Elasto-Plastic Analysis with an Adaptive FEM-BEM Coupling Method

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Abstract

The purpose of this paper is to present an adaptive finite element-boundary element coupling method for elasto-plastic analysis. The finite element method is utilized in regions where plastic material behavior is expected to develop, whereas substantial parts of the bounded/unbounded linear elastic body are approximated using the boundary element method. In order to obtain a computationally efficient coupling method, considerable attention is devoted to the generation and adaption of the finite element and boundary element discretizations, according to the state of computation. The proposed method is computationally efficient since it employs smaller finite element regions. Moreover, the method is practically advantageous. Unlike available coupling approaches, the adaptive finite element-boundary element coupling method does not demand predefinition and manual localization of the finite element and boundary element sub-domains.

Keywords: FEM; BEM; Adaptive Coupling; Elasto-Plasticity

1. Introduction

The finite element method (FEM) and the boundary element method (BEM) are eminent computational techniques for obtaining approximate solutions to the partial differential equations that evolve in scientific and engineering applications. Each method has its own range of applications where it is most efficient and neither is superlative for all applications.

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The BEM reduces the dimension and simplifies the modeling considerably because of its boundary only approach. It is efficient, accurate and relatively easy to use in treatment of bounded/unbounded domains with linear elastic material behavior. The approximation of singularities can be handled easily by the BEM. However, the BEM is not the preferable approach when inhomogeneities and non-linearities in the analysis domains exist. To tackle such problems, the BEM requires special formulations, which do not make the method fully general. The FEM is usually the method of choice in dealing with problems involving inhomogeneities and non-linearities in bounded domains. For these categories of problems, the FEM is a more robust and mature technology. Thus, if the problem of interest includes local inhomogeneities and/or non-linearities only in a portion of the bounded/unbounded domain, the concept of solving it in an adjacent sub-domains, employing the most suited solution technique for each of them, is, by all means, appropriate. In this way, we are lead to the coupling of FEM and BEM (FEM-BEM coupling). Examples include the detailed analysis of stresses in the surroundings of pressure holes in an unbounded domain. The FEM can be employed to capture the plastic behavior at the vicinity of the opening. The remaining unbounded linear elastic region may be best represented by the BEM. The same is true for many problems in solid mechanics.

Among the first authors who started to combine the FEM and BEM are Zienkiewicz, Kelly and Bettes [1, 2], signifying a "*Marriage a la mode - the best of both worlds*". Since then a large number of papers devoted to the topic have appeared, see, e.g. references [3-22], not to mention many others.

The symmetric coupling of BEM and FEM goes back to Costabel [3,4]. With the Symmetric Galerkin BEM, FEM-like stiffness matrices can be produced which are suitable for FEM-BEM coupling, see, e.g. references [3,4,5,7,12,13,14,20]. Iterative substructuring solvers for symmetric coupled boundary and finite element equations have been developed by Langer [7], Haase et al. [10], Hsiao et al. [23], and Steinbach [24] for elliptic boundary value problems in bounded and unbounded, two and three-dimensional domains. Parallel implementations showed high performance on several platforms [10]. Langer and Steinbach [25] introduced the boundary element tearing and interconnecting (BETI) methods as boundary element counterparts of the well-established finite element tearing and Interconnecting (FETI) methods, see e.g., references [26,27] . Langer and Steinbach [15] introduced the coupled finite and boundary element tearing and interconnecting methods (FETI/BETI) as a logical continuance of the BETI technique.

Brink et al. [8] investigated a coupling of mixed finite elements and Galerkin boundary elements in linear elasticity, taking into account adaptive mesh refinement based on a posteriori error estimators. Carstensen et al. [9] presented an h-adaptive FEM-BEM coupling algorithm (mesh refinement of the boundary elements and the finite elements) for the solution of viscoplastic and elasto-plastic interface problems. Mund and Stephan [11] derived a posteriori error estimate for nonlinear-coupled FEM-BEM equations by using hierarchical basis techniques. They presented an algorithm for adaptive error control which allows independent refinements of the finite elements and the boundary elements.

In boundary element analysis, Astrinidis et al. [28] presented adaptive discretization schemes that are based on a stress smoothing error criterion in the case of two-dimensional elastic analysis, and on a total strain smoothing error criterion in the case of two-dimensional elasto-plasticity. Rebeiro et al. [29] developed a pure BEM procedure to automatically generate the internal cells to compute domain integrals in the plastic region. The discretization of the internal cells progressively generated only in the zones where plasticity occurs. Maischak and Stephan [30] showed convergence for the boundary element approximation, obtained by the hp-version, for elastic contact problems, and derive a-posteriori error estimates together with error indicators for adaptive hp-algorithms.

Available coupling approaches demand manual localization of the FEM and BEM sub-domains. The FEM and BEM sub-domains are defined a priori and remain unchanged during the computation. Inevitability could do with preliminary expert knowledge about the problem at hand. Besides, a predefined FEM sub-domain may result in whichever under/overestimation of the nonlinear region where the FEM is employed. In the former case, inaccurate solutions is obtained to the problem at hand while for the later the computational cost is higher than necessary.

This paper presents an adaptive FEM-BEM coupling method for solving problems in elasto-plasticity. Materials of von-Mises type are considered for this study. The method facilitates an automatic generation of the FEM mesh to cover regions where plasticity occurs. In order to obtain an initial estimate of the regions sensible for FEM discretization, the adaptive coupling method follows a linear hypothetical elastic computation. Energetic methods [31-34] are then utilized to account for relaxation and redistribution of stresses that occur due to plastic deformation. A final estimate of the regions sensible for FEM discretization is then derived. The FEM mesh is automatically generated over the estimated regions. Consequently, the BEM mesh is generated to best represent the remaining linear elastic region. In order to ensure a compatible coupling between the BEM and FEM, the interface is constructed reflecting the current state of computation. A coupled FEM-BEM stress analysis involving elasto-plastic deformations is then conducted.

