A Refined Difference Field Theory for Symbolic Summation

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Abstract

In this article we present a refined summation theory based on Karr's difference field approach. The resulting algorithms find sum representations with optimal nested depth. For instance, the algorithms have been applied successively to evaluate Feynman integrals from Perturbative Quantum Field Theory.

Key words: Symbolic summation, difference fields, nested depth

1. Introduction

Over the past few years rapid progress has been made in the field of symbolic summation. The beginning was made by Gosper's telescoping algorithm [9] for hypergeometric terms and Zeilberger's extension of it to creative telescoping [38]. An algebraic clarification of Gosper's setting has been carried out by Paule [19]. Meanwhile various important variations or generalizations have been developed, like for q–hypergeometric terms [20], the mixed case [3], or the ∂-finite case [7].

In particular, Karr's telescoping algorithm [11, 12] based on his theory of difference fields provides a fundamental general framework for symbolic summation. His algorithm is, in a sense, the summation counterpart to Risch's algorithm [23, 24] for indefinite integration. Karr introduced the so-called $\Pi\Sigma$-extensions, in which parameterized first order linear difference equations can be solved in full generality; see below. As a consequence, Karr's algorithm cannot only deal with telescoping and creative telescoping over (q-)hypergeometric terms, but also over rational terms consisting of arbitrarily nested sums and products. More generally, it turned out that also parameterized linear difference equations can be solved in such difference fields [31]. This enables to solve recurrence relations with coefficients in terms of indefinite nested sums and products; it also gives rise to algorithms for rather general classes, for instance, holonomic sequences [29].

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An important general aspect of using difference field methods is the following: In order to exploit the full power of the algorithmic machinery, it is necessary to find for a summand, given in terms of indefinite nested sums and products, a "good" representation in a suitable $\Pi\Sigma^*$-field extension $\mathbb{E}$ of $\mathbb{F}$. Based on the results of [11] Karr comes to the following somehow misleading conclusion [11, p. 349]:

> *Loosely speaking, if f is summable in $\mathbb{E}$, then part of it is summable in $\mathbb{F}$, and the rest consists of pieces whose formal sums have been adjoined to $\mathbb{F}$ in the construction of $\mathbb{E}$. This makes the construction of extension fields in which f is summable somewhat uninteresting and justifies the tendency to look for sums of $f \in \mathbb{F}$ only in $\mathbb{F}$.*

In other words, following Karr's point of view, one either succeeds to express a given sum of f in $\mathbb{F}$, or, if one fails, one adjoins the sum formally to $\mathbb{F}$ which leads to a bigger field $\mathbb{E}$. But, it turns out that Karr's theory of difference field extensions can be refined. Namely, as shown below, his strategy in general produces sum representations that are not optimal with respect to simplification; see, e.g., Examples 4 and 12.

As a measure of simplification we introduce the notion of nested depth. And the main part of this article deals with the problem of finding sum representations which are optimal with respect to this property. Based on results of [27, 28, 33] we develop a refined version of Karr's summation theory, which leads to the definition of the so called depth-optimal $\Pi\Sigma^*$-extensions. Various important properties hold in such extensions which are relevant in symbolic summation. Moreover, an efficient telescoping algorithm which computes sum representations with optimal nested depth is presented. Throughout this article all these ideas will be illustrated by one guiding example, namely the identity

$$\sum_{k=1}^{K} \frac{\sum_{i=1}^{k} x^{i-1}\binom{m+i-1}{m}}{k+m} = \left(\sum_{k=1}^{K} \frac{1}{k+m}\right)\left(\sum_{k=1}^{K} x^{k-1}\binom{m+k-1}{m}\right) - \sum_{k=1}^{K} \binom{m-k-1}{m} x^{k-1} \sum_{i=1}^{k-1} \frac{1}{m+i}, \quad (1)$$

which was needed in [22] to generalize identities from statistics.

We stress that our algorithms are of particular importance to simplify d'Alembertian solutions [1, 25], a subclass of Liouvillian solutions [10], of a given recurrence; for applications see, e.g., [26, 8, 34, 35]. Furthermore, we obtain a refined version of creative telescoping which can find recurrences with smaller order; for applications see, e.g., [21, 16, 13, 18]. In addition, we show how our algorithms can be used to compute efficiently algebraic relations of nested sums, like harmonic sums [6, 36]

$$S_{m_1,\dots,m_r}(n) = \sum_{i_1=1}^{n} \frac{\operatorname{sign}(m_1)^{i_1}}{i_1^{|m_1|}} \sum_{i_2=1}^{i_1} \frac{\operatorname{sign}(m_2)^{i_2}}{i_2^{|m_2|}} \cdots \sum_{i_r=1}^{i_{r-1}} \frac{\operatorname{sign}(m_r)^{i_r}}{i_r^{|m_r|}}, \quad (2)$$

$m_1, \dots, m_r \in \mathbb{Z} \setminus \{0\}$. We illustrate by concrete examples [4, 17] from Perturbative Quantum Field Theory how our algorithms can evaluate efficiently Feynman diagrams.

The general structure of this article is as follows. In Section 2 we introduce the basic summation problems in difference fields. In Section 3 we present in summarized form our refined summation theory of depth-optimal $\Pi\Sigma^*$-extensions in which the central results are supplemented by concrete examples. Some first properties of depth-optimal $\Pi\Sigma^*$-extensions are proven then in Section 4. After considering a variation of Karr's reduction technique in Section 5 we are ready to design algorithms to construct depth-optimal $\Pi\Sigma^*$-extensions in Section 6. As a consequence we can prove the main results, stated in Section 3, in Section 7. Finally, we present applications from particle physics in Section 8.

2. Refined Telescoping in $\Pi\Sigma^*$-extensions

Let $\mathbb{F}$ be a *difference field* with field automorphism $\sigma : \mathbb{F} \to \mathbb{F}$. Note that

$$\mathrm{const}_\sigma\mathbb{F} := \{c \in \mathbb{F} \mid \sigma(c) = c\}$$

forms a subfield of $\mathbb{F}$; we call $\mathbb{K} := \mathrm{const}_\sigma\mathbb{F}$ the *constant field*[1] of the difference field $(\mathbb{F}, \sigma)$. Subsequently, we consider the following two problems:

- SR (*Sequence Representation*): **Given** sequences $f_1(k), \dots, f_n(k) \in \mathbb{K}^{\mathbb{N}}$; try to **construct** an appropriate difference field $(\mathbb{F}, \sigma)$ with elements $f_1, \dots, f_n \in \mathbb{F}$ where the shift-behavior $f_i(k+1)$ for $1 \leq i \leq n$ is reflected by $\sigma(f_i)$.
- PT (*Parameterized Telescoping*): **Given** $(\mathbb{F}, \sigma)$ with $\mathbb{K} = \mathrm{const}_\sigma\mathbb{F}$ and $f_1, \dots, f_n \in \mathbb{F}$; **find** all $c_1, \dots, c_n \in \mathbb{K}$ and $g \in \mathbb{F}$ such that

$$\sigma(g) - g = c_1 f_1 + \dots + c_n f_n. \tag{3}$$

Then reinterpreting such a solution $g \in \mathbb{F}$ with $c_1, \dots, c_n \in \mathbb{K}$ in terms of a sequence $g(k)$ gives

$$g(k+1) - g(k) = c_1 f_1(k) + \dots + c_n f_n(k)$$

which then holds in a certain range $a \leq k \leq b$. Hence, summing this equation over k gives

$$g(b+1) - g(a) = c_1 \sum_{k=a}^{b} f_1(k) + \dots + c_n \sum_{k=a}^{b} f_n(k).$$

If we restrict to $n = 1$ in (3) and search for a solution with $c_1 = 1$, we solve the telescoping problem: **Given** $f \in \mathbb{F}$; **find** $g \in \mathbb{F}$ such that

$$\sigma(g) - g = f. \tag{4}$$

Moreover, Zeilberger's creative telescoping [38] can be formulated by translating $f(N + i - 1, k)$ into $f_i \in \mathbb{F}$ for a parameter N which occurs in the constant field $\mathbb{K}$.

Karr's summation theory [11, 12] treats these problems in the so-called $\Pi\Sigma$-difference fields. In our work we restrict to $\Pi\Sigma^*$-extensions [25] being slightly less general but covering all sums and products treated explicitly in Karr's work. Those fields are introduced by difference field extensions. A difference field $(\mathbb{E}, \sigma)$ is a *difference field extension* of a difference field $(\mathbb{F}, \sigma')$ if $\mathbb{F}$ is a subfield of $\mathbb{F}$ and $\sigma(f) = \sigma'(f)$ for all $f \in \mathbb{F}$; since σ and σ' agree on $\mathbb{F}$, we usually do not distinguish between them anymore.

Definition 1. A difference field extension $(\mathbb{F}(t), \sigma)$ of $(\mathbb{F}, \sigma)$ is a Σ^**-extension* (resp. Π-*extension*), if t is transcendental over $\mathbb{F}$, $\sigma(t) = t + a$ (resp. $\sigma(t) = a\,t$) for some $a \in \mathbb{F}^*$ and $\mathrm{const}_\sigma\mathbb{F}(t) = \mathrm{const}_\sigma\mathbb{F}$; if it is clear from the context, we say that t is a Σ^*-extension (resp. a Π-extension). A $\Pi\Sigma^*$*-extension* is either a Π-extension or a Σ^*-extension.
A Π-*extension (resp.* Σ^**-extension/*$\Pi\Sigma^*$*-extension)* $(\mathbb{F}(t_1)\dots(t_e), \sigma)$ of $(\mathbb{F}, \sigma)$ is a tower of such Π-extensions (resp. Σ^*-extensions/$\Pi\Sigma^*$-extensions). Such an extension if defined over $\mathbb{H}$ if $\mathbb{H}$ is a subfield of $\mathbb{F}$ and for all $1 \leq i \leq e$, $\sigma(t_i)/t_i$ or $\sigma(t_i) - t_i$ is in $\mathbb{H}(t_1, \dots, t_{i-1})$; note: $\mathbb{H}(t_1, \dots, t_{i-1}) = \mathbb{H}$, if $i = 1$.
A $\Pi\Sigma^*$-extension $(\mathbb{F}(t_1)\dots(t_e), \sigma)$ of $(\mathbb{F}, \sigma)$ is called *generalized d'Alembertian*, or in short *polynomial*, if $\sigma(t_i) - t_i \in \mathbb{F}[t_1, \dots, t_{i-1}]$ or $\sigma(t_i)/t_i \in \mathbb{F}$ for all $1 \leq i \leq e$.
A $\Pi\Sigma^*$*-field* $(\mathbb{F}, \sigma)$ *over* $\mathbb{K}$ is a $\Pi\Sigma^*$-extension $(\mathbb{F}, \sigma)$ of $(\mathbb{K}, \sigma)$ with $\mathrm{const}_\sigma\mathbb{K} = \mathbb{K}$.

[1] All fields have characteristic zero. As a consequence, the constant field contains the rational numbers.

2.1. A solution of problem PT.

Karr derived an algorithm [11] that solves the following more general problem which under the specialization $a_1 = 1$ and $a_2 = -1$ gives (3).

PFDE (*Parameterized First order Difference Equations*): **Given** $\mathbf{0} \neq (a_1, a_2) \in \mathbb{F}^2$ and $\boldsymbol{f} = (f_1, \dots, f_n) \in \mathbb{F}^n$; **find** all $(c_1, \dots, c_n, g) \in \mathbb{K}^n \times \mathbb{F}$ such that

$$a_1\, \sigma(g) + a_2\, g = c_1\, f_1 + \dots + c_n\, f_n. \tag{5}$$

Remark 2. Karr's algorithm or our simplified version [31] can be applied if $(\mathbb{F}, \sigma)$ is a $\Pi\Sigma^*$-extension of $(\mathbb{G}, \sigma)$ where $(\mathbb{G}, \sigma)$ satisfies certain properties; see [14]. As a consequence, we obtain algorithms for problem PFDE if $(\mathbb{G}, \sigma)$ is given as follows:

(1) $\mathbb{K} = \mathrm{const}_\sigma \mathbb{G}$: As worked out in [30, Thm. 3.2,Thm. 3.5], we obtain a complete algorithm, if $\mathbb{K}$ is as a rational function field over an algebraic number field.

(2) The free difference field $(\mathbb{G}, \sigma)$ with $\mathbb{G} = \mathbb{K}(\dots, x_{-1}, x_0, x_1, \dots)$, $\sigma(x_i) = x_{i+1}$ for $i \in \mathbb{Z}$, and $\mathbb{K}$ is as in **(1)**. In this setting generic sequences can be treated; see [14, 13].

(3) The radical difference field $(\mathbb{K}(k)(\dots, x_{-1}, x_0, x_1, \dots), \sigma)$ with $\sigma(k) = k + 1$ and $\sigma(x_i) = x_{i+1}$ where $x_i^2 = k$; $\mathbb{K}$ is given as in **(1)**. This allows to handle $\sqrt{k}$; see [15].

(4) $(\mathbb{G}, \sigma)$ is a $\Pi\Sigma^*$-extension of one of the difference fields described in (1)–(3).

2.2. A naive approach for problem SR.

Sum-product expressions can be represented in $\Pi\Sigma^*$-fields with the following

Theorem 3 ([11])**.** *Let $(\mathbb{F}(t), \sigma)$ be a difference field extension of $(\mathbb{F}, \sigma)$ with $\sigma(t) = at+f$.*

(1) *t a Σ^*-extension iff $a = 1$ and there is no $g \in \mathbb{F}$ s.t.* (4)*.*

(2) *t is a Π-extension iff $t \neq 0$, $f = 0$ and there are no $g \in \mathbb{F}^*, m > 0$ s.t. $\sigma(g) = a^m g$.*

Consequently, we are allowed to adjoin a sum formally by a Σ^*-extension if and only if there does not exist a solution of the telescoping problem. The product case works similarly; for further information and problematic cases we refer to [30].

Example 4. We try to simplify the left hand side of (1) by telescoping, or equivalently, by representing (1) in a $\Pi\Sigma^*$-field. For simplicity of representation, it will be convenient to rewrite this expression as

$$\sum_{k=1}^{K} \frac{1}{k+m} \overbrace{\sum_{i=1}^{k} \frac{i}{x(m+i)} x^i \binom{m+i}{m}}^{=s(k)}. \tag{6}$$

(1) We start with the difference field $(\mathbb{Q}(x, m), \sigma)$ with $\sigma(c) = c$ for all $c \in \mathbb{Q}(x, m)$, i.e., $\mathbb{K} = \mathbb{Q}(m, x)$ is the constant field. Since there is no $g \in \mathbb{K}$ with $\sigma(g) - g = 1$, we can define the Σ^*-extension $(\mathbb{K}(k), \sigma)$ of $(\mathbb{K}, \sigma)$ with $\sigma(k) = k + 1$.

(2) Since there are no $n > 0$ and $g \in \mathbb{K}(k)^*$ with $\sigma(g) = x^n g$ (for algorithms see [11]), we can define the Π-extension $(\mathbb{K}(k)(q), \sigma)$ of $(\mathbb{K}(k), \sigma)$ with $\sigma(q) = xq$. Similarly, we introduce the Π-extension $(\mathbb{K}(k)(q)(b), \sigma)$ of $(\mathbb{K}(k)(q), \sigma)$ with $\sigma(b) = \frac{1+m+k}{k+1} b$. By construction, σ reflects the shift in k with $S_k x^k = x x^k$ and $S_k \binom{m+k}{m} = \frac{1+m+k}{k+1} \binom{m+k}{m}$.

(3) Next, we try to simplify $s(k)$ by telescoping. Since we fail, i.e., there is no $g \in \mathbb{K}(k)(q)(b)$ with $\sigma(g) - g = qb = \sigma(\frac{kqb}{(m+k)x})$, we add the Σ^*-extension $(\mathbb{K}(k)(q)(b)(s), \sigma)$ of $(\mathbb{K}(k)(q)(b), \sigma)$ with $\sigma(s) = s + qb$; note that $S_k s(k) = s(k+1) = s(k) + x^k \binom{m+k}{m}$.

(4) Finally, we look for a $g \in \mathbb{K}(k)(q)(b)(s)$ such that

$$\sigma(g) - g = \frac{s + q\,b}{1 + k + m}. \tag{7}$$

Since there is none, see Example 28, we adjoin the sum (6) in form of the Σ^*-extension $(\mathbb{K}(k)(q)(b)(s)(S), \sigma)$ of $(\mathbb{K}(k)(q)(b)(s), \sigma)$ with $\sigma(S) = S + \frac{s+q\,b}{1+k+m}$.

Summarizing, using this straight-forward approach the sum (6) could not be simplified: the two nested sum-quantifier is reflected by the nested definition of $(\mathbb{K}(k)(q)(b)(s)(S), \sigma)$.

