Solving and Factoring Boundary Problems for Linear Ordinary Differential Equations in Differential Algebras ⋆

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Abstract

We present a new approach for expressing and solving boundary problems for linear ordinary differential equations in the language of differential algebras. Starting from an algebra with a derivation and integration operator, we construct a ring of linear integro-differential operators that is expressive enough for specifying regular boundary problems with arbitrary Stieltjes boundary conditions as well as their solution operators.

Based on these structures, we define a new multiplication on regular boundary problems in such a way that the resulting Green's operator is the reverse composition of the constituent Green's operators. We provide also a method for lifting any factorization of the underlying differential operator to the level of boundary problems. Since this method only needs the computation of initial value problems, it can be used as an effective alternative for computing Green's operators in case one knows how to factor the given differential operators.

Key words: Linear boundary value problems; Ordinary differential equations; Green's operators; Factorization; Differential algebra; Noncommutative Gröbner bases.

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1. Introduction

In this paper, we develop a new approach for handling *boundary problems* in the language of differential algebras, restricting ourselves to the case of linear boundary problems for ordinary differential equations. (We reserve the traditional word "boundary value problem" for the particular type of boundary problems that have only point evaluations, i.e. local boundary conditions in the terminology of Section 5.) The algebraic language we build up allows

- to state boundary problems in a natural algebraic language,
- to express their solution operators in the same language,
- to compute the solution operators from a fundamental system,
- to multiply boundary problems corresponding to the solution operators,
- to lift factorizations of differential operators to boundary problems.

The present paper extends the ideas from (Rosenkranz, 2005) and (Rosenkranz et al., 2003) in several aspects: Boundary problems can now be formulated and solved in any differential algebra that meets some natural conditions (Theorem 21), the case of variable coefficients is fully included, and a new monoid structure on boundary problems provides an elegant description and an alternative computation method for the corresponding solution operators.

For developing an appropriate notion of boundary problem in a given differential algebra, it will be useful to have a look at the *classical setting* of (Stakgold, 1979, p. 203) dealing with a two-point boundary problem on a finite interval $I = [a, b]$. Disregarding weak solutions and ill-posed problems for simplicity, the general idea is that a differential equation

$$u^{(n)}(x) + c_{n-1}(x)\, u^{(n-1)}(x) + \ldots + c_1(x)\, u'(x) + c_0(x)\, u(x) = f(x) \qquad (1)$$

with coefficient functions $c_{n-1}, \ldots, c_1, c_0 \in C^\infty(I)$ and forcing function $f \in C^\infty(I)$ is supplemented with additional conditions that determine the solution $u \in C^\infty(I)$ uniquely. In certain cases, these may be initial conditions, but in general one has to deal with constraints that combine the values and derivatives of u at both endpoints a and b. In the context of a linear differential equation like (1), it is natural to restrict oneself to linear conditions of the form

$$p_{n-1}\, u^{(n-1)}(a) + \ldots + p_0\, u(a) + q_{n-1}\, u^{(n-1)}(b) + \ldots + q_0\, u(b) = e, \qquad (2)$$

where the p_i, q_i and e are given complex numbers. For obvious reasons, boundary conditions of the form (2) are known as two-point boundary conditions; note that they include initial conditions as the special case where all the q_i vanish. In order to obtain a well-posed boundary problem, one imposes n suitable linear boundary conditions (2) on a given linear differential equation (1). Since all differential equations, operators and conditions will be linear in this paper, we will from now on drop the attribute "linear."

Classical boundary problems (1) (2) have a rich structure. First of all, it is clear that one can decompose the solution of (1), (2) into a solution of the semi-inhomogeneous problem (obtained from (2) by setting $e = 0$) and a solution of the semi-homogeneous problem (obtained from (1) by setting $f = 0$). Since we assume fundamental systems are available, the latter problem reduces to linear algebra, and we can concentrate on the semi-inhomogeneous problem. Thus we assume from now on homogenous boundary conditions.

A second crucial observation is that the solution u depends linearly on the forcing function f. In fact, the assumption of a well-posed boundary problem means that there is a unique u for every given f, so there is a solution operator $G\colon C^\infty(I) \to C^\infty(I)$ with $u = Gf$. This so-called *Green's operator* G is linear: If u_1 is the solution for a forcing function f_1 and u_2 the solution for a forcing function f_2, it is clear that u_1+u_2 fulfills the (homogeneous!) boundary conditions (2), while substituting $u_1 + u_2$ into the left-hand side of the differential equation (1) obviously leads to $f_1(x) + f_2(x)$ on its right-hand side. Then $u_1 + u_2$ is "a" solution for the forcing function $f_1 + f_2$, so by uniqueness also

$$u_1 + u_2 = G(f_1 + f_2).$$

Since $u_1 = Gf_1$ and $u_2 = Gf_2$, this means that G is linear.

Taking advantage of this linear structure, it is possible to compute the Green's operator G rather than a particular solution u belonging to some fixed forcing function f. We may view this as solving the parametrized differential equation (1) together with boundary conditions (2). There is also a practical reason why it is useful to have the Green's operator: The forcing function f is often more likely to change (e.g. as the "source term" in heat conduction), whereas the shape of the differential equation (its left-hand side) and boundary conditions remain fixed. In the classical setting, it turns out that $G\colon C^\infty(I) \to C^\infty(I)$ can be represented in the form of an integral operator

$$Gf(x) = \int_a^b g(x,\xi) f(\xi)\, d\xi$$

with a uniquely determined *Green's function* $g \in C^{n-2}(I^2)$. So once g is found, one can compute each desired solution u by a single integration.

Now let us describe our strategy of rebuilding this scenario in a (moderately general) *differential algebra*. In the place of $C^\infty(I)$, we take a differential algebra $\mathcal{F}$ as our starting point. Obviously, a differential equation (1) is then given by

$$Tu = f \tag{3}$$

with a differential operator $T \in \mathcal{F}[\partial]$, and one has to find the solution $u \in \mathcal{F}$ in terms of a given forcing function $f \in \mathcal{F}$. (In order to gain flexibility, we will actually consider differential operators $T \in \mathcal{F}_0$ for a suitable subalgebra $\mathcal{F}_0 \leq \mathcal{F}$; see Definition 13.) A boundary condition is given by

$$\beta_1 u, \ldots, \beta_n u = 0 \tag{4}$$

for suitable functionals $\beta_1, \ldots, \beta_n \in \mathcal{F}^*$, where $\mathcal{F}^*$ denotes the dual space of $\mathcal{F}$. We will allow rather general boundary conditions of the so-called Stieltjes type (see Definition 9), including not only two-point conditions like (2) but also global conditions involving integrals.

At this point, we would like to make a general remark on point *evaluation* in differential algebra. This is a topic not often considered (within the given algebraic setting), despite its undisputed importance in the applications. The problem is that the elements of a differential algebra (or differential ring or differential field) are abstractions of a function that are not meant for being "evaluated." Abraham Robinson (1961) has addressed this discrepancy by introducing what he called localized differential rings. Working in the much wider scope of polynomial differential equations, he has developed a solvability criterion for initial value problems. According to our knowledge, his ideas have not found much resonance. For a more practical perspective on initial value problems for differential-algebraic equations, see the recent survey (Pritchard and Sit, 2007), containing a method for determining admissible initial conditions. Our own point of view is to consider boundary conditions in their natural context: as functionals of the aforementioned type.

Hence the need for a differential algebra—they provide a vector space structure together with the structure of a differential ring. In fact, we need more than that (Section 2): Since we want to express the Green's operator of a boundary problem (3)(4), we need an explicit operator $\int$ for denoting *integration*, just like ∂ is is used for denoting differentiation. Obviously, we must stipulate that

$$\partial \textstyle\int = 1,$$

while further analysis will make it clear that we must also require $\int$ to fulfill the so-called Baxter axiom, an algebraic version of integration by parts. As we shall see, this necessarily excludes differential fields from the admissible differential algebras $\mathcal{F}$. We are thus led to the following crucial observation (Proposition 4): Despite their extremely useful role for example in the Galois theory of linear differential equations (van der Put and Singer, 2003), differential fields are inadequate for treating initial/boundary conditions along with the differential equations. In some sense, this result is not unexpected: Point evaluations correspond to maximal ideals, while in a field the only proper ideal is the trivial one.

We call the resulting structure $(\mathcal{F}, \partial, \int)$ an integro-differential algebra. They induce a natural algebra of *integro-differential operators* $\mathcal{F}[\partial, \int]$, just like $(\mathcal{F}, \partial)$ alone induces the algebra of differential operators $\mathcal{F}[\partial]$. We introduce a suitable rewrite system for these operators (Section 3), enabling their convenient symbolic manipulation. Our rewrite system is both noetherian and confluent (Proposition 8), and the corresponding normal forms have a natural description (Proposition 12). The advantage of the $\mathcal{F}[\partial, \int]$ language is that it provides a uniform frame for stating initial/boundary problems as well as for deriving and expressing their Green's operators.

The departure from differential fields has the consequence that inhomogeneous differential equations cannot be reduced to homogenous ones, as explained in (van der Put and Singer, 2003, Exercise 1.14.1). Hence we have to resort to an algebraic version of the well-known "variation of the constant" method for solving even *initial value problems* (Section 4) and this necessitates a mild condition on the solutions of inhomogeneous first-order differential equations. This condition basically requires that exponential solutions exist and behave as normal: they have a reciprocal.

For treating *boundary problems* (3) (4) in a convenient fashion, we specify them as pairs

$$(T, \mathcal{B}) \quad \text{with} \quad T \in \mathcal{F}_0[\partial] \quad \text{and} \quad \mathcal{B} = [\beta_1, \ldots, \beta_n] \leq \mathcal{F}^*.$$

Using this setup, we will show (Section 5) that they have a Green's operator that can be expressed in $\mathcal{F}[\partial, \int]$, and we sketch how one can compute it. For a concrete implementation in the classical C^∞ setting, see the previous article (Rosenkranz, 2005). Generalizing the idea of a boundary problem as "surjective linear map plus linear functionals as side conditions", we have developed an abstract treatment in general vector spaces in our forthcoming paper (Regensburger and Rosenkranz, 2007).

The algebraic treatment of boundary problems applied in this paper does not only allow for a symbolic solution, it is also a natural setting for exposing an important structure connecting boundary problems amongst themselves (Section 6): It turns out that the composition structure of Green's operators is reflected in a *monoid structure* on the boundary problems, arising as a semi-direct product of $\mathcal{F}_0[\partial]$ and the additive structure of subspaces in $\mathcal{F}^*$.

Finally (Section 7), we will show how to factor given boundary problems $(T, \mathcal{B})$ into smaller ones. While *factorization* of linear ordinary differential operators is an important topic in symbolic computation (Grigoriev, 1990; van der Put and Singer, 2003; Schwarz, 1989; Tsarev, 1996), it neglects the presence of boundary conditions (possibly addressed in a post-processing step). We will show how every factorization of the differential operator T gives rise to various factorizations of $(T, \mathcal{B})$, whose full classification is stated. In order to lift a factorization of T to the level of boundary problems, one only needs to solve an initial value problem. Hence one may employ factorization as a tool for computing the Green's operator G. In the extreme case of splitting T into linear factors, one obtains G as a composition of first-order Green's operators, which can be computed easily. (In practical examples, one will often be content with a partial factorization.)

Some remarks on *notation*. We write $\mathbb{N}$ for the set of all natural numbers including zero. The variable n ranges over $\mathbb{N}$. All algebras are assumed to be commutative and with identity. The zero-dimensional subspace of any vector space will be denoted by $O = \{0\}$. We write $[f_1, \ldots, f_n]$ for the subspace generated by the vectors $f_1, \ldots, f_n$ of some vector space $\mathcal{F}$. For subsets $\mathcal{F}_0 \subseteq \mathcal{F}$ and $\mathcal{B}_0 \subseteq \mathcal{F}^*$, the so-called orthogonal is defined as

$$\begin{aligned} \mathcal{F}_0^\perp &= \{\varphi \in \mathcal{F}^* \mid \forall_{f \in \mathcal{F}_0}\, \varphi(f) = 0\} \leq \mathcal{F}^*, \\ \mathcal{B}_0^\perp &= \{f \in \mathcal{F} \mid \forall_{\varphi \in \mathcal{B}_0}\, \varphi(f) = 0\} \leq \mathcal{F}; \end{aligned}$$

see Section 5 for more details.

2. Integration in Differential Algebras

Let $(\mathcal{F}, \partial)$ be a *differential algebra* over a field K, so $\partial \colon \mathcal{F} \to \mathcal{F}$ is a K-linear map fulfilling the Leibniz rule $\partial(fg) = f\,\partial(g) + g\,\partial(f)$. For convenience, we may assume $K \leq \mathcal{F}$, and we write f' as a shorthand for $\partial(f)$. Furthermore, we will assume that K has characteristic zero except stated otherwise (even though some definitions and results would make sense in positive characteristic). Then we may also assume $\mathbb{Q} \leq K$, so that $\mathcal{F}$ is what is sometimes called a Ritt algebra.

The skew ring of (formal) *differential operators* over the differential algebra $\mathcal{F}$ is denoted by $\mathcal{F}[\partial]$ as in van der Put and Singer (2003). The addition in $\mathcal{F}[\partial]$ is obvious, while the multiplication is determined by the rule $\partial f = f\partial + f'$ for all $f \in \mathcal{F}$. Each $T \in \mathcal{F}[\partial]$ acts on $\mathcal{F}$ as an (actual) differential operator $T \colon \mathcal{F} \to \mathcal{F}$. The identity operator of $\mathcal{F}[\partial]$ is denoted by $\partial^0 = 1$ just like the unit element $1 \in \mathcal{F}$; it will be clear from the context which of the two is meant.

