POSITIVITY OF CERTAIN SUMS OVER JACOBI KERNEL POLYNOMIALS

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ABSTRACT. We present a computer-assisted proof of positivity of sums over kernel polynomials for ultraspherical Jacobi polynomials.

1. INTRODUCTION

In this paper we show positivity of sums over Jacobi kernel polynomials $k_j^\alpha(x,0)$ on the interval $[-1,1]$ where we consider ultraspherical Jacobi polynomials $P_n^{(\alpha,\alpha)}(x)$ with $\alpha \in [-\frac{1}{2},\frac{1}{2}]$. This problem originated in a new convergence proof for a certain finite element scheme in the course of which Schöberl [8] was led to conjecture that the inequality

$$\sum_{j=0}^{n}(4j+1)(2n-2j+1)P_{2j}(0)P_{2j}(x) \geq 0 \tag{1}$$

holds for $-1 \leq x \leq 1$ and $n \geq 0$, where $P_n(x)$ denotes the nth Legendre polynomial. Relation (1) corresponds to setting $\alpha = 0$ in the inequality of Theorem 1 that will be proven below. Asymptotics seem to be difficult even for this special case [4]. In this paper we describe an approach that makes heavy use of computer algebra algorithms. Based on treating the special cases $\alpha = \pm\frac{1}{2}$ we will determine a decomposition of the given sum into expressions that can be estimated from below. For this proof we will use the Mathematica packages `SumCracker` [6] and `GeneratingFunctions` [7]. Both implementations, as well as a variety of other algorithms for symbolic summation are available at

`http://www.risc.uni-linz.ac.at/research/combinat/software/`

In the following section we introduce kernel polynomials and formulate the conjectured inequality. We also briefly outline the background from which the original problem (1) emerged. In section 3 we will show positivity for the special cases $\alpha = \pm\frac{1}{2}$ of $P_n^{(\alpha,\alpha)}(x)$ being Chebyshev polynomials. This proof motivates a decomposition of the given sum in the remaining case $-\frac{1}{2} < \alpha < \frac{1}{2}$, Lemma 5 in section 4, which allows finding a lower bound in closed form whose positivity can be verified using `SumCracker`'s `ProveInequality` command.

2. MOTIVATION

When constructing a smoothing operator for a high order finite element scheme, Schöberl [8] considered an integral operator that serves as point evaluation when applied to polynomials up to a given degree n. More precisely, he wanted to find a family of polynomials $\{\phi_n\}$ such

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that

$$\int_{-1}^{1} \phi_n(x)v(x)\,dx = v(0), \tag{2}$$

for all polynomials v with $\deg v \leq n$. Moreover, he wanted $\{\phi_n\}$ to satisfy the following norm estimate

$$\|\phi_n\|_{L^1} = \int_{-1}^{1} |\phi_n(x)|\,dx \leq C,$$

where the constant C is independent of n. Property (2) led to considering so-called kernel polynomials.

Let $\{p_j(x)\}$ be a given sequence of polynomials defined on a real interval $[a,b]$ and being orthogonal with respect to some weight function $w(x)$, which is a nondecreasing function with an infinite number of points of increase in the interval $[a,b]$. Then the kernel polynomial sequence is defined as

$$k_n(x,y) = \sum_{j=0}^{n} \frac{1}{h_j} p_j(x)p_j(y),$$

where $h_n = \int_a^b p_n(x)^2\, w(x)\,dx$. Kernel polynomials have the reproducing property

$$\int_a^b k_n(x,y)q(x)\,w(x)\,dx = q(y)$$

for all polynomials $q(x)$ with degree less or equal to n. From the three term recurrence relation for the $p_n(x)$ one easily obtains a compact expression for these kernel polynomials, namely

$$k_n(x,y) = c(n)\,\frac{p_{n+1}(x)p_n(y) - p_{n+1}(y)p_n(x)}{x-y},$$

where $c(n)$ depends on the leading coefficients of $p_{n+1}(x), p_n(x)$ and h_n, for more details see e.g. [1, 9].