2.3. A refined approach for problem SR*: The depth of nested sums and products*

Let $(\mathbb{F}, \sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{G}, \sigma)$ with $\mathbb{F} := \mathbb{G}(t_1)\dots(t_e)$ where $\sigma(t_i) = a_i\,t_i$ or $\sigma(t_i) = t_i + a_i$ for $1 \le i \le e$. The *depth function for elements of* $\mathbb{F}$ *over* $\mathbb{G}$, $\delta_{\mathbb{G}} : \mathbb{F} \to \mathbb{N}_0$, is defined as follows.

(1) For any $g \in \mathbb{G}$, $\delta(g) := 0$.

(2) If $\delta_{\mathbb{G}}$ is defined for $(\mathbb{G}(t_1)\dots(t_{i-1}), \sigma)$ with $i > 1$, we define $\delta_{\mathbb{G}}(t_i) := \delta_{\mathbb{G}}(a_i) + 1$; for $g = \frac{g_1}{g_2} \in \mathbb{G}(t_1)\dots(t_i)$, with $g_1, g_2 \in \mathbb{G}[t_1, \dots, t_i]$ coprime, we define

$$\delta_{\mathbb{G}}(g) := \max(\{\delta_{\mathbb{G}}(t_i) \mid t_i \text{ occurs in } g_1 \text{ or } g_2\} \cup \{0\}).$$

For $\boldsymbol{f} = (f_1, \dots, f_n) \in \mathbb{F}^n$, $\delta_{\mathbb{G}}(\boldsymbol{f}) := \max_{1 \le i \le n} \delta_{\mathbb{G}}(f_i)$. The *depth of* $(\mathbb{F}, \sigma)$, in short $\delta_{\mathbb{G}}(\mathbb{F})$, is given by $\delta_{\mathbb{G}}((0, t_1, \dots, t_e))$. Similarly, the *extension depth* of a $\Pi\Sigma^*$-extension $(\mathbb{F}(x_1)\dots(x_r), \sigma)$ of $(\mathbb{F}, \sigma)$ is $\delta_{\mathbb{G}}((0, x_1, \dots, x_r))$. This extension is *ordered* if $\delta_{\mathbb{G}}(x_1) \le \dots \le \delta_{\mathbb{G}}(x_r)$; if $\mathrm{const}_\sigma \mathbb{F} = \mathbb{F}$, we call $(\mathbb{F}(x_1)\dots(x_r), \sigma)$ an *ordered* $\Pi\Sigma^*$-field.

Example 5. In the $\Pi\Sigma^*$-extension $(\mathbb{K}(k)(q)(b)(s)(S), \sigma)$ of $(\mathbb{K}, \sigma)$ from Example 4 the depth (function) is given by $\delta_{\mathbb{K}}(k) = 1, \delta_{\mathbb{K}}(q) = 1, \delta_{\mathbb{K}}(b) = 2, \delta_{\mathbb{K}}(s) = 3$, and $\delta_{\mathbb{K}}(S) = 4$.

Throughout this article the depth is defined over the ground field $(\mathbb{G}, \sigma)$; we set $\delta := \delta_{\mathbb{G}}$. We might use the depth function without mentioning $\mathbb{G}$. Then we assume that the corresponding difference fields are $\Pi\Sigma^*$-extensions of $(\mathbb{G}, \sigma)$. Moreover, note that the definition of δ depends on the particular way the extension field $\mathbb{F}$ is build from $\mathbb{G}$.

Example 6. We consider the sum (6) and take the $\Pi\Sigma^*$-field $(\mathbb{K}(k)(q)(b)(s), \sigma)$ with

$$\sigma(k) = k + 1, \qquad \sigma(q) = x\,q, \qquad \sigma(b) = \frac{1 + m + k}{k + 1} b, \qquad \sigma(s) = s + q\,b, \tag{8}$$

which we introduced in Example 4. Now we proceed differently: We compute the $\Pi\Sigma^*$-extension $(\mathbb{K}(k)(q)(s)(h)(H), \sigma)$ of $(\mathbb{K}(k)(q)(s), \sigma)$ with

$$\sigma(h) = h + \frac{1}{1 + k + m}, \qquad \sigma(H) = H - bqh \tag{9}$$

in which we find the solution $g = sh + H$ of (7); for details see Example 10. Note that $\delta(h) = 2$ and $\delta(H) = 3$, in particular, $\delta(g) = 3$. Reinterpreting g as a sequence and checking initial values produces (1). We emphasize that this way we have reduced the depth in (1) since g and the summand $f = \frac{s+q\,b}{1+k+m}$ in (7) have the same depth $\delta(g) = \delta(f)$.

This example motivates us to consider the following refined telescoping problem.

DOT (*Depth Optimal Telescoping*): **Given** a $\Pi\Sigma^*$-extension $(\mathbb{F}, \sigma)$ of $(\mathbb{G}, \sigma)$ and $f \in \mathbb{F}$; **find**, if possible, a Σ^*-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{F}, \sigma)$ with $g \in \mathbb{E}$ s.t. (4) and[2] $\delta(g) = \delta(f)$.

[2] Note that for any $\Pi\Sigma^*$-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{F}, \sigma)$ and any $g \in \mathbb{E}$ as in (4) we have $\delta(g) \ge \delta(f)$. Note also that it suffices to restrict to extensions with $\delta(\mathbb{E}) = \delta(f)$.

Example 7. Our goal is to encode the harmonic sums $S_{4,2}(k)$ and $S_{2,4}(k)$ in a $\Pi\Sigma^*$-field.

(1) First we express $S_{4,2}(k) = \sum_{i=1}^{k} \frac{S_2(i)}{i^4}$ with the inner sum $S_2(k) = \sum_{j=1}^{k} \frac{1}{j^2}$ in the $\Pi\Sigma^*$-field $(\mathbb{Q}(k)(s_2)(s_{4,2}), \sigma)$ with $\sigma(k) = k+1$, $\sigma(s_2) = s_2 + \frac{1}{(k+1)^2}$ and $\sigma(s_{4,2}) = s_{4,2} + \frac{\sigma(s_2)}{(k+1)^4}$, i.e., s_2 and $s_{4,2}$ represent $S_2(k)$ and $S_{4,2}(k)$, respectively; note that we failed to express $S_{4,2}(k)$ in an extension with depth $< \delta(s_{4,2}) = 3$.

(2) Next, we consider $S_{2,4}(k) = \sum_{i=1}^{k} \frac{S_4(i)}{i^2}$. We start with $S_4(k)$ and construct the Σ^*-extension $(\mathbb{Q}(s_2)(s_{4,2})(s_4), \sigma)$ of $(\mathbb{Q}(s_2)(s_{4,2}), \sigma)$ with $\sigma(s_4) = s_4 + \frac{1}{(k+1)^4}$. Finally, we treat the sum $S_{2,4}(k)$ and look for a g such that $\sigma(g) - g = \frac{\sigma(s_4)}{(k+1)^2}$.

The naive approach: Since there is no $g \in \mathbb{Q}(k)(s_2)(s_{4,2})(s_4)$, we take the Σ^*-extension $(\mathbb{Q}(k)(s_2)(s_{4,2})(s_4)(s_{2,4}), \sigma)$ of $(\mathbb{Q}(s_2)(s_{4,2})(s_4), \sigma)$ with $\sigma(s_{2,4}) = s_{2,4} + \frac{\sigma(s_4)}{(k+1)^2}$.

The refined approach: We can compute the Σ^*-extension $(\mathbb{Q}(k)(s_2)(s_{4,2})(s_4)(s_6), \sigma)$ of $(\mathbb{Q}(k)(s_2)(s_{4,2})(s_4), \sigma)$ with $\sigma(s_6) = s_6 + \frac{1}{(k+1)^6}$ in which we find the solution $g = s_6 + s_2\, s_4 - s_{4,2}$. Note that this alternative solution has the same depth, namely $\delta(s_{4,2}) = \delta(g) = 3$, but the underlying $\Pi\Sigma^*$-field is simpler. As result, we obtain

$$S_{2,4}(N) = S_6(N) + S_2(N)S_4(N) - S_{4,2}(N). \tag{10}$$

To sum up, the following version of telescoping is relevant.

DOT^*: **Given** a $\Pi\Sigma^*$-extension $(\mathbb{F}, \sigma)$ of $(\mathbb{G}, \sigma)$ and $f \in \mathbb{F}$; **find** a Σ^*-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{F}, \sigma)$ with *minimal* extension depth such that (4) for some $g \in \mathbb{E}$.

We shall refine Karr's theory such that we can find a common solution to DOT and DOT^*.

3. A refined summation theory: Depth-optimal $\Pi\Sigma^*$-extensions

Definition 8. A difference field extension $(\mathbb{F}(s), \sigma)$ of $(\mathbb{F}, \sigma)$ with $\sigma(s) = s + f$ is called *depth-optimal* Σ^*-extension, in short Σ^δ-extension, if there is no Σ^*-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{F}, \sigma)$ with extension depth $\leq \delta(f)$ and $g \in \mathbb{E}$ such that (4). A $\Pi\Sigma^*$-extension $(\mathbb{F}(t_1)\dots(t_e), \sigma)$ of $(\mathbb{F}, \sigma)$ is depth-optimal, in short a $\Pi\Sigma^\delta$-extension, if all Σ^*-extensions[3] are depth-optimal. A $\Pi\Sigma^\delta$-field consists of Π- and Σ^δ-extensions.

Our main result is that problems SR, DOT and DOT^* can be solved algorithmically in $\Pi\Sigma^\delta$-extensions. Moreover, we will derive various properties that are of general relevance to the field of symbolic summation and that do not hold for $\Pi\Sigma^*$-extensions in general.

In all our Results 1–9, stated below and proved in Section 7, we suppose that $(\mathbb{F}, \sigma)$ is a $\Pi\Sigma^\delta$-extension of $(\mathbb{G}, \sigma)$ and $\delta = \delta_{\mathbb{G}}$. From an algorithmic point of view we assume that $(\mathbb{G}, \sigma)$ is σ-computable:

Definition 9. A difference field $(\mathbb{G}, \sigma)$ is *σ-computable*, if one can execute the usual polynomial arithmetic of multivariate polynomials over $\mathbb{G}$ (including factorization), and if one can solve problem PFDE algorithmically in any $\Pi\Sigma^*$-extension $(\mathbb{F}, \sigma)$ of $(\mathbb{G}, \sigma)$.

For instance, $(\mathbb{G}, \sigma)$ can be any of the fields given in Remark 2. In our examples we restrict to the case $\mathrm{const}_\sigma\mathbb{G} = \mathbb{G}$, i.e., $(\mathbb{F}, \sigma)$ is a $\Pi\Sigma^*$-field over $\mathbb{G}$.

[3] Note that Σ^δ-extensions are Σ^*-extension by Theorem 3.1. Note also that Π-extensions are not refined here; this gives room for further investigations; see, e.g., [30].

1. Construction. Problem SR can be handled algorithmically in $\Pi\Sigma^\delta$-extensions.

Result 1. For any $f \in \mathbb{F}$ there is a Σ^δ-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{F}, \sigma)$ with $g \in \mathbb{E}$ s.t. (4); if $(\mathbb{F}, \sigma)$ is a polynomial extension of $(\mathbb{G}, \sigma)$, $(\mathbb{E}, \sigma)$ can be constructed as a polynomial extension of $(\mathbb{G}, \sigma)$. If $(\mathbb{G}, \sigma)$ is σ-computable, $(\mathbb{E}, \sigma)$ and $g \in \mathbb{E}$ can be given explicitly.

Example 10. The $\Pi\Sigma^*$-field $(\mathbb{K}(k)(q)(b)(s), \sigma)$ with (8) is depth-optimal since DOT with $\mathbb{F} = \mathbb{K}(k)(q)(b)$ and $f = qb$ has no solution. Moreover, $(\mathbb{K}(k)(q)(b)(s)(h)(H), \sigma)$ is a Σ^δ-extension of $(\mathbb{K}(k)(q)(b)(s), \sigma)$ with (9). With the solution $g = sh + H$ of (7) we represent the sum (6) in a $\Pi\Sigma^\delta$-field; for algorithmic details see Example 53.

2. Reordering. Let $(\mathbb{G}(t_1) \dots (t_e), \sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{G}, \sigma)$ with $\sigma(t_i) = a_i\, t_i$ or $\sigma(t_i) = t_i + a_i$ for $1 \leq i \leq e$. If there is a permutation $\tau \in S_e$ with $a_{\tau(i)} \in \mathbb{G}(t_{\tau(1)}) \dots (t_{\tau(i-1)})$ for all $1 \leq i \leq e$, $(\mathbb{G}(t_{\tau(1)}) \dots (t_{\tau(e)}), \sigma)$ is again a $\Pi\Sigma^*$-extension of $(\mathbb{G}, \sigma)$ and $\mathbb{G}(t_1) \dots (t_e)$ is isomorphic to $\mathbb{G}(t_{\tau(1)}) \dots (t_{\tau(e)})$ as fields. In short, we say that $(\mathbb{G}(t_1) \dots (t_e), \sigma)$ can be *reordered to* $(\mathbb{G}(t_{\tau(1)}) \dots (t_{\tau(e)}), \sigma)$; on the field level we identify such fields. Clearly, by definition of nested depth there is always a reordering that brings a given field to its ordered form, i.e., $\delta(t_{i-1}) \leq \delta(t_i)$ for $1 \leq i \leq e$.

Note that reordering of $\Pi\Sigma^\delta$-extensions without destroying depth-optimality is not so obvious: Putting Σ^*-extensions in front or removing them, might change the situation of problem DOT^*. But one of our main results says that reordering indeed does not matter.

Result 2. Any possible reordering of $(\mathbb{F}, \sigma)$ is again a $\Pi\Sigma^\delta$-extension of $(\mathbb{G}, \sigma)$.

Example 11. Let $(\mathbb{Q}(k)(s_2)(s_{4,2})(s_4)(s_6), \sigma)$ be the $\Pi\Sigma^\delta$-field from Example 7. Then, e.g., the ordered $\Pi\Sigma^*$-field $(\mathbb{Q}(k)(s_2)(s_4)(s_6)(s_{4,2}), \sigma)$ is depth-optimal.

3. Depth–stability. The following example illustrates the importance of Result 3.

Example 12. Let $(\mathbb{Q}(k)(s_2)(s_{4,2})(s_4)(s_{2,4}), \sigma)$ be the $\Pi\Sigma^*$-field from Example 7 which is not depth-optimal. We find the solution $g = s_{2,4} + s_{4,2} - s_2\, s_4$ of (4) with $f := \frac{1}{(k+1)^6}$. Hence, $3 = \delta(g) > \delta(f) + 1 = 2$. In other words, we obtained the identity $S_6(k) = S_{2,4}(k) + S_{4,2}(k) - S_2(k) S_4(k)$ where the depth is increased by telescoping; compare (10).

Result 3. For any $f, g \in \mathbb{F}$ as in (4) we have

$$\delta(f) \leq \delta(g) \leq \delta(f) + 1. \tag{11}$$

We remark that Result 3 can be exploited algorithmically: In order to find all solutions of (4), one only has to take into account those extensions with depth $\leq \delta(f) + 1$.

4. Extension–stability. The most crucial property is the following: Suppose we are given a Σ^δ-extension $(\mathbb{S}, \sigma)$ of $(\mathbb{F}, \sigma)$. Then we can embed any Σ^*-extension $(\mathbb{H}, \sigma)$ of $(\mathbb{F}, \sigma)$ in a Σ^δ-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{S}, \sigma)$ without increasing the depth.

Example 13. The $\Pi\Sigma^*$-field $(\mathbb{F}, \sigma)$ from Example 11 with $\mathbb{F} = \mathbb{Q}(k)(s_2)(s_{4,2})$ is depth-optimal, and $(\mathbb{S}, \sigma)$ with $\mathbb{S} = \mathbb{F}(s_6)$ is a Σ^δ-extension of $(\mathbb{F}, \sigma)$. Now consider in addition the $\Pi\Sigma^*$-extension $(\mathbb{H}, \sigma)$ of $(\mathbb{F}, \sigma)$ with $\mathbb{H} = \mathbb{F}(s_4)(s_{2,4})$; see Example 7. Then we can take the Σ^δ-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{S}, \sigma)$ with $\mathbb{E} = \mathbb{S}(s_4)$, see Example 11, and we can define the field homomorphism $\tau:\mathbb{H}\to\mathbb{E}$ with $\tau(f)=f$ for all $f\in\mathbb{F}$, $\tau(s_4)=s_4$ and $\tau(s_{2,4})=s_6+s_2\,s_4-s_{4,2}$. By construction, τ is injective and $\sigma_{\mathbb{E}}(\tau(h))=\tau(\sigma_{\mathbb{H}}(h))$ for all $h\in\mathbb{H}$. In other words, we have embedded $(\mathbb{H},\sigma)$ in $(\mathbb{E},\sigma)$ with (12) for all $a\in\mathbb{H}$.