Our goal is to solve inhomogeneous differential equations by Green's operators. The simplest such equation is obviously $u' = f$, and its solution operators $\int$ are exactly the sections of the differential operator ∂. (Note that a derivation need not possess a section: In the algebra of univariate differential polynomials, the differential indeterminate clearly cannot be a derivative.) Since $\mathcal{F}$ exhibits a vector space structure and since we are only interested in linear solution operators where we think of u as fixed by suitable side conditions (confer our remarks in Section 1), it is straightforward to classify all *sections* of ∂.

Proposition 1. *Every section* $\int : \mathcal{F} \to \mathcal{F}$ *of the derivation* $\partial : \mathcal{F} \to \mathcal{F}$ *corresponds to a unique projector* $P : \mathcal{F} \to \mathcal{F}$ *with* $P = 1 - \int \partial$*, and to a unique direct sum* $\mathcal{F} = \mathcal{C} \dotplus \mathcal{I}$ *with* $\mathcal{C} = \mathrm{Ker}(\partial) = \mathrm{Im}(P)$ *and* $\mathcal{I} = \mathrm{Im}(\int) = \mathrm{Ker}(P)$.
If $\int$ *is any fixed section of* ∂*, every projector* P *with* $\mathrm{Im}(P) = \mathrm{Ker}(\partial)$ *induces a section* $(1 - P)\int$*, and every section of* ∂ *arises uniquely in this way.*

Proof. See (Regensburger and Rosenkranz, 2007) or (Nashed and Votruba, 1976, p. 17). □

We refer to the elements of $\mathcal{I} = \mathrm{Im}(\int)$ as the *initialized* functions (with respect to $\int$), while those of $\mathcal{C} = \mathrm{Ker}(\partial)$ are usually known as the *constants* (with respect to ∂). (In the prototypical case of $\mathcal{F} = C^\infty(\mathbb{R})$, the initialized functions are those that can be written as $F(x) = \int_\alpha^x f(\xi)\, d\xi$ for an integrand $f \in C^\infty(\mathbb{R})$ and an initialization point $\alpha \in \mathbb{R}$; hence they are exactly the functions F that fulfill the initial condition $F(\alpha) = 0$.)

For solving inhomogeneous differential equations $Tu = f$ of higher order, one must expect to iterate the section $\int$. In general, this could lead to "nested integrals" of arbitrary complexity. But we know from the classical C^∞ setting (see Section 1) that the Green's operator G can always be expressed by a single integration, with the so-called *Green's function* g as its integral kernel. The essential role of Green's functions is to resolve nested integrals, whereas the passage from an operator $G : C^\infty(I) \to C^\infty(I)$ to a function $g \in C^{n-2}(I^2)$ is quite immaterial (from our viewpoint).

In order to capture this behavior in abstract differential algebras we need an identity on sections that lets us resolve nested integrals (eventually leading to the $\int f \int$ rule in Table 1). Such an identity is given by the so-called *Baxter axiom* (of weight zero), asserting

$$(\textstyle\int f)(\int g) = \int (f \int g) + \int (g \int f) \tag{5}$$

for all $f, g \in \mathcal{F}$; see (Guo, 2002; Baxter, 1960; Rota, 1969) for more details. One sees immediately that (5) is an algebraic version of integration by parts, rewritten in such a way that it does not need refer to any derivation. A Baxter algebra $(\mathcal{F}, \int)$ is then a K-algebra $\mathcal{F}$ with a K-linear operation $\int$ fulfilling the Baxter axiom (5).

Let us note one important consequence of (5) at this point. Writing x as an abbreviation for $\int 1$, we obtain $x^2/2 = \int\int 1$ and inductively $x^n/n! = \int \cdots \int 1$ with n iterates of $\int$. Hence the powers $u = x^k$ with $k < n$ are solutions of $u^{(n)} = f$, and one checks immediately that they are all linearly independent. This means that $\mathrm{Ker}(\partial^n)$ contains $[1, x, \ldots, x^{n-1}]$ as an n-dimensional subspace. Since n can be chosen arbitrarily high, we see that $\mathcal{F}$ in fact contains (an isomorphic copy of) the polynomial ring $K[x]$. Note that $K[x] \leq \mathcal{F}$ is both a differential algebra an a Baxter algebra (this will be called an integro-differential algebra in Definition 6). In particular, we see that $\mathcal{F}$ is necessarily infinite-dimensional.

What we shall actually need is the differential form of the Baxter axiom (eventually leading to the $\int f \partial$ rule in Table 1), a slightly stronger variant that cannot be stated in pure Baxter algebras since it involves the derivation. It claims that one has

$$\textstyle\int fg = f \int g - \int (f' \int g) \tag{6}$$

for all $f, g \in \mathcal{F}$. Note that (6) is what most people do when they actually apply integration by parts. The pure version (5) follows immediately by substituting $\int f$ for f in (6). In fact, we can easily characterize what makes the *differential Baxter axiom* stronger than the pure one.

Lemma 2. *A section $\int$ of ∂ fulfills the differential Baxter axiom* (6) *iff it fulfills the pure Baxter axiom* (5) *and the homogeneity condition $\int cf = c \int f$ for all $c \in \mathcal{C}$ and $f \in \mathcal{F}$.*

Proof. Assume $\int$ fulfills (6). Then $\int$ also fulfills (5) as observed above, while substituting a constant $c \in \mathcal{C}$ for f in (6) gives homogeneity.

Conversely, assume that $\int$ fulfills (5) and the homogeneity condition. The latter hypothesis means that (6) is satisfied for all $f \in \mathcal{C}$. Now consider $f \in \mathcal{I}$ so that $\int f' = f$. Substituting f' for f in (5), we see that (6) is also satisfied for these $f \in \mathcal{I}$. But then the general case of $f \in \mathcal{F}$ follows via the direct sum $\mathcal{F} = \mathcal{C} \dotplus \mathcal{I}$. $\square$

Example 3. Most "natural" integro-differential algebras actually fulfill (6), so apparently we have to contrive a somewhat artificial example for seeing that (6) is really stronger that (5). As usual, let K be any field of characteristic zero. Then $(R[x], \partial)$ with $R = K[y]/y^4$ and $\partial f = f_x$ is clearly a differential algebra over the ground field K. Defining

$$\textstyle\int f = \displaystyle\int_0^x f(\xi, y)\, d\xi + f(0,0)\, y^2, \tag{7}$$

we obtain a K-linear map $\int \colon R[x] \to R[x]$. Since the second term vanishes under ∂, we see immediately that $\int$ is a section of ∂. For verifying the Baxter axiom 5, let us write $\fint$ for the ordinary integral in (7) and compute

$$\begin{aligned}
&(\textstyle\int f)(\int g) = (\fint f)(\fint g) + y^2 \fint \big(g(0,0)\, f + f(0,0)\, g\big) + f(0,0)\, g(0,0)\, y^4,\\
&\textstyle\int (f \int g) = \int f (\fint g + g(0,0)\, y^2) = \fint (f \fint g) + y^2 \fint \big(g(0,0)\, f\big).
\end{aligned}$$

Since $y^4 \equiv 0$ and the ordinary integral $\fint$ fulfills the Baxter axiom (5), this implies immediately that $\int$ does also. Hence we conclude that $(R[x], \partial, \int)$ is an integro-differential algebra. However, it does not fulfill the stronger axiom (6), because the homogeneity condition is violated: Observe that $\mathrm{Ker}(\partial) = R$, so in particular we should have $\int y \cdot 1 = y \cdot \int 1$. But one checks immediately that the left-hand side actually yields xy, while the right-hand side yields $xy + y^3$.

We call a section $\int$ of the derivation ∂ an *integral* if it satisfies the differential Baxter axiom (6). Using the characterization of sections in Proposition 1, we can now classify the integrals either by their projectors or by their induced direct sum of $\mathcal{F}$.

Proposition 4. *A section $\int \colon \mathcal{F} \to \mathcal{F}$ of the derivation $\partial \colon \mathcal{F} \to \mathcal{F}$ is an integral iff its projector $P \colon \mathcal{F} \to \mathcal{F}$ is multiplicative and iff $\mathcal{I} = \mathrm{Im}(\int)$ is an ideal.*

Proof. Assume first $\int$ is an integral for ∂, let $P = 1 - \int \partial$ be its projector and $\mathcal{F} = \mathcal{C} \dotplus \mathcal{I}$ the corresponding direct sum with $\mathcal{C} = \mathrm{Ker}(\partial) = \mathrm{Im}(P)$ and $\mathcal{I} = \mathrm{Im}(\int)$, according to Proposition 1. We must prove $P(fg) = P(f)\, P(g)$ for all $f, g \in \mathcal{F}$. Substituting g' for g in (6), we obtain

$$\begin{aligned}
0 &= \textstyle\int f g' - f \int g' + \int (f' \int g') = \int f g' - f(g - Pg) + \int (f'(g - Pg))\\
&= \textstyle\int f g' + \int f' g - fg + f\, Pg - (\int f')\, Pg,
\end{aligned}$$

where we have used the homogeneity of $\int$ in the last step. But then

$$P(fg) = fg - \textstyle\int (f'g + fg') = (f - \int f')\, Pg = Pf\, Pg,$$

as claimed.

Assume conversely that P is multiplicative, and take $f, G \in \mathcal{F}$ arbitrary. Expanding the definition of P and using the Leibniz law gives

$$P(fG) = (1 - \textstyle\int\partial)fG = fG - \int f'G - \int fG'$$

and

$$Pf\,PG = (f - \textstyle\int f')(G - \int G') = fG - G\int f' - f\int G' + (\int f')(\int G');$$

equating the two expressions, we obtain

$$(\textstyle\int f')(\int G') + \int f'G + \int fG' = G\int f' + f\int G',$$

which yields indeed (6) upon specializing to $G = \int g$.

Let us now prove that $\mathcal{I}$ is an ideal under the assumption that P is multiplicative. Since P is a projector along $\mathcal{I}$, we have $PG = 0$ iff $G \in \mathcal{I}$. Hence $G \in \mathcal{I}$ implies $P(fG) = Pf\,PG = 0$, and $fG \in \mathcal{I}$ as claimed.

Finally, we assume $\mathcal{I}$ is an ideal and prove P multiplicative. Taking $f, g \in \mathcal{F}$ arbitrary, we set $f_0 = Pf \in \mathcal{C}$ and $g_0 = Pg \in \mathcal{C}$. Then $f_1 = f - f_0 \in \mathcal{I}$ and likewise $g_1 = g - g_0 \in \mathcal{I}$, so we obtain

$$P(fg) = P(f_0g_0) + P(f_0g_1) + P(f_1g_0) + P(f_1g_1) = f_0g_0 = Pf\,Pg$$

since all of $f_0g_1, f_1g_0, f_1g_1 \in \mathcal{I}$ vanish under P while $f_0g_0 \in \mathcal{C}$ is fixed by P. $\square$

As an *example*, take $\mathcal{F} = C^\infty[0,1]$ with its usual derivation D. The operators A and $-B$, defined as in (Rosenkranz, 2005, p. 176) by

$$Af(x) = \int_0^x f(\xi)\,d\xi \quad \text{and} \quad Bf(x) = \int_x^1 f(\xi)\,d\xi,$$

are integrals corresponding respectively to the projectors $L\colon f \mapsto f(0)$ and $R\colon f \mapsto f(1)$. By contrast, the operator C, defined by

$$C = A - RAA$$

and used for regularizing an ill-posed problem in (Rosenkranz, 2005, p. 192), is just a section for D but not an integral. Indeed, its projector $RA = \int_0^1$ is not multiplicative, unlike L and R.

Since in this paper we want to attack boundary problems for ordinary differential equations, we will from now on restrict the underlying differential algebra $(\mathcal{F}, \partial)$ accordingly. The solution space of an ordinary differential equation (in the C^∞ setting) has dimension equal to the order of the equation. Hence let us call $\mathcal{F}$ an *ordinary* differential algebra if

$$\dim \operatorname{Ker}(\partial) = 1.$$

Note that here our terminology deviates from Kolchin (1973, p. 58), where it simply refers to having a single derivation. (So in Kolchin's sense, the differential algebra $(C^\infty(\mathbb{R}^2), \partial_x + \partial_y)$ discussed at the end of this section would be addressed as "ordinary", while we prefer to exclude this case because $\dim \operatorname{Ker}(\partial) = \infty$.)

Having an ordinary differential algebra $\mathcal{F}$ has several important *consequences*. First of all, it is clear that we have now $K = \mathcal{C}$, so $\mathcal{F}$ is an algebra over its own field of constants. But then a section $\int$ is automatically homogeneous over $\mathcal{C}$, so the pure Baxter axiom (5) and its differential version (6) coincide. As mentioned above, the Baxter axiom implies that $\operatorname{Ker}(\partial^n)$ contains $[1, x, \ldots, x^{n-1}]$ as a subspace. Now we can use the identity

$$\operatorname{Ker}(T^2) = G \operatorname{Ker}(T) \dotplus \operatorname{Ker}(T),$$

valid for any epimorphism T on a vector space and any section G of T; see (Regensburger and Rosenkranz, 2007) for a more general statement and proof. This yields $\dim \operatorname{Ker}(\partial^n) = n$ by induction and hence also

$$\operatorname{Ker}(\partial^n) = [1, x, \ldots, x^{n-1}], \tag{8}$$

the familiar polynomial space.

Finally, the *projector* $P\colon \mathcal{F} \to \mathcal{F}$ can be regarded as a normalized linear functional. (In any vector space, a projector onto a one-dimensional subspace $[w]$ can be written as $P(v) = \varphi(v)\, w$ where φ is a unique linear functional with $\varphi(w) = 1$. Since $\mathcal{F}$ is moreover an algebra, a projector onto $K = [1]$ is canonically described by the functional φ with $\varphi(1) = 1$.)