In the following we consider only kernel polynomials for Jacobi polynomials of the form $P_n^{(\alpha,\alpha)}(x)$ which we will denote by $k_n^\alpha(x,y)$. They are orthogonal with respect to the weight function $w(x) = (1-x^2)^\alpha$ and can be expressed as [1]

$$k_n^\alpha(x,y) = \frac{c_n^\alpha}{x-y}[P_{n+1}^{(\alpha,\alpha)}(x)P_n^{(\alpha,\alpha)}(y) - P_n^{(\alpha,\alpha)}(x)P_{n+1}^{(\alpha,\alpha)}(y)], \tag{3}$$

where

$$c_n^\alpha = 2^{-2\alpha-1}\frac{\Gamma(n+2)\Gamma(n+2\alpha+2)}{\Gamma(n+\alpha+1)\Gamma(n+\alpha+2)}.$$

If we choose ϕ_n to be the Legendre kernel polynomials $k_n^0(x,0)$ then condition (2) is satisfied because of the reproducing property with respect to the L^2-inner product $\int_{-1}^1 f(x)g(x)\,dx$ corresponding to the constant weight function $w(x) = (1-x)^0 \equiv 1$. But for this candidate we do not know uniform bounds in the L^1-norm. In addition numerical computations indicate that the $k_n^0(x,0)$ are not uniformly bounded in the L^1-norm at all. So Schöberl was led to consider a modified ansatz using so-called gliding averages [3],

$$\phi_n(x) = \frac{1}{n+1}\sum_{j=n}^{2n} k_j^0(x,0). \tag{4}$$

Here ϕ_n is a polynomial of degree $2n$ satisfying (2). Defining the sum

$$S(n,x) = \frac{1}{n+1}\sum_{j=0}^{n} k_j^0(x,0), \tag{5}$$

we can write ϕ_n in the form

$$\phi_n(x) = \frac{2n+1}{n+1}S(2n,x) - \frac{n}{n+1}S(n-1,x).$$

Schöberl conjectured that (5) is positive for even indices, i.e. $S(2n,x) \geq 0$. Once this result is established one can bound the L^1-norm of ϕ_n for odd n immediately via

$$\|\phi_n\|_{L^1} \leq \frac{2n+1}{n+1}\int_{-1}^{1} S(2n,x)\,dx + \frac{n}{n+1}\int_{-1}^{1} S(n-1,x)\,dx = \frac{3n+1}{n+1} \leq 3, \quad n \text{ odd}.$$

Here we only needed to invoke the positivity of $S(2k,x)$ and its constant preserving property. After applying the triangle inequality we can omit the absolute values and evaluate each of the integrals over $S(2n,x)$ and $S(n-1,x)$ to 1. Having only an estimate for ϕ_{2n+1} at hand clearly is no obstruction to the application we have in mind since one can always raise the degree of the smoothing operator by one, if needed.

Trying to prove that $S(2n,x) \geq 0,\ x \in [-1,1]$, we observed that this inequality seems to remain valid if we define $S(n,x)$ more generally as a sum over Jacobi kernel polynomials k_n^α with $\alpha \in [-\frac{1}{2}, \frac{1}{2}]$. Consequently we define

$$S_n^\alpha(x,y) := \sum_{j=0}^{n} k_j^\alpha(x,y).$$

In this notation we have $S(n,x) = (n+1)\,S_n^0(x,0)$. In the remainder of this paper we will prove the extended conjecture formulated in the following theorem.

Theorem 1. *Let $k_n^\alpha(x,y)$ be the nth kernel polynomial for Jacobi polynomials $P_j^{(\alpha,\alpha)}(x)$. Then*

$$S_{2n}^\alpha(x,0) \geq 0 \qquad \text{for} \qquad -\frac{1}{2} \leq \alpha \leq \frac{1}{2}, \quad -1 \leq x \leq 1, \quad n \geq 0.$$

Note that for odd degrees, i.e. $S_{2n+1}^\alpha(x,0)$, the sums are not positive. The proof of Theorem 1 will be split into two parts. In section 3 we will consider the cases $\alpha = \pm\frac{1}{2}$, corresponding to the Chebyshev polynomials of the first and second kind, respectively. The proof of these special cases motivates a decomposition of the sum $S_{2n}^\alpha(x,0)$ which is the key to proving Theorem 1 for the remaining part where $-\frac{1}{2} < \alpha < \frac{1}{2}$.

3. Chebyshev polynomials of first and second kind ($\alpha = \pm\frac{1}{2}$)

Jacobi polynomials $P_n^{(-1/2,-1/2)}(x)$ can be identified with Chebyshev polynomials of the first kind $T_n(x)$. The sum $S_n^{-1/2}(x,y)$ is called Fejér kernel and positivity is well known for all $n \geq 0$ and for all x, y in the unit square $[-1,1]^2$, for a short proof see e.g. [10]. Hence we only have to consider the case $\alpha = \frac{1}{2}$.