The remainder of this paper is organized as follows. Section 2 shows and briefly summarizes the symmetric Galerkin BEM in linear elasticity and FEM in elasto-plasticity. Next the adopted conventional (direct) FEM-BEM coupling equations are described. In the sequence, Section 3 presents the proposed adaptive FEM-BEM coupling method for elasto-plastic analysis. In Section 4, we elaborate more on the initial and final estimates of the regions sensible for FEM discretization. Finally, two numerical examples that highlight the potentialities of the adaptive FEM-BEM coupling method are presented in Section 5.

2. Problem Formulation and Basic Equations

2.1 Symmetric Galerkin BEM in linear elasticity

The Galerkin boundary element method for the symmetric formulation of boundary integral equations is an efficient and powerful tool to solve boundary value problems in linear elasticity see, e.g. references [35-41], not to mention many others.

Let $\Omega \subset \mathbb{R}^n\ (n = 2, 3)$ be a bounded domain with a Lipschitz boundary $\Gamma = \partial\Omega$. We consider a mixed boundary value problem in linear elasticity, to determine the displacement field $u(x)$ for $x \subset \Omega$,

$$\begin{aligned} &\sigma_{ij,j}(u) + f_i(x) = 0 && \text{for } x \in \Omega, \\ &u_i(x) = g_i(x) && \text{for } x \in \Gamma_D, \\ &t_i(x) = \sigma_{ij}(u)\, n_j(x) = h_i(x) && \text{for } x \in \Gamma_N. \end{aligned} \tag{1}$$

The stress tensor $\sigma_{ij}(u)$ is related to the strain tensor $\varepsilon_{kl}(u)$ by Hooke's law $\sigma_{ij}(u) = C_{ijkl}\varepsilon_{kl}(u)$. For isotropic elastostatics and assuming a homogeneous material behavior with constant parameters (Young modulus E and Poisson ratio ν), the system of boundary integral equations may be written as follows

$$\begin{pmatrix} \underline{u} \\ \underline{t} \end{pmatrix} = \begin{pmatrix} \frac{1}{2}I - K & V \\ D & \frac{1}{2}I + K' \end{pmatrix} \begin{pmatrix} \underline{u} \\ \underline{t} \end{pmatrix} \tag{2}$$

where V, K, K' and D denote the single layer potential, double layer potential, its adjoint and hypersingular integral operators, respectively, defined by:

$$
\begin{aligned}
(V\underline{t})(x) &= \int_{\Gamma} U^{*}(x,y)\,\underline{t}\,(y)ds_y, \\
(K\underline{u})(x) &= \int_{\Gamma} T^{*}(x,y)\,\underline{u}(y)ds_y, \\
(K'\underline{t})(x) &= \int_{\Gamma} T_x U^{*}(x,y)\,\underline{t}(y)ds_y, \\
(D\underline{u})(x) &= -T_x \int_{\Gamma} T^{*}(x,y)\,\underline{u}(y)ds_y.
\end{aligned}
\tag{3}
$$

The definition of all the boundary potential is based on a fundamental solution which is given by the Kelvin solution

$$
U^{*}_{ij}(x,y) = \frac{1}{4(n-1)}\frac{1}{E}\frac{1+\nu}{1-\nu}\left[(3-4\nu)Z(x,y) + \frac{(x_i - y_i)(x_j - y_j)}{|x-y|^n}\right]
\tag{4}
$$

for $i,j = 1,....,n$ where $Z(x,y) = -log|x-y|$ for $n=2$ and $Z(x,y) = \dfrac{1}{|x-y|}$ for $n=3$.

The i-th boundary stress vector component is given by the operator $(T_x \underline{u}(x))_i = \sigma_{ij}(\underline{u},x)n_j(x)$ and $T^{*}(x,y) = T_y U^{*}(x,y)$.

In order to find the complete Cauchy data $[u,t]_\Gamma$, the first integral equation for $x \in \Gamma_D$ and the second one for $x \in \Gamma_N$ are rewritten as [35,37,41]

$$
(V\underline{t})_i(x) = \frac{1}{2}g_i(x) + (K\underline{u})_i(x) \text{ for } x \in \Gamma_{D,i}
\tag{5}
$$

$$
(D\underline{u})_i(x) = \frac{1}{2}h_i(x) - (K'\underline{t})_i(x) \text{ for } x \in \Gamma_{N,i}
\tag{6}
$$

The standard Galerkin discretization of (5) and (6) yield the skew symmetric and positive definite system of linear equations

$$
\begin{pmatrix} V_h & -K_h \\ K_h^T & D_h \end{pmatrix}\begin{pmatrix} \underline{t}^h \\ \underline{u}^h \end{pmatrix} = \begin{pmatrix} \underline{f}_1 \\ \underline{f}_2 \end{pmatrix},
\tag{7}
$$

where the block matrices in (7) result from discretization of the corresponding parts of the boundary.

In typical applications in linear elastostatics, the Dirichlet part Γ_D is often smaller than the Neumann part Γ_N where the boundary tractions are prescribed. Therefore, the inverse of the discrete single layer potential V_h may be computed using some direct method such as a Cholesky decomposition to obtain

$$\underline{t}^h = V_h^{-1}[\underline{f}_1 + K_h \underline{u}^h]\cdot \tag{8}$$

Inserting (8) into the second of (7) yields the Schur complement system

$$[D_h + K_h^T V_h^{-1} K_h]\,\underline{u}^h = \underline{f}_2 - K_h^T V_h^{-1}\underline{f}_1\cdot \tag{9}$$

The Schur complement system (9) is symmetric and positive definite and is suitable for coupling with FEM. System (9) may be rewritten as

$$[{}_B K][{}_B \underline{u}] = [{}_B \underline{f}]. \tag{10}$$

where the subscript B stand for the BEM sub-domain.