A *unit functional* on $\mathcal{F}$ is an $\eta \in \mathcal{F}^*$ with the normalization $\eta(1) = 1$, a *character* is a multiplicative functional (and hence also a unit functional). Let us write $\mathcal{U}(\mathcal{F}^*)$ for the set of all unit functionals and $\mathcal{M}(\mathcal{F}^*)$ for the space of all characters. As we have seen, every section $\int$ corresponds to a unique $\eta = 1 - \int\partial \in \mathcal{U}(\mathcal{F}^*)$. If $\int$ is moreover an integral, Proposition 4 tells us that $\eta \in \mathcal{M}(\mathcal{F}^*)$, and $\mathcal{I}$ is an ideal with $\mathcal{F} = K \dotplus \mathcal{I}$.

Direct summands for the ground field of an algebra are also known as *augmentation ideals*, while the corresponding projector η is then called its *augmentation*; see (Cohn, 2003, p. 132). Augmentation ideals are always maximal ideals: Since η induces the direct sum $\mathcal{F} = K \dotplus \mathcal{I}$, we have an isomorphism of vector spaces

$$\tilde{\eta}\colon \mathcal{F}/\mathcal{I} \cong K \quad \text{with} \quad \tilde{\eta}(f + \mathcal{I}) = \eta(f)$$

But $\tilde{\eta}$ is also a morphism of rings because the character $\eta\colon \mathcal{F} \to K$ is one, so $\mathcal{F}/\mathcal{I}$ is a field isomorphic to K, and $\mathcal{I}$ a maximal ideal. Let us summarize these results.

Corollary 5. *In an ordinary differential algebra $(\mathcal{F}, \partial)$, a section $\int$ of ∂ is an integral iff its unit functional is a character and iff $\mathcal{I} = \operatorname{Im}(\int)$ is an augmentation ideal.*

Note that the augmentation ideal $\mathcal{I}$ corresponding to an integral is in general not a differential ideal of $\mathcal{F}$. We can see this in the standard example $\mathcal{F} = C^\infty[0, 1]$ by taking $\int = A$, say. Then $\mathcal{I}$ consists of all $f \in \mathcal{F}$ with $f(0) = 0$, so $\mathcal{I}$ is *not differentially closed* since $x \mapsto x \in \mathcal{I}$ but $x \mapsto 1 \notin \mathcal{I}$.

We have now gathered the main ingredients that we need for treating boundary problems: a differential algebra with an integral. We call such structures *integro-differential algebras*, treated in greater generality in the recent preprint (Guo and Keigher, 2007), which came to our attention only after completing this article. (The situation considered in (Guo and Keigher, 2007) is more general in three respects: The algebras are over unital commutative rings rather than fields, they may be noncommutative, and they may have nonzero weight. Their interest stems mainly from combinatorial investigations of tree-like structures, where the weight is usually nonzero.)

Definition 6. We call $(\mathcal{F}, \partial, \int)$ an *integro-differential algebra* if $(\mathcal{F}, \partial)$ is a differential algebra and $\int$ is an integral for ∂. Its associated *evaluation* is the character $1 - \int\partial$. If $\dim \operatorname{Ker}(\partial) = 1$, we speak of an *ordinary* integro-differential algebra.

One may readily verify that an *analytic algebra* $(\mathcal{F}, \partial, {\textstyle\int^*}, {\textstyle\int_*})$ in the sense of (Rosenkranz, 2005, p. 182) is equivalent to a pair of integro-differential algebras $(\mathcal{F}, \partial, {\textstyle\int^*})$ and $(\mathcal{F}, \partial, -{\textstyle\int_*})$. Writing $\ldots^{\leftarrow}$ and $\ldots^{\rightarrow}$ for the respective evaluations of ${\textstyle\int^*}$ and ${\textstyle\int_*}$, one finds the relation

$$({\textstyle\int^*} f)^{\rightarrow} = {\textstyle\int^*} f + {\textstyle\int_*} f = ({\textstyle\int_*} f)^{\leftarrow}$$

for $f \in \mathcal{F}$. This relation may be interpreted as requiring ${\textstyle\int_*}$ to be the dual of ${\textstyle\int^*}$, with respect to the inner product

$$\langle f|g\rangle = ({\textstyle\int^*} + {\textstyle\int_*})\, fg.$$

In the standard example $\mathcal{F} = C^\infty[0,1]$, this gives the usual L^2 inner product $\langle f|g\rangle = \int_0^1 f(x)g(x)\,dx$.

Before restricting ourselves to ordinary integro-differential algebras in the rest of the paper, let us conclude this section with a non-trivial example of an integro-differential algebra with an *infinite-dimensional constant space*. Consider $\mathcal{F} = C^\infty(\mathbb{R}^2)$ with the derivation $\partial u = u_x + u_y$. Finding sections for ∂ means to solve the partial differential equation $u_x + u_y = f$. Its general solution is given by

$$u(x,y) = \int_\alpha^x f(t, t-x+y)\,dt + g(y-x),$$

where $g \in C^\infty(\mathbb{R})$ and $\alpha \in \mathbb{R}$ are arbitrary. In order to ensure a linear section, one has to choose $g = 0$, arriving at

$${\textstyle\int} f = \int_\alpha^x f(t, t-x+y)\,dt.$$

Using a simple change of variables, one may immediately verify that ${\textstyle\int}$ satisfies the Baxter axiom (5), so $(\mathcal{F}, {\textstyle\int})$ is really a Baxter algebra. We see also that $\mathcal{C} = \mathrm{Ker}(\partial)$ is given by the functions $(x,y) \mapsto g(x-y)$ with arbitrary $g \in C^\infty(\mathbb{R})$, while $\mathcal{I} = \mathrm{Im}({\textstyle\int})$ consists of those functions $f \in C^\infty(\mathbb{R}^2)$ that satisfy $f(\alpha, y) = 0$ for all $y \in \mathbb{R}$. The evaluation η maps a function $f \in C^\infty(\mathbb{R}^2)$ to the function $(x,y) \mapsto f(\alpha, \alpha - x + y)$. Obviously η is multiplicative, so $(\mathcal{F}, \partial, {\textstyle\int})$ is indeed an integro-differential algebra by Corollary 5.

3. Integro-Differential Operators

From now on, let $(\mathcal{F}, \partial, {\textstyle\int})$ be an ordinary integro-differential algebra over a field K with associated evaluation $\mathtt{E}$. We define the *integro-differential operators* $\mathcal{F}[\partial, {\textstyle\int}]$ as the K-algebra generated by the symbols ∂ and ${\textstyle\int}$, the "functions" $f \in \mathcal{F}$ and the multiplicative "functionals" $\varphi \in \mathcal{M}(\mathcal{F}^*)$, modulo the reduction rules given in Table 1. Every integro-differential operator can be written as a linear combination of "monomials" (a coefficient times a monomial will be called a "term").

In the rules of Table 1 as well as in the rest of this paper, we use the notation $E \cdot f$ for the *action* of E on a function f, where E is an element of the free algebra in the above generators. It is an easy matter to check that the reductions of Table 1 are fulfilled in $(\mathcal{F}, \partial, {\textstyle\int})$, so we may regard $\cdot$ as an action of $\mathcal{F}[\partial, {\textstyle\int}]$ on $\mathcal{F}$. In particular, $\tilde{f} \cdot f$ now denotes the product of functions $\tilde{f}, f \in \mathcal{F}$.

We remark that Table 1 is to be understood as including *implicit rules* for ${\textstyle\int\int}$, ${\textstyle\int}\partial$ and ${\textstyle\int}\varphi$ by substituting $f = 1$ in the rules for ${\textstyle\int} f {\textstyle\int}$, ${\textstyle\int} f \partial$ and ${\textstyle\int} f \varphi$, respectively. Moreover, one obtains the *derived rule* $\mathtt{E}{\textstyle\int} = 0$ from the definition of the evaluation $\mathtt{E}$. Note that $\mathcal{F}[\partial]$ is a subalgebra of $\mathcal{F}[\partial, {\textstyle\int}]$ with the same induced action on $\mathcal{F}$.

$\tilde{f} f$	$\to$	$\tilde{f} \cdot f$	∂f	$\to$	$\partial \cdot f + f \partial$	$\int f \int$	$\to$	$(\int \cdot f) \int - \int (\int \cdot f)$
$\tilde{\varphi} \varphi$	$\to$	φ	$\partial \varphi$	$\to$	0	$\int f \partial$	$\to$	$f - \int(\partial \cdot f) - (\mathtt{E} \cdot f)\, \mathtt{E}$
φf	$\to$	$(\varphi \cdot f)\, \varphi$	$\partial \int$	$\to$	1	$\int f \varphi$	$\to$	$(\int \cdot f)\, \varphi$

Table 1. Reduction Rules for Integro-Differential Operators

Example 7. The *analytic polynomials* of (Rosenkranz, 2005, p. 176) are also an important special case of integro-differential operators (the restriction to $K = \mathbb{C}$ imposed there is not essential). They are constructed on top of an analytic algebra $(\mathcal{F}, \partial, \int^*, \int_*)$ with corresponding evaluations $\ldots^{\leftarrow}$ and $\ldots^{\rightarrow}$ as explained in Section 3. As usual, we can express one integral by the other, yielding either $-\int_* = (1 - ^{\rightarrow}) \int^*$ or $-\int^* = (1 - ^{\leftarrow}) \int_*$. Choosing randomly the first alternative, we work with the integro-differential ring $(\mathcal{F}, \partial, \int^*)$. Up to notational details, the analytic polynomials over $(\mathcal{F}, \partial, \int^*, \int_*)$ are then the subalgebra of $\mathcal{F}[\partial, \int^*]$ generated by the operators

$$\begin{aligned} D &= \partial, & L &= {}^{\leftarrow}, \\ A &= \textstyle\int^*, & R &= {}^{\rightarrow}, \\ B &= \textstyle\int_*, = (1 - ^{\leftarrow}) \int^* & \lceil f \rceil &= f, \end{aligned}$$

using the same names as in the cited article. We use also the abbreviation $F = A + B$ for the operator of definite integration (note that F is a non-multiplicative unit functional).

Note that for analytic polynomials, the multipliers $\lceil f \rceil$ are restricted to *basis elements* $f \in \mathcal{F}$; similar restrictions could be made here. The point is that a system of normal forms on $\mathcal{F}[\partial, \int]$ presupposes a canonical simplifier on the free algebra generated by ∂ and $\int$, the functions $f \in \mathcal{F}$ and the functionals $\varphi \in \mathcal{M}(\mathcal{F}^*)$. Expansion with respect to fixed bases on $\mathcal{F}$ and $\mathcal{M}(\mathcal{F}^*)$ provides such a canonical simplifier, but there may also be others. Hence we take the viewpoint that the free algebra is equipped with *some* canonical simplifier (the "ground simplifier"), and the confluence result of the following proposition has to be understood relative to this ground simplifier.

Proposition 8. *The rewrite system of Table 1 is noetherian and confluent.*

Proof. By the Diamond Lemma 1.2 from (Bergman, 1978), it suffices to ensure the following two facts: First we must construct a partial wellorder $>$ on the word monoid in the generators of $\mathcal{F}[\partial, \int]$ such that $>$ is compatible with the monoid structure and the reduction system in Table 1. Second we have to prove that all ambiguities of the reduction system are resolvable. For defining the partial wellorder, we put $\partial > f$ for all functions f and extend this to words by the graded lexicographic construction. The resulting partial order is clearly well-founded (since it is on the generators) and compatible with the monoid structure (by its grading). It is also compatible with the reduction system because all rules reduce the word length except for the Leibniz rule, which is compatible because $\partial > f$.

For proving that the ambiguities of Table 1 are resolvable, note first that we have no inclusion ambiguities while there are exactly 14 overlap ambiguities. For overlapping rules $ww_1 \to p_1$ and $w_2 w \to p_2$ to be resolvable, their S-polynomial $p_2 w_1 - w_2 p_1$ must reduce to zero. This is indeed the case, as one can check by an easy calculation (using also the axioms of integro-differential algebra for $\mathcal{F}$). As a representative example, let us reassure ourselves that the S-polynomial from the rules for $ww_1 = {\textstyle\int} f\partial$ and $w_2w = {\textstyle\int} g {\textstyle\int}$ indeed reduces to

$$
\begin{aligned}
&({\textstyle\int} \cdot g){\textstyle\int} f\partial - {\textstyle\int} \,({\textstyle\int} \cdot g)\, f\partial - {\textstyle\int} gf + {\textstyle\int} g{\textstyle\int} f' + {\textstyle\int} g\,(\mathtt{E}\cdot g)\,\mathtt{E}\\
&\quad = ({\textstyle\int} \cdot g)f - ({\textstyle\int} \cdot g){\textstyle\int} f' - ({\textstyle\int} \cdot g)\,(\mathtt{E}\cdot f)\,\mathtt{E} - ({\textstyle\int} \cdot g)f + {\textstyle\int} \partial \cdot (({\textstyle\int} \cdot g)\cdot f)\\
&\qquad\quad + (\mathtt{E}\cdot(({\textstyle\int} \cdot g)\cdot f)\,\mathtt{E} - {\textstyle\int} (g\cdot f) + ({\textstyle\int} \cdot g){\textstyle\int} f' - {\textstyle\int} ({\textstyle\int} \cdot g)f' + (\mathtt{E}\cdot f)({\textstyle\int} \cdot g)\,\mathtt{E}\\
&\quad = {\textstyle\int} \partial \cdot (({\textstyle\int} \cdot g)\cdot f) + (\mathtt{E}\cdot(({\textstyle\int} \cdot g)\cdot f)\,\mathtt{E} - {\textstyle\int} (g\cdot f) - {\textstyle\int} ({\textstyle\int} \cdot g)f'\\
&\quad = {\textstyle\int} (g\cdot f) + {\textstyle\int} ({\textstyle\int} \cdot g)f' + 0 - {\textstyle\int} (g\cdot f) - {\textstyle\int} ({\textstyle\int} \cdot g)f'\\
&\quad = 0,
\end{aligned}
$$

as it should. $\square$

In other words, the ideal generated by the polynomials occurring on the left-hand sides of Table 1 form a *noncommutative Gröbner basis*. For theory of Gröbner bases, we refer to (Buchberger, 1965, 1970, 1998) and for its noncommutative extension to (F. Mora, 1986), (T. Mora, 1994), (Ufnarovski, 1998).