For $\alpha = \frac{1}{2}$ Jacobi polynomials $P_n^{(\alpha,\alpha)}(x)$ are called Chebyshev polynomials of the second kind and commonly denoted by $U_n(x)$. Their kernel polynomials are

$$k_n^{1/2}(x,y) = \frac{1}{\pi(x-y)}[U_{n+1}(x)U_n(y) - U_n(x)U_{n+1}(y)].$$

SumCracker yields a closed form for $S_{2n}^{1/2}(x,y)$, namely,

$$S_n^{1/2}(x,y) = \frac{1}{\pi(x-y)^2}[U_{n+1}(x)(xU_n(y) - U_{n+1}(y)) + U_n(x)(yU_{n+1}(y) - U_n(y)) + 1]. \tag{6}$$

Remark 2. *Here we used the* `Crack` *command which takes an expression and returns a reformulation in "smaller" terms. A "human" proof of this identity which only uses the Chebyshev three term recurrence will be given later in this section.*

To prove that $S_{2n}^{1/2}(x,0) \geq 0$ we proceed as follows. Since $U_{2n+1}(0) = 0$ and $U_{2n}(0) = (-1)^n$ we have that

$$S_{2n}^{1/2}(x,0) = \frac{1}{\pi x^2}[1 + (-1)^n x\, U_{2n+1}(x) - (-1)^n U_{2n}(x)].$$

Inspection of the first few polynomials $S_{2n}^{1/2}(x,0)$ suggests that

$$S_{4m}^{1/2}(x,0) = p_{2m}(x)^2 \qquad \text{and} \qquad S_{4m+2}^{1/2}(x,0) = (1-x^2)q_{2m}(x)^2,$$

where $p_{2m}(x), q_{2m}(x)$ are polynomials of degree $2m$ satisfying the relation $q_n(x)S_1^{1/2}(x,0) = (p_{n+1}(x) - p_n(x))^2$. To verify this claim we use the `GuessRE` command of Mallinger's `GeneratingFunctions` package that tries to guess a holonomic recurrence equation given the first few terms of a sequence. Applying this function to $p_m(x)$ yields the following rewriting of $S_{2n}^{1/2}(x,0)$.

Lemma 3. *For $m \geq 0$ and $-1 \leq x \leq 1$ we have*

$$S_{4m}^{1/2}(x,0) = \frac{2}{\pi x^2}T_{2m+1}(x)^2,$$

and

$$S_{4m+2}^{1/2}(x,0) = \frac{1}{2\pi x^2(1-x^2)}(T_{2m+3}(x) - T_{2m+1}(x))^2,$$

where $T_m(x)$ are the Chebyshev polynomials of the first kind.

Proof. The closed forms for $S_{4m}^{1/2}(x,0)$ and $S_{4m+2}^{1/2}(x,0)$ can be verified immediately with Kauers' `SumCracker` package. For this purpose we use an algorithm that decides zero equivalences of a given admissible sequence, for details see [6],

In[1]:= $\mathbf{ZeroSequenceQ[}\boldsymbol{x}\mathbf{ChebyshevU[4}\boldsymbol{m}\mathbf{+1,}\boldsymbol{x}\mathbf{] - ChebyshevU[4}\boldsymbol{m}\mathbf{,}\boldsymbol{x}\mathbf{] + 1}$
$\quad\quad \mathbf{- 2ChebyshevT[2}\boldsymbol{m}\mathbf{+1,}\boldsymbol{x}\mathbf{]^2]}$

Out[1]= True

In[2]:= $\mathbf{ZeroSequenceQ[-}\boldsymbol{x}\mathbf{ChebyshevU[4}\boldsymbol{m}\mathbf{+3,}\boldsymbol{x}\mathbf{] + ChebyshevU[4}\boldsymbol{m}\mathbf{+2,}\boldsymbol{x}\mathbf{] + 1}$
$\quad\quad \mathbf{- (ChebyshevT[2}\boldsymbol{m}\mathbf{+3,}\boldsymbol{x}\mathbf{] - ChebyshevT[2}\boldsymbol{m}\mathbf{+1,}\boldsymbol{x}\mathbf{])^2/(2(1-}\boldsymbol{x}\mathbf{^2))]}$

Out[2]= True

□

From these representations it is obvious that the sums $S_{2n}^{1/2}(x,0)$ are non-negative. For Chebyshev polynomials of the second kind the closed form representation (6) for the sum $S_n^{1/2}(x,y)$ exists, yet for arbitrary $\alpha > -1$ this is not the case. Still, examining a derivation of (6) using only the three term recurrence satisfied by $U_n(x)$ indicates how to continue dealing with general Jacobi polynomials $P_n^{(\alpha,\alpha)}(x)$, $-\frac{1}{2} < \alpha < \frac{1}{2}$.