2.2 FEM in linear elasticity

Consider the solid, in which the internal stresses σ, the distributed loads/unit volume f and the external applied tractions h form an equilibrating field, to undergo an arbitrary virtual displacement δu which results in compatible strains $\delta\varepsilon$ and internal displacements δd. Then the principle of virtual work requires that [42]

$$\int_\Omega \delta\varepsilon^T \sigma\, d\Omega - \int_\Omega \delta d^T f\, d\Omega - \int_{\Gamma_N} \delta d^T h\, d\Gamma = 0. \tag{11}$$

The normal finite element discretizing procedure leads to the following expressions for the displacements and strains within any element

$$\delta d = N\delta u, \qquad \delta\varepsilon = B\delta u, \tag{12}$$

where N and B are the usual matrix of shape functions and the elastic strain-displacement matrix, respectively. The element assembly process gives

$$\int_\Omega \delta u^T (B^T\sigma - N^T f)\, d\Omega - \int_{\Gamma_N} \delta u^T N^T h\, d\Gamma = 0, \tag{13}$$

where the volume integration over the solid is the sum of the individual element contributions. Since (13) must hold true for any arbitrary δu then

$$\int_\Omega B^T \sigma\, d\Omega - (\int_\Omega N^T f\, d\Omega + \int_{\Gamma_N} N^T h\, d\Gamma) = 0. \tag{14}$$

Substituting for $\sigma = C\varepsilon$, using Equation (14), yields

$$Ku = (\int_\Omega N^T f\, d\Omega + \int_{\Gamma_N} N^T h\, d\Gamma), \tag{15}$$

where the stiffness matrix is given by $K = \int_{\Omega} B^{T} C\, B\, d\Omega$. The final system of the assembled finite element equations in elasticity may now be written as

$$[{}_F K][{}_F \underline{u}] = [{}_F \underline{f}], \tag{16}$$

where the subscript F stand for the FEM sub-domain.

2.3 Coupled FEM-BEM in linear elasticity

For a numerical representation of an arbitrary domain Ω with known boundary conditions specified at the entire boundary $\Gamma = \Gamma_N \cup \Gamma_D$ the FEM and BEM are used. The domain is decomposed into two sub-domains, namely, ${}_F\Omega$ and ${}_B\Omega$ with the FEM-BEM coupling interface Γ_C.

The stiffness matrix ${}_B K$ can be interpreted as the element stiffness matrix of a finite macro element, computed by the BEM. Combining Equations (10) and (16) while satisfying the continuity conditions along the FEM-BEM interface results in

$$\begin{bmatrix} {}_F K_{FF} & {}_F K_{FC} & \\ {}_F K_{CF} & {}_F K_{CC} + {}_B K_{CC} & {}_B K_{CB} \\ & {}_B K_{BC} & {}_B K_{BB} \end{bmatrix} \begin{bmatrix} {}_F \underline{u}_F \\ \underline{u}_C \\ {}_B \underline{u}_B \end{bmatrix} = \begin{bmatrix} {}_F \underline{f}_F \\ {}_F \underline{f}_C + {}_B \underline{f}_C \\ {}_B \underline{f}_B \end{bmatrix}, \tag{17}$$

where the subscripts $(\,)_F$ and $(\,)_B$ indicate the displacement vectors (force vectors) not associated with the FEM and BEM sub-domains interface, respectively. Subscript $(\,)_C$ indicates those associated with the interface Γ_C. The matrix presented in (17) is symmetric and positive definite.

2.4 FEM in elasto-plasticity

Some force terms in (14) may be a function of displacement, u, or stress may be a nonlinear function of strain, ε, as a result of material non-linearity such as plasticity. In all of these cases, a nonlinear solution is required. Equation (14) will not be generally satisfied at any stage of computation, and thus the equilibrium equation can be restated in the form of a residual (or out-of-balance) force vector, ψ, given by (see references [42,43] for further details on computational aspects)

$$\psi = \int_{\Omega} B^T \sigma\, d\Omega - \left(\int_{\Omega} N^T f\, d\Omega + \int_{\Gamma_N} N^T h\, d\Gamma\right) \neq 0. \tag{18}$$

If a material nonlinear only analysis is performed, the integrals in the above equation are computed with respect to the initial configuration (Lagrangian FEM formulation).

For an elasto-plastic situation the material stiffness is continuously varying, and instantaneously the incremental stress-strain relationship is given by

$$d\sigma = D_{ep}\, d\varepsilon, \tag{19}$$

where D_{ep} is the elasto-plastic stress-strain matrix.

Solution procedure involves the incremental form of (18), namely

$$\Delta\psi = \int_{\Omega} B^T \Delta\sigma\, d\Omega - \left(\int_{\Omega} N^T \Delta f\, d\Omega + \int_{\Gamma_N} N^T \Delta h\, d\Gamma\right). \tag{20}$$

Substituting for $\Delta\sigma$, using (19), results in

$$\Delta\psi = K_T \Delta u - \left(\int_{\Omega} N^T \Delta f\, d\Omega + \int_{\Gamma_N} N^T \Delta h\, d\Gamma\right). \tag{21}$$

where the tangent stiffness matrix is given by $K_T = \int_{\Omega} B^T D_{ep} B\, d\Omega$.

Equation (21) may now be written as

$$\Delta\psi = K_T \Delta u - \Delta f\ . \tag{22}$$

For the solution of (22), and for each load increment, the incremental nodal displacements and stresses are calculated. The updated stresses are then brought down to the yield surface and are used to calculate the equivalent nodal forces. These nodal forces can be compared with the externally applied loads to form a system of residual forces, which is brought sufficiently close to zero through an iterative process, before moving to the next load increment.

2.5 Coupled FEM-BEM in elasto-plasticity

In this paper, the symmetric Galerkin boundary element method as summarized in Section 2.1 is utilized for the coupling of the FEM and BEM. Elasto-plastic problems with limited spread of plastic strains lend themselves to a coupled approach, where the FEM is utilized in regions where plastic material behavior is expected to develop, whereas the complementary bounded/unbounded linear elastic region is approximated using the symmetric Galerkin BEM. Combining Equations (10) and (22) while satisfying the continuity conditions along the FEM-BEM interface, results in

$$\begin{bmatrix} \Delta_F \underline{\psi}_F \\ \Delta\underline{\psi}_C \\ \Delta_B \underline{\psi}_B \end{bmatrix} = \begin{bmatrix} {}_F K_{T\,FF} & {}_F K_{T\,FC} & \\ {}_F K_{T\,CF} & {}_F K_{T\,CC} + {}_B K_{CC} & {}_B K_{CB} \\ & {}_B K_{BC} & {}_B K_{BB} \end{bmatrix} \begin{bmatrix} \Delta_F \underline{u}_F \\ \Delta\underline{u}_C \\ \Delta_B \underline{u}_B \end{bmatrix} - \begin{bmatrix} \Delta_F \underline{f}_F \\ \Delta_B \underline{f}_C + \Delta_F \underline{f}_C \\ \Delta_B \underline{f}_B \end{bmatrix}. \tag{23}$$

For each load increment, the global equation systems (23) are solved.