Comparing the analytic polynomials in (Rosenkranz, 2005, p. 183) with the reduction system of Table 1, we would like to emphasize the gain in *simplicity and economy*: Despite their higher generality, the integro-differential operators of $\mathcal{F}[\partial, {\textstyle\int}]$ require just 9 instead of 36 identities! Consequently, this confluence proof (resolving 14 overlaps) can still be done by hand, while the automatically generated confluence proof for the analytic polynomials (resolving 233 overlaps) contains 2000 lines; see (Rosenkranz, 2005, p.184f) for a small fragment of it.

Having a noetherian and confluent rewrite system, every integro-differential operator has a normal form. In order to describe these normal forms explicitly, it is useful to single out a particular portion of the operators that will also turn out to be play a distinguished role in specifying *boundary conditions* (see Section 5).

Definition 9. The elements of the right ideal

$$\mathrm{Cond}(\mathcal{F}^*) = \mathcal{M}(\mathcal{F}^*)\,\mathcal{F}[\partial, {\textstyle\int}]$$

are called *Stieltjes boundary conditions* over $\mathcal{F}$; if there is no danger of ambiguity, we will henceforth just speak of "boundary conditions."

We will now describe the *normal forms* in $\mathcal{F}[\partial, {\textstyle\int}]$, starting with a simple observation on reducibility (in general not describing normal forms), which is then used for characterizing the normal forms of boundary conditions.

Lemma 10. *Every integro-differential operator in $\mathcal{F}[\partial, {\textstyle\int}]$ can be reduced to a linear combination of monomials $f\varphi{\textstyle\int} g\psi\partial^i$, where $i \geq 0$ and each of $f, \varphi, {\textstyle\int}, g, \psi$ may also be absent.*

Proof. Call a monomial consisting only of functions and functionals "algebraic." Using the left column of Table 1, it is immediately clear that all such monomials can be reduced to f or φ or $f\varphi$. Now let w be an arbitary monomial in the generators of $\mathcal{F}[\partial, {\textstyle\int}]$. By using the middle column of Table 1, we may assume that all occurrences of ∂ are moved to the right, so that all monomials have the form $w = w_1 \cdots w_n\partial^i$ with $i \geq 0$ and each of $w_1, \dots, w_n$ either a function, a functional or $\textstyle\int$. We may further assume that there is at most one occurrence of $\textstyle\int$ among the $w_1, \dots, w_n$. Otherwise the monomials $w_1 \cdots w_n$ contain $\textstyle\int \tilde{w} \textstyle\int$, where each $\tilde{w}$ is an algebraic monomial. But then we can reduce

$$\textstyle\int \tilde{w} \textstyle\int = (\textstyle\int f\varphi)\textstyle\int = (\textstyle\int \cdot f)\varphi\textstyle\int$$

by using the corresponding rule of Table 1. Applying these reductions repeatedly, we arrive at algebraic monomials left and right of $\textstyle\int$ (or just a single algebraic monomial if $\textstyle\int$ is absent). $\square$

Proposition 11. *Every boundary condition $\beta \in \mathrm{Cond}(\mathcal{F}^*)$ has a normal form*

$$\beta = \sum_{\varphi \in \mathcal{M}(\mathcal{F}^*)} \left(\sum_{i \in \mathbb{N}} a_{\varphi,i}\, \varphi \partial^i + \varphi \textstyle\int f_\varphi \right)$$

with $a_{\varphi,i} \in K$ and $f_\varphi \in \mathcal{F}$ almost all zero.

Proof. By Lemma 10, every $\beta \in \mathrm{Cond}(\mathcal{F}^*)$ is a linear combination of monomials having the form

$$\beta_0 = \chi f \varphi \textstyle\int g \psi \partial^i \quad \text{or} \quad \beta_0 = \chi f \varphi \partial^i \tag{9}$$

where each of f, g, φ, ψ may also be missing. Using the left column of Table 1, the prefix $\chi f \varphi$ can be reduced to a scalar multiple of a functional, so we may as well assume that f and φ are not present; this finishes the right-hand case of (9). For the remaining case $\beta_0 = \chi \textstyle\int g \psi \partial^i$, assume first that ψ is present. Then we have

$$\chi\,(\textstyle\int g\psi) = \chi\,(\textstyle\int \cdot\, g)\,\psi = (\chi \cdot \textstyle\int \cdot\, g)\,\chi\psi = (\chi \cdot \textstyle\int \cdot\, g)\,\psi,$$

so β_0 is again a scalar multiple of $\psi\partial^i$, and we are done. Finally, assume we have $\beta_0 = \chi \textstyle\int g \partial^i$. If $i = 0$, this is already a Stieltjes normal form. Otherwise we obtain

$$\beta_0 = \chi\,(\textstyle\int g \partial)\,\partial^{i-1} = (\chi \cdot g)\,\chi\partial^{i-1} - \chi \textstyle\int g' \partial^{i-1} - (\mathtt{E} \cdot g)\,\mathtt{E}\partial^{i-1},$$

where the first and the last summand are in the required normal form, while the middle summand is to be reduced recursively, eventually leading to a middle term in Stieltjes normal form $-\chi \textstyle\int g' \partial^0 = -\chi \textstyle\int g'$. $\square$

The Stieltjes boundary conditions have the additional benefit of allowing a simple description of the normal forms for all integro-differential operators. For obtaining a smooth formulation, let us call the elements of $\mathcal{F}[\partial] \subset \mathcal{F}[\partial, \textstyle\int]$ *differential operators*; they have a straight-forward normal form. Analogously, we write $\mathcal{F}[\textstyle\int] \subset \mathcal{F}[\partial, \textstyle\int]$ for the subalgebra of *integral operators*, generated by the functions and $\textstyle\int$ modulo the Baxter rule (uppermost in the right column of Table 1). Using Lemma 10, it is clear that the normal forms of integral operators are linear combinations of $f \textstyle\int g$ with $f, g \in \mathcal{F}$.

Finally, we write $\mathcal{F}[\mathtt{E}] = \mathcal{F}\ \mathrm{Cond}(\mathcal{F}^*) \subseteq \mathcal{F}[\partial, \textstyle\int]$ for the left $\mathcal{F}$-submodule generated by $\mathrm{Cond}(\mathcal{F}^*)$, called the *Stieltjes boundary operators* (briefly "boundary operators"). Note that $\mathcal{F}[\mathtt{E}]$ includes $\mathrm{Cond}(\mathcal{F}^*)$ as well as all finite-dimensional projection operators P along linearly independent Stieltjes conditions $\varphi_1, \dots, \varphi_n$. In fact, P is determined by choosing a complement $[f_1, \dots, f_n]$ to $[\varphi_1, \dots, \varphi_n]^\perp$, and the (f_i) can be chosen biorthogonal to the (φ_i) such that

$$P = \sum_{i=1}^{n} f_i\, \varphi_i; \tag{10}$$

see for example (Regensburger and Rosenkranz, 2007, Prop. 2) and (Köthe, 1969, p. 71). From the representation (10) it is immediately clear that $P \in \mathcal{F}[\mathtt{E}]$. All elements of $\mathcal{F}[\mathtt{E}]$ have the normal form (10), except that the (f_i) need not be biorthogonal to the (φ_i).

It turns out now that every monomial of an integro-differential operator is either a *differential operator* or an *integral operator* or a *boundary operator*.

Proposition 12. *Up to term ordering, every normal form of $\mathcal{F}[\partial, \textstyle\int]$ with respect to the reduction system of Table 1 can be written uniquely as a sum $T + G + B$ having the following normal-form summands: a differential operator $T \in \mathcal{F}[\partial]$, an integral operator $G \in \mathcal{F}[\textstyle\int]$, and a boundary operator $B \in \mathcal{F}[\mathtt{E}]$.*

Proof. Inspection of Table 1 confirms that all integro-differential operators having the described sum representation $T+G+P$ are indeed in normal form. Let us now prove that every $E \in \mathcal{F}[\partial, \textstyle\int]$ has such a representation. It is sufficient to consider the monomials E_0 of E. If E_0 starts with a functional, we obtain a boundary condition by Proposition 11; so assume this is not the case. From Lemma 10 we know that

$$E_0 = f\varphi \textstyle\int g\psi\partial^i \quad \text{or} \quad E_0 = f\varphi\partial^i,$$

where each of φ, g, ψ may also be missing. But $E_0 \in \mathcal{F}[\mathtt{E}]$ unless φ is missing, so we may actually assume

$$E_0 = f \textstyle\int g\psi\partial^i \quad \text{or} \quad E_0 = f\partial^i.$$

The right-hand case yields $E_0 \in \mathcal{F}[\partial]$. If ψ is present in the other case, we may reduce $\int g\psi$ to $(\int \cdot\, g)\,\psi$, and we obtain again $E_0 \in \mathcal{F}[\mathtt{E}]$. Hence we are left with $E_0 = f \int g\partial^i$. Now assume $i > 0$, since otherwise we have $E_0 \in \mathcal{F}[\int]$ immediately; then we can reduce

$$\begin{aligned} E_0 &= f\,(\textstyle\int g\partial)\,\partial^{i-1} = f\Big(g - \textstyle\int (\partial \cdot g) - (\mathtt{E} \cdot g)\,\mathtt{E}\Big)\partial^{i-1} \\ &= (fg)\,\partial^{i-1} - f \textstyle\int (\partial \cdot g)\,\partial^{i-1} - (\mathtt{E} \cdot g)\, f\, \mathtt{E}\,\partial^{i-1}, \end{aligned}$$

where the first term is obviously in $\mathcal{F}[\partial]$ and the last one in $\mathcal{F}[\mathtt{E}]$. The middle term may be reduced recursively until the exponent of ∂ has dropped to zero, leading to a term in $\mathcal{F}[\int]$. □

4. Initial Value Problems

Up to now we have not discussed the existence of solutions for differential equations, except for two particularly simple cases: the homogeneous differential equation $u^{(n)} = 0$, whose general solution is given by $[1, x, \ldots, x^{n-1}]$ as stated in (8), and the inhomogeneous equation $u^{(n)} = f$, which has $\int f$ as one particular solution. In order to have some finer control on which differential equations we want to have solutions, we will allow to specify the *coefficients* of the pertinent linear differential operators separately. (In Differential Galois Theory, one usually works with differential fields, where one can study extensions in a much more convenient manner. As we have seen above, though, this route is not accessible for us here.)

Definition 13. A differential subalgebra $\mathcal{F}_0 \leq \mathcal{F}$ is called *saturated* for a differential algebra $\mathcal{F}$ if $\dim \operatorname{Ker}(T) = n$ for every monic $T \in \mathcal{F}_0[\partial]$ with $\deg T = n$ and if all nonzero solutions for $u' = au$ with $a \in \mathcal{F}_0$ are invertible in $\mathcal{F}$. In this context, we call $\mathcal{F}$ the *ground algebra* and $\mathcal{F}_0$ the *coefficient algebra*. (If $\mathcal{F}_0$ coincides with $\mathcal{F}$, we simply speak of a saturated integro-differential algebra.)

Some remarks on this definition are in order. First of all, we point out that we need $\mathcal{F}_0$ to be differentially closed such that we can multiply and factor within $\mathcal{F}_0[\partial]$, which will be needed for multiplying and factoring boundary problems in Section 6 and Section 7, respectively. The first condition on solvability ensures that *homogeneous equations* $Tu = 0$ have a fundamental system with the appropriate number of solutions, while the second condition means that the exponentials behave as expected. Note also that $\mathcal{F}$ must be an ordinary differential algebra as soon as it possesses a saturated coefficient algebra.

Let us give some *examples* of integro-differential algebras with saturated coefficient algebras.

Example 14. The prototypical example is furnished by $C^\infty(I)$ where $I = [a, b]$ is some interval. As a coefficient algebra, one may take either $C^\infty(I)$ itself or any differential subalgebras like $\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{C}[x]$. Similarly, one may take analytic functions $C^\omega(I)$ and its differential subalgebras. Less demanding, the exponential polynomials, defined in (Rosenkranz, 2005, p. 176), can be taken as a ground algebra with $\mathbb{C}$ as a coefficient algebra.

Example 15. For any field K of characteristic 0, the formal power series $K[[z]]$ are a saturated integro-differential algebra, with derivation and integration defined as usual. This may also be inferred from the next example by the isomorphism described there.

Example 16. Let K be an arbitrary field (note that we are explicitly including the case of positive characteristic in this example). Then the algebra $H(K)$ Hurwitz series (Keigher, 1997) over K is defined as the K-vector space of infinite K-sequences with the multiplication defined as

$$(a_n) \cdot (b_n) = \left(\sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i} \right)_n$$

for all $(a_n), (b_n) \in H(K)$. If one introduces derivation and integration by

$$\begin{aligned} \partial\,(a_0, a_1, a_2, \dots) &= (a_1, a_2, \dots), \\ {\textstyle\int}\,(a_0, a_1, \dots) &= (0, a_0, a_1, \dots), \end{aligned}$$

the Hurwitz series form an integro-differential algebra $(H(K), \partial, \int)$, as explained in (Keigher and Pritchard, 2000; Guo, 2002).

Note that as an additive group, $H(K)$ coincides with the formal power series $K[[z]]$, but its multiplicative structure differs: We have an isomorphism

$$\sum_{n=0}^{\infty} a_n\, z^n \;\mapsto\; (n!\, a_n)$$

from $K[[z]]$ to $H(K)$ if and only if K has characteristic zero. The point is that one can integrate every element of $H(K)$, whereas the formal power series z^{p-1} does not have an antiderivative in $K[[z]]$ if K has characteristic p.