More precisely, $\tau:\mathbb{F}\to\mathbb{F}'$ is called a *σ-monomorphism/σ-isomorphism* for $(\mathbb{F},\sigma)$ and $(\mathbb{F}',\sigma')$ if τ is a field monomorphism/isomorphism with $\sigma'(\tau(a))=\tau(\sigma(a))$ for all $a\in\mathbb{F}$. Let $(\mathbb{F},\sigma)$ and $(\mathbb{F}',\sigma)$ be difference field extensions of $(\mathbb{H},\sigma)$. An *$\mathbb{H}$-monomorphism/$\mathbb{H}$-isomorphism* $\tau:\mathbb{F}\to\mathbb{F}'$ is a σ-monomorphism/σ-isomorphism with $\tau(a)=a$ for all $a\in\mathbb{H}$.

Result 4. Let $(\mathbb{S},\sigma)$ be a Σ^{δ}-extension of $(\mathbb{F},\sigma)$. Then for any Σ^*-extension $(\mathbb{H},\sigma)$ of $(\mathbb{F},\sigma)$ with extension depth d there is a Σ^{δ}-extension $(\mathbb{E},\sigma)$ of $(\mathbb{S},\sigma)$ with extension depth $\leq d$ and an $\mathbb{F}$-monomorphism $\tau:\mathbb{H}\to\mathbb{E}$ such that

$$\delta(\tau(a))\leq\delta(a) \tag{12}$$

for all $a\in\mathbb{H}$. Such $(\mathbb{S},\sigma)$ and τ can be constructed explicitly if $(\mathbb{G},\sigma)$ is σ-computable.

5. Depth–optimal transformation. By Result 5 any $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$ can be transformed to a $\Pi\Sigma^{\delta}$-extension with the same or an improved depth-behavior. Hence the refinement to $\Pi\Sigma^{\delta}$-extensions does not restrict the range of applications; on the contrary, the refinement to $\Pi\Sigma^{\delta}$-extensions can lead only to better depth behavior.

Result 5. Let $(\mathbb{E},\sigma)$ be a $\Pi\Sigma^*$-extension (resp. Σ^*-extension) of $(\mathbb{F},\sigma)$. Then there is a $\Pi\Sigma^{\delta}$-extension (resp. Σ^{δ}-extension) $(\mathbb{D},\sigma)$ of $(\mathbb{F},\sigma)$ and an $\mathbb{F}$-monomorphism $\tau:\mathbb{E}\to\mathbb{D}$ s.t. (12) for all $a\in\mathbb{E}$. $(\mathbb{D},\sigma)$ and τ can be given explicitly if $(\mathbb{G},\sigma)$ is σ-computable.

6. Product–freeness. Π-extensions are irrelevant for problem DOT.

Result 6. Let $f\in\mathbb{F}$ and let $(\mathbb{E},\sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$ with $g\in\mathbb{E}$ s.t. (4). Then there is a Σ^{δ}-extension $(\mathbb{S},\sigma)$ of $(\mathbb{F},\sigma)$ with a solution $g'\in\mathbb{S}$ of (4) s.t. $\delta(g')\leq\delta(g)$.

7. Alternative definition. Thus we obtain the following equivalent definition.

Result 7. A Σ^*-extension $(\mathbb{F}(s),\sigma)$ of $(\mathbb{F},\sigma)$ with $\sigma(s)=s+f$ is depth-optimal iff there is no $\Pi\Sigma^*$-extension $(\mathbb{E},\sigma)$ of $(\mathbb{F},\sigma)$ with extension depth $\leq\delta(f)$ and $g\in\mathbb{E}$ s.t. (4).

8. A common solution to DOT and DOT^* can be found by Result 1 and

Result 8. Let $f\in\mathbb{F}$ and let $(\mathbb{E},\sigma)$ with $\mathbb{E}=\mathbb{F}(s_1)\dots(s_e)$ be a Σ^{δ}-extension of $(\mathbb{F},\sigma)$ with extension depth $\mathfrak{d}$ and with $g\in\mathbb{E}$ such that (4). Then the following holds.
(1) For any $\Pi\Sigma^*$-extension $(\mathbb{H},\sigma)$ of $(\mathbb{F},\sigma)$ and any solution $g'\in\mathbb{H}$ of (4), $\delta(g)\leq\delta(g')$.
(2) Suppose that $\delta(s_e)=\mathfrak{d}$ and that $g\in\mathbb{E}\setminus\mathbb{F}(s_1)\dots(s_{e-1})$. Then for any $\Pi\Sigma^*$-extension $(\mathbb{H},\sigma)$ of $(\mathbb{F},\sigma)$ with a solution $g'\in\mathbb{H}$ of (4) the extension depth is $\geq\mathfrak{d}$.

9. Refined parameterized telescoping.

Result 9. Let $\boldsymbol{f}\in\mathbb{F}^n$ with $\mathfrak{d}:=\delta(\boldsymbol{f})$. Then there is a Σ^{δ}-extension $(\mathbb{E},\sigma)$ of $(\mathbb{F},\sigma)$ with extension depth $\leq\mathfrak{d}$ such that: For any $\Pi\Sigma^*$-extension $(\mathbb{H},\sigma)$ of $(\mathbb{F},\sigma)$ with extension depth $\leq\mathfrak{d}$ and any $g\in\mathbb{H}$, $\boldsymbol{c}\in\mathbb{K}^n$ s.t. (3) there is a $g'\in\mathbb{E}$ s.t. $\sigma(g')-g'=\boldsymbol{c}\,\boldsymbol{f}$ and $\delta(g')\leq\delta(g)$. If $(\mathbb{G},\sigma)$ is σ-computable, $(\mathbb{E},\sigma)$ can be constructed explicitly.

Example 14. For $F_m(k) = \binom{m}{k} S_1(k)$ we take the $\Pi\Sigma^\delta$-field $(\mathbb{Q}(m)(k)(b)(s_1), \sigma)$ over $\mathbb{Q}(m)$ with $\sigma(k) = k+1$, $\sigma(b)\frac{m-k}{k+1}b$ and $\sigma(s_1) = s_1 + \frac{1}{k+1}$; then $F_m(k)$ and $F_{m+1}(k) = \frac{m+1}{m-k+1}F_m(k)$ can be represented by $\boldsymbol{f} = (f_1, f_2) = (bs_1, \frac{m+1}{m-k+1}bs_1)$, respectively. We set $\mathbb{G} := \mathbb{Q}(m)(k)(b)$ and $\delta := \delta_{\mathbb{G}}$, and get $\delta(\boldsymbol{f}) = 1$. Then we find the Σ^δ-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{G}(s_1), \sigma)$ with $\mathbb{E} := \mathbb{G}(s_1)(h)$, $\sigma(h) = h + \frac{b}{k+1}$ and $\delta(h) = 1$ which fulfills the properties from Result 9. In particular, we get $(c_1, c_2) = (2, -1)$ and $g = \frac{kbs_1}{m-k+1} - h$ s.t. (3). Hence, for $g(k) = \frac{k}{m-k+1}\binom{m}{k}S_1(k) - \sum_{i=1}^{k} \frac{1}{m-i+1}\binom{m}{i}$, we get $g(k+1) - g(k) = 2F_m(k) - F_{m+1}(k)$. Summation over k from 0 to m gives $2S(m) - S(m+1) = -\sum_{i=1}^{m} \frac{1}{m-i+1}\binom{m}{i}$ for

$$S(m) = \sum_{k=0}^{m} S_1(k)\binom{m}{k}.$$

Together with $\sum_{i=1}^{m} \frac{1}{m-i+1}\binom{m}{i} = \frac{-1+2^{m+1}}{m+1}$, we arrive at the recurrence relation

$$S(m+1) - 2S(m) = -\frac{-1+2^{m+1}}{m+1}$$

and can read off the closed form $S(m) = 2^m\big(S_1(m) - \sum_{i=1}^{m} \frac{1}{i2^i}\big)$. Note: only a recurrence of order two is produced for $S(m)$ by using standard creative telescoping; see [34, Sec. 2.4].

How to proceed. We will prove Results 1–9 as follows. In Section 4 we first show weaker versions of Results 2–4; there we impose that all $\Pi\Sigma^\delta$-extensions are ordered. After general preparation in Section 5, these results allow us to produce Theorem 40 (cf. Result 1) in Section 6. Given all these properties, we will show our Results 1–9 in full generality in Section 7.

3.2. Recalling basic properties of $\Pi\Sigma^*$*-extensions*

Let $(\mathbb{F}, \sigma)$ be a difference field with $\mathbb{K} = \text{const}_\sigma\mathbb{F}$, $\boldsymbol{f} = (f_1, \dots, f_n) \in \mathbb{F}^n$ and $p \in \mathbb{F}$. We write $\sigma(\boldsymbol{f}) := (\sigma(f_1), \dots, \sigma(f_n))$ and $\boldsymbol{f}\,p := (f_1\,p, \dots, f_n\,p)$. Let $\mathbb{V}$ be a subspace of $\mathbb{F}$ over $\mathbb{K}$. The *solution space* for $\boldsymbol{0} \neq \boldsymbol{a} = (a_1, a_2) \in \mathbb{F}^2$ and $\boldsymbol{f}$ in $\mathbb{V}$ is defined by

$$\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{V}) := \{(c_1, \dots, c_n, g) \in \mathbb{K}^n \times \mathbb{V} \,|\, (5) \text{ holds}\}.$$

It forms a subspace of the $\mathbb{K}$-vector space $\mathbb{K}^n \times \mathbb{F}$. In particular, note that the dimension is at most $n+1$; see [11]. If $\boldsymbol{a} = (1, -1)$, we write in short

$$\mathrm{V}(\boldsymbol{f}, \mathbb{V}) := \mathrm{V}((1, -1), \boldsymbol{f}, \mathbb{V}).$$

Thus finding bases of $\mathrm{V}(\boldsymbol{f}, \mathbb{V})$ or $\mathrm{V}(\boldsymbol{a}, \boldsymbol{f}, \mathbb{V})$ solves problem PT or PFDE, respectively.

Let $\mathbb{F}(t)$ be a rational function field. For a polynomial $p \in \mathbb{F}[t]$ the degree is denoted by $\deg(p)$; we set $\deg(0) = -1$. We define

$$\mathbb{F}(t)_{(\mathrm{r})} := \{\tfrac{p}{q} \,|\, p, q \in \mathbb{F}[t], \deg(p) < \deg(q)\}; \qquad \mathbb{F}[t]_k := \{p \in \mathbb{F}[t] \,|\, \deg(p) \leq k\}$$

for $k \geq -1$. Let $\boldsymbol{f} = (f_1, \dots, f_n) \in \mathbb{F}[t]^n$. Then $\text{coeff}(f_i, r)$ gives the r-th coefficient of $f_i \in \mathbb{F}[t]$. Moreover, we define $\text{coeff}(\boldsymbol{f}, r) = (\text{coeff}(f_1, r), \dots, \text{coeff}(f_n, r))$.

Extensions and reordering. We will exploit the following fact frequently: If $(\mathbb{G}(t), \sigma)$ is a $\Pi\Sigma^*$-extension of $(\mathbb{G}, \sigma)$ and $(\mathbb{G}(t)(t_1)\dots(t_e), \sigma)$ is a $\Pi\Sigma^*$-extension of $(\mathbb{G}(t), \sigma)$ over $\mathbb{G}$, we can reorder $(\mathbb{G}(t)(t_1)\dots(t_e), \sigma)$ to the $\Pi\Sigma^*$-extension $(\mathbb{G}(t_1)\dots(t_e)(t), \sigma)$ of $(\mathbb{G}, \sigma)$. We call a difference field $(\mathbb{G}'(t_1)\dots(t_e), \sigma)$ a $\Pi\Sigma^*$-extension (resp. Σ^*-extension/Π-extension) of $(\mathbb{G}(t_1)\dots(t_e), \sigma)$ if there is a $\Pi\Sigma^*$-extension (resp. Σ^*-extension/Π-extension)

$(\mathbb{G}(t_1)\dots(t_e)(x_1)\dots(x_r),\sigma)$ of $(\mathbb{G}(t_1)\dots(t_e),\sigma)$ over $\mathbb{G}$ such that we get the difference field $(\mathbb{G}'(t_1)\dots(t_e),\sigma)$ by reordering the difference field $(\mathbb{G}(t_1)\dots(t_e)(x_1)\dots(x_r),\sigma)$. Note that here $\mathbb{G}'$ could be $\mathbb{G}(x_1)\dots(x_r)$, but additional reordering might be possible. In a nutshell, we enlarge the ground field $\mathbb{G}$ by additional $\Pi\Sigma^*$-extensions to the field $\mathbb{G}'$ where $(\mathbb{G}'(t_1)\dots(t_e),\sigma)$ is still a $\Pi\Sigma^*$-extension of $(\mathbb{G}(t_1)\dots(t_e),\sigma)$.

Basic properties. The next properties follow from Karr's theory [11, 12].

Lemma 15 ([12, Lemmas 4.1,4.2])**.** *Let $(\mathbb{H}(x),\sigma)$ with $\sigma(x)=\alpha x+\beta$ be a $\Pi\Sigma^*$-extension of $(\mathbb{H},\sigma)$. Let $a,f\in\mathbb{H}$ and suppose there is a solution $g\in\mathbb{H}(x)$ with $\sigma(g)-ag=f$, but no solution in $\mathbb{H}$. If x is a Π-extension, then $f=0$ and $a=\frac{\sigma(h)}{h}\alpha^m$ for some $h\in\mathbb{H}^*$ and $m\neq 0$; if x is a Σ^*-extension, then $f\neq 0$ and $a=1$.*

Corollary 16. *Let $(\mathbb{S},\sigma)$ be a Σ^*-extension of $(\mathbb{F},\sigma)$. If $(\mathbb{F}(t),\sigma)$ is a Π-extension of $(\mathbb{F},\sigma)$ with $\sigma(t)=a\,t$, then $(\mathbb{S}(t),\sigma)$ is a Π-extension of $(\mathbb{S},\sigma)$ with $\sigma(t)=a\,t$.*

Proof. Write $\mathbb{S}=\mathbb{F}(s_1)\dots(s_e)$ with the Σ^*-extensions s_i. If $e=0$, nothing has to be shown. Suppose that $(\mathbb{S}(t),\sigma)$ is not a Π-extension of $(\mathbb{S},\sigma)$. Then we find a $g\in\mathbb{F}(s_1)\dots(s_e)$ with $\sigma(g)/g=a^m$ for some $m>0$ by Theorem 3.2. By Lemma 15 it follows that $g\in\mathbb{F}$. Hence $(\mathbb{F}(t),\sigma)$ is not a Π-extension of $(\mathbb{F},\sigma)$ by Theorem 3.2. $\square$

Proposition 17. *Let $(\mathbb{E},\sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$ with $\mathbb{K}:=\text{const}_\sigma\mathbb{F}$ and $f\in\mathbb{F}$.*

(1) *If there is a $g\in\mathbb{E}\setminus\mathbb{F}$ such that (4), then there is no $g\in\mathbb{F}$ such that (4).*

(2) *Let $\mathbb{E}=\mathbb{F}(t_1)\dots(t_e)$ with $\sigma(t_i)-t_i\in\mathbb{F}$ or $\frac{\sigma(t_i)}{t_i}\in\mathbb{F}$ for $1\leq i\leq e$. If $g\in\mathbb{E}$ s.t. (4), then $g=\sum_{i=1}^e c_i\,t_i+w$ where $c_i\in\mathbb{K}$ and $w\in\mathbb{F}$; moreover, $c_i=0$, if $\sigma(t_i)/t_i\in\mathbb{F}$.*

Proof. **(1)** Assume there are such $g'\in\mathbb{F}$ and $g\in\mathbb{E}\setminus\mathbb{F}$. Then $\sigma(g-g')=(g-g')$, and thus $\text{const}_\sigma\mathbb{E}\neq\text{const}_\sigma\mathbb{F}$, a contradiction that $(\mathbb{E},\sigma)$ is a $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$. **(2)** is a special case of [12, Result, page 314]; see also [25, Thm. 4.2.1]. $\square$

Proposition 18. *Let $(\mathbb{F},\sigma)$, $(\mathbb{F}',\sigma')$ be difference fields with a σ-isomorphism $\tau:\mathbb{F}\to\mathbb{F}'$.*