So, let again $\alpha = \frac{1}{2}$. In order to derive (6), we show that $S_n^{1/2}(x, y)$ rewritten according to (3) as the sum

$$S_n^{1/2}(x, y) = \frac{1}{\pi(x-y)} \sum_{j=0}^{n} [U_{j+1}(x)U_j(y) - U_j(x)U_{j+1}(y)],$$

is a sum representation which telescopes to the right hand side of (6). Because of symmetry it suffices to consider only one part of the sum. For the first part, `SumCracker` yields

$$(x-y)\sum_{j=0}^{n} U_{j+1}(x)U_j(y) = \frac{1}{2}\left(2xU_{n+1}(x)U_n(y) - U_n(x)U_n(y) - U_{n+1}(x)U_{n+1}(y) + 1\right),$$

which suggests that

$$(x-y)U_{j+1}(x)U_j(y) = \frac{1}{2}\Delta_j(2xU_j(x)U_{j-1}(y) - U_{j-1}(x)U_{j-1}(y) - U_j(x)U_j(y)) =: \frac{1}{2}\Delta_j G_j(x, y),$$

where Δ_j denotes the difference operator $\Delta_j[\psi(j)] = \psi(j+1) - \psi(j)$. The correctness of this identity can be verified by straight-forward calculation using the three term recurrence for Chebyshev polynomials,

$$U_n(x) - 2xU_{n+1}(x) + U_{n+2}(x) = 0, \qquad U_0(x) = 1,\ U_1(x) = 2x. \tag{7}$$

Namely, first we use (7) to rewrite $2x\,U_j(x)$ and then, to involve y, we use the same recurrence relation to replace $U_{j-1}(y) + U_{j+1}(y)$. This way we obtain,

$$\begin{aligned} G_{j+1}(x, y) - G_j(x, y) &= 2xU_j(y)U_{j+1}(x) - U_{j+1}(x)U_{j+1}(y) \\ &\quad - 2xU_{j-1}(y)U_j(x) + U_{j-1}(x)U_{j-1}(y) \\ &= 2xU_j(y)U_{j+1}(x) - U_{j+1}(x)U_{j+1}(y) - U_{j-1}(y)U_{j+1}(x) \\ &= 2(x-y)U_{j+1}(x)U_j(y). \end{aligned} \tag{8}$$

We note that this telescoping property is due to the fact that Chebyshev polynomials satisfy a three term recurrence with constant coefficients. Consequently this procedure cannot be performed the same way for Jacobi polynomials $P_n^{(\alpha,\alpha)}(x)$, $\alpha \neq \pm\frac{1}{2}$. However mimicking the steps of the proof above one obtains a decomposition of $S_{2n}^{\alpha}(x, 0)$, $-\frac{1}{2} < \alpha < \frac{1}{2}$, that makes the problem better treatable with our methods.

Remark 4. *Because of the fact that Chebyshev polynomials of first and second kind satisfy the same recurrence relation but with different starting values, a closed form for $S_n^{-1/2}(x, y)$ can be computed completely analogously.*

4. Jacobi polynomials $P_n^{(\alpha,\alpha)}(x)$ with $-\frac{1}{2} < \alpha < \frac{1}{2}$

In this section we will prove Theorem 1, i.e. the positivity of $S_{2n}^{\alpha}(x, 0)$, $-\frac{1}{2} < \alpha < \frac{1}{2}$, where the sum representation according to (3) is given by

$$S_n^{\alpha}(x, y) = \frac{1}{x-y}\sum_{j=0}^{n} c_j^{\alpha}[P_{j+1}^{(\alpha,\alpha)}(x)P_j^{(\alpha,\alpha)}(y) - P_j^{(\alpha,\alpha)}(x)P_{j+1}^{(\alpha,\alpha)}(y)], \tag{9}$$