Defining the exponential function $\exp = (1, 1, 1, \dots)$, we have immediately $\partial \exp = \exp$. One can introduce a composition $f \circ g$ for $f, g \in H(K)$ whenever g has vanishing constant term, and the usual chain rule is satisfied for this composition (Keigher and Pritchard, 2000). Then the first-order homogeneous equation $u' = au$ with $a \in H(K)$ is solved by

$$u = c\, \exp \circ ({\textstyle\int} a),$$

which is easily seen to be invertible in $H(K)$. By Corollary 4.3 in (Keigher and Pritchard, 2000), we know also that all monic homogeneous differential equations of order n have an n-dimensional kernel. Hence $H(K)$ is a saturated integro-differential algebra.

From now on throughout the rest of this paper, we assume $(\mathcal{F}, \partial, \int)$ is an integro-differential algebra with a saturated coefficient algebra $\mathcal{F}_0$. As before, we write $\mathtt{E}$ for the associated evaluation. Having integrals, it is natural to expect that we can also solve *inhomogeneous equations*. As we shall see now, it is always possible to find such a particular solution, but we can be more specific than that.

We formulate the *initial value problem* for a monic differential operator $T \in \mathcal{F}_0[\partial]$ and evaluation $\eta \in \mathcal{M}(\mathcal{F}^*)$ as follows: Given a forcing function $f \in \mathcal{F}$, find $u \in \mathcal{F}$ such that

$$\begin{aligned} &Tu = f \\ &\eta u, \eta u', \dots, \eta u^{(n-1)} = 0. \end{aligned} \tag{11}$$

is satisfied. Problems of this kind can be solved uniquely.

Proposition 17. *Under the above assumptions, the initial value problem* (11) *has a unique solution $u \in \mathcal{F}$ for every $f \in \mathcal{F}$.*

Proof. We can use the usual technique of reformulating (11) as a system of linear first-order differential equations with companion matrix $A \in \mathcal{F}_0^{n \times n}$, then we apply the well-known variation-of-constants formula (Coddington and Levinson, 1955, p. 74). To this end, we pick a fundamental system $u_1, \dots, u_n \in \mathcal{F}$ for T and compute the Wronskian matrix

$$W = \begin{pmatrix} u_1 & \dots & u_n \\ u_1' & \dots & u_n' \\ \vdots & \ddots & \vdots \\ u_1^{(n-1)} & \dots & u_n^{(n-1)}. \end{pmatrix},$$

Note that $d = \det W$ satisfies the first-order differential equation $d' = ad$, where $a = \operatorname{trc} A \in \mathcal{F}_0$; see for example Exercise 1.14.5 in (van der Put and Singer, 2003), noting that one does not need a differential field here. Since $\mathcal{F}_0$ is saturated for $\mathcal{F}$, the determinant d must be invertible and hence W a regular matrix. By Proposition 1, the operator $⨍ = (1-\eta)\int$ is the section of ∂ with evaluation $1 - ⨍\partial = \eta$. We extend the action of the operators $⨍, \partial, \eta$ componentwise to $\mathcal{F}^n$. Setting now

$$\hat{u} = (W ⨍ W^{-1})\,\hat{f}$$

with $\hat{f} = (0, \dots, 0, f)^\top \in \mathcal{F}^n$, one may readily check that $\hat{u} \in \mathcal{F}^n$ is a solution of the first-order system $\hat{u}' = A\hat{u} + \hat{f}$ with initial condition $\eta\hat{u} = 0$, where A is the companion matrix of T. Writing u for the first component of $\hat{u}$, we have a solution of (11).

For proving uniqueness, assume u is a solution of (11) for $f = 0$; we must show $u = 0$. We may expand $u = c_1 u_1 + \dots + c_n u_n$ in terms of the fundamental system $u_1, \dots, u_n$ with suitable coefficients $c_1, \dots, c_n \in K$. Then the initial conditions of (11) may be summarized by $\eta(Wc) = 0$ with the coefficient vector $c = (c_1, \dots, c_n)^\top \in K^n$. But $\eta(Wc) = \eta(W)c$ because η is linear, and $\det \eta(W) = \eta(\det W)$ since it is moreover multiplicative; hence $\eta(W) \in K^{n \times n}$ is regular with $c = \eta(W)^{-1} 0 = 0$ and $u = 0$, as required. $\square$

Every integro-differential ring $(\mathcal{F}, \partial, \int)$ comes with a distinguished evaluation $\eta = \mathtt{E}$, so we may speak of the initial value problem associated with any monic $T \in \mathcal{F}_0[\partial]$. If $u \in \mathcal{F}$ is the unique solution to the corresponding inital value problem with forcing function f, we obtain an operator $T^\blacklozenge \colon \mathcal{F} \to \mathcal{F}$ with $u = T^\blacklozenge f$, which we shall call the *fundamental right inverse* of T. Notation and terminology are in accordance with (Rosenkranz, 2005), where the evaluation $\mathtt{E}\colon C^\infty[a, b] \to C^\infty[a, b]$ is given by $u \mapsto u(a)$. Inspecting the proof of Proposition 17, one can see that u may in fact be obtained from f by the operation of a suitable integro-differential operator of $\mathcal{F}[\partial, \int]$. This holds in particular for the initial value problem with evaluation $\eta = \mathtt{E}$.

Fact 18. *For every monic $T \in \mathcal{F}_0[\partial]$, the fundamental right inverse can be realized as an integro-differential operator $T^\blacklozenge \in \mathcal{F}[\partial, \int]$.*

5. Boundary Problems

The main purpose of $\mathcal{F}[\partial, \int]$ is to provide a unified language for expressing *boundary problems* as well as their *solutions*. As explained in Section 1, a boundary problem of order n is typically formulated as follows: Given a forcing function $f \in \mathcal{F}$, we have to find $u \in \mathcal{F}$ such that

$$\begin{aligned} &Tu = f \\ &\beta_1 u = \ldots = \beta_n u = 0 \end{aligned} \tag{12}$$

for a monic differential operator $T \in \mathcal{F}_0[\partial]$ with $\deg T = n$ and boundary conditions $\beta_1, \ldots, \beta_n \in \mathcal{F}^*$. The differential operator is evidently in $\mathcal{F}[\partial, \int]$, and the same is true about the boundary conditions if we stick to the restriction of Definition 9. The solution is usually expressed as $u = Gf$, where $G\colon \mathcal{F} \to \mathcal{F}$ is the so-called Green's operator of the boundary problem (12). As we shall see in Theorem 21, the Green's operator G can also be expressed as the action of an element of $\mathcal{F}[\partial, \int]$.

We think of the boundary conditions $\beta_1, \ldots, \beta_n \in \mathcal{F}^*$ of (12) as specifying a *space of admissible functions*

$$\mathcal{A} = \{\beta_1, \ldots, \beta_n\}^\perp \leq \mathcal{F}.$$

Obviously we may replace the boundary conditions $\beta_1, \ldots, \beta_n \in \mathcal{F}^*$ by other boundary conditions $\tilde\beta_1, \ldots, \tilde\beta_n \in \mathcal{F}^*$ such that $\tilde\beta_i = c_{i1}\beta_1 + \ldots + c_{in}\beta_n$ for a regular matrix $(c_{ij}) \in K^{n\times n}$, leading to the same space of admissible functions $\mathcal{A} = \{\tilde\beta_1, \ldots, \tilde\beta_n\}^\perp$. This means that the admissible functions may be described invariantly as $\mathcal{A} = \mathcal{B}^\perp$ in terms of $\mathcal{B} = [\beta_1, \ldots, \beta_n] = [\tilde\beta_1, \ldots, \tilde\beta_n]$. Such a finite-dimensional subspace $\mathcal{B} \leq \mathcal{F}^*$ will be called a *space of boundary conditions*.

The operators $\ldots^\perp$ on $\mathcal{F}$ and $\mathcal{F}^*$ create an order-reversing *Galois connection* between the complemented modular lattices $\mathbb{P}(\mathcal{F})$ and $\bar{\mathbb{P}}(\mathcal{F}^*)$, where $\mathbb{P}(\ldots)$ denotes the full subspace lattice and $\bar{\mathbb{P}}(\ldots)$ the sublattice of all orthogonally closed subspaces (the latter means that $\ldots^{\perp\perp}$ acts as the identity). Specifically, we have

$$\mathcal{B}^\perp = \{u \in \mathcal{F} \mid \forall_{\beta \in \mathcal{B}}\ \beta(u) = 0\}$$

for space of functions satisfying the boundary conditions in $\mathcal{B}$ and

$$\mathcal{A}^\perp = \{\beta \in \mathcal{F}^* \mid \forall_{u \in \mathcal{A}}\ \beta(u) = 0\}$$

for the space of boundary conditions satisfied by the functions in $\mathcal{A}$. We are thus in a similar situation as in algebraic geometry, where affine varieties play the role of $\mathbb{P}(\mathcal{F})$

while radical ideals correspond to $\bar{\mathbb{P}}(\mathcal{F}^*)$. Having an order-reversing lattice isomorphism between $\mathbb{P}(\mathcal{F})$ and $\bar{\mathbb{P}}(\mathcal{F}^*)$, we can take advantage of relations like

$$(\mathcal{F}_1 \cap \mathcal{F}_2)^\perp = \mathcal{F}_1^\perp + \mathcal{F}_2^\perp \quad \text{and} \quad \mathcal{F}_1 \dotplus \mathcal{F}_2 = \mathcal{F} \ \Leftrightarrow \ \mathcal{F}_1^\perp \dotplus \mathcal{F}_2^\perp = \mathcal{F}^*,$$

which will turn out to be important for manipulating boundary problems (see Section 6). We refer to our forthcoming article (Regensburger and Rosenkranz, 2007) for an abstract approach along these lines.

For our present purposes, however, we are interested in an *algorithmic treatment* of boundary conditions and their associated spaces of admissible functions. Hence we have to restrict the set of available primitives in such a way that all the relevant operations—like the ones occurring in the relations above—are algorithmically feasible but still generous enough to be applicable to the boundary problems under consideration. (If we want to build what is called a regular boundary problem, further restrictions have to be imposed on the boundary conditions; see Section 6 on this.)

In the usual setting, a boundary problem of order n is accompanied by so-called two-point boundary conditions (Stakgold, 1979, p. 203), which have the form

$$\beta u = \sum_{i=0}^{n-1} a_i \, u^{(i)}(0) + b_i \, u^{(i)}(1)$$

with $a_0, \ldots, a_{n-1}, b_0, \ldots, b_{n-1} \in K$. Obviously, we may view

$$\beta = \sum_{i=0}^{n-1} a_i \, LD^i + b_i \, RD^i$$

as an element of $\mathcal{F}[\partial, \textstyle\int]$ if we adopt the classical setting $\mathcal{F} = C^\infty[0,1]$ with its usual interpretation as an analytic algebra. Note that $L, R \in \mathcal{M}(\mathcal{F}^*)$. In a general integro-differential algebra $\mathcal{F}$, we will allow any linear combination of conditions having the form $\varphi\partial^i$ with $\varphi \in \mathcal{M}(\mathcal{F}^*)$; let us speak of a *point condition* in this case.

We use the more general setting of *Stieltjes conditions*, more commonly used for systems of linear ordinary differential equations; see for example (Brown and Krall, 1974, 1977). Assuming again a boundary problem of order n in the usual setting, such a condition takes the form

$$\beta u = \sum_{i=0}^{n-1} a_i \, u^{(i)}(0) + b_i \, u^{(i)}(1) + \int_0^1 f(\xi) \, u(\xi) \, d\xi,$$

where the sum part gives a point condition as before while the kernel $f \in \mathcal{F}$ provides an *integral condition*. Note that the normal forms in Proposition 11 are exactly in this form. A boundary condition is called global if the kernel $f \neq 0$ and local otherwise.

There are at least three *reasons* for considering Stieltjes conditions: First of all, they are interesting in themselves because certain boundary problems are naturally expressed in terms of global side conditions (for example, specifying the heat radiated through the boundary). This in also true for regularizing ill-posed problems and computing their generalized Green's function (Rosenkranz, 2005, p. 191). A second reason for introducing Stieltjes conditions will become manifest in Section 7: Factoring a boundary problem leads to factor problems with global conditions, even for a problem having only local conditions (see also Example 23). Finally, a third advantage of Stieltjes conditions is that they have a natural algebraic characterization as we have see in Definition 9.

Let us return to *spaces of boundary conditions*, now in the following precise sense: We write $\mathfrak{B}_n$ for the set of all n-dimensional subspaces $\mathcal{B} = [\beta_1, \dots, \beta_n] \leq \mathcal{F}^*$ generated by n linearly independent Stieltjes boundary conditions $\beta_1, \dots, \beta_n \in \mathrm{Cond}(\mathcal{F}^*)$; note that $[] = O \in \mathfrak{B}_0$. Since all finite-dimensional subspaces of $\mathcal{F}^*$ are orthogonally closed, the Galois connection restricts to an order-reversing lattice ismorphism—this is no longer the case when dealing with linear boundary problems for partial differential equations (again we refer to (Regensburger and Rosenkranz, 2007) for details).

Then $\mathfrak{B} = \bigcup_n \mathfrak{B}_n$ is closed under the operation $+$ of constructing the sum of vector spaces, thus yielding an abelian monoid $(\mathfrak{B}, +)$ to be called the *monoid of boundary conditions*. Specifically, the sum of an m-dimensional and an n-dimensional space of boundary conditions gives

$$[\beta_1, \dots, \beta_m] + [\tilde{\beta}_1, \dots, \tilde{\beta}_n] = [\beta_1, \dots, \beta_m, \tilde{\beta}_1, \dots, \tilde{\beta}_n] = [\gamma_1, \dots, \gamma_k],$$

with dimension $k \leq m+n$. In order to compute linearly independent boundary conditions $\gamma_1, \dots, \gamma_k$, one may apply the following evident strategy.