3. Adaptive FEM-BEM Coupling Method

As pointed out previously, it is not useful to predefine the FEM and BEM sub-domains in an elasto-plastic FEM-BEM coupling analysis. Predefinition of the FEM and BEM sub-domains, will result in whichever under/overestimation of the regions where plastic material behavior is going to develop (regions where the FEM is employed). In order to obtain a computationally efficient coupling method (to avoid inaccurate or costly computations), we propose in this section an adaptive FEM-BEM coupling method that automatically generate and progressively adapt the finite and boundary element sub-domains. The section discusses the preliminaries of our coupling method, general features of the adaptive concept and the progressive adaption of the FEM and BEM sub-domains.

The basic steps of implementation of our adaptive concept in elasto-plastic FEM-BEM coupling analysis are summarized as follows (Figure 1)

1. Load increment and BEM elastic analysis with an initial BEM discretization

   A hypothetical elastic stress state is determined.

2. Detection of regions sensible for FEM discretization (Figure 2)

   In order to obtain an initial estimate of the regions sensible for FEM discretization, the adaptive coupling method follows the linear elastic computation of step 1. Violation to the yield condition provides an initial estimate of the regions sensible for discretization by FEM. Simple fast post-calculations based on energetic methods are then utilized to account for relaxation and redistribution of stresses that occur due to plastic deformation. A final estimate of the regions sensible for FEM discretization is then derived.

3. Automatic generation of FEM discretization (consequently the BEM sub-domain discretization) for the current state of computation

   Particular regions that fulfill the proposed criterion are discretized by the FEM. Consequently, the BEM mesh is generated so as to best represent the remaining bounded/unbounded linear elastic region. In order to ensure the compatible coupling between the BEM and FEM sub-domains, the interface is constructed reflecting the current situation

   It may be useful to reuse the BEM internal points as finite element nodes for the FEM discretization, as they are conveniently distributed in the particular area of interest. This will result in a reduction of the complexity of data management and ease of the automatic generation and adaption of the FEM sub-domain.

4. Coupled FEM-BEM stress analysis involving elasto-plastic deformations is then conducted
5. Next load increment requires a repetition of steps 1-4.

In our adaptive method, the user needs not to predefine the FEM and BEM sub-domains. In Section 4, we elaborate more on the initial and final estimates of the regions where plastic material behavior is expected to develop (regions where the FEM is employed).

4. Estimate of regions sensible for FEM discretization

The adaptive FEM-BEM coupling method given in Section 3 follows two criteria for indicating regions to be discretized by the FEM. The first criterion is violation to yield condition (elastic prediction). It gives an initial estimate of regions sensible for discretization by the FEM. A linear elastic analysis is conducted with an initial BEM discretization (Figure 2). Hypothetical stress values are computed at predefined points inside the BEM sub-domain. If the hypothetical linear stress state violates the yielding condition at a particular region, it is estimated (initially) as sensible for discretization by the FEM.

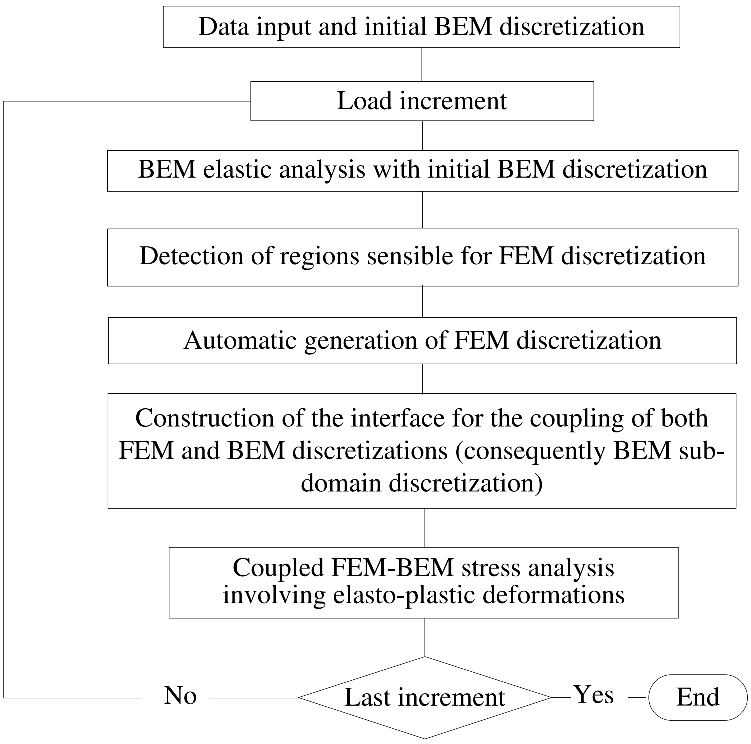

Figure 1: Adaptive FEM-BEM coupling method for elasto-plastic analysis.