(1) *Let $(\mathbb{F}(t),\sigma)$ and $(\mathbb{F}'(t'),\sigma')$ be Σ^*-extensions of $(\mathbb{F},\sigma)$ and $(\mathbb{F}',\sigma')$, respectively, with $g\in\mathbb{F}'(t')\setminus\mathbb{F}'$ such that $\sigma'(g)-g=\tau(\sigma(t)-t)$. Then there is a σ-isomorphism $\tau':\mathbb{F}(t)\to\mathbb{F}'(t')$ with $\tau'(t)=g$ and $\tau'(f)=\tau(f)$ for all $f\in\mathbb{F}$.*

(2) *Let $(\mathbb{F}(t),\sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$ with $\sigma(t)=\alpha\,t+\beta$. Then there is a $\Pi\Sigma^*$-extension $(\mathbb{F}'(t'),\sigma)$ of $(\mathbb{F}',\sigma)$ with $\sigma(t')=\tau(\alpha)t'+\tau(\beta)$. Moreover, there is the σ-isomorphism $\tau':\mathbb{E}\to\mathbb{E}'$ where $\tau'(t)=t'$ and $\tau'(a)=\tau(a)$ for all $a\in\mathbb{F}$.*

(3) *Let $(\mathbb{E},\sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$ Then there is a $\Pi\Sigma^*$-extension $(\mathbb{E}',\sigma)$ of $(\mathbb{F}',\sigma)$ with a σ-isomorphism $\tau':\mathbb{E}\to\mathbb{E}'$ where $\tau'(a)=\tau(a)$ for all $a\in\mathbb{F}$.*

Proof. **(1)** Let $\beta:=\sigma(t)-t\in\mathbb{F}$, and let $g\in\mathbb{F}'(t')\setminus\mathbb{F}'$ such that $\sigma'(g)-g=\tau(\beta)$. By Proposition 17.2 there are a $0\neq c\in\text{const}_\sigma\mathbb{F}$ and a $w\in\mathbb{F}'$ such that $g=c\,t'+w$. Since t' is transcendental over $\mathbb{F}'$, also g is transcendental over $\mathbb{F}'$. Therefore we can define the field isomorphism $\tau':\mathbb{F}(t)\to\mathbb{F}'(g)$ with $\tau'(t)=g$ and $\tau'(f)=\tau(f)$ for all $f\in\mathbb{F}$. We have $\tau'(\sigma(t))=\tau'(t+\beta)=g+\tau(\beta)=\sigma'(g)=\sigma'(\tau'(t))$ and thus τ' is a σ-isomorphism. Since $\mathbb{F}'(g)=\mathbb{F}(t')$, the first part is proven. **(2)** Let $(\mathbb{F}(t),\sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$ with $\sigma(t)=\alpha\,t+\beta$. Since τ is a σ-isomorphism, there is a $\Pi\Sigma^*$-extension $(\mathbb{F}'(t'),\sigma)$ of $(\mathbb{F}',\sigma')$ with $\sigma(t')=\tau(\alpha)t'+\tau(\beta)$ by Theorem 3.1. We can construct the field isomorphism $\tau':\mathbb{F}(t)\to\mathbb{F}'(t')$ with $\tau'(t)=t'$ and $\tau'(a)=\tau(a)$ for all $a\in\mathbb{F}$. Since $\sigma(\tau'(t))=\tau'(\sigma(t))$, τ' is a σ-isomorphism. Iterative application of **(2)** shows **(3)**. $\square$

4. Preparing the stage I: Properties of ordered $\Pi\Sigma^\delta$-extensions

We will show the first properties of depth-optimal $\Pi\Sigma^*$-extensions; some of the following results and proofs are simplified and streamlined versions of [28]

Proposition 19. *A $\Pi\Sigma^*$-extension $(\mathbb{G}(t_1)\dots(t_e),\sigma)$ of $(\mathbb{G},\sigma)$ with $\delta(t_i)\leq 2$, $\sigma(t_1)=t_1+1$ and $\text{const}_\sigma\mathbb{G}=\mathbb{G}$ is depth-optimal.*

Proof. t_1 is depth-optimal. Suppose that t_k is not depth-optimal with $2\leq k\leq e$. Set $f:=\sigma(t_k)-t_k\in\mathbb{F}:=\mathbb{G}(t_1)\dots(t_{k-1})$. Then there is a $\Pi\Sigma^*$-extension $(\mathbb{F}(x_1)\dots(x_r),\sigma)$ of $(\mathbb{F},\sigma)$ with $\delta(x_i)=1$ and $g\in\mathbb{F}(x_1)\dots(x_r)\setminus\mathbb{F}$ such that (4). By Prop. 17.2, $q_j:=\sigma(x_j)-x_j\in\mathbb{G}$ for some x_j. Then $\sigma(q_jt_1)-q_jt_1=q_j$; a contradiction to Theorem 3.1. $\square$

Lemma 20. *Let $(\mathbb{E},\sigma)$ with $\mathbb{E}=\mathbb{F}(t_1)\dots(t_e)$ be an ordered $\Pi\Sigma^\delta$-extension of $(\mathbb{F},\sigma)$ with $\mathfrak{d}:=\delta(\mathbb{F})$ and $\delta(t_1)>\mathfrak{d}$. Let $f\in\mathbb{F}$ with $\delta(f)<\mathfrak{d}$. Then for any $g\in\mathbb{E}$ with (4), $\delta(g)\leq\mathfrak{d}$.*

Proof. Suppose we have (4) with $g\in\mathbb{E}$ and $m:=\delta(g)>\mathfrak{d}$. Hence g depends on one of the t_k, i.e., let $k\geq 1$ and $\mathbb{H}:=\mathbb{F}(t_1)\dots(t_{k-1})$ such that $g\in\mathbb{H}(t_k)\setminus\mathbb{H}$. By Proposition 17.2, $g=c\,t_k+h$ where $h\in\mathbb{H}$, $0\neq c\in\text{const}_\sigma\mathbb{F}$ and $\beta:=\sigma(t_k)-t_k\in\mathbb{H}$. Since the extension is ordered, $\delta(t_k)=m$. By Proposition 17.1 there is no $g'\in\mathbb{H}$ with $\sigma(g')-g'=f$. Therefore by Theorem 3.1 one can construct a Σ^*-extension $(\mathbb{H}(s),\sigma)$ of $(\mathbb{H},\sigma)$ with $\sigma(s)=s+f$ where $\delta(s)=\delta(f)+1\leq\mathfrak{d}<m$. Note that $\sigma(g')-(g')=\beta$ with $g':=(s-h)/c\in\mathbb{H}(s)$. Hence t_k is not depth-optimal, a contradiction. Therefore $\delta(g)\leq\mathfrak{d}$. $\square$

Theorem 21 (cf. Result 3)**.** *Let $(\mathbb{F},\sigma)$ be an ordered $\Pi\Sigma^\delta$-extension of $(\mathbb{G},\sigma)$ and $f\in\mathbb{F}^*$. If $g\in\mathbb{F}$ as in (4), then (11).*

Proof. Since $\delta(\sigma(g))=\delta(g)\geq\delta(\sigma(g)-g)$, $\delta(g)\geq\delta(f)$. If $\delta(\mathbb{F})=\delta(f)$, then $\delta(g)=\delta(f)$. Otherwise, since $(\mathbb{F},\sigma)$ is ordered, we can split $\mathbb{F}$ into the $\Pi\Sigma^\delta$-extension $(\mathbb{F},\sigma)$ of $(\mathbb{H},\sigma)$ with $\mathbb{F}=\mathbb{H}(t_1)\dots(t_e)$ $(e\geq 0)$ and the $\Pi\Sigma^\delta$-extension $(\mathbb{H},\sigma)$ of $(\mathbb{G},\sigma)$ where $\delta(t_i)>\delta(f)+1$ for each $1\leq i\leq e$ and $\delta(\mathbb{H})=\delta(f)+1$. By Lemma 20, $\delta(g)\leq\delta(f)+1$. $\square$

Lemma 22. *Let $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ be a $\Pi\Sigma^\delta$-extension of $(\mathbb{F},\sigma)$ and $(\mathbb{F}(t_1)\dots(t_e)(x),\sigma)$ be a Σ^*-extension of $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ with $\delta(x)<\delta(t_i)$ for all $1\leq i\leq e$. By reordering, $(\mathbb{F}(x)(t_1)\dots(t_e),\sigma)$ is a $\Pi\Sigma^\delta$-extension of $(\mathbb{F}(x),\sigma)$.*

Proof. We show the lemma by induction. If $e=0$, nothing has to be shown. Consider $(\mathbb{F}(t_1)\dots(t_e)(x),\sigma)$ as claimed above with $e>0$. Then by the induction assumption $(\mathbb{F}(t_1)(x)(t_2)\dots(t_e),\sigma)$ is a $\Pi\Sigma^\delta$-extension of $(\mathbb{F}(t_1)(x),\sigma)$. Note that $(\mathbb{F}(x)(t_1),\sigma)$ is a $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$. If t_1 is a Π-extension, we are done. Otherwise, suppose that t_1 is a Σ^*-extension with $f:=\sigma(t_1)-t_1\in\mathbb{F}$ which is not depth-optimal. Then there is a Σ^*-extension $(\mathbb{H},\sigma)$ of $(\mathbb{F}(x),\sigma)$ with extension depth $<\delta(t_1)$ and $g\in\mathbb{H}$ such that (4). Since $\delta(x)<\delta(t_1)$, $(\mathbb{H},\sigma)$ is a Σ^*-extension of $(\mathbb{F},\sigma)$ with extension depth $<\delta(t_1)$. A contradiction that $(\mathbb{F}(t_1),\sigma)$ is a Σ^δ-extension of $(\mathbb{F},\sigma)$. $\square$

The following two propositions will be heavily used in Section 6.

Proposition 23. *Let $(\mathbb{E},\sigma)$ with $\mathbb{E}=\mathbb{F}(t_1)\dots(t_e)$ be an ordered $\Pi\Sigma^\delta$-extension of $(\mathbb{F},\sigma)$ where $\mathfrak{d}:=\delta(\mathbb{F})$ and $\delta(t_i)>\mathfrak{d}$. Suppose that $(\mathbb{F}(x_1)\dots(x_r),\sigma)$ is a Σ^*-extension of $(\mathbb{F},\sigma)$ with $\beta_i:=\sigma(x_i)-x_i$ and $\delta(x_i)\leq\mathfrak{d}$ for $1\leq i\leq r$. Then the following holds.*

(1) *There is the* Σ^**-extension* $(\mathbb{E}(x_1)\dots(x_r),\sigma)$ *of* $(\mathbb{E},\sigma)$ *with* $\sigma(x_i) = x_i + \beta_i$ *for all* i *with* $1 \leq i \leq r$.

(2) *In particular, by reordering, we get the* $\Pi\Sigma^*$*-extension* $(\mathbb{F}(x_1)\dots(x_r)(t_1)\dots(t_e),\sigma)$ *of* $(\mathbb{F},\sigma)$*;* $(\mathbb{F}(x_1)\dots(x_r)(t_1)\dots(t_e),\sigma)$ *is a* $\Pi\Sigma^\delta$*-extension of* $(\mathbb{F}(x_1)\dots(x_r),\sigma)$.

Proof. **(1)** Let $i \geq 1$ be minimal s.t. $(\mathbb{E}(x_1)\dots(x_i),\sigma)$ is not a Σ^*-extension of $(\mathbb{E},\sigma)$. Take $g \in \mathbb{E}(x_1)\dots(x_{i-1})$ s.t. $\sigma(g) - g = \beta_i$. Hence $\delta(g) \leq \mathfrak{d}$, i.e., $g \in \mathbb{F}(x_1)\dots(x_{i-1})$ by Lemma 20; this contradicts Thm. 3.1. Iterative application of Lemma 22 proves **(2)**. $\square$

Proposition 24. *Let* $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ *be a* $\Pi\Sigma^\delta$*-extension of* $(\mathbb{F},\sigma)$ *with* $\delta(t_i) = \mathfrak{d}$ *for* $1 \leq i \leq e$ *and* $\delta(\mathbb{F}) \leq \mathfrak{d}$. *For* $\tau \in S_e$*,* $(\mathbb{F}(t_{\tau(1)})\dots(t_{\tau(e)}),\sigma)$ *is a* $\Pi\Sigma^\delta$*-extension of* $(\mathbb{F},\sigma)$.

Proof. Let $e \geq 1$ ($e = 0$ is trivial); take u with $1 \leq u \leq e$ such that $(\mathbb{F}(t_{\tau(1)})\dots(t_{\tau(u)}),\sigma)$ is not a Σ^δ-extension of $(\mathbb{F}(t_{\tau(1)})\dots(t_{\tau(u-1)}),\sigma)$ with $f := \sigma(t_{\tau(u)}) - t_{\tau(u)} \in \mathbb{F}$. Choose a Σ^*-extension $(\mathbb{F}(t_{\tau(1)})\dots(t_{\tau(u-1)})(s_1)\dots(s_r),\sigma)$ of $(\mathbb{F}(t_{\tau(1)})\dots(t_{\tau(u-1)}),\sigma)$ with $\delta(s_i) < \mathfrak{d}$ and $g \in \mathbb{F}(t_{\tau(1)})\dots(t_{\tau(u-1)})(s_1)\dots(s_r)$ such that (4). Note that $(\mathbb{F}(s_1)\dots(s_r),\sigma)$ is a Σ^*-extension of $(\mathbb{F},\sigma)$ by reordering; hence by Prop. 23.2, $(\mathbb{F}(t_1)\dots(t_e)(s_1)\dots(s_r),\sigma)$ is a Σ^*-extension of $(\mathbb{F}(t_1)\dots(t_e),\sigma)$. Let $S = \{\tau(i) | 1 \leq i < u \wedge \sigma(t_{\tau(i)}) - t_{\tau(i)} \in \mathbb{F}\}$. By Prop. 17.2, $g = \sum_{i \in S} c_i t_i + h$ where $c_i \in \text{const}_\sigma\mathbb{F}$ and $h \in \mathbb{F}(s_1)\dots(s_r)$. Let $v \in S$ be maximal such $c_v \neq 0$; if $c_i = 0$ for all $i \in S$, set $v = 0$. If $v < \tau(u)$, $g \in \mathbb{F}(x_1)\dots(x_r)(t_1)\dots(t_w)$ for $w := \tau(u) - 1$. Otherwise, $\sigma(g') - g' = f_v$ with $g' = \frac{1}{c_v}(t_{\tau(u)} - h - \sum_{i \in S\setminus\{v\}} c_i t_i) \in \mathbb{F}(x_1)\dots(x_r)(t_1)\dots(t_w)$ for $w := v - 1$. Summarizing, $(\mathbb{F}(t_1)\dots(t_{w+1}),\sigma)$ is not a Σ^δ-extension of $(\mathbb{F}(t_1)\dots(t_w),\sigma)$, a contradiction. $\square$

Theorem 25 (cf. Res. 4)**.** *Let* $(\mathbb{F},\sigma)$ *be a* $\Pi\Sigma^*$*-extension of* $(\mathbb{G},\sigma)$ *and let* $(\mathbb{S},\sigma)$ *be a* Σ^**-extension of* $(\mathbb{F},\sigma)$ *which can be brought to an ordered* $\Pi\Sigma^\delta$*-extension of* $(\mathbb{G},\sigma)$*. Then for any* Σ^**-extension* $(\mathbb{H},\sigma)$ *of* $(\mathbb{F},\sigma)$ *with extension depth* d *there is a* Σ^**-extension* $(\mathbb{E},\sigma)$ *of* $(\mathbb{S},\sigma)$ *with extension depth* $\leq d$ *and an* $\mathbb{F}$*-monomorphism* $\tau : \mathbb{H} \to \mathbb{E}$ *s.t.* (12) *for* $a \in \mathbb{H}$.

Proof. Let $(\mathbb{D},\sigma)$ be an ordered $\Pi\Sigma^\delta$-extension of $(\mathbb{F},\sigma)$ that we get by reordering the Σ^*-extension $(\mathbb{S},\sigma)$ of $(\mathbb{F},\sigma)$. Moreover, let $(\mathbb{H},\sigma)$ be a Σ^*-extension of $(\mathbb{F},\sigma)$ with extension depth d, i.e., $\mathbb{H} := \mathbb{F}(t_1)\dots(t_e)$. Suppose that $\delta(t_i) \leq \delta(t_{i+1})$, otherwise we can reorder it without loosing any generality. We show that there is a Σ^*-extension $(\mathbb{E},\sigma)$ of $(\mathbb{D},\sigma)$ with extension depth $\leq d$ and an $\mathbb{F}$-monomorphism $\tau : \mathbb{H} \to \mathbb{E}$ with $\delta(\tau(a)) \leq \delta(a)$ for $a \in \mathbb{H}$. Then reordering of $(\mathbb{D},\sigma)$ proves the statement for the extension $(\mathbb{S},\sigma)$ of $(\mathbb{F},\sigma)$. If $e = 0$, i.e., $\mathbb{H} = \mathbb{F}$, the statement is proven by taking $(\mathbb{E},\sigma) := (\mathbb{D},\sigma)$ with the $\mathbb{F}$-monomorphism $\tau : \mathbb{F} \to \mathbb{D}$ where $\tau(a) = a$ for all $a \in \mathbb{F}$.