Fact 19. *There is an effective method for computing a basis for an arbitrary $\mathcal{B} \in \mathfrak{B}$: Given generators $\beta_1, \dots, \beta_l$ of $\mathcal{B}$, find linearly independent $\gamma_1, \dots, \gamma_k$ such that $\mathcal{B} = [\gamma_1, \dots, \gamma_k]$.*

Proof. Expand each of $\beta_1, \dots, \beta_l$ in the K-basis of normal-form monomials as given by Proposition 11. Although the number of such basis elements is infinite, the expansions of $\beta_1, \dots, \beta_l$ will only use finitely many of them, say, $m_1, \dots, m_r$. This yields an $l \times r$ matrix (a_{ij}) over K such that $\beta_i = a_{i1} m_1 + \ldots + a_{ir} m_r$ for all $i \in \{1, \dots, l\}$. Reducing the matrix (a_{ij}) to row echelon and discarding the zero rows leads to the desired K-basis $\gamma_1, \dots, \gamma_k$ of $\mathcal{B}$. $\square$

Let us write $\mathfrak{D}_n$ for the set of all monic $T \in \mathcal{F}_0[\partial]$ with $\det T = n$ and set $\mathfrak{D} = \bigcup_n \mathfrak{D}_n$. In this paper, we will only be concerned with boundary problems (12) that are *regular* in the sense that they have a unique solution u for each forcing function f. Below we will reformulate the condition of regularity directly in terms of the differential operator and the space of boundary conditions.

Note that we do not require well-posedness. Following Hadamard, a *well-posed* problem (Engl et al., 1996, p. 86) must be regular as well as stable (meaning the solution u depends continuously on the data f). Our approach is purely algebraic, so we do not care about stability (which would first of all require a topology on $\mathcal{F}$). For example, the following boundary problem in $\mathcal{F} = C^\infty[0,1]$ is regular but not well-posed, at least not when in the common setting of the Banach space $(\mathcal{F}, \|\cdot\|_\infty)$: Given f, find u such that $u' - u = f$ and $u''(0) = 0$. In this case, the solution exists and is unique; in fact, it is given by $u(x) = \int_0^x f(\xi)\, d\xi - (f(0) + f'(0))\, e^x$, so the Green's operator is $e^x - e^x L - e^x LD$. Incidentally, this example illustrates another unusual feature of our setting—we do not restrict the derivatives in the boundary/initial conditions to orders below the order of the differential equation (even though it will often be reasonable to make such a restriction).

Definition 20. A *boundary problem* of order n is a pair $(T, \mathcal{B})$ with $T \in \mathfrak{D}_n$ and $\mathcal{B} \in \mathfrak{B}_n$. We write $\mathfrak{P}_n$ for the set of all such boundary problems, setting $\mathfrak{P} = \bigcup_n \mathfrak{P}_n$. A boundary problem $(T, \mathcal{B})$ is called *regular* if $\mathrm{Ker}(T) \dotplus \mathcal{B}^\perp = \mathcal{F}$.

As explained in (Regensburger and Rosenkranz, 2007), the requirement of the direct sum is equivalent to $\mathrm{Ker}(T) \cap \mathrm{Ker}(\mathcal{B}) = O$ and also to $\mathrm{Ker}(T) + \mathrm{Ker}(\mathcal{B}) = \mathcal{F}$ since we have insisted on $\deg T = \dim \mathcal{B}$ in the current setting. It is moreover equivalent to regularity in the sense discussed above and to the following algorithmic *determinant criterion*: If $u_1, \ldots, u_n$ is any basis of $\mathrm{Ker}(T)$ and $\beta_1, \ldots, \beta_n$ any basis of $\mathrm{Ker}(\mathcal{B})$, the problem $(T, \mathcal{B})$ is regular iff

$$\begin{vmatrix} \beta_1(u_1) & \cdots & \beta_1(u_n) \\ \vdots & \ddots & \vdots \\ \beta_n(u_1) & \cdots & \beta_n(u_n) \end{vmatrix} \neq 0.$$

This test may be found in (Kamke, 1967, p. 184) for the special case of two-point boundary conditions, but it is a general property of linear epimorphisms with finite-dimensional kernel (see (Regensburger and Rosenkranz, 2007) for more details). Since in this paper we consider only regular boundary problems, we will henceforth suppress the attribute "regular."

The *Green's operator* G of a boundary problem $(T, \mathcal{B})$ is defined by the two conditions $TG = 1$ and $\mathrm{Im}(G) = \mathcal{B}^\perp$. If $\deg T = n$, the space of boundary conditions $\mathcal{B}$ can be described by n basis elements $\beta_1, \ldots, \beta_n$, and we can rewrite the two conditions in the following traditional form: Given $f \in \mathcal{F}$, find $u \in \mathcal{F}$ such that

$$\begin{aligned} &Tu = f, \\ &\beta_1 u, \ldots, \beta_n u = 0; \end{aligned}$$

the Green's operator G is then given by the mapping $f \mapsto u$. Since every boundary problem $(T, \mathcal{B})$ has a unique such Green's operator G, we can introduce the notation $(T, \mathcal{B})^{-1}$ for it. In (Rosenkranz, 2005), we have explained how to compute from a fundamental system of T the Green's operator of a two-point boundary problem $(T, \mathcal{B})$ over the analytic algebra $C^\infty[a, b]$. This result generalizes to our present setting.

Theorem 21. *Every boundary problem $(T, \mathcal{B}) \in \mathfrak{P}$ has a Green's operator that can be written as integro-differential operator $G \in \mathcal{F}[\partial, \textstyle\int]$.*

Proof. The decomposition method explained in (Rosenkranz, 2005) is also valid in our case; based on the algebraic generalized inverse (Nashed and Votruba, 1976; Engl and Nashed, 1981), it also carries over to the general setting described in (Regensburger and Rosenkranz, 2007). Thus we have $G = (1 - P)\, T^\blacklozenge$, where P is the projector onto $\mathrm{Ker}(T)$ along $\mathcal{B}$ and $T^\blacklozenge$ the fundamental right inverse of T. (In fact, we could take any right inverse of T, but $T^\blacklozenge$ is a canonical choice.)

Setting $\mathcal{B} = [\varphi_1, \ldots, \varphi_n]$ and $\mathrm{Ker}(T) = [f_1, \ldots, f_n]$ with the (φ_i) biorthogonal to the (f_i), the projector P is characterized by Equation (10), as explained there; in particular, it is then clear that $P \in \mathcal{F}[\mathtt{E}] \subseteq \mathcal{F}[\partial, \textstyle\int]$. As noted in Fact 18, we have also $T^\blacklozenge \in \mathcal{F}[\partial, \textstyle\int]$. Hence we can conclude that $G = (1 - P)\, T^\blacklozenge \in \mathcal{F}[\partial, \textstyle\int]$, as claimed. □

We observe that the above proof is *essentially constructive*: Given a "reasonable" subring of $\mathcal{F}[\partial, \textstyle\int]$, the computation of P is pure linear algebra while that of $T^\blacklozenge$ amounts to solving a linear homogeneous linear differential equation. (A "reasonable subring" should be a computable ring with a canonical ground simplifier such that every boundary problem of interest is expressible. One example is furnished by the analytic polynomials of

Example 7; see (Rosenkranz, 2005, 2003) for the corresponding algorithm for computing Green's operators.)

In fact, the *factorization method* described in Section 7 provides an alternative approach for computing Green's operators. The crucial point is that multiplying boundary problems corresponds to composing their Green's operators in reverse order (see Proposition 24). In the case of differential operators with constant coefficients, one can in principle express any Green's operator as a product of first-order Green's operators (which have a straight-forward formula in terms of the corresponding solutions).

6. Multiplying Boundary Problems

Every integro-differential operator $E \in \mathcal{F}[\partial, \textstyle\int]$ acts on $\mathfrak{B}$ via its adjoint E^* and thus leads to the right action

$$\mathcal{B} \cdot E = \{\beta \circ E \mid \beta \in \mathcal{B}\} = E^*(\mathcal{B});$$

if $\mathcal{B}$ is generated by n (not necessarily linearly independent) boundary conditions $\beta_1, \dots, \beta_n$, this gives

$$[\beta_1, \dots, \beta_n] \cdot E = [\beta_1 \circ E, \dots, \beta_n \circ E].$$

Considering $\mathcal{F}[\partial, \textstyle\int]$ as a multiplicative monoid, this gives a contravariant *monoid representation* $\Phi\colon \mathcal{F}[\partial, \textstyle\int] \to \mathrm{Hom}^*(\mathfrak{B})$ with $\Phi_E(\mathcal{B}) = \mathcal{B} \cdot E$ and $\mathrm{Hom}^*(\dots)$ the monoid of anti-homomorphisms. Using differential operators $T \in \mathcal{F}_0[\partial]$ even yields monomorphisms

$$\begin{aligned} \Phi_T\colon \quad & \mathfrak{B} \to \mathfrak{B} \\ & \mathcal{B} \mapsto \mathcal{B} \cdot T \end{aligned} \tag{13}$$

because each T has a right inverse like $T^\blacklozenge$. Moreover, a basis $\beta_1, \dots, \beta_n$ of $\mathcal{B}$ is transported into a basis $\beta_1 \circ T, \dots, \beta_n \circ T$ of $\mathcal{B} \cdot T$. Writing $\mathrm{Mono}^*(\dots)$ for the monoid of anti-monomorphisms, this means we have a contravariant monoid representation $\Phi\colon \mathcal{F}_0[\partial] \to \mathrm{Mono}^*(\mathfrak{B})$ that restricts to $\Phi\colon \mathfrak{D}_n \to \mathrm{Mono}^*(\mathfrak{B}_n)$.

A *semi-direct product* may be defined for monoids just as for groups (Cohn, 1982, p. 277); the resulting structure is clearly again a monoid. In the case of monoids, one has to distinguish semi-direct products and reverse semi-direct products (Eilenberg, 1974); see also (Regensburger and Rosenkranz, 2007) for more details. In our case, we construct the reverse semi-direct product $\mathfrak{D} \ltimes_\Phi \mathfrak{B} = (\mathfrak{D} \times \mathfrak{B}, \cdot_\Phi)$ with multiplication

$$(\tilde{T}, \tilde{\mathcal{B}}) \cdot_\Phi (T, \mathcal{B}) = (\tilde{T}T, \tilde{\mathcal{B}} \cdot T + \mathcal{B}) = (\tilde{T}T, \Phi_T(\tilde{\mathcal{B}}) + \mathcal{B}) \tag{14}$$

As it turns out, the boundary problems form a submonoid of $\mathfrak{D} \ltimes_\Phi \mathfrak{B}$ since they are closed under this product, henceforth just written as $\cdot$.

Lemma 22. *The set $\mathfrak{P} \subseteq \mathfrak{D} \times \mathfrak{B}$ is closed under the multiplication* (14).

Proof. The neutral element of $\mathfrak{P}$ is given by the degenerate boundary problem $(1, O)$, as one checks immediately: While $1 \in \mathfrak{D}_0$ and $O \in \mathfrak{B}_0$, the regularity condition is fulfilled because $\mathrm{Ker}(1) = O$ and $\mathrm{Ker}(O) = \mathcal{F}$. (Written out in the classical notation, this is the following "problem": Given $f \in \mathcal{F}$, find $u \in \mathcal{F}$ such that $u = f$ without further boundary conditions.)

For the remaining proof, we refer to (Regensburger and Rosenkranz, 2007), where the construction of the monoid of boundary problems is carried out in a more general abstract setting; moreover, it follows also from Proposition 24. The intuition behind the proof is clear: In the composite boundary problem (14), the differentiated right-hand boundary conditions $\mathcal{B} \cdot \tilde{T}$ are linearly independent of the left-hand ones since the former are of a higher order. $\square$

Example 23. Let us carry out a *simple multiplication* in the monoid $(\mathfrak{P}, \cdot)$, working with the analytic polynomials of Example 7 over the ground algebra $\mathcal{F} = C^\infty[0, 1]$. We claim that

$$(D, [F]) \cdot (D, [L]) = (D^2, [L, R]). \tag{15}$$

Indeed, we have $[F] \cdot D = [FD] = [AD + BD] = [(1 - L) + (-1 + R)] = [R - L]$ and $[F] \cdot D + [L] = [L, R]$, so (15) follows. Written in classical notation, we have multiplied the boundary problems

$$\boxed{\begin{array}{l} u' = f \\ \int_0^1 u(\xi)\, d\xi = 0 \end{array}} \cdot \boxed{\begin{array}{l} u' = f \\ u(0) = 0 \end{array}} = \boxed{\begin{array}{l} u'' = f \\ u(0) = u(1) = 0 \end{array}}.$$

We see at this point that global conditions are necessary for the converse process: If we want to factor the boundary problem (see Section 7) on the right-hand side, we cannot have two-point boundary conditions in the left-hand factor.

As mentioned after Theorem 21, we can compute Green's operators from the constituent Green's operators in a factorization, and in Section 7 we will present a method for producing such factorizations from a given factorization of the underlying differential operator. But of course this presupposes that the product of boundary problems *corresponds to the composition* of their Green's operators in reverse order. In fact, this is why the composition of boundary problems was defined in the way it is. For the precise statement, let us write $\mathfrak{G}$ for the monoid generated by all Green's operators for boundary problems in $\mathfrak{P}$.

Proposition 24. *The transformation $(T, \mathcal{B}) \mapsto (T, \mathcal{B})^{-1}$ is an anti-isomorphism from the monoid $\mathfrak{P}$ to the monoid $\mathfrak{G}$. In other words, every Green's operator corresponds to exactly one boundary problem, and we have*

$$(\mathcal{P}_1 \mathcal{P}_2)^{-1} = \mathcal{P}_2^{-1} \mathcal{P}_1^{-1}$$

for all $\mathcal{P}_1, \mathcal{P}_2 \in \mathfrak{P}$.