with $c_j^{\alpha} = 2^{-2\alpha-1}\frac{\Gamma(j+2)\Gamma(j+2\alpha+2)}{\Gamma(j+\alpha+1)\Gamma(j+\alpha+2)}$. To this end we need several intermediate results starting with a suitable decomposition of $S_n^{\alpha}(x, y)$ which will be obtained by following the steps of the derivation (8). For this we will invoke the three term recurrence [1, 9]

$$(n+2)(n+2\alpha+2)P_{n+2}^{(\alpha,\alpha)}(x) = (n+\alpha+2)(2n+2\alpha+3)\,x\,P_{n+1}^{(\alpha,\alpha)}(x) - (n+\alpha+1)(n+\alpha+2)P_n^{(\alpha,\alpha)}(x) \tag{10}$$

for $n \geq 0$ with initial values $P_{-1}^{(\alpha,\alpha)}(x) = 0$, $P_0^{(\alpha,\alpha)}(x) = 1$. With this relation we obtain for all $j \geq 0$

$$\begin{aligned}
&(x-y)c_j^\alpha P_{j+1}^{(\alpha,\alpha)}(x)P_j^{(\alpha,\alpha)}(y) \\
&= x\, c_j^\alpha P_{j+1}^{(\alpha,\alpha)}(x)P_j^{(\alpha,\alpha)}(y) - \frac{c_j^\alpha}{(j+\alpha+1)(2j+2\alpha+1)}P_{j+1}^{(\alpha,\alpha)}(x) \\
&\quad \times [(j+\alpha)(j+\alpha+1)P_{j-1}^{(\alpha,\alpha)}(y) + (j+1)(j+2\alpha+1)P_{j+1}^{(\alpha,\alpha)}(y)] \\
&= x\, c_j^\alpha P_{j+1}^{(\alpha,\alpha)}(x)P_j^{(\alpha,\alpha)}(y) - c_j^\alpha \frac{(j+1)(j+2\alpha+1)}{(j+\alpha+1)(2j+2\alpha+1)}P_{j+1}^{(\alpha,\alpha)}(x)P_{j+1}^{(\alpha,\alpha)}(y) \\
&\quad - c_j^\alpha \frac{(j+\alpha)(j+\alpha+1)}{(2j+2\alpha+1)(j+1)(j+2\alpha+1)}P_{j-1}^{(\alpha,\alpha)}(y) \\
&\quad \times [x(2j+2\alpha+1)P_j^{(\alpha,\alpha)}(x) - (j+\alpha)P_{j-1}^{(\alpha,\alpha)}(x)] \\
&= xc_j^\alpha P_{j+1}^{(\alpha,\alpha)}(x)P_j^{(\alpha,\alpha)}(y) - xc_{j-1}^\alpha P_j^{(\alpha,\alpha)}(x)P_{j-1}^{(\alpha,\alpha)}(y) \\
&\quad - c_j^\alpha \frac{(j+1)(j+2\alpha+1)}{(j+\alpha+1)(2j+2\alpha+1)}P_{j+1}^{(\alpha,\alpha)}(x)P_{j+1}^{(\alpha,\alpha)}(y) \\
&\quad + c_j^\alpha \frac{(j+\alpha)^2(j+\alpha+1)}{(j+1)(j+2\alpha+1)(2j+2\alpha+1)}P_{j-1}^{(\alpha,\alpha)}(x)P_{j-1}^{(\alpha,\alpha)}(y).
\end{aligned}$$

Now we plug this identity into Definition (9), set $y = 0$ and substitute $n \mapsto 2n$. This gives

$$\begin{aligned}
x^2 S_{2n}^\alpha(x,0) = &\sum_{j=0}^{2n} x\Delta_j[c_{j-1}^\alpha P_j^{(\alpha,\alpha)}(x)P_{j-1}^{(\alpha,\alpha)}(0)] \\
&- 2\sum_{j=0}^{2n} c_j^\alpha \frac{(j+1)(j+2\alpha+1)}{(j+\alpha+1)(2j+2\alpha+1)}P_{j+1}^{(\alpha,\alpha)}(x)P_{j+1}^{(\alpha,\alpha)}(0) \\
&+ 2\sum_{j=0}^{2n} c_j^\alpha \frac{(j+\alpha)^2(j+\alpha+1)}{(j+1)(j+2\alpha+1)(2j+2\alpha+1)}P_{j-1}^{(\alpha,\alpha)}(x)P_{j-1}^{(\alpha,\alpha)}(0),
\end{aligned}$$