Otherwise, suppose that $\mathbb{H} := \mathbb{H}'(t)$ with $\mathbb{H}' := \mathbb{F}(t_1)\dots(t_{e-1})$ and $\beta := \sigma(t) - t \in \mathbb{H}'$ where $d' := \delta(t_{e-1})$ and $d := \delta(t) \geq d'$. Moreover, assume that there is a Σ^*-extension $(\mathbb{E}',\sigma)$ of $(\mathbb{D},\sigma)$ with extension depth $\leq d'$ together with an $\mathbb{F}$-monomorphism $\tau : \mathbb{H}' \to \mathbb{E}'$ such that (12) for all $a \in \mathbb{H}'$. Now define $f := \tau(\beta) \in \mathbb{E}$. By assumption

$$\delta(f) \leq \delta(\beta) < d. \tag{13}$$

Case 1: Suppose that there is no $g \in \mathbb{E}'$ as in (4). Then we can construct the Σ^*-extension $(\mathbb{E}'(y),\sigma)$ of $(\mathbb{E}',\sigma)$ with $\sigma(y) = y + f$ by Theorem 3.1 and can define the $\mathbb{F}$-monomorphism $\tau' : \mathbb{H}(t) \to \mathbb{E}'(y)$ such that $\tau'(a) = \tau(a)$ for all $a \in \mathbb{H}'$ and $\tau'(t) = y$. With (13) we get $\delta(y) = \delta(f) + 1 \leq d$. By the induction assumption, $\delta(\tau'(a)) \leq \delta(a)$ for all $a \in \mathbb{H}'(t)$. Clearly, the Σ^*-extension $(\mathbb{E}'(y),\sigma)$ of $(\mathbb{D},\sigma)$ has extension depth $\leq d$.

Case 2: Suppose there is a $y \in \mathbb{E}'$ with $\sigma(y) - y = f$. Since $(\mathbb{E}', \sigma)$ is a Σ^*-extension of $(\mathbb{D}, \sigma)$ with extension depth $\leq d'$ $(d' \leq d)$ we can apply Lemma 22 and obtain by reordering of $(\mathbb{E}', \sigma)$ an ordered $\Pi\Sigma^\delta$-extension $(\mathbb{G}(z_1)\dots(z_l)(x_1)\dots(x_u), \sigma)$ of $(\mathbb{G}(z_1)\dots(z_l), \sigma)$ where $\delta(\mathbb{G}(z_1)\dots(z_l)) \leq d$ and $\delta(x_j) > d$ for all $1 \leq j \leq u$. Hence with (13) we can apply Lemma 20 and it follows that $\delta(y) \leq d$, i.e.,

$$\delta(y) \leq \delta(t). \tag{14}$$

Since τ is a monomorphism, there is no g in the image $\tau(\mathbb{H}')$ such that (4). Since $(\tau(\mathbb{H}')(y), \sigma)$ is a difference field (it is a sub-difference field of $(\mathbb{E}', \sigma)$), y is transcendental over $\tau(\mathbb{H}')$ by Theorem 3.1. In particular, we get the $\mathbb{F}$-monomorphism $\tau' : \mathbb{H}'(t) \to \mathbb{E}'$ with $\tau'(a) = \tau(a)$ for all $a \in \mathbb{H}'$ and $\tau'(t) = y$. With (14) and our induction assumption it follows that $\delta(\tau'(a)) \leq \delta(a)$ for all $a \in \mathbb{H}'(t)$. This completes the induction step. $\square$

5. Preparing the stage II: A variation of Karr's reduction

We modify Karr's reduction for problem PT: Given a $\Pi\Sigma^*$-extension $(\mathbb{H}(t), \sigma)$ of $(\mathbb{H}, \sigma)$ and $\boldsymbol{f} \in \mathbb{H}(t)$; find a basis B of $\mathbb{V} := \mathrm{V}(\boldsymbol{f}, \mathbb{H}(t))$, as follows: First split $\boldsymbol{f} \in \mathbb{H}(t)^n$ by polynomial division in the form $\boldsymbol{f} = \boldsymbol{h} + \boldsymbol{p}$ such that $\boldsymbol{h} \in \mathbb{H}(t)^n_{(\mathrm{r})}$ and $\boldsymbol{p} \in \mathbb{H}[t]^n$; in short we write

$$\boldsymbol{f} = \boldsymbol{h} + \boldsymbol{p} \in \mathbb{H}(t)^n_{(\mathrm{r})} \oplus \mathbb{H}[t]^n. \tag{15}$$

Then the following lemma, a direct consequence of [33, Lemma 3.1], is crucial.

Lemma 26. *Let* $(\mathbb{H}(t), \sigma)$ *be a* $\Pi\Sigma^*$*-extension of* $(\mathbb{H}, \sigma)$ *with* $\sigma(t) = \alpha\, t + \beta$ *and* (15)*. Then* $\boldsymbol{c} \in \mathbb{K}^n$, $g = \rho + g' \in \mathbb{H}(t)_{(r)} \oplus \mathbb{H}[t]$ *solve* (3) *iff* $\sigma(\rho) - \rho = \boldsymbol{c}\boldsymbol{h}$ *and* $\sigma(g') - g' = \boldsymbol{c}\boldsymbol{p}$.

Note that we get a first strategy: Find bases for $\mathrm{V}(\boldsymbol{h}, \mathbb{H}(t)_{(\mathrm{r})})$ and $\mathrm{V}(\boldsymbol{p}, \mathbb{H}[t])$, and afterwards combine the solutions accordingly to get a basis of $\mathrm{V}(\boldsymbol{f}, \mathbb{H}(t))$. As it will turn out, the following version, presented in Figure 1, is more appropriate: First solve the rational problem (RP); if there is no solution, there is no solution for the original problem. Otherwise plug in the rational solutions into our ansatz (3) and continue to find the polynomial solutions (problem PP); for details see Remark 27.

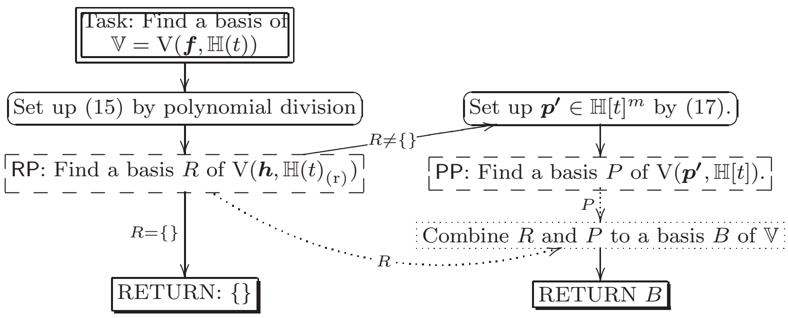

Fig. 1. The rational reduction.

Remark 27. Find a basis $R = \{(d_{i1}, \dots, d_{in}, \rho_i) \,|\, 1 \leq i \leq m\}$ of $\mathrm{V}(\boldsymbol{h}, \mathbb{H}(t)_{(\mathrm{r})})$; note that $m \leq n+1$. If $R = \{\}$, then $\mathrm{V}(\boldsymbol{f}, \mathbb{H}(t)) = \{\boldsymbol{0}\}$ by Lemma 26. Otherwise, define $\boldsymbol{D} = (d_{ij}) \in \mathbb{K}^{m \times n}$ and $\boldsymbol{\rho} = (\rho_1, \dots, \rho_m)$. Then we look for all $\boldsymbol{e} \in \mathbb{K}^m$ and $g' \in \mathbb{H}[t]$ such that

$$\sigma(\boldsymbol{e}\boldsymbol{\rho} + g') - (\boldsymbol{e}\boldsymbol{\rho} + g') = \boldsymbol{e}\boldsymbol{D}\boldsymbol{f}. \tag{16}$$

Since $\boldsymbol{Df} = \boldsymbol{Dh} + \boldsymbol{Dp}$ and $\sigma(\boldsymbol{e\rho}) - (\boldsymbol{e\rho}) = \boldsymbol{eDh}$ by construction, problem (16) is equivalent to looking for all $\boldsymbol{e} \in \mathbb{K}^m$ and $g' \in \mathbb{H}[t]$ s.t. $\sigma(g') - g' = \boldsymbol{eDp}$. Hence, set up

$$\boldsymbol{p'} := \boldsymbol{D}\,\boldsymbol{p} \tag{17}$$

where $\boldsymbol{p'} \in \mathbb{H}[t]^m$ and find a basis $P = \{(e_{i1}, \dots, e_{im}, g'_i) \mid 1 \leq i \leq l\}$ of $\mathbb{V}' = \mathrm{V}(\boldsymbol{p'}, \mathbb{H}[t])$. Note that $P \neq \{\}$, since there is the trivial solution $\sigma(1) - 1 = 0$; define $\boldsymbol{E} = (e_{ij})$ and $\boldsymbol{g'} = (g'_1, \dots, g'_l)$. Then with (16) it follows that $\sigma(\boldsymbol{E\rho} + \boldsymbol{p'}) - (\boldsymbol{E\rho} + \boldsymbol{g'}) = \boldsymbol{EDf}$. Thus, if we define $(g_1, \dots, g_l) := \boldsymbol{E\rho} + \boldsymbol{g'}$ and $(c_{i,j}) := \boldsymbol{ED} \in \mathbb{K}^{l \times n}$, we get a set of generators $B = \{(c_{i1}, \dots, c_{in}, g_i) \mid 1 \leq i \leq l\}$ that spans a subspace of $\mathbb{V} := \mathrm{V}(\boldsymbol{f}, \mathbb{H}(t))$. By simple linear algebra arguments it follows that B is a basis of $\mathbb{V}$.

Example 28. Consider the ordered $\Pi\Sigma^\delta$-field $(\mathbb{K}(k)(q)(b)(s), \sigma)$ over $\mathbb{K} = \mathbb{Q}(x, m)$ with (8) and $\boldsymbol{f} = (\frac{bq+s}{1+k+m})$. By (15) we have $\boldsymbol{h} = (0)$ and $\boldsymbol{p} = \boldsymbol{f}$. Following the reduction of Figure 1, we need a basis R of $\mathrm{V}(\boldsymbol{h}, \mathbb{K}(k)(q)(b)(s)_{(\mathrm{r})})$; obviously $R = \{(1, 0)\}$. Thus $\boldsymbol{p'} = \boldsymbol{p} = \boldsymbol{f}$ by (17). In Example 32 we will show that $P = \{(0, 1)\}$ is a basis of $\mathrm{V}(\boldsymbol{f}, \mathbb{K}(k)(q)(b)[s]))$. By Remark 27 a basis of $\mathrm{V}(\boldsymbol{f}, \mathbb{K}(k)(q)(b)(s))$ is $\{(0, 1)\}$.

Remark. In [11] the reversed strategy was proposed: First consider the polynomial and afterwards the rational problem. Related to problems DOT and DOT*, the following remark is in place. In Lemma 36 we will show that the solutions of RP are independent of the type of extension that are needed to solve problems DOT,DOT*. Thus, we will consider problem RP first. If there is no solution, we can stop; see Corollary 37. Otherwise, we will attack problems DOT,DOT on $\boldsymbol{p'}$ which is usually simpler ($m \leq n$) than $\boldsymbol{p}$.

The next lemma will be needed in Section 6.3 to solve problems DOT,DOT* efficiently.

Lemma 29. *Let $(\mathbb{H}(t), \sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{H}, \sigma)$ and $\boldsymbol{f} \in \mathbb{H}(t)^n$. If $a \in \mathbb{H}$ occurs in $\boldsymbol{f}$, then the reduction in Figure 1 can be applied so that a occurs in $\boldsymbol{p'}$.*

Proof. Let $a \in \mathbb{H}$ occur in the ith position of $\boldsymbol{f}$. Write $\boldsymbol{f} = \boldsymbol{h} + \boldsymbol{p} \in \mathbb{H}(t)^n_{(\mathrm{r})} \oplus \mathbb{H}[t]^n$. Then the ith entry in $\boldsymbol{h}$ is zero. Hence, we can take a basis R for $\mathrm{V}(\boldsymbol{h}, \mathbb{H}(t)_{(\mathrm{r})})$ where the ith unit-vector is in R. Applying (17) it follows that a occurs in $\boldsymbol{p'}$. □

5.1. *The rational problem* RP

Subproblem RP has been solved in [11, Sections 3.4, 3.5]. Alternatively, this task can be accomplished by computing a basis R' of $\mathbb{V}' := \mathrm{V}(\boldsymbol{h}, \mathbb{H}(t))$ with the algorithms presented in [31]. Namely, by [11, Cor. 1,2] it follows that $\mathbb{V}' = \mathrm{V}(\boldsymbol{h}, \mathbb{H}(t)_{(\mathrm{r})}) \oplus (\{0\}^n \times \mathbb{K})$. Hence a basis R for $\mathrm{V}(\boldsymbol{h}, \mathbb{H}(t)_{(\mathrm{r})})$ can be derived by simple manipulation of the basis R'.
We remark that both approaches can be solved algorithmically if $(\mathbb{H}, \sigma)$ is σ-computable.

5.2. *The polynomial problem* PP

As in Karr's reduction [11] we bound the degree of the polynomial solutions.

Lemma 30 ([11],Cor. 1,2)**.** *Let $(\mathbb{H}(t), \sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{H}, \sigma)$ and let $\boldsymbol{p'} = (p_1, \dots, p_m) \in \mathbb{H}[t]^m$. Then $\mathrm{V}(\boldsymbol{p'}, \mathbb{H}[t]_b) = \mathrm{V}(\boldsymbol{p'}, \mathbb{H}[t])$ for*

$$b := \begin{cases} \max(\deg(p_1), \dots, \deg(p_m), 1) & \text{if } t \text{ is a } \Pi\text{-extension} \\ \max(\deg(p_1), \dots, \deg(p_m), 1) + 1 & \text{if } t \text{ is a } \Sigma^*\text{-extension.} \end{cases} \tag{18}$$

Thus we set up $r := b \geq 0$ and $\boldsymbol{f_r} := \boldsymbol{p'} \in \mathbb{H}[t]_r^m$ and look for a basis B_r of $\mathbb{V}_r := \mathrm{V}(\boldsymbol{f_r}, \mathbb{H}[t]_r)$. We will accomplish this task by solving instances of problem PFDE in $(\mathbb{H}, \sigma)$; see also [11, Thm. 12] or [31, Sec. 3.3]. Note that this is possible if $(\mathbb{H}, \sigma)$ is σ-computable.

If $r = 0$, we are already in the base case. Otherwise, let $r > 0$. Then we try to get all $g = \sum_{i=0}^{r} g_i t^i \in \mathbb{H}[t]_r$ and $\boldsymbol{c} \in \mathbb{K}^m$ such that $\sigma(g) - g = \boldsymbol{c} \boldsymbol{f_r}$ as follows. First, we derive the possible leading coefficients g_r in $(\mathbb{H}, \sigma)$, then we plug in the resulting solutions into $\sigma(g) - g = \boldsymbol{c} \boldsymbol{f_r}$ and look for the remaining $g = \sum_{i=0}^{r-1} g_i t^i$ by recursion. The technical details are given in Remark 31, and a graphical illustration is presented in Figure 2: Here the task of finding a basis of $\mathbb{V}_r$ is reduced to finding a basis of the "leading coefficients" (problem CP) and to finding a basis of the "remaining coefficients" $\mathbb{V}_{r-1}$.

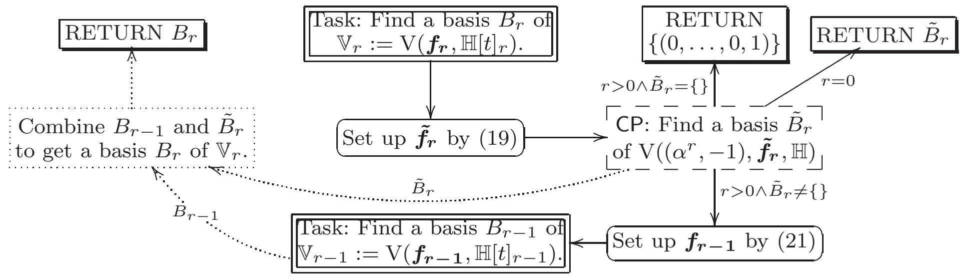

Fig. 2. The polynomial reduction.