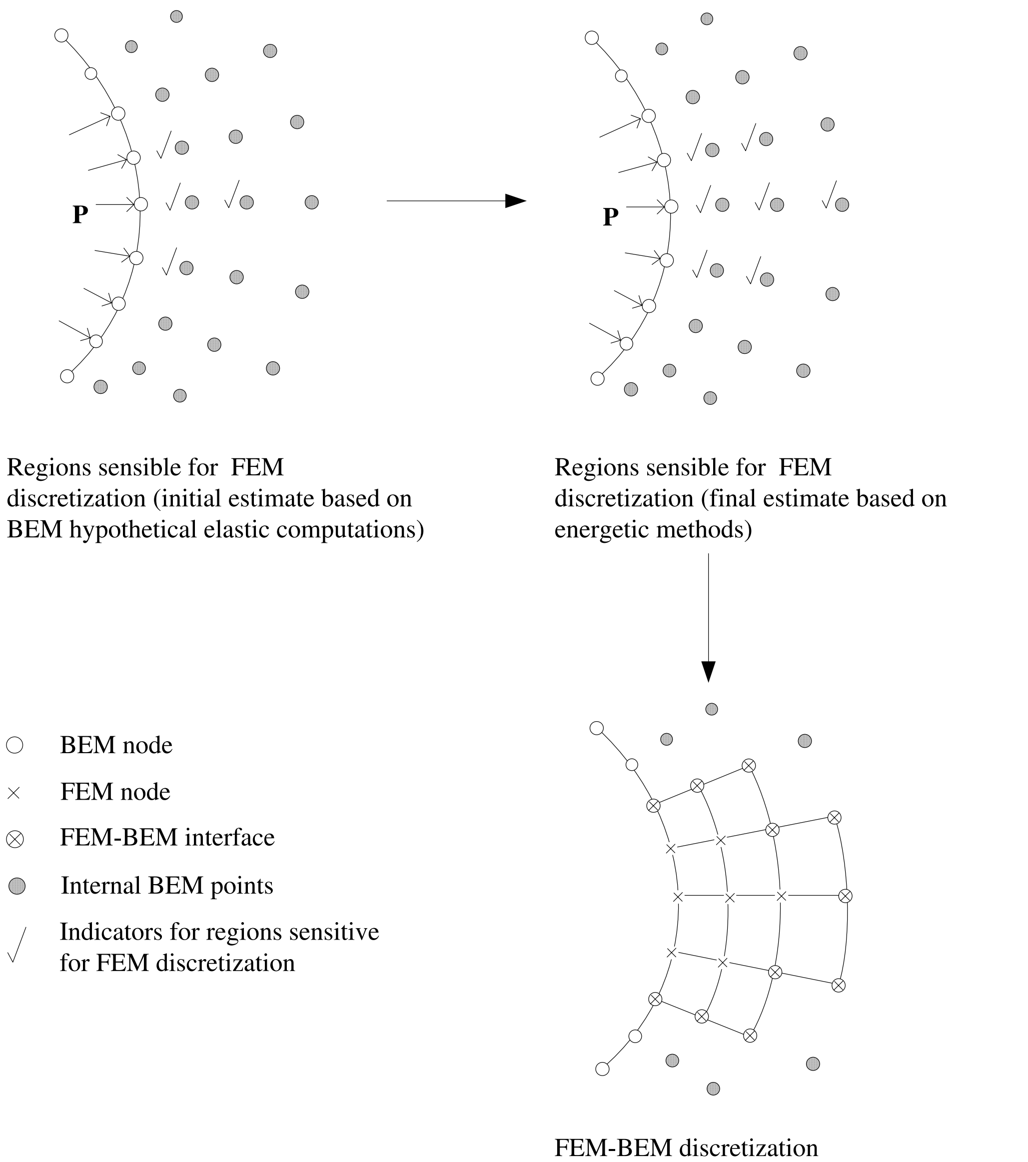

Figure 2: Initial and final estimates of the FEM region and FEM-BEM discretization.

The second criterion carries out simple fast post-calculations based on energetic methods, e.g., Neuber's and strain energy density methods. This will account for relaxation and redistribution of stresses that occur due to plastic deformation. It gives a final estimate of the FEM sub-domains.

The simplicity of linear elastic analysis and the difficulties associated with non-linear elasto-plastic analysis have motivated some researchers to attempt solving elasto-plastic problems by adapting a modified form of available elastic solutions [31-34]. There are variants of this approach with names such as: generalized local stress strain [44], iterative elastic [45], pseudo-elastic [31] and elastic compensation [45]. The idea is not new. It is mainly utilized in analysis of notches and in limit load analysis for design considerations. Iterative linear elastic analyses are conducted with updated material properties at each iteration termed as “effective material properties”. The schemes for updating the effective material properties include: projection, arc length, and energy methods. It should be mentioned, however, that we are not interested here in carrying out an iterative elastic analysis. Our aim is to determine the zones that are sensible for FEM discretization.

Let us consider materials of von-Mises type obeying a bilinear strain hardening rule. Neuber’s and strain energy density methods (Figure. 3) are energy equivalence between the hypothetical elastic and the elasto-plastic calculations of the same geometry submitted to the same loading [31-34]. For uni-dimensional states of stress, it is assumed that the product stress x strain in elasticity is locally identical to the same product calculated by means of an elasto-plastic analysis.

For tri-dimensional states of stress, the fundamental hypothesis may be written as [31-34]

$$(\sigma_{ij}\varepsilon_{ij})_{\text{elasto-plastic}} = (\sigma_{ij}\varepsilon_{ij})_{\text{hyp elastic}} \tag{24}$$

where $(.)_{\text{hyp elastic}}$ corresponds to hypothetical elastic state of stresses.

The energy density balance, Equation (24), is obtained by using the defined quantities appropriately for the actual elasto-plastic stress-strain state and the hypothetical elastic stress-strain state. However, a local method leads to a violation of equilibrium. Thus a proportionality factor is to be introduced in order to account for the stress relaxation and redistribution due to plastic deformations.

