i=1}^{\infty}\frac{B(N,i)}{(i+N+1)^2}S_1(i) = (-1)^N\frac{2S_{1,-2}(N)+S_{-3}(N)+\zeta_2S_1(N)+\zeta_3}{N(N+1)}+\frac{\zeta_2-S_2(N)}{(N+1)^2}\ , \tag{A.4}$$

$$\begin{aligned}\sum_{i=1}^{\infty}\frac{B(N,i)}{(N+i+2)^2}S_1(i) &= \frac{(-1)^N}{N(N+1)(N+2)}\Big(2\zeta_3+4S_{1,-2}(N)+2S_{-3}(N)+2S_1(N)\zeta_2\\ &\quad+\frac{2\zeta_2}{N+1}+\frac{4S_{-2}(N)}{N+1}\Big)+\frac{\zeta_2-S_2(N)}{(N+2)^2}+\frac{3N^2+N-8}{N(N+1)^4(N+2)^2}\ , \end{aligned}\tag{A.5}$$

$$\begin{aligned}\sum_{i=1}^{\infty}\frac{B(N,i)}{i}S_2(N+i) &= \frac{7}{10}\zeta_2^2-2S_{-2}(N)\zeta_2+S_2(N)\zeta_2-3S_4(N)-2S_{-2}(N)^2-S_2(N)^2\\ &\quad+(-1)^N\frac{\zeta(2)+2S_{-2}(N)}{N^2}+\frac{S_2(N)}{N^2}\ . \end{aligned}\tag{A.6}$$

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