Remark 31. The main task is to find a basis B_r of $\mathrm{V}(\boldsymbol{f_r}, \mathbb{H}[t]_r)$. First, define

$$\tilde{\boldsymbol{f}}_{\boldsymbol{r}} := \mathrm{coeff}(\boldsymbol{f_r}, r) \tag{19}$$

where $\tilde{\boldsymbol{f}}_{\boldsymbol{r}} \in \mathbb{H}^m$. Then find a basis $\tilde{B}_r = \{(c_{i1}, \dots, c_{im}, w_i)\}_{1 \leq i \leq \lambda}$ of $\mathrm{V}((\alpha^r, -1), \tilde{\boldsymbol{f}}_{\boldsymbol{r}}, \mathbb{H})$. If $\tilde{B}_r = \{\}$, then $\boldsymbol{c} = \boldsymbol{0}$ and $g \in \mathbb{H}[t]_{r-1}$ are the only choices, i.e., $B_r = \{(0, \dots, 0, 1)\}$. Otherwise, if $\tilde{B}_r \neq \{\}$, define

$$\boldsymbol{C} := (c_{ij}) \in \mathbb{K}^{\lambda \times m}, \quad \boldsymbol{g} := (w_1 t^r, \dots, w_\lambda t^r) \in t^r \mathbb{H}^\lambda, \tag{20}$$

$$\boldsymbol{f_{r-1}} := \boldsymbol{C} \boldsymbol{f_r} - (\sigma(\boldsymbol{g}) - \boldsymbol{g}). \tag{21}$$

By construction, $\boldsymbol{f_{r-1}} \in \mathbb{H}[t]_{r-1}^\lambda$. Now we proceed as follows. Find all $h \in \mathbb{H}[t]_{r-1}$ and $\boldsymbol{d} \in \mathbb{K}^\lambda$ such that

$$\sigma(h + \boldsymbol{d}\,\boldsymbol{g}) - (h + \boldsymbol{d}\,\boldsymbol{g}) = \boldsymbol{d}\,\boldsymbol{C}\,\boldsymbol{f_r} \tag{22}$$

which is equivalent to $\sigma(h) - h = \boldsymbol{d}\,\boldsymbol{f_{r-1}}$. Therefore the subtask is to find a basis B_{r-1} of $\mathbb{V}_{r-1} := \mathrm{V}(\boldsymbol{f_{r-1}}, \mathbb{H}[t]_{r-1})$; note that $B_{r-1} \neq \{\}$, since $\sigma(1) - 1 = 0$.

Given $B_{r-1} = \{(d_{i1}, \dots, d_{i\lambda}, h_i)\}_{1 \leq i \leq l}$ and $\tilde{B}_r$, a basis for $\mathrm{V}(\boldsymbol{f_r}, \mathbb{H}[t]_r)$ can be constructed as follows. Define $\boldsymbol{D} := (d_{ij}) \in \mathbb{K}^{l \times \lambda}$, $\boldsymbol{h} := (h_1, \dots, h_l) \in \mathbb{H}[t]_{r-1}^l$, $\boldsymbol{E} = (e_{ij}) = \boldsymbol{C}\boldsymbol{D} \in \mathbb{K}^{l \times m}$, and $(g_1, \dots, g_l) := (p_1, \dots, p_l) + \boldsymbol{D}\boldsymbol{q} \in \mathbb{H}[t]_r^l$. Then by (22) the set $B_r = \{(e_{i1}, \dots, e_{im}, g_i) \,|\, 1 \leq i \leq l\}$ spans a subspace of $\mathbb{V}_r = \mathrm{V}(\boldsymbol{p}, \mathbb{H}[t])$. By simple linear algebra arguments it follows that B_r is a basis of $\mathbb{V}_r$.

Example 32. Given the $\Pi\Sigma^*$-field $(\mathbb{H}(s), \sigma)$ over $\mathbb{K}$ with $\mathbb{H} = \mathbb{K}(k)(q)(b)$ and $\boldsymbol{p'} = (\frac{bq+s}{1+k+m})$ from Example 28, we compute a basis of $\mathrm{V}(\boldsymbol{p'}, \mathbb{H}[s])$ as follows. We start the reduction of Figure 2 with $r := 2$, see (18), and $\boldsymbol{f_r} := \boldsymbol{p'} = (\frac{bq+s}{1+k+m})$.

$r = 2$**:** By (19) we get $\tilde{\boldsymbol{f}}_{\boldsymbol{2}} = (0)$; a basis of $\mathrm{V}(\tilde{\boldsymbol{f}}_{\boldsymbol{2}}, \mathbb{H})$ is $\tilde{B}_2 = \{(1, 0), (0, 1)\}$.

$r = 1$**:** We get $\boldsymbol{f_{r-1}} = (\frac{bq+s}{1+k+m}, -b^2q^2 - 2bqs)$ by (21) and $\tilde{\boldsymbol{f}}_{\boldsymbol{1}} = (\frac{1}{1+k+m}, -2bq)$ by (19).

By another reduction in $\mathbb{H}$ we compute the basis $\tilde{B}_1 = \{(0, 0, 1)\}$ of $\mathrm{V}(\tilde{\boldsymbol{f}}_{\boldsymbol{1}}, \mathbb{H})$.

$r = 0$**:** $\boldsymbol{f_0} = (0)$ by (21). Clearly, $B_0 = \{(1,0),(0,1)\}$ is a basis of $\mathrm{V}(\boldsymbol{f_0}, \mathbb{H})$.
Finally, we combine $\tilde{B}_2$, $\tilde{B}_1$ and B_0 and get the basis $B_1 = \{(0,0,1)\}$, $B_2 = \{(0,1)\}$ of $\mathrm{V}(\boldsymbol{f_1}, \mathbb{H}[s]_1)$, $\mathrm{V}(\boldsymbol{f_1}, \mathbb{H}[s]_2)$, respectively. Thus B_2 is a basis of $\mathrm{V}(\boldsymbol{p'}, \mathbb{K}(k)(q)(b)[s])$.
We emphasize that the summand $\frac{1}{1+k+m}$ of h given in (9) occurs in $\tilde{\boldsymbol{f}}_{\boldsymbol{1}}$. This observation is crucial for our refined summation algorithm; see Example 55.

Remark 33. If $r = 0$, or if $r > 0$ and t is a Σ^*-extension ($\alpha = 1$), problem CP is nothing else than problem PT in the ground field $(\mathbb{H}, \sigma)$. Hence, we can apply again the reductions presented in the Figures 1 and 2 to the subfield $\mathbb{H}$. More precisely, if $(\mathbb{G}(t_1)\dots(t_e), \sigma)$ is a $\Pi\Sigma^*$-extension of a σ-computable $(\mathbb{G}, \sigma)$, we reduce problem PT to PT in the fields below whenever possible, and change to the more general situation PFDE only when it is necessary. This strategy will be the basis to construct Σ^δ-extensions in Section 6.

The following lemmata are needed for our refined algorithms. Lemma 34 is immediate by construction; it is used to prove Corollary 39. Lemma 35 is crucial in Section 6.3.

Lemma 34. *Let $(\mathbb{F}(x_1)\dots(x_e)(t), \sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{F}, \sigma)$ with $\sigma(t) = \alpha\, t + \beta$ where $\alpha, \beta \in \mathbb{F}$. Set $\mathbb{H} := \mathbb{F}(x_1)\dots(x_e)$, let $r > 0$, $\boldsymbol{f_r} \in \mathbb{F}[t]_r^m$ and $\tilde{\boldsymbol{f}}_{\boldsymbol{r}} \in \mathbb{F}^m$ with (19). If the coefficients with the monomial t^r in $\mathbb{V} := \mathrm{V}(\boldsymbol{f_r}, \mathbb{H}[t]_r)$ are free of the x_i, it suffices to take a basis of $\mathrm{V}((\alpha^r, -1), \tilde{\boldsymbol{f}}_{\boldsymbol{r}}, \mathbb{F})$ to get a basis of $\mathbb{V}$ following the reduction in Figure 2.*

Lemma 35. *Let $(\mathbb{H}(t), \sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{H}, \sigma)$ with $\sigma(t) = \alpha t + \beta$ and $\boldsymbol{p'} \in \mathbb{H}[t]^m$. Suppose we succeed in reducing the problem to the base case $r = 0$ with $\boldsymbol{f_0} \in \mathbb{F}^\lambda$.*
(1) *The reduction can be applied s.t. all entries of $\boldsymbol{p'}$ which are in $\mathbb{F}$ also occur in $\boldsymbol{f_0}$.*
(2) *Moreover, if $\alpha = 1$, we can guarantee that β occurs in $\boldsymbol{f_0}$.*

Proof. **(1)** Suppose that $a \in \mathbb{F}$ occurs in the ith position of $\boldsymbol{p'}$. Define $b \geq 0$ by (18) and set $r := b$. If $r = 0$, nothing has to be shown. Otherwise, suppose that $r > 0$. Since $a \in \mathbb{H}$, the ith entry in $\tilde{\boldsymbol{f}}_{\boldsymbol{r}}$, defined by (19), is zero. Hence we can choose a basis $\tilde{B}_r$ of $\mathrm{V}((\alpha^r, -1), \tilde{\boldsymbol{f}}_{\boldsymbol{r}}, \mathbb{H})$ where the ith unit-vector is in $\tilde{B}_r$ and all other elements of $\tilde{B}_r$ have zero in the ith position. With the corresponding $\boldsymbol{C}$ and (20) we get $\boldsymbol{f_{r-1}}$ where a pops up. This construction can be done for all such entries of $\boldsymbol{p'}$ which are in $\mathbb{F}$. If we continue with this refined reduction to the case $\boldsymbol{f_0}$, all $\mathbb{F}$-entries of $\boldsymbol{p'}$ occur in $\boldsymbol{f_0}$.
(2) Assume that t is a Σ^*-extension. Hence $r := b > 0$ by (18). Suppose we are in the reduction for $r = 1$. Since $\mathrm{V}(\tilde{\boldsymbol{f}}_{\boldsymbol{1}}, \mathbb{H})$ contains $\boldsymbol{b} := (0, \dots, 0, -1)$, we can choose a basis $\tilde{B}_1$ with $\boldsymbol{b} \in \tilde{B}_1$. By construction of $\boldsymbol{f_0}$ it follows that $\beta = \sigma(t) - t$ occurs in $\boldsymbol{f_0}$. $\square$

5.3. Key properties for refined algorithms

We focus on the problem when extensions do not contribute to solutions of PFDE.

Lemma 36. *Let $(\mathbb{H}(t), \sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{H}, \sigma)$ and $(\mathbb{H}(t)(x_1)\dots(x_e), \sigma)$ be a $\Pi\Sigma^*$-ext. of $(\mathbb{H}(t), \sigma)$ over $\mathbb{H}$. For $\boldsymbol{h} \in \mathbb{H}(t)^n_{(r)}$, $\mathrm{V}(\boldsymbol{h}, \mathbb{H}(x_1)\dots(x_e)(t)_{(r)}) = \mathrm{V}(\boldsymbol{h}, \mathbb{H}(t)_{(r)})$.*

Proof. Suppose we find an additional solution in a $\Pi\Sigma^*$-extension $(\mathbb{H}(t)(x_1)\dots(x_e), \sigma)$ of $(\mathbb{H}(t), \sigma)$ over $\mathbb{H}$, i.e., there is a $g \in \mathbb{H}(x_1)\dots(x_e)(t)_{(\mathrm{r})}$ such that g depends on x_e and $\sigma(g) - g = \boldsymbol{ch}$ for some $\boldsymbol{c} \in \mathbb{K}^n$. Take such a solution and define $f := \boldsymbol{ch} \in \mathbb{H}(t)_{(\mathrm{r})}$. Now reorder the extension to the $\Pi\Sigma^*$-extension $(\mathbb{H}(t)(x_1)\dots(x_e), \sigma)$ of $(\mathbb{H}, \sigma)$. With Proposition 17.2 it follows that $g = dx_e + w$ for some $d \in \mathbb{K}^*$ and $w \in \mathbb{H}(t)(x_1)\dots(x_{e-1})$. Write $w = \frac{w_1}{w_2}$ with $w_1, w_2 \in \mathbb{H}(x_1)\dots(x_e)[t]$; $w_2 \neq 0$. Then $g = \frac{dx_e w_2 + w_1}{w_2}$. Since w_1 and w_2 are free of x_e, $\deg_t(dx_e w_2 + w_1) \geq \deg_t(w_2)$, i.e., $g \notin \mathbb{F}(x_1)\dots(x_e)(t)_{(\mathrm{r})}$. $\square$

Corollary 37. *Let $(\mathbb{H}(t),\sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{H},\sigma)$ and let $(\mathbb{H}(t)(x_1)\dots(x_e),\sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{H}(t),\sigma)$ over $\mathbb{H}$. Let $\boldsymbol{f}\in\mathbb{F}(t)^n$ and take $\mathbb{V}:=\mathrm{V}(\boldsymbol{f},\mathbb{H}(t))$ and $\mathbb{V}':=\mathrm{V}(\boldsymbol{f},\mathbb{H}(t)(x_1)\dots(x_e))$. Write $\boldsymbol{f}=\boldsymbol{h}+\boldsymbol{p}$ as in* (15).

(1) *If $\mathrm{V}(\boldsymbol{h},\mathbb{H}(t)_{(r)})=\{\}$, then $\mathbb{V}=\mathbb{V}'=\{0\}^n\times\mathbb{K}$.*

(2) *Otherwise, define $\boldsymbol{p'}\in\mathbb{H}[t]^m$ and b by* (17) *and* (18)*. Then $\mathbb{V}=\mathbb{V}'$ iff $\mathrm{V}(\boldsymbol{p'},\mathbb{H}[t]_b)=\mathrm{V}(\boldsymbol{p'},\mathbb{H}(x_1)\dots(x_r)[t]_b)$. If R, P are bases of $\mathrm{V}(\boldsymbol{f},\mathbb{H}(t)_{(r)})$ and $\mathrm{V}(\boldsymbol{f},\mathbb{H}(x_1)\dots(x_r)[t]_b)$, respectively, we get a basis of $\mathrm{V}(\boldsymbol{f},\mathbb{H}(x_1)\dots(x_r)(t))$ as given in Remark 27.*

Proof. By Lemma 36 we have $\mathrm{V}(\boldsymbol{h},\mathbb{H}(t)_{(\mathrm{r})})=\mathrm{V}(\boldsymbol{h},\mathbb{H}(x_1)\dots(x_e)(t)_{(\mathrm{r})})$, i.e., R is also a basis of $\mathrm{V}(\boldsymbol{h},\mathbb{H}(x_1)\dots(x_e)(t))$. Thus, if $R=\{\}$, then $\mathbb{V}=\mathbb{V}'=\{0\}^n\times\mathbb{K}$; see Fig. 1. This proves **(1)**. Otherwise, let $\boldsymbol{p'}$ and b be as assumed. Note that b bounds the polynomial solutions in $\mathbb{H}[t]$ *and* $\mathbb{H}(x_1)\dots(x_e)[t]$ by Lemma 30. Hence, if $\mathrm{V}(\boldsymbol{p'},\mathbb{H}[t]_b)=\mathrm{V}(\boldsymbol{p'},\mathbb{H}(x_1)\dots(x_e)[t]_b)$, then by Remark 27 we get $\mathbb{V}=\mathbb{V}'$. Conversely, if $\mathrm{V}(\boldsymbol{p'},\mathbb{H}[t]_b)\subsetneq\mathrm{V}(\boldsymbol{p'},\mathbb{H}(x_1)\dots(x_e)[t]_b)$, then $\mathbb{V}\subsetneq\mathbb{V}'$ by construction. This proves **(2)**. □

Consequently, if one wants to find an extension with additional solutions, one has to focus on problem CP; see Fig. 2. With Lemma 38 we can refine this observation in Corollary 39.