Proof. By the definition of the monoid structure of $\mathfrak{P}$, we have

$$(T_1, \mathcal{B}_1)(T_2, \mathcal{B}_2) = (T_1 T_2, \mathcal{B}_1 \cdot T_2 + \mathcal{B}_2).$$

We must show that the Green's operator of the right-hand boundary problem is given by $G_2 G_1$, if we set $G_1 = (T_1, \mathcal{B}_1)^{-1}$ and $G_2 = (T_2, \mathcal{B}_2)^{-1}$. Clearly we have

$$TG = (T_1 T_2)(G_2 G_1) = T_1 (T_2 G_2) G_1 = T_1 G_1 = 1;$$

let us now show $\operatorname{Im}(G_2 G_1) = (\mathcal{B}_1 \cdot T_2 + \mathcal{B}_2)^\perp$. Consider first $u = G_2 G_1 f$. We have $\beta(u) = 0$ for all $\beta \in \mathcal{B}_2$ since $\operatorname{Im}(G_2) = \mathcal{B}_2^\perp$ and $\beta(T_2 u) = \beta(G_1 f) = 0$ for all $\beta \in \mathcal{B}_1$ since $\operatorname{Im}(G_1) = \mathcal{B}_1^\perp$, so $u \in (\mathcal{B}_1 \cdot T_2 + \mathcal{B}_2)^\perp$ as claimed. Now assume conversely $u \in \mathcal{B}_1 \cdot T_2^\perp$ as well as $u \in \mathcal{B}_2^\perp$. The latter condition means $u = G_2 v$ for some v, while the former condition implies $v \in \mathcal{B}_1^\perp$ and hence $v = G_1 f$ for some f. Hence we have indeed $u = G_2 G_1 f$.

Now for the uniqueness of the Green's operators. Consider two boundary problems $(T, \mathcal{B}), (\bar{T}, \bar{\mathcal{B}}) \in \mathfrak{P}$ with the same Green's operator G. Then we obtain from $TG = 1$ and $\bar{T}G = 1$ that $(T - \bar{T})G = 0$, so $T - \bar{T}$ vanishes on the infinite-dimensional space $\mathrm{Im}(G) \leq \mathcal{F}$. Assume now $T \neq \bar{T}$ for a contradiction. Then $T - \bar{T}$ is a nonzero differential operator over a saturated coefficient algebra $\mathcal{F}_0$, so it has a finite-dimensional kernel and hence cannot vanish on all of $\mathrm{Im}(G)$. Hence we have indeed $T = \bar{T}$. Finally, we have also $\mathcal{B}^\perp = \mathrm{Im}(G) = \bar{\mathcal{B}}^\perp$ and hence $\mathcal{B} = \bar{\mathcal{B}}$. $\square$

7. Factoring Boundary Problems

In this section we will study how to split boundary problems into smaller ones. It turns out that every factorization of a differential operator can be "lifted" to the level of boundary (Theorem 28).

Definition 25. A boundary problem $(T_2, \mathcal{B}_2) \in \mathfrak{P}$ is called a right factor of a boundary problem $(T, \mathcal{B}) \in \mathfrak{P}$ if T_2 is a right factor of T and $\mathcal{B}_2$ a subspace of $\mathcal{B}$.

Proposition 26. *Let $(T, \mathcal{B}) \in \mathfrak{P}$ be a boundary problem and $T = T_1 T_2$ a factorization of its differential operator. Then $(T, \mathcal{B})$ has a right factor $(T_2, \mathcal{B}_2) \in \mathfrak{P}$.*

Proof. Set $\deg T_1 = m$ and $\deg T_2 = n$. Choose a basis

$$u_1, \ldots, u_m, u_{m+1}, \ldots, u_{m+n} \in \mathcal{F}$$

of $\mathrm{Ker}(T)$ such that $u_1, \ldots, u_m$ is a basis of $\mathrm{Ker}(T_2)$, and choose any basis

$$\beta_1, \ldots, \beta_{m+n} \in \mathrm{Cond}(\mathcal{F}^*)$$

of $\mathcal{B}$. Since $(T, \mathcal{B})$ is a regular problem, the matrix

$$B = \begin{pmatrix} \beta_1(u_1) & \ldots & \beta_1(u_m) & \beta_1(u_{m+1}) & \ldots & \beta_1(u_{m+n}) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \beta_{m+n}(u_1) & \ldots & \beta_{m+n}(u_m) & \beta_{m+n}(u_{m+1}) & \ldots & \beta_{m+n}(u_{m+n}) \end{pmatrix}$$

is regular. Hence we may use row operations to annul all entries below the upper left $(m \times m)$ block. (We might also annul the entries below the diagonal or even reduce the whole matrix B to row echelon form, but this is more than we need at this point.) These operations are realized by left-multiplying B with a suitable matrix $P \in \mathrm{GL}(K, m+n)$ such that the upper left is transformed into a regular matrix

$$B_2 = \begin{pmatrix} \tilde{\beta}_1(u_1) & \ldots & \tilde{\beta}_1(u_m) \\ \vdots & \ddots & \vdots \\ \tilde{\beta}_m(u_1) & \ldots & \tilde{\beta}_m(u_m) \end{pmatrix}$$

with new upper boundary conditions

$$\tilde{\beta}_i = \sum_{j=1}^{m+n} P_{ij} \beta_j \qquad (i = 1, \ldots, m).$$

But the regularity of B_2 means that $(T_2, \mathcal{B}_2) \in \mathfrak{P}$ if we define $\mathcal{B}_2 = [\tilde{\beta}_1, \dots, \tilde{\beta}_m]$, while $\mathcal{B}_2$ is obviously a subspace of $\mathcal{B}$. □

A refined analysis of Proposition 26 (see (Regensburger and Rosenkranz, 2007) for the detailed statement and proof in an abstract setting) leads to a full *classification of all the right factors* $(T_2, \mathcal{B}_2) \in \mathfrak{P}$ of a given boundary problem $(T, \mathcal{B}) \in \mathfrak{P}$. The bottom line is that every right factor $(T_2, \mathcal{B}_2)$ corresponds to a direct summand $\mathcal{L}_2 = \mathcal{B}_2^\perp \cap \operatorname{Ker}(T)$ of $\operatorname{Ker}(T_2)$ in $\operatorname{Ker}(T)$, while every such $\mathcal{L}_2$ gives rise to a right factor $(T_2, \mathcal{B}_2)$ with $\mathcal{B}_2 = \mathcal{B} \cap \mathcal{L}_2^\perp$ such that one obtains a bijection between right factors of $(T, \mathcal{B})$ and direct summands of $\operatorname{Ker}(T_2)$ in $\operatorname{Ker}(T)$. Moreover, $(T_2, \mathcal{B}_2)$ is regular iff $\mathcal{B}_2 \dotplus \mathcal{B} \cap \operatorname{Ker}(T_2)^\perp = \mathcal{B}$.

When referring to $\mathcal{P}_2 = (T_2, \mathcal{B}_2)$ as a right factor of $\mathcal{P} = (T, \mathcal{B})$, we are actually anticipating that there is also a *left factor* $\mathcal{P}_1 = (T_1, \mathcal{B}_1)$ such that their product yields $\mathcal{P}$. This is indeed the case, as we will see in Proposition 27. But what is immediately clear is that if $\mathcal{P}_1$ exists, it is uniquely determined by $\mathcal{P}$ alone. Indeed, we know from Proposition 24 that $G = G_2 G_1$, where G, G_1, G_2 denote the Green's operators respectively of $\mathcal{P}$, $\mathcal{P}_1$, $\mathcal{P}_2$. But this implies $G_1 = T_2 G$ and hence $\mathcal{B}_1 = \operatorname{Im}(T_2 G)^\perp$.

Apart from the existence question, the disturbing feature of this representation is that it presuppose knowledge of the Green's operator G and thus defeats the plan of exploiting a factorization of the given boundary problem for determining its Green's operator from that of its factors. The next proposition remedies this flaw: it turns out that all we need is an *arbitrary right inverse* H_2 of the differential operator T_2. Now we could take $H_2 = G_2$, but this would still need the computation of a Green's operator (albeit of a smaller size). A more reasonable choice is of course $H_2 = T_2^\blacklozenge$, thus reducing the task of computing Green's operators to solving initial value problems. (The fundamental right inverse is a canonical choice here, but there are still many other possibilities. In specific settings, it may be algorithmically advantageous to choose other right inverses of T_2.)

Proposition 27. *Given $(T, \mathcal{B}) \in \mathfrak{P}$ with $T = T_1 T_2$, there is a unique $(T_1, \mathcal{B}_1) \in \mathfrak{P}$ such that every right factor $(T_2, \mathcal{B}_2) \in \mathfrak{P}$ of $(T, \mathcal{B})$ satisfies $(T, \mathcal{B}) = (T_1, \mathcal{B}_1) \cdot (T_2, \mathcal{B}_2)$ and such that its space of boundary conditions is given by*

$$\mathcal{B}_1 = (\operatorname{Ker}(T_2)^\perp \cap \mathcal{B}) \cdot H_2,$$

where H_2 is an arbitrary right inverse of T_2. Hence $\mathcal{B}_1 = \mathcal{B} \cdot G_2$ if G_2 is the Green's operator of any right factor $(T_2, \mathcal{B}_2)$.

Proof. Set $m = \deg T_1$ and $n = \deg T_2$. We have already seen that if $(T_1, \mathcal{B}_1)$ exists, it is unique with $\mathcal{B}_1 = \operatorname{Im}(T_2 G)^\perp$. Since $T_2 G$ is a right inverse of T_1, we have also $\operatorname{Ker}(T_1) \dotplus \operatorname{Im}(T_2 G) = \mathcal{F}$. But this means $(T_1, \mathcal{B}_1) \in \mathfrak{P}$ if only we can ensure that $\mathcal{B}_1$ has a basis of Stieltjes conditions. And this follows immediately once we have proved

$$\operatorname{Im}(T_2 G)^\perp = (\operatorname{Ker}(T_2)^\perp \cap \mathcal{B}) \cdot H_2 \tag{16}$$

since when $\mathcal{B}$ is generated by Stieltjes conditions, its intersection with $\operatorname{Ker}(T_2)^\perp$ is generated by certain linear combinations of them, while right-multiplication by H_2 still yields Stieltjes conditions because $\operatorname{Cond}(\mathcal{F}^*)$ was defined as a right ideal in $\mathcal{F}[\partial, \textstyle\int]$.

Using (16), we obtain also $(T, \mathcal{B}) = (T_1, \mathcal{B}_1) \cdot (T_2, \mathcal{B}_2)$. For that, it suffices to ensure that

$$(\operatorname{Ker}(T_2)^\perp \cap \mathcal{B}) \cdot H_2 T_2 = \operatorname{Ker}(T_2)^\perp \cap \mathcal{B} \tag{17}$$

since the regularity of $(T_2, \mathcal{B}_2)$ is equivalent to $\mathcal{B}_2 \dotplus \mathcal{B} \cap \operatorname{Ker}(T_2)^\perp = \mathcal{B}$ as mentioned after Proposition 26. For proving (17), we apply the stronger result $\beta \mapsto \beta \circ H_2 T_2$ leaves $\operatorname{Ker}(T_2)^\perp \cap \mathcal{B}$ pointwise invariant, which follows from the fact that $1 - H_2 T_2$ is a projector onto $\operatorname{Ker}(T_2)$.

It remains to prove (16). First assume $\beta(T_2 G u) = 0$ for all $u \in \mathcal{F}$. Then we have $\beta = \tilde{\beta} \circ H_2$ if we set $\tilde{\beta} = \beta \circ T_2$, and it suffices to show $\tilde{\beta} \in \operatorname{Ker}(T_2)^\perp$ and $\tilde{\beta} \in \mathcal{B} = \operatorname{Im}(G)^\perp$. But the former is immediate from the definition of $\tilde{\beta}$ while the latter follows since $\tilde{\beta}(Gu) = \beta(T_2 G u) = 0$ by hypothesis. Conversely, let us now assume $\tilde{\beta} \in \operatorname{Ker}(T_2)^\perp \cap \mathcal{B}$ and show $\tilde{\beta} \circ H_2 \in \operatorname{Im}(T_2 G)^\perp$. Indeed, we have

$$(\tilde{\beta} \circ H_2)(T_2 G u) = \tilde{\beta}(H_2 T_2 G u) = \tilde{\beta}\big(Gu\big) - \tilde{\beta}\big((1 - H_2 T_2)\, Gu\big) = 0,$$

because the left term in the sum vanishes by the hypothesis $\tilde{\beta} \in \mathcal{B} = \operatorname{Im}(G)^\perp$ and the right term by the hypothesis $\tilde{\beta} \in \operatorname{Ker}(T_2)^\perp$ and the fact that $1 - H_2 T_2$ is a projector onto $\operatorname{Ker}(T_2)$.

The constructive method for computing $\mathcal{B}_1 = (\operatorname{Ker}(T_2)^\perp \cap \mathcal{B}) \cdot H_2$ is the same as in the proof of Proposition 26. Using the row-operation matrix $P \in \mathrm{GL}(K, m+n)$ constructed there (the original version creating zeroes only in the lower left block), we compute the new lower boundary conditions

$$\tilde{\beta}_i = \sum_{j=1}^{m+n} P_{ij} \beta_j \qquad (i = m+1, \ldots, m+n)$$

to obtain a basis $\tilde{\beta}_{m+1} \circ H_2, \ldots, \tilde{\beta}_{m+n} \circ H_2$ of $\mathcal{B}_1$. $\square$

Putting together Proposition 26 and Proposition 27, we have now established the following *Factorization Theorem for Boundary Problems*, our main result in this section.