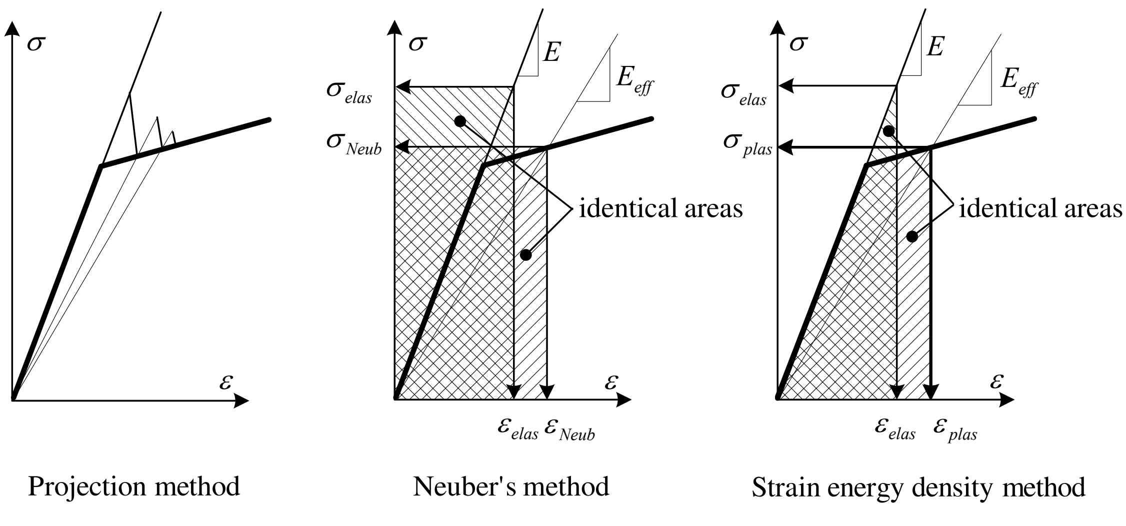

Figure 3: Projection, Neuber’s and strain energy density methods.

In this study, we propose a simple, at the same time fast and effective, method for a final estimate of regions sensible for FEM discretization.

From a virtual work principle we may utilize a global formulation

$$(\int_{\Omega} \sigma_{ij} \varepsilon_{ij}^{*} dV)_{\text{elasto-plastic}} \cong (\int_{\Omega} \sigma_{ij} \varepsilon_{ij}^{*} dV)_{\text{hyp elastic}} . \tag{25}$$

The total hypothetical elastic strain energy may be written as

$$U_{\text{hyp elastic}} = \int_{\Omega} (\sigma_{ij} \varepsilon_{ij})_{\text{hyp elastic}} dV . \tag{26}$$

Next, we define the total strain energy that is vulnerable for redistribution due to plastic deformations, U_{dist}, as

$$U_{\text{dist}} = \int_{\Omega} ((\sigma_{ij} \varepsilon_{ij})_{\text{hyp elastic}} - \sigma_y \varepsilon_y)_x \kappa_x dV , \tag{27}$$

where $\kappa_x = 1$ if $((\sigma_{ij} \varepsilon_{ij})_{\text{hyp elastic}} - \sigma_y \varepsilon_y)_x > 0$, otherwise $\kappa_x = 0$ and σ_y is the uniaxial yield strength.

Subsequently, we define a "pseudo" value of the material yield strength, $\sigma_{y\ \text{pseudo}}$. This pseudo value is evaluated as follows (see Figure 4)

$$\frac{U_{\text{dist}}}{U_{\text{hyp elastic}}} = \frac{c(\sigma_y - \sigma_{y\ \text{pseudo}})}{\sigma_y} , \tag{28}$$

where c is a constant that depends on the geometry of the stress-strain curve (Figure 4). Finally, the hypothetical elastic state of stresses is checked against yielding with the pseudo value of the yield strength. Regions that violate the modified yielding condition are determined. A final estimate of the FEM sub-domain is obtained. It should be noted that the pseudo value of yield strength is only utilized for the purpose of obtaining the regions sensible for FEM discretization.

The basic steps of proposed post-calculations are summarized as

1. compute the total hypothetical elastic strain energy, $U_{\text{hyp elastic}}$, Equation (26)
2. evaluate the strain energy that is vulnerable for redistribution, U_{dist}, Equation (27)
3. determine a pseudo value of the yielding strength, $\sigma_{y\ \text{pseudo}}$, Equation (28)
4. check for regions that violate the pseudo yielding condition. A final estimate of the FEM sub-domain is obtained.

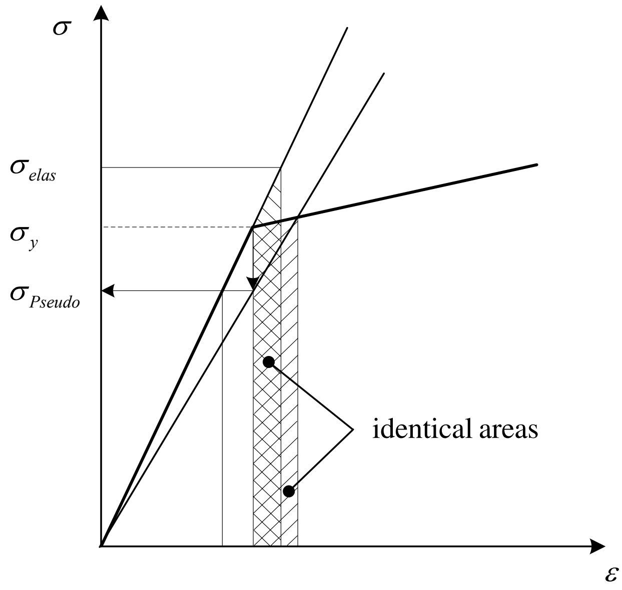

Figure 4: Pseudo material yield strength.

The procedure outlined with its inherent assumptions, provides a simple, at the same time fast and effective, method for a final estimate of the FEM and BEM sub-domains. A usual FEM-BEM coupling analysis is then conducted (see Section 2.5) while utilizing the finally estimated FEM and BEM regions.

5. Numerical Examples

In this section we present two numerical examples that highlight the potentialities of the adaptive FEM-BEM coupling method presented in Sections 3 and 4.

The first numerical example (Figure 5) serves as a benchmark problem in computational plasticity [46]. The benchmark problem is a stretched steel plate (width=height=200 mm) with a circular central hole (radius r=10 mm) under plane strain condition. A surface load P is applied to the plate's upper and lower edges. The applied tractions $\mathrm{P} = 100\ \mathrm{N/mm}^2$ are scaled with the load factor λ. The elastic material properties of the plate are described by Young's modulus ($E = 206.9\ \mathrm{GPa}$) and Poisson's ratio ($\nu = 0.29$). Material of von-Mises type is considered ($\sigma_y = 450\ \ \mathrm{MPa}$), with no hardening effect ($H = 0.$), as a yield function and plane strain loading conditions. Due to symmetry, only one quarter of the problem is modeled.