The first sum can easily be simplified by telescoping, the second and third sums can be combined by shifting summation indices. We also use the fact that ultraspherical Jacobi polynomials $P_n^{(\alpha,\alpha)}$ of odd degree vanish at $x = 0$. Thus with

$$g_{2n}^\alpha(x,0) = c_{2n}^\alpha \left[x\, P_{2n+1}^{(\alpha,\alpha)}(x) - 2\frac{2n+\alpha+1}{4n+2\alpha+3}P_{2n}^{(\alpha,\alpha)}(x)\right] P_{2n}^{(\alpha,\alpha)}(0)$$

and

$$f_{2n}^\alpha(x,0) = 2(4\alpha^2-1)\sum_{j=0}^{n} \frac{(2j+\alpha+1)c_{2j}^\alpha}{(2j+1)(2j+2\alpha+1)(4j+2\alpha-1)(4j+2\alpha+3)}P_{2j}^{(\alpha,\alpha)}(0)P_{2j}^{(\alpha,\alpha)}(x)$$

we obtain

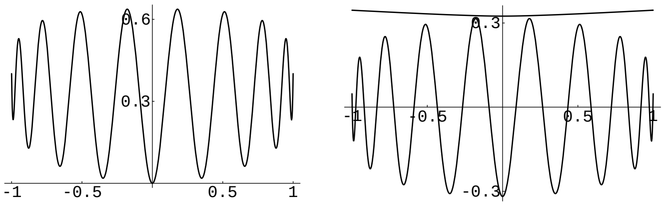

FIGURE 1. $x^2 S_{2n}^0(x,0)$ and $f_{2n}^0(x,0), g_{2n}^0(x,0)$ for $n = 8$

Lemma 5.

$$x^2 S_{2n}^{\alpha}(x,0) = f_{2n}^{\alpha}(x,0) + g_{2n}^{\alpha}(x,0), \qquad -\frac{1}{2} < \alpha < \frac{1}{2},\ -1 \le x \le 1,\ n \ge 0.$$

As can be seen from figure 1, $g_{2n}^{\alpha}(x)$ contains the main oscillations whereas in $f_{2n}^{\alpha}(x)$ they are dampenend out. In order to prove non-negativity of $S_{2n}^{\alpha}(x,0)$ we will show that $f_{2n}^{\alpha}(x,0) + g_{2n}^{\alpha}(x,0) \ge 0$. This will be achieved by estimating the sum $f_{2n}^{\alpha}(x,0)$ from below. Adding this lower bound to $g_{2n}^{\alpha}(x,0)$ can then be shown to be positive with `SumCracker`'s `ProveInequality` command.

The first step is to define, more generally, f_n^{α} for arguments $x, y \in [-1,1]$ by

$$f_n^{\alpha}(x,y) = 2(2\alpha-1)(2\alpha+1) \sum_{j=0}^{n} \frac{(j+\alpha+1)c_j^{\alpha}}{(j+1)(j+2\alpha+1)(2j+2\alpha-1)(2j+2\alpha+3)} P_j^{(\alpha,\alpha)}(x) P_j^{(\alpha,\alpha)}(y).$$

This definition is consistent with that of $f_{2n}^{\alpha}(x,0)$ above. The coefficient of the Jacobi polynomials inside the sum is positive for $j \ge 1$, hence we have

$$\sum_{j=1}^{n} \frac{(j+\alpha+1)c_j^{\alpha}}{(j+1)(j+2\alpha+1)(2j+2\alpha-1)(2j+2\alpha+3)} [P_j^{(\alpha,\alpha)}(x) - P_j^{(\alpha,\alpha)}(y)]^2 \ge 0,$$

which is equivalent to

$$\begin{aligned} -\sum_{j=0}^{n} \frac{(j+\alpha+1)c_j^{\alpha}}{(j+1)(j+2\alpha+1)(2j+2\alpha-1)(2j+2\alpha+3)} P_j^{(\alpha,\alpha)}(x) P_j^{(\alpha,\alpha)}(y) \ge \\ -\sum_{j=0}^{n} \frac{(j+\alpha+1)c_j^{\alpha}}{(j+1)(j+2\alpha+1)(2j+2\alpha-1)(2j+2\alpha+3)} P_j^{(\alpha,\alpha)}(x)^2 \\ -\sum_{j=0}^{n} \frac{(j+\alpha+1)c_j^{\alpha}}{(j+1)(j+2\alpha+1)(2j+2\alpha-1)(2j+2\alpha+3)} P_j^{(\alpha,\alpha)}(y)^2 \end{aligned}$$