Lemma 38. *Let $(\mathbb{H}(t),\sigma)$ be a Π-extension of $(\mathbb{H},\sigma)$ and let $(\mathbb{H}(t)(x_1)\dots(x_e),\sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{H}(t),\sigma)$ over $\mathbb{H}$. Set $\alpha:=\sigma(t)/t\in\mathbb{H}$. Then for $\boldsymbol{f}\in\mathbb{H}$ and $r>0$, $\mathrm{V}((\alpha^r,-1),\boldsymbol{f},\mathbb{H}(x_1)\dots(x_e))=\mathrm{V}((\alpha^r,-1),\boldsymbol{f},\mathbb{H})$.*

Proof. Suppose that $\mathrm{V}((\alpha^r,-1),\boldsymbol{f},\mathbb{H}(x_1)\dots(x_e))\supsetneq\mathrm{V}((\alpha^r,-1),\boldsymbol{f},\mathbb{H})$. Then there are $\boldsymbol{c}\in\mathbb{K}^n$ and $g\in\mathbb{H}(x_1)\dots(x_j)\setminus\mathbb{H}(x_1)\dots(x_{j-1})$ for some $j\geq 1$ such that $\alpha^r\sigma(g)-g=\boldsymbol{cf}=:f$. Note: there is no $g_0\in\mathbb{H}(t)(x_1)\dots(x_{j-1})$ with $\alpha^r\sigma(g_0)-g_0=f$. (Otherwise, $\alpha^r\sigma(g-g_0)-(g-g_0)$. Since $g\neq g_0$, we get $\sigma(g')/g'=\alpha^r$ for $g'=1/(g-g_0)$. But, by reordering we get the Π-extension $(\mathbb{H}(x_1)\dots(x_{j-1})(t),\sigma)$ of $(\mathbb{H}(x_1)\dots(x_{j-1}),\sigma)$, a contradiction by Theorem 3.2.) Thus we can apply Lemma 15: If x_j is a Π-extension with $\sigma(x_j)=a\,x_j$ for some $a\in\mathbb{H}(x_1)\dots(x_{j-1})$, $\alpha^r=\frac{\sigma(h)}{h}a^m$ for some $m\neq 0$ and $h\in\mathbb{H}(x_1)\dots(x_{j-1})$. Hence, $\alpha^r=\sigma(g)/g$ with $g=h\,x_j^m$. Otherwise, if x_j is a Σ^*-extension, $\alpha^r=1$. Summarizing, $\sigma(g)/g=\alpha^r$ for some $g\in\mathbb{H}(x_1)\dots(x_j)$. Since $(\mathbb{H}(x_1)\dots(x_j)(t),\sigma)$ is a Π-extension of $(\mathbb{H}(x_1)\dots(x_j),\sigma)$, this contradicts Theorem 3.2. □

Corollary 39. *Let $(\mathbb{H}(t),\sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{H},\sigma)$ and take a $\Pi\Sigma^*$-extension $(\mathbb{H}(t)(x_1)\dots(x_l)(y_1)\dots(y_k),\sigma)$ of $(\mathbb{H}(t),\sigma)$ over $\mathbb{H}$ where for $1\leq i\leq k$ we have $\frac{\sigma(y_i)}{y_i}$ or $\sigma(y_i)-y_i\in\mathbb{H}(x_1)\dots(x_l)$. Let $r>0$, $\boldsymbol{f_r}\in\mathbb{H}[t]_r^m$, and set $\mathbb{V}:=\mathrm{V}(\boldsymbol{f_r},\mathbb{H}[t]_r)$ and $\mathbb{V}':=\mathrm{V}(\boldsymbol{f_r},\mathbb{H}(x_1)\dots(x_l)(y_1)\dots(y_k)[t]_r)$. Define $\tilde{\boldsymbol{f}}_{\boldsymbol{r}}$ and $\boldsymbol{f_{r-1}}$ as in* (19) *and* (21).

(1) *$\sigma(t)-t\in\mathbb{F}$: If $\mathrm{V}(\tilde{\boldsymbol{f}}_{\boldsymbol{r}},\mathbb{H}(x_1)\dots(x_l))=\mathrm{V}(\tilde{\boldsymbol{f}}_{\boldsymbol{r}},\mathbb{H})$ and $\mathrm{V}(\boldsymbol{f_{r-1}},\mathbb{H}(x_1)\dots(x_l)(y_1)\dots(y_k)[t]_{r-1})=\mathrm{V}(\boldsymbol{f_{r-1}},\mathbb{H}[t]_{r-1})$, then $\mathbb{V}'=\mathbb{V}$. Given the basis B_{r-1} of $\mathrm{V}(\boldsymbol{f_{r-1}},\mathbb{H}(x_1)\dots(x_l)(y_1)\dots(y_k)[t]_{r-1})$ and the basis $\tilde{B}_r$ of $\mathrm{V}(\tilde{\boldsymbol{f}}_{\boldsymbol{r}},\mathbb{H}(x_1)\dots(x_l))$, one gets a basis of $\mathbb{V}'$ as stated in Remark 31.*

(2) *$\alpha:=\frac{\sigma(t)}{t}\in\mathbb{F}$: If $\mathrm{V}(\boldsymbol{f_{r-1}},\mathbb{H}(x_1)\dots(x_l)(y_1)\dots(y_k)[t]_{r-1})=\mathrm{V}(\boldsymbol{f_{r-1}},\mathbb{H}[t]_{r-1})$, then $\mathbb{V}'=\mathbb{V}$. Given the basis B_{r-1} of $\mathrm{V}(\boldsymbol{f_{r-1}},\mathbb{H}(x_1)\dots(x_l)(y_1)\dots(y_k)[t]_{r-1})$ and the basis $\tilde{B}_r$ of $\mathrm{V}((\alpha^r,-1),\tilde{\boldsymbol{f}}_{\boldsymbol{r}},\mathbb{H})$, one gets a basis of $\mathbb{V}'$ as stated in Remark 31.*

Proof. **(1)** t is a Σ^*-extension: By Prop. 17.2 the leading coefficients with degree r in the solutions of $\mathbb{V}'$ are free of $y_1,\dots,y_k$. Hence, by Lemma 34 it suffices to take a basis of $\mathrm{V}(\tilde{\boldsymbol{f}}_{\boldsymbol{r}},\mathbb{H}(x_1)\dots(x_l))$ to get a basis of $\mathbb{V}'$ following Remark 31. Thus, if $\mathrm{V}(\boldsymbol{f_{r-1}},\mathbb{H}[t]_{r-1})=$

$\mathrm{V}(\boldsymbol{f_{r-1}}, \mathbb{H}(x_1)\dots(x_l)(y_1)\dots(y_k)[t]_{r-1})$ and $\mathrm{V}(\tilde{\boldsymbol{f}}_{\boldsymbol{r}}, \mathbb{H}(x_1)\dots(x_l)) = \mathrm{V}(\tilde{\boldsymbol{f}}_{\boldsymbol{r}}, \mathbb{H})$, then $\mathbb{V} = \mathbb{V}'$ by Remark 31. Given B_{r-1}, $\tilde{B}_r$ from above one gets a basis of $\mathbb{V}'$.
(2) If $\sigma(t) = \alpha\, t$, then $\mathrm{V}((\alpha^r, -1), \tilde{\boldsymbol{f}}_{\boldsymbol{r}}, \mathbb{H}(x_1)\dots(x_l)(y_1)\dots(y_k)) = \mathrm{V}((\alpha^r, -1), \tilde{\boldsymbol{f}}_{\boldsymbol{r}}, \mathbb{H})$ by Lemma 38. Hence, if $\mathrm{V}(\boldsymbol{f_{r-1}}, \mathbb{H}(x_1)\dots(x_l)(y_1)\dots(y_k)[t]_{r-1}) = \mathrm{V}(\boldsymbol{f_{r-1}}, \mathbb{H}[t]_{r-1})$, $\mathbb{V} = \mathbb{V}'$ by Remark 31. Given the bases B_{r-1}, $\tilde{B}_r$ as stated above, one gets a basis of $\mathbb{V}'$. $\square$

6. Constructing $\Pi\Sigma^{\delta}$-extensions

Subsequently, we prove the following theorem which will establish Result 1 in Section 7.

Theorem 40. *Let $(\mathbb{F}, \sigma)$ be an ordered $\Pi\Sigma^{\delta}$-extension of $(\mathbb{G}, \sigma)$ and $f \in \mathbb{F}$. Then there is a Σ^*-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{F}, \sigma)$ such that $(\mathbb{E}, \sigma)$ can be brought to an ordered $\Pi\Sigma^{\delta}$-extension of $(\mathbb{G}, \sigma)$ and such that there is a $g \in \mathbb{E}$ as in (4). If $(\mathbb{G}, \sigma)$ is σ-computable, such an $(\mathbb{E}, \sigma)$ and g can be given explicitly.*

In order to accomplish this task, we consider the following more general situation.

Definition 41. Let $(\mathbb{F}, \sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{G}, \sigma)$ and $\boldsymbol{f} \in \mathbb{F}^n$. Then $(\mathbb{F}, \sigma)$ is $(\boldsymbol{f}, \mathfrak{d})$*–complete*, if for any $\Pi\Sigma^*$-extension[4] $(\mathbb{F}(x_1)\dots(x_u), \sigma)$ of $(\mathbb{F}, \sigma)$ with extension depth $\leq \mathfrak{d}$ we have $\mathrm{V}(\boldsymbol{f}, \mathbb{F}(x_1)\dots(x_u)) = \mathrm{V}(\boldsymbol{f}, \mathbb{F})$.

That is to say, we show the following theorem.

Theorem 42. *Let $(\mathbb{F}, \sigma)$ be an ordered $\Pi\Sigma^{\delta}$-extension of $(\mathbb{G}, \sigma)$, $\mathfrak{d} \geq 0$ and $\boldsymbol{f} \in \mathbb{F}^n$. Then there is a Σ^*-extension $(\mathbb{S}, \sigma)$ of $(\mathbb{F}, \sigma)$ which is $(\boldsymbol{f}, \mathfrak{d})$–complete and which can be brought to an ordered $\Pi\Sigma^{\delta}$-extension of $(\mathbb{G}, \sigma)$. If $(\mathbb{G}, \sigma)$ is σ-computable, then such an $(\mathbb{S}, \sigma)$ and a basis of $\mathrm{V}(\boldsymbol{f}, \mathbb{S})$ can be given explicitly.*

Then Theorem 40 is implied by the following lemma.

Lemma 43. *Let $(\mathbb{F}, \sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{G}, \sigma)$ and $\boldsymbol{f} \in \mathbb{F}^n$. If $(\mathbb{F}, \sigma)$ is $(\boldsymbol{f}, \delta(\boldsymbol{f})+1)$–complete, then $\dim \mathrm{V}(\boldsymbol{f}, \mathbb{F}) = n + 1$.*

Proof. Suppose $\dim \mathbb{V} < n + 1$, i.e., there is a $\boldsymbol{c} \in \mathbb{K}^n$ such that there is no $g \in \mathbb{F}$ with $\sigma(g) - g = \boldsymbol{cf} =: f$. Thus there is the Σ^*-extension $(\mathbb{F}(s), \sigma)$ of $(\mathbb{F}, \sigma)$ with $\sigma(s) = s + f$ and $\delta(s) \leq \delta(\boldsymbol{f}) + 1$. Hence $(\mathbb{F}, \sigma)$ is not $(\boldsymbol{f}, \delta(\boldsymbol{f}) + 1)$–complete. $\square$

Namely, we conclude by Theorem 42 that there is a Σ^*-extension $(\mathbb{S}, \sigma)$ of $(\mathbb{F}, \sigma)$ which is $((f), \delta(\mathbb{F}) + 1)$–complete and which can be brought to an ordered $\Pi\Sigma^{\delta}$-extension of $(\mathbb{G}, \sigma)$. Hence by Lemma 43 there is a $g \in \mathbb{S}$ such that (4).

In most applications one works with a $\Pi\Sigma^*$-field over $\mathbb{G}$, with $\sigma(k) = k + 1$ for some $k \in \mathbb{F}$. In this case, the following shortcut can be applied; the proof is similar to Prop. 19.

Lemma 44. *Let $(\mathbb{F}, \sigma)$ be a $\Pi\Sigma^*$-field over $\mathbb{G}$ and $\boldsymbol{f} \in \mathbb{F}^n$. If $\sigma(g) - g \in \mathrm{const}_{\sigma}\mathbb{G}^*$ for some $g \in \mathbb{F}$, then $(\mathbb{F}, \sigma)$ is $(\boldsymbol{f}, 1)$–complete.*

[4] Note that for later applications we could restrict to the case that all x_i are Σ^*-extensions.

6.1. A constructive proof

The proof of Theorem 42 will be obtained by refining the reduction of Section 5. Namely, let $\mathfrak{d} > 0$, let $(\mathbb{E}, \sigma)$ with $\mathbb{E} = \mathbb{F}(t_1) \dots (t_e)$ be an ordered $\Pi\Sigma^{\delta}$-extension of $(\mathbb{G}, \sigma)$ where $e = 0$ or $\delta(t_1) \geq \delta(\mathbb{F})$, and let $\boldsymbol{f} \in \mathbb{E}^n$. Then loosely speaking, we will obtain an $(\boldsymbol{f}, \mathfrak{d})$–complete extension $(\mathbb{S}, \sigma)$ of $(\mathbb{E}, \sigma)$ by constructing step by step a tower of extensions, say $\mathbb{F} = \mathbb{F}_0 \leq \mathbb{F}_1 \leq \mathbb{F}_2 \leq \dots \leq \mathbb{F}_l$, where $(\mathbb{F}_i, \sigma)$ is a Σ^{δ}-extension of $(\mathbb{F}_{i-1}, \sigma)$ for $1 \leq i \leq l$. Within this construction problem CP in Figure 2 will be refined to the following subproblem: we are given a vector $\boldsymbol{f'}$ with entries from $\mathbb{F}_{i-1}$, and we have to enrich $\mathbb{F}_{i-1}$ by Σ^{δ}-extensions to $\mathbb{F}_i$ such that $\mathbb{F}_i$ becomes $(\boldsymbol{f'}, \mathfrak{d} - 1)$-compete. Note that during this extension process it is crucial that $(\mathbb{F}_i(t_1) \dots (t_e), \sigma)$ forms a $\Pi\Sigma^{\delta}$-extension of $(\mathbb{F}_i, \sigma)$ for each $1 \leq i \leq l$. In order to get a grip on this situation, we introduce the following definition, which reduces to Definition 45 when $e = 0$.

Definition 45. Let $(\mathbb{E}, \sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{F}, \sigma)$ with $\mathbb{E} = \mathbb{F}(t_1) \dots (t_e)$ and $\boldsymbol{f} \in \mathbb{F}^n$. Then $(\mathbb{E}, \sigma)$ is $(\boldsymbol{f}, \mathfrak{d}, \mathbb{F})$–*complete*, if for any $\Pi\Sigma^*$-extension $(\mathbb{E}(x_1) \dots (x_u), \sigma)$ of $(\mathbb{E}, \sigma)$ over $\mathbb{F}$ with extension depth $\leq \mathfrak{d}$ we have $\mathrm{V}(\boldsymbol{f}, \mathbb{F}(x_1) \dots (x_u)) = \mathrm{V}(\boldsymbol{f}, \mathbb{F})$.

Subsequently, we prove the following theorem which implies Theorems 42 and 40.

Theorem 46. *Let $\mathfrak{d} \geq 0$ and let $(\mathbb{F}(t_1) \dots (t_e), \sigma)$ be an ordered $\Pi\Sigma^{\delta}$-extension of $(\mathbb{G}, \sigma)$; if $e > 0$, $\delta(t_1) \geq \mathfrak{d}$. Let $\boldsymbol{f} \in \mathbb{F}^n$. Then there is a Σ^*-extension*[5] *$(\mathbb{F}'(t_1) \dots (t_e), \sigma)$ of $(\mathbb{F}(t_1) \dots (t_e), \sigma)$ with extension depth $\leq \mathfrak{d}$ such that $(\mathbb{F}'(t_1) \dots (t_e), \sigma)$ is $(\boldsymbol{f}, \mathfrak{d}, \mathbb{F}')$–complete and such that $(\mathbb{F}'(t_1) \dots (t_e), \sigma)$ is an ordered $\Pi\Sigma^{\delta}$-extension of $(\mathbb{G}, \sigma)$. If $(\mathbb{G}, \sigma)$ is σ-computable, such an $(\mathbb{S}, \sigma)$ and a basis of $\mathrm{V}(\boldsymbol{f}, \mathbb{F}'(t_1) \dots (t_e))$ can be given explicitly.*

We will show Theorem 46 by induction on the depth $\mathfrak{d}$. The base case $\mathfrak{d} = 0$ is covered by Lemma 47.1; the proof of Lemma 47 is immediate with Lemma 43.