Theorem 28. *Given a problem $(T, \mathcal{B}) \in \mathfrak{P}$, every factorization $T = T_1 T_2$ of the differential operator can be lifted to a factorization $(T, \mathcal{B}) = (T_1, \mathcal{B}_1) \cdot (T_2, \mathcal{B}_2)$ of the boundary problem with $(T_1, \mathcal{B}_1), (T_2, \mathcal{B}_2) \in \mathfrak{P}$ and $\mathcal{B}_2 \leq \mathcal{B}$.*

We conclude this section with an example of a fourth-order boundary problem arising in mechanics; see (Kamke, 1967, p. 525).

Example 29. In its traditional formulation, this it the boundary problem $\mathcal{P}$ specified by

$$\boxed{\begin{aligned} &u'''' + 4u = f,\\ &u(0) = u(1) = u'(0) = u'(1) = 0. \end{aligned}}$$

Using the language of analytic polynomials (see Example 7), this means that

$$\mathcal{P} = (D^4 + 4, [L, R, LD, RD]),$$

and we see immediately that here one can employ the natural factorization $D^4 + 4 = (D^2 - 2i)(D^2 + 2i)$. Using the basis functions $u_{\pm\pm} = e^{\pm 1 \pm i}$ for the kernel of $D^4 + 4$, we choose the boundary conditions for the right factor $D^2 + 2i$ in such a way that its Green's operator G_1 has a convenient formulation (this is not necessary in principle but keeps expressions shorter). By the generic second-order formula from (Stakgold, 1979, p. 195), also derived

	u_{--}	u_{-+}	u_{+-}	u_{++}
a_{--}	$(1+i)(e^2-e^{2i})$	$2i(1-e^2)$	$2(e^{2i}-1)$	$(1-i)(2-e^2-e^{2i})$
b_{--}	$(1+i)(e^2-e^{2i})$	$2i(1-e^2)$	$2(e^{2i}-1)$	$(1-i)(e^{-2}+e^{-2i}-2)$
a_{++}	$(1-i)(e^{-2}+e^{-2i}-2)$	$2(1-e^{-2i})$	$2i(e^{-2}-1)$	$(1+i)(e^{-2i}-e^{-2})$
b_{++}	$(1-i)(2-e^{2i}-e^2)$	$2(1-e^{-2i})$	$2i(e^{-2}-1)$	$(1+i)(e^{-2i}-e^{-2})$

Table 2. Coefficients for G_2

in (Rosenkranz, 2005, p. 196), we are led to $\mathcal{P}_1 = (D^2+2i, [(i-1)\,L-LD, (1-i)\,R-RD])$ or

$$\boxed{\begin{aligned} &u''+2i\,u=f,\\ &(i-1)\,u(0)-u'(0)=(1-i)\,u(1)-u'(1)=0\end{aligned}}$$

in traditional formulation.

Boundary problem $\mathcal{P}_1$ can now be solved easily by the generic second-order formula. Alternatively, one could also apply the machinery from Theorem 21 or a factorization into first-order problems as explained at the end of Section 5. In any case, one arrives at the Green's operator

$$G_1=\frac{1+i}{4}\big(\lceil u_{+-}\rceil A\,\lceil u_{-+}\rceil+\lceil u_{-+}\rceil B\,\lceil u_{+-}\rceil\big),$$

acting on a function $f\in C^\infty[0,1]$ according to

$$G_1f(x)=\frac{1+i}{4}\left(\int_0^x e^{(1-i)(x-\xi)}f(\xi)\,d\xi+\int_x^1 e^{(i-1)(x-\xi)}f(\xi)\,d\xi\right).$$

We can now use the Green's operator G_1 of boundary problem $\mathcal{P}_1$ for describing the boundary conditions of the (unique!) left factor $\mathcal{P}_2$ in the factorization $\mathcal{P}=\mathcal{P}_2\mathcal{P}_1$. One may easily verify that

$$\mathcal{P}_2=(D^2-2i,[F\lceil u_{+-}\rceil,F\lceil u_{-+}\rceil])$$

or

$$\boxed{\begin{aligned} &u''-2i\,u=f\\ &\textstyle\int_0^1 e^{(1-i)\xi}\,f(\xi)\,d\xi=\int_0^1 e^{(i-1)\xi}\,f(\xi)\,d\xi=0\end{aligned}} \tag{18}$$

in traditional formulation. Since this is not a two-point boundary problem, let us go through its solution in some detail. According to Theorem 21, we must first compute the projector P onto $\operatorname{Ker}(D^2-2i)$ characterized by the fact that $\operatorname{Im}(1-P)$ is the space of boundary conditions of (18). In other words, we have to find a basis $(\hat{u}_{+-},\hat{u}_{-+})$ of $\operatorname{Ker}(D^2-2i)$ that is biorthogonal to $(F\lceil u_{+-}\rceil, F\lceil u_{-+}\rceil)$; then the projector is given by $P=\lceil\hat{u}_{+-}\rceil\,F\,\lceil u_{+-}\rceil+\lceil\hat{u}_{-+}\rceil\,F\,\lceil u_{-+}\rceil$. Carrying out the computation (which involves inverting a 2×2 matrix and four definite integrals) leads to

$$\begin{aligned}\hat{u}_{+-}&=\frac{(e^2-1)\,u_{--}-(e^{-2i}-1)i\,u_{++}}{\Delta},\\ \hat{u}_{-+}&=\frac{(e^{2i}-1)i\,u_{--}-(e^{-2}-1)\,u_{++}}{\Delta},\end{aligned}$$

	u_{--}	u_{-+}	u_{+-}	u_{++}
a_{--}	$i(e^{2i}-e^2)$	$(1-i)(1-e^2)$	$(1+i)(1-e^{2i})$	$e^2+e^{2i}-2$
b_{--}	$i(e^{2i}-e^2)$	$(1-i)(1-e^2)$	$(1+i)(1-e^{2i})$	$2-e^{-2}-e^{-2i}$
a_{-+}	$(1-i)(1-e^2)$	e^2-e^{-2i}	$i(2-e^2-e^{-2i})$	$(1+i)(e^{-2i}-1)$
b_{-+}	$(1-i)(1-e^2)$	e^2-e^{-2i}	$i(e^{-2}+e^{2i}-2)$	$(1+i)(e^{-2i}-1)$
a_{+-}	$(1+i)(1-e^{2i})$	$i(e^{-2}+e^{2i}-2)$	$e^{2i}-e^{-2}$	$(1-i)(e^{-2}-1)$
b_{+-}	$(1+i)(1-e^{2i})$	$i(2-e^2-e^{-2i})$	$e^{2i}-e^{-2}$	$(1-i)(e^{-2}-1)$
a_{++}	$2-e^{-2}-e^{-2i}$	$(1+i)(e^{-2i}-1)$	$(1-i)(e^{-2}-1)$	$i(e^{-2}-e^{-2i})$
b_{++}	$e^2+e^{2i}-2$	$(1+i)(e^{-2i}-1)$	$(1-i)(e^{-2}-1)$	$i(e^{-2}-e^{-2i})$

Table 3. Coefficients for G

where $\Delta = \cos 2 + \cosh 2 - 2$. The next step is to determine the fundamental right inverse of $D^2 - 2i$. A straightforward computation yields

$$H_2 = \frac{i-1}{4}\Big(\lceil u_{--}\rceil\, A\, \lceil u_{++}\rceil - \lceil u_{++}\rceil\, A\, \lceil u_{--}\rceil\Big).$$

Now we can compute the Green's operator of boundary problem $\mathcal{P}_2$ as $G_2 = (1-P)H_2$. Using the normalization engine for analytic polynomials decribed in (Rosenkranz, 2005), we arrive at

$$G_2 = \frac{1}{8\Delta}\Big(\lceil u_{--}\rceil\, A\, \lceil a_{--}\rceil + \lceil u_{--}\rceil\, B\, \lceil b_{--}\rceil \\ + \lceil u_{++}\rceil\, A\, \lceil a_{++}\rceil + \lceil u_{++}\rceil\, B\, \lceil b_{++}\rceil\Big),$$

where each of $a_{--}, b_{--}, a_{++}, b_{++}$ is a linear combination of the four functions $u_{--}, u_{-+}, u_{+-}, u_{++}$ according to Table 2.

According to Proposition 24, the Green's operator G of the full boundary problem $\mathcal{P}$ is given by G_1G_2. Its explicit form, obtained by noncommutative multiplication and subsequent normalization, is given here for reference; often one will actually prefer the factored representation in terms of G_1 and G_2. We have

$$G = \frac{1+i}{32\Delta}\Big(\lceil u_{--}\rceil\, A\, \lceil a_{--}\rceil + \ldots + \lceil u_{++}\rceil\, B\, \lceil b_{++}\rceil,$$

similar to G_2 in structure, but now with four additional summands coming from u_{-+} and u_{+-}. The eight functions $a_{--}, \ldots, a_{++}$ are again linear combinations of the type before, with the coefficients given in Table 3.

8. Conclusion

Factoring a differential equation reduces the order in the resulting problems and thus aids in solving the given equation. Since differential equations usually come together with boundary conditions, they must be incorporated in an additional step (typically viewed as external to differential algebra). The theory presented in this paper extends the technique of factorization for linear ordinary differential equations in such a way that the boundary conditions become an integral part, leading to an algorithmic machinery for *factoring and solving* (not necessarily self-adjoint) *boundary problems* over integro-differential algebras. The implementation of these algorithms will be described in a subsequent paper.

Let us now discuss some possibilities of *extending* our approach into various directions: partial differential equations, linear systems of ordinary differential equations, difference equations, polynomial boundary conditions, semilinear boundary problems, dual pairings and duality theory, analytical aspects, and localization.

In this paper, we have restricted ourselves to ordinary differential equations (and thus to ordinary integro-differential algebras in the sense of Definition 6). This is convenient since–relative to given fundamental systems—it allows us to compute Green's operators in closed form. But the concept of multiplying (and hence factoring) boundary problems, as defined in (14), may be transferred to a much *more general setting*, see (Regensburger and Rosenkranz, 2007).

It can in particular be applied to *linear partial differential equations*, where one can exploit suitable results about factoring linear partial differential operators (Grigoriev and Schwarz, 2007, 2005, 2004; Tsarev, 1998). As a first example, we have factored the one-dimensional inhomogeneous wave equation on a bounded interval into two first-order "boundary problems." This example, stated in (Regensburger and Rosenkranz, 2007) in some detail, will be treated in a future paper, in which we plan to investigate partial differential equations. Part of the work will be the development of symbolic algorithms for first-order partial differential equations (typical factor problems!) in non-trivial geometries. Since factorization will typically end up with (symbolically) irreducible boundary problems, it becomes more important to address stability issues here: Well-posed boundary problems should be factored into well-posed blocks, if possible (Engl et al., 1996).

Going into a different direction, one can also apply our methodology of multiplying and factoring boundary problems to *systems* of linear ordinary differential equations. We expect that the solution theory (now using "Green's matrices" instead of Green's functions) as well as the algorithms will essentially carry over to this setting.

Everything considered in this paper was directed towards the continuous case of linear differential equations, but we expect the discrete case of linear *difference equations* to be tractable in principle by the same methods, except for the well-known complications arising from a skew Leibniz rule and a Baxter axiom with weight unity instead of zero; see Example 1.6 in (Guo, 2002). As pointed out in Section 2, the concept of integro-differential algebras generalizes naturally to this situation (Guo and Keigher, 2007).

By contrast, the restriction to linear differential equations seems to be quite rigid: we do not see how to translate our ideas to nonlinear differential equations. What could be considered, though, is the case of linear differential equations with *polynomial boundary conditions*, a case that is also of interest in applications. (A classical example is given by the heat equation with radiation on the boundary, described by the Stefan-Boltzmann law: The normal derivative of the temperature is proportional to its fourth power.) Although the solution operator of such a problem is necessarily nonlinear, we hope that one can adapt some of our ideas by handling the boundary conditions through ideals instead of linear subspaces.

In this article, we have worked with the (algebraic) dual of the vector space structure of the underlying differential algebra. We think that our approach could in principle be transferred to a setting where a *dual pairing* is given instead of the canonical bilinear form; this would include important topological vector spaces like C^k and L^p. Of course, this requires a modification of the composition structure, leading to a category rather than a monoid of boundary problems as described in (Regensburger and Rosenkranz, 2007). The advantage might be that one gains topological assertions relating various operators (like the differential and the Green's operators) and spaces (like images and kernels).

Speaking of duality, one should also mention that the usual *duality theory* of linear boundary problems (Coddington and Levinson, 1955, Chapter 11) can be transferred to "classical" Stieltjes boundary conditions (on real or complex-valued functions); see for example (Brown, 1975). The idea is that every boundary problem should have a dual or "adjoint" problem whose solution operator is the "transpose" of the original problem. The adjoint problem is often useful for characterizing certain aspects of a given primal problem (e.g. solvability by the Fredholm alternative).

We have not yet exploited the factorized representation of Green's operators for *characterizing Green's functions* (possibly restricted to the well-posed case for avoiding distributions). This may be done from two different perspectives: From an algebraic viewpoint, one might proceed in a manner similar to the Galois theory of linear ordinary differential equations; from an analytic viewpoint, the singular value decomposition would be of interest.

Finally, we should also mention that we have also treated singular boundary problems, where one needs a modified Green's function/operator as in the example from Section 3.5 in (Rosenkranz, 2005). This leads to a *localization* in the ring of Green's operators—differential operators appear as the "reciprocals" of suitable integral operators. In this manner, one obtains a noncommutative generalization of the Mikusiński calculus that allows a symbolic treatment of boundary problems just like the ordinary Mikusiński calculus does for initial value problems (Mikusiński, 1959). These ideas will be discussed in a future paper.

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