Since $(1-2\alpha)(1+2\alpha)$ is positive for $-\frac{1}{2} < \alpha < \frac{1}{2}$ we can multiply both sides of the last inequality with this factor to obtain

Lemma 6. *Let* $-\frac{1}{2} < \alpha < \frac{1}{2}$. *Then*

$$f_n^{\alpha}(x,y) \ge \frac{1}{2}(f_n^{\alpha}(x,x) + f_n^{\alpha}(y,y)), \qquad n \ge 0,$$

for all $x, y \in [-1,1]$.

This lower bound has the advantage that we can find a closed form for $f_n^\alpha(x,x)$. Kauers' package `Crack` delivers closed form expressions for specific values of α, and guessing suggests the identity

Lemma 7.

$$f_n^\alpha(x,x) = 2c_n^\alpha\left[\frac{(n+1)(n+2\alpha+1)}{(n+\alpha+1)(2n+2\alpha+1)}P_{n+1}^{(\alpha,\alpha)}(x)^2\right.$$
$$\left.-x\,P_n^{(\alpha,\alpha)}(x)P_{n+1}^{(\alpha,\alpha)}(x) + \frac{n+\alpha+1}{2n+2\alpha+3}P_n^{(\alpha,\alpha)}(x)^2\right],$$

for all $n \geq 0,\ -1 \leq x \leq 1$ *and* $\alpha > -1$.

The key point is discovering this identity. Once it has been found its validity can be proven fairly easily.

Proof. By telescoping and by $f_{-1}^\alpha(x,x) = 0$ it suffices to show that

$$\frac{2(4\alpha^2-1)c_{j-1}^\alpha}{(j+\alpha)(2j+2\alpha-1)(2j+2\alpha+3)}P_j^{(\alpha,\alpha)}(x)^2 = f_j^\alpha(x,x) - f_{j-1}^\alpha(x,x) = \Delta_j[f_{j-1}^\alpha(x,x)].$$

Since $\frac{c_j^\alpha}{c_{j-1}^\alpha} = \frac{(j+1)(j+2\alpha+1)}{(j+\alpha)(j+\alpha+1)}$ we have

$$\begin{aligned}
&\frac{1}{2c_{j-1}^\alpha}(f_j^\alpha(x,x) - f_{j-1}^\alpha(x,x))\\
&= \frac{4\alpha^2-1}{(j+\alpha)(2j+2\alpha-1)(2j+2\alpha+3)}P_j^{(\alpha,\alpha)}(x)^2\\
&\quad + \frac{(j+1)(j+2\alpha+1)}{(j+\alpha)(j+\alpha+1)}\left[\frac{(j+1)(j+2\alpha+1)}{(j+\alpha+1)(2j+2\alpha+1)}P_{j+1}^{(\alpha,\alpha)}(x) - xP_j^{(\alpha,\alpha)}(x)\right]P_{j+1}^{(\alpha,\alpha)}(x)\\
&\quad + \left[xP_j^{(\alpha,\alpha)}(x) - \frac{j+\alpha}{2j+2\alpha+1}P_{j-1}^{(\alpha,\alpha)}(x)\right]P_{j-1}^{(\alpha,\alpha)}(x).
\end{aligned}$$

By the Jacobi recurrence relation (10) the expressions in the last two rows cancel. □

Figure 2 illustrates how the functions $g_{2n}^\alpha(x,0),\ f_{2n}^\alpha(x,0)$ and $\frac{1}{2}(f_{2n}^\alpha(x,x) + f_{2n}^\alpha(0,0))$ are related. Now we collect the previous lemmas to give a proof of Theorem 1.

Proof of Theorem 1. The cases $\alpha = \pm\frac{1}{2}$ are covered by the results of section 3. For $\alpha = -\frac{1}{2}$ Theorem 1 follows from well known results on the Fejèr kernel [10] and positivity of $S_{2n}^{1/2}(x,0)$ is obvious from the rewriting stated in Lemma 3.