The problem is solved by means of the adaptive coupling method presented in Sections 3 and 4. The loads are applied incrementally. Figure 6 shows the initial (elastic prediction) and final (post calculations based on energetic methods) estimates of the regions sensible for discretization by the FEM. Figure 6 further shows the yielded regions obtained using the adaptive coupled FEM-BEM method for the selected values of λ. It should be noted that the coupled FEM-BEM solutions are obtained via an automatically generated FEM and BEM discretization for the particular values of λ. The FEM discretization is generated over the regions that are finally estimated as sensible for FEM discretization, while the BEM mesh is generated to represent the remaining linear elastic region (Figure 6). The results clearly show that the adaptive FEM-BEM coupled method employs smaller FEM sub-domains. Moreover, the method is practically advantageous as it does not necessitate a predefined and manually localized FEM and BEM sub-domains.

In the second example we consider an unbounded plate with a rectangular hole (width=height=2) under uniform pressure, $\mathrm{P}=100$. The applied uniform pressure is scaled with the load factor λ which is assumed to be as high as 10. We assume plane strain loading conditions with the elastic material parameters $E = 206.9 \text{ x } 10^3$ and $\nu = 0.29$. Material of von-Mises type is considered ($\sigma_y = 450$), with no hardening effect. Figure 7 shows selected calculation results ($\lambda = 4, 6, 8 \text{ and } 10$). Figure 7 gives the initial and final estimates of the zones sensible for FEM discretization, and the yielded regions obtained using the adaptive FEM-BEM coupling method. The results clearly show the advantages of the adaptive coupled FEM-BEM models in terms of efficiency.

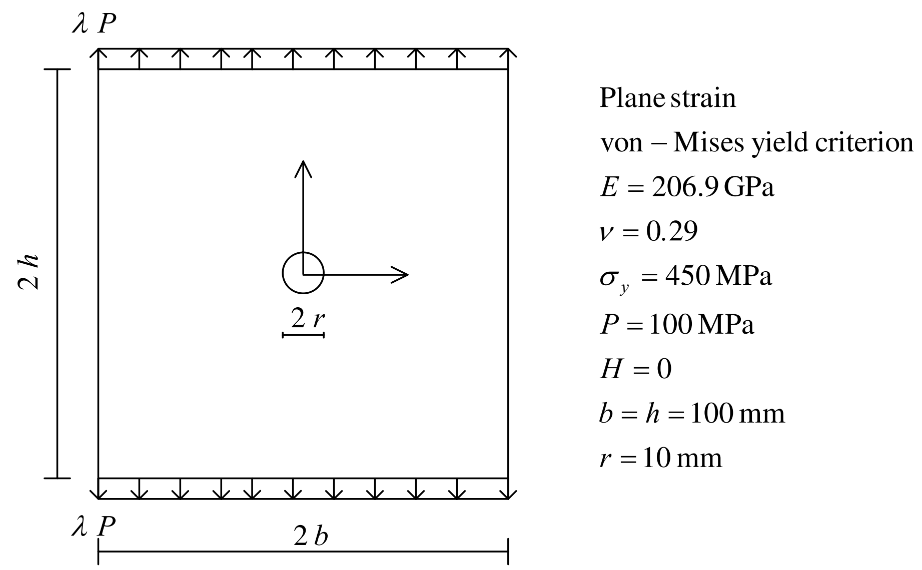

Figure 5: Stretched steel plate with a circular hole in plane strain state.