Next we consider $-\frac{1}{2} < \alpha < \frac{1}{2}$. With the decomposition given in Lemma 5 and the lower bound from Lemma 6 we have

$$x^2S_{2n}^\alpha(x,0) = g_{2n}^\alpha(x,0) + f_{2n}^\alpha(x,0) \geq g_{2n}^\alpha(x,0) + \frac{1}{2}(f_{2n}^\alpha(x,x) + f_{2n}^\alpha(0,0)).$$

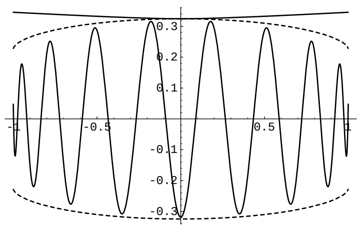

FIGURE 2. $g_{2n}(x,0)$, $f_{2n}(x,0)$, dashed: $\pm\frac{1}{2}(f_{2n}(x,x)+f_{2n}(0,0))$

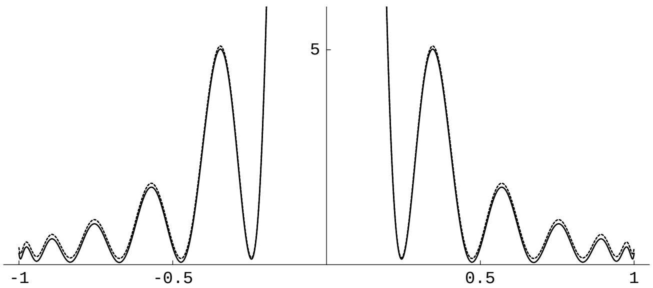

FIGURE 3. $[g_{2n}(x,0)+\frac{1}{2}(f_{2n}(x,x)+f_{2n}(0,0))]/x^2$, dotted: $2S^0_{2n}(x,0)$ for $n=12$.

To finish the proof it suffices to show positivity of the latter expression. We plug in the closed form stated in Lemma 7 and simplify to

$$\begin{aligned} &\frac{1}{c^\alpha_{2n}}[g^\alpha_{2n}(x,0)+\frac{1}{2}(f^\alpha_{2n}(x,x)+f^\alpha_{2n}(0,0))] \\ =\;& \frac{(2n+1)(2n+2\alpha+1)}{(2n+\alpha+1)(4n+2\alpha+1)}P^{(\alpha,\alpha)}_{2n+1}(x)^2 - x\,P^{(\alpha,\alpha)}_{2n+1}(x)[P^{(\alpha,\alpha)}_{2n}(x)-P^{(\alpha,\alpha)}_{2n}(0)] \\ &+\frac{2n+\alpha+1}{4n+2\alpha+3}[P^{(\alpha,\alpha)}_{2n}(x)-P^{(\alpha,\alpha)}_{2n}(0)]^2. \qquad (11) \end{aligned}$$

We use the `ProveInequality` command of `SumCracker` in the following way:

In[3]:= $\mathbf{ProveInequality}[\frac{(2n+1)(2n+2\alpha+1)}{(2n+\alpha+1)(4n+2\alpha+1)}\mathbf{JacobiP}[2n+1,\alpha,\alpha,x]^2$
$-x\mathbf{JacobiP}[2n+1,\alpha,\alpha,x](\mathbf{JacobiP}[2n,\alpha,\alpha,x]-\mathbf{JacobiP}[2n,\alpha,\alpha,0])$
$+\frac{2n+\alpha+1}{4n+2\alpha+3}(\mathbf{JacobiP}[2n,\alpha,\alpha,x]-\mathbf{JacobiP}[2n,\alpha,\alpha,0])^2\geq 0,$
$\mathbf{Using}\to\{-1\leq x\leq 1,\ -\frac{1}{2}<\alpha<\frac{1}{2}\},\mathbf{Variable}\to n,\mathbf{From}\to 0]//\mathbf{Timing}$

Out[3]= $\{5358.25\text{Second},\text{True}\}$

This command constructs an inductive proof using cylindrical algebraic decomposition [2, 6, 5], which is also where the main computational effort lies.

□

5. Conclusion

The condition on α above cannot be removed if we want positivity of (11) for $n \geq 0$. It seems though that this expression stays non-negative for n greater some lower bound, possibly depending on α.

An obvious open problem is to give a "human" proof of the positivity of the expression in (11).

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