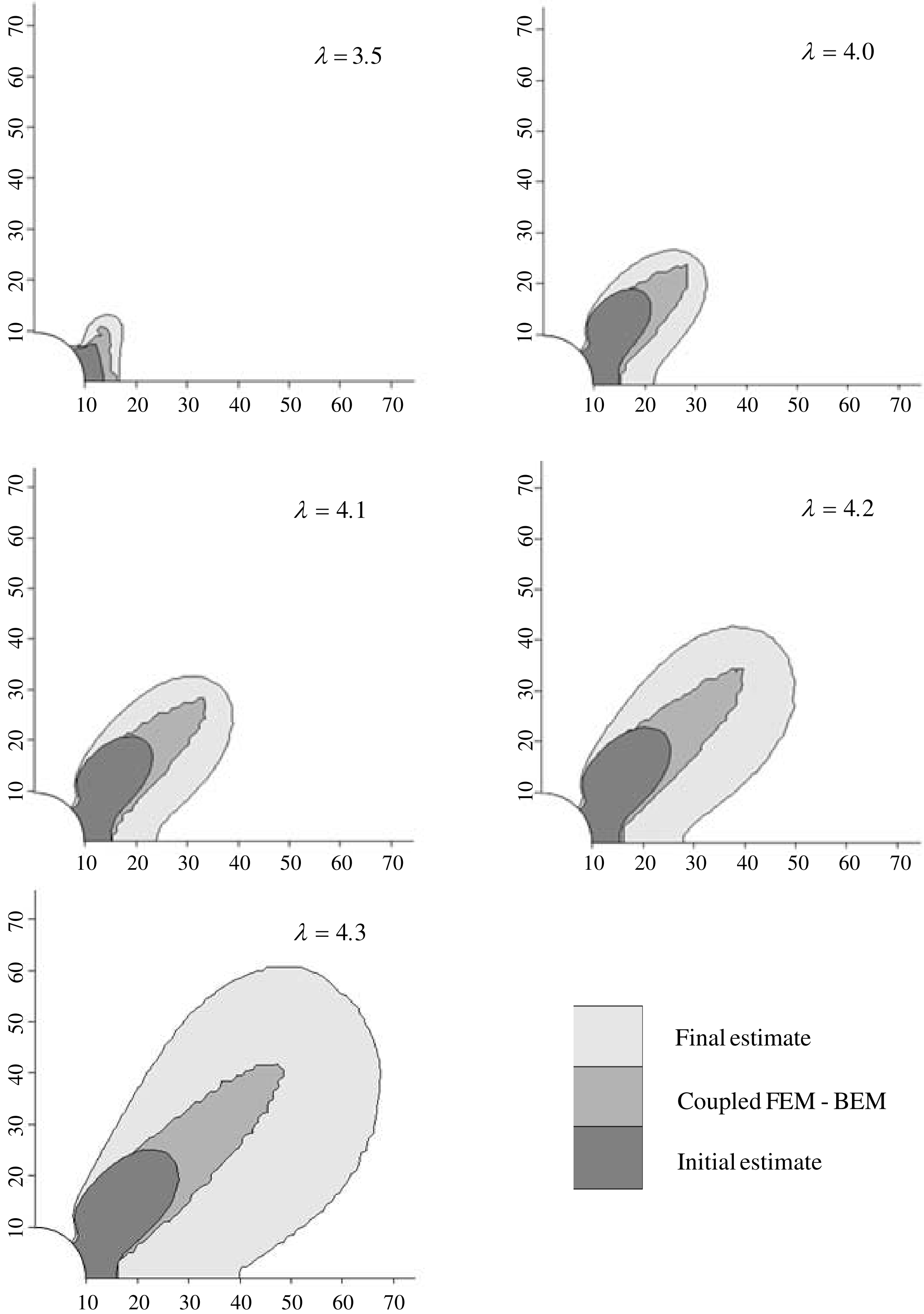

Figure 6: Initial and final estimates of FEM sub-domains and yielded regions (adaptive FEM-BEM coupling method) for selected values of the load factor, λ.

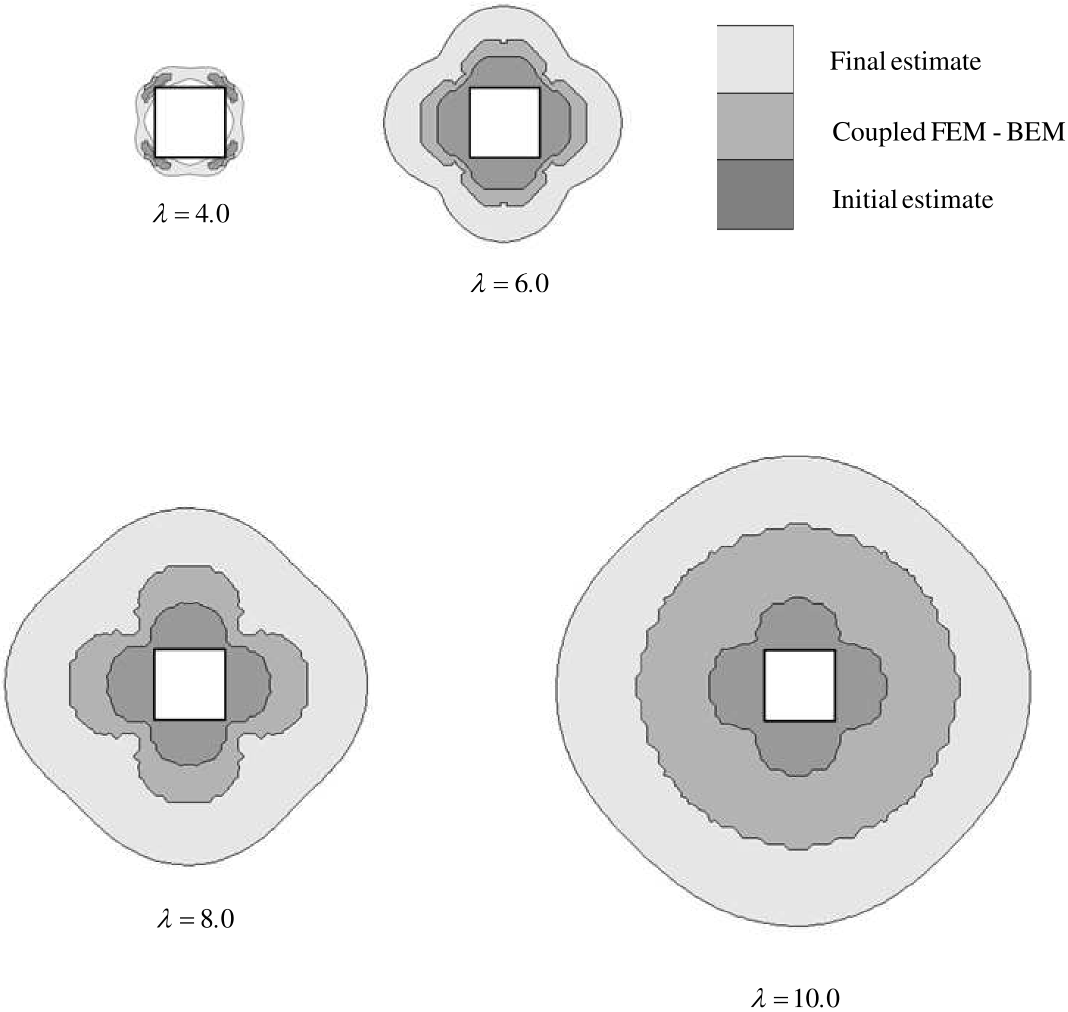

Figure 7: Initial and final estimates of FEM sub-domains and computed results via an adaptive FEM-BEM coupling method (Example 2).

Conclusions

This paper deals with FEM-BEM coupling for elasto-plastic analysis. The paper proposes the use of simple, and at the same time fast, post-calculations, based on energetic methods which follow a simple hypothetical elastic boundary element computation, in order to give fast and helpful estimation of the FEM and BEM sub-domains. The FEM mesh is automatically generated over the estimated regions. Consequently, the BEM mesh is generated so as to best represent the remaining linear elastic region. Furthermore, the FEM and BEM discretization are progressively adapted according to the state of computation. The present adaptive coupling method is practically advantageous as it does not necessitate predefinition and manual localization of the FEM and BEM sub-domains. Moreover, the method is computationally efficient as it substantially decreases the size of FEM meshes, which plainly leads to reduction of required system resources and gain in efficiency. The numerical results confirm the effectiveness of the proposed method.

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