Lemma 47. *Let $(\mathbb{E}, \sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{G}, \sigma)$, let $\boldsymbol{f} \in \mathbb{F}^n$ and set $\mathbb{V} := \mathrm{V}(\boldsymbol{f}, \mathbb{F})$. Then the following holds.*
(1) *$(\mathbb{E}, \sigma)$ is $(\boldsymbol{f}, 0, \mathbb{F})$–complete.*
(2) *If $\dim \mathbb{V} = n + 1$ (or $\dim \mathrm{V}(\boldsymbol{f}, \mathbb{E}) = n + 1$), $(\mathbb{E}, \sigma)$ is $(\boldsymbol{f}, i, \mathbb{F})$–complete for all $i \geq 0$.*
(3) *$(\mathbb{F}, \sigma)$ is $(\boldsymbol{f}, \delta(\boldsymbol{f}) + 1, \mathbb{F})$–complete iff $\dim \mathrm{V}(\boldsymbol{f}, \mathbb{F}) = n + 1$.*

In the following let $\mathfrak{d} > 0$ and let $(\mathbb{F}(t_1) \dots (t_e), \sigma)$ be an ordered $\Pi\Sigma^{\delta}$-extension of $(\mathbb{G}, \sigma)$; if $e > 0$, then $\delta(t_1) \geq \mathfrak{d}$.

Simplification I. Note that it suffices to restrict to the case that $\delta(t_i) = \mathfrak{d}$ for $1 \leq i \leq e$. Otherwise, let $r \geq 0$ be maximal such that $\delta(t_r) = \mathfrak{d}$. Then we show that there exists such a Σ^*-extension $(\mathbb{F}'(t_1) \dots (t_r), \sigma)$ of $(\mathbb{F}(t_1) \dots (t_r), \sigma)$ as required. Finally, by Proposition 23 we get the desired $\Pi\Sigma^{\delta}$-extension $(\mathbb{F}'(t_1) \dots (t_r) \dots (t_e), \sigma)$ of $(\mathbb{G}, \sigma)$.

The induction step uses another induction on the number of extensions in $\mathbb{F}$ with depth $\mathfrak{d}$. The base case and the induction step of this "internal induction" are considered in Sections 6.1.1 and 6.1.2, respectively.

[5] For further remarks on this construction see page 9.

6.1.1. The completion phase

The case $\delta(\mathbb{F}) < \mathfrak{d}$ (including $\mathbb{F} = \mathbb{G}$) is covered by the following consideration.

Simplification II. We can assume that $\mathfrak{d} = \delta(\mathbb{F}) + 1$ by Lemma 47: If we find such a Σ^*-extension $\mathbb{F}'(t_1)\dots(t_e)$ of $(\mathbb{F}(t_1)\dots(t_e), \sigma)$ which is $(\boldsymbol{f}, \delta(\mathbb{F}) + 1)$–complete, then $\dim \mathrm{V}(\boldsymbol{f}, \mathbb{F}'(t_1)\dots(t_e)) = n + 1$, and thus $\mathbb{F}'(t_1)\dots(t_e)$ is $(\boldsymbol{f}, i)$–complete for any $i \geq 0$. With this preparation the following lemma gives the key idea.

Lemma 48. *Let $\mathfrak{d} > 0$ and let $(\mathbb{E}, \sigma)$ with $\mathbb{E} := \mathbb{F}(t_1)\dots(t_e)$ be an ordered $\Pi\Sigma^\delta$-extension of $(\mathbb{F}, \sigma)$ with $\delta(\mathbb{F}) < \mathfrak{d}$ and $\delta(t_i) = \mathfrak{d}$ for $1 \leq i \leq e$. Let $\boldsymbol{f} = (f_1, \dots, f_n) \in \mathbb{F}^n$ and suppose that $(\mathbb{E}, \sigma)$ is $(\boldsymbol{f}, \mathfrak{d}-1)$–complete. Then any Σ^*-extension $(\mathbb{E}(s_1)\dots(s_r), \sigma)$ of $(\mathbb{E}, \sigma)$ with $\sigma(s_i) - s_i \in \{f_1, \dots, f_n\}$ for $1 \leq i \leq r$ is depth-optimal; in particular, $\delta(s_i) = \mathfrak{d}$.*

Proof. Let $(\mathbb{E}(s_1)\dots(s_r), \sigma)$ be such a Σ^*-extension of $(\mathbb{E}, \sigma)$ with constant field $\mathbb{K}$. First note that $\delta(s_i) = \mathfrak{d}$ for all $1 \leq i \leq r$: If there is an s_v with $\delta(s_v) < \mathfrak{d}$ and $\sigma(s_v) = s_v + f_j$, then by reordering we get the Σ^*-extension $(\mathbb{E}(s_v), \sigma)$ of $(\mathbb{E}, \sigma)$ with extension depth $< \mathfrak{d}$; a contradiction that $(\mathbb{E}, \sigma)$ is $(\boldsymbol{f}, \mathfrak{d} - 1)$–complete. Now suppose that s_u, $1 \leq u \leq r$, is not depth-optimal with $\sigma(s_u) = s_u + f_j$; set $\mathbb{H} := \mathbb{E}(s_1)\dots(s_{u-1})$. Then there is a Σ^*-extension $(\mathbb{H}(x_1)\dots(x_v), \sigma)$ of $(\mathbb{H}, \sigma)$ with $\delta(x_i) \leq \delta(f_j) < \mathfrak{d}$ for $1 \leq i \leq v$ such that there is a $g \in \mathbb{H}(x_1)\dots(x_v) \setminus \mathbb{H}$ with $\sigma(g) - g = f_j$. The Σ^*-extension $(\mathbb{E}(x_1)\dots(x_v)(s_1)\dots(s_{u-1}), \sigma)$ of $(\mathbb{E}, \sigma)$ is obtained by reordering (note that $\delta(x_i) < \mathfrak{d}, \delta(s_i) = \mathfrak{d}$). By Prop. 17.2, $g = \sum_{i=1}^{u-1} c_i s_i + g'$ where $c_i \in \mathbb{K}$, $g' \in \mathbb{E}(x_1)\dots(x_v) \setminus \mathbb{E}$. Hence $\sigma(g') - g' = \boldsymbol{c}\boldsymbol{f}$ for some $\boldsymbol{c} \in \mathbb{K}^n$, i.e., $(\mathbb{E}, \sigma)$ is not $(\boldsymbol{f}, \mathfrak{d}-1)$–complete, a contradiction. □

Namely, by our induction assumption we apply Theorem 46 and take a Σ^*-extension $(\mathbb{H}(t_1)\dots(t_e), \sigma)$ of $(\mathbb{F}(t_1)\dots(t_e), \sigma)$ with extension depth $< \mathfrak{d}$ which is an ordered $\Pi\Sigma^\delta$-extension $(\mathbb{G}, \sigma)$ and which is $(\boldsymbol{f}, \mathfrak{d} - 1)$–complete. Then we adjoin step by step Σ^*-extensions such that for all $1 \leq j \leq n$ there is a g with $\sigma(g) - g = f_j$ as follows:

$i := 0$. FOR $1 \leq j \leq n$ DO
  IF $\delta(f_j) = \mathfrak{d} - 1$ and $\nexists g \in \mathbb{H}(s_1)\dots(s_i)$ s.t. $\sigma(g) = g + f_j$ THEN adjoin the Σ^*-extension
    $(\mathbb{H}(s_1)\dots(s_{i+1}), \sigma)$ of $(\mathbb{H}(s_1)\dots(s_i), \sigma)$ with $\sigma(s_{i+1}) = s_{i+1} + f_j$; i:=i+1. FI
OD

Finally, we get a Σ^*-extension, say $(\mathbb{H}(t_1)\dots(t_e)(s_1)\dots(s_r), \sigma)$ of $(\mathbb{F}(t_1)\dots(t_e), \sigma)$; note that this extension process can be constructed explicitly if $(\mathbb{G}, \sigma)$ is σ-computable. We complete the base case (of the internal induction) by the following arguments.

- By Lemma 48 $(\mathbb{H}(t_1)\dots(t_e)(s_1)\dots(s_r), \sigma)$ is an ordered $\Pi\Sigma^\delta$-extension of $(\mathbb{G}, \sigma)$. By Prop. 24 we get the Σ^δ-extension $(\mathbb{H}'(t_1)\dots(t_e), \sigma)$ of $(\mathbb{H}, \sigma)$ with $\mathbb{H}' := \mathbb{H}(s_1)\dots(s_r)$. Hence $(\mathbb{H}'(t_1)\dots(t_e), \sigma)$ is an ordered $\Pi\Sigma^\delta$-extension of $(\mathbb{G}, \sigma)$.
- Since $\dim \mathrm{V}(\boldsymbol{f}, \mathbb{H}'(t_1)\dots(t_e)) = n + 1$, $(\mathbb{H}'(t_1)\dots(t_e), \sigma)$ is $(\boldsymbol{f}, \mathfrak{d}, \mathbb{H}')$–complete by Lemma 47; if $(\mathbb{G}, \sigma)$ is σ-computable, a basis of $\mathrm{V}(\boldsymbol{f}, \mathbb{H}')$ can be given explicitly.

Example 49. Take the ordered $\Pi\Sigma^\delta$-field $(\mathbb{F}(b), \sigma)$ over $\mathbb{K}$ with $\mathbb{F} = \mathbb{K}(k)(q)$ and (8); let $\boldsymbol{f} = (\frac{1}{1+k+m})$. By Lemma 44 $(\mathbb{F}(b), \sigma)$ is $(\boldsymbol{f}, 1)$–complete. Since there is no $g \in \mathbb{F}(b)$ with $\sigma(g) - g = \frac{1}{1+k+m}$, we get the Σ^δ-extension $(\mathbb{F}(b)(h), \sigma)$ of $(\mathbb{F}(b), \sigma)$ with $\sigma(h) = h + \frac{1}{1+k+m}$ by Lemma 48. By Proposition 24 we obtain the ordered $\Pi\Sigma^\delta$-field $(\mathbb{F}(h)(b), \sigma)$. By construction, $\mathbb{V} := \mathrm{V}(\boldsymbol{f}, \mathbb{F}(h)(b)) = \{(1, h), (0, 1)\}$, i.e., $\dim \mathbb{V} = 2$. Therefore $(\mathbb{F}(h)(b), \sigma)$ is $(\boldsymbol{f}, 2, \mathbb{F}(h))$–complete. Note: Since $(\mathbb{F}(b)(s), \sigma)$ with $\sigma(s) = s + q\,b$ is an ordered $\Pi\Sigma^\delta$-field with $\delta(s) > \delta(h)$, we get the ordered $\Pi\Sigma^\delta$-field $(\mathbb{F}(h)(b)(s), \sigma)$ by Proposition 23.

Example 50. Take the ordered $\Pi\Sigma^\delta$-field $(\mathbb{F}(s),\sigma)$ with $\mathbb{F}=\mathbb{K}(k)(q)(h)(b)$ from Example 49; let $\boldsymbol{f}=(-bqh,bq)$. First note that $(\mathbb{F}(s),\sigma)$ is $(\boldsymbol{f},2)$–complete and that there is no $g\in\mathbb{F}(s)$ with $\sigma(g)-g=-bqh$; see Example 63. Hence we can construct the Σ^δ-extension $(\mathbb{F}(s)(H),\sigma)$ of $(\mathbb{F}(s),\sigma)$ with $\sigma(H)=s-bqH$; by reordering we get the ordered $\Pi\Sigma^\delta$-field $(\mathbb{F}(H)(s),\sigma)$. A basis of $\mathrm{V}(\boldsymbol{f},\mathbb{F}(H)(s))$ is $\{(1,0,H),(0,1,s),(0,0,1)\}$. Clearly, $(\mathbb{F}(H)(s),\sigma)$ is $(\boldsymbol{f},3,\mathbb{F}(H))$–complete.

6.1.2. The reduction phase

We suppose that $\delta(\mathbb{F})\geq\mathfrak{d}>0$, i.e., $\mathbb{F}=\mathbb{H}(t)$ where $(\mathbb{H}(t),\sigma)$ is a $\Pi\Sigma^\delta$-extension of $(\mathbb{H},\sigma)$ with $\delta(t)\geq\mathfrak{d}$ and $\delta(t)\geq\delta(\mathbb{H})\geq\delta(t)-1$. As above, $(\mathbb{H}(t)(t_1)\dots(t_e),\sigma)$ is a $\Pi\Sigma^*$-extension of $(\mathbb{H}(t),\sigma)$; in particular with the Simplification I: if $e>0$, then

$$\delta(t)=\delta(t_1)=\dots=\delta(t_e). \tag{23}$$

With the following definition and Corollary 37 we obtain Corollary 52.

Definition 51. Let $(\mathbb{E},\sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{H},\sigma)$ with $\mathbb{E}=\mathbb{H}(t)(t_1)\dots(t_e)$ and $\boldsymbol{f}\in\mathbb{H}[t]_r^n$. Then $(\mathbb{E},\sigma)$ is $(\boldsymbol{f},\mathfrak{d},\mathbb{H}[t]_r)$*–complete*, if for any $\Pi\Sigma^*$-extension $(\mathbb{E}(x_1)\dots(x_u),\sigma)$ of $(\mathbb{E},\sigma)$ over $\mathbb{F}$ with extension depth $\leq\mathfrak{d}$ we have $\mathrm{V}(\boldsymbol{f},\mathbb{H}(x_1)\dots(x_u)[t]_r)=\mathrm{V}(\boldsymbol{f},\mathbb{H}[t]_r)$.

Corollary 52. *Let $\mathfrak{d}>0$ and let $(\mathbb{H}(t)(t_1)\dots(t_e),\sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{H},\sigma)$ with $\delta(t)\geq\mathfrak{d}$; if $e>0$, then* (23). *Let $\boldsymbol{f}\in\mathbb{H}(t)^n$, and define $\boldsymbol{h}$ and $\boldsymbol{p}$ by* (15). *Let R be a basis of $\mathrm{V}(\boldsymbol{h},\mathbb{H}(t)_{(r)})$ and set $\mathbb{V}:=\mathrm{V}(\boldsymbol{f},\mathbb{H}(t))$.*
(1) *If $R=\{\}$, $(\mathbb{H}(t)(t_1)\dots(t_e),\sigma)$ is $(\boldsymbol{f},\mathbb{H}(t),\mathfrak{d})$–complete; $\mathbb{V}=\{0\}^n\times\mathbb{K}$. Otherwise:*
(2) *Take $\boldsymbol{p'}\in\mathbb{H}[t]^m,b$ by* (17), (18). *If $(\mathbb{H}'(t)(t_1)\dots(t_e),\sigma)$ is a $\Pi\Sigma^*$-extension of $(\mathbb{H}(t_1)\dots(t_e),\sigma)$ that is $(\boldsymbol{p'},\mathbb{H}'[t]_b,\mathfrak{d})$-complete, $(\mathbb{H}'(t)(t_1)\dots(t_e),\sigma)$ is $(\boldsymbol{f},\mathbb{H}'(t),\mathfrak{d})$-complete. If P is a basis of $\mathrm{V}(\boldsymbol{f},\mathbb{H}'[t]_b)$, a basis of $\mathbb{V}$ can be constructed by Remark 27.*

Example 53. Consider the ordered $\Pi\Sigma^\delta$-field $(\mathbb{H}(s),\sigma)$ with $\mathbb{H}:=\mathbb{K}(k)(q)(b)$ and (8); let $\boldsymbol{f}=(\frac{bq+s}{1+k+m})$. As in Example 28 we get $\boldsymbol{h}=(0)$, $\boldsymbol{p}=\boldsymbol{f}$, $R=\{(1,0),(0,1)\}$, $\boldsymbol{p'}=\boldsymbol{f}$ and $r=2$. In Examples 55 and 56 we will construct the Σ^*-extension $(\mathbb{H}'(s),\sigma)$ of $(\mathbb{H}(s),\sigma)$ with $\mathbb{H}':=\mathbb{K}(k)(q)(h)(b)(H)$ and (9) where $(\mathbb{H}'(s),\sigma)$ is an ordered $\Pi\Sigma^\delta$-field which is $(\boldsymbol{f},3,\mathbb{H}'[s]_2)$–complete; we obtain the basis $P=\{(1,sh+H),(0,1)\}$ of $\mathrm{V}(\boldsymbol{p'},\mathbb{H}'[s]_2))$. Hence $(\mathbb{H}'(s),\sigma)$ is $(\boldsymbol{f},3,\mathbb{H}')$-complete by Corollary 52; a basis of $\mathrm{V}(\boldsymbol{f},\mathbb{H}'(s))$ is P.

Let $r:=b>0$ and set $\mathbb{H}_r:=\mathbb{H}$ and $\boldsymbol{f_r}:=\boldsymbol{p'}\in\mathbb{H}[t]_r^m$. Define $\tilde{\boldsymbol{f}}_{\boldsymbol{r}}$ as in (19). If t is a Π-extension, we can apply Corollary 54 (see below) and obtain the reduction $r\to r-1$.

Otherwise, if t is a Σ^*-extension, the following preprocessing step is necessary. By the induction assumption we can apply Theorem 46 and get a Σ^*-extension of $(\mathbb{H}_r,\sigma)$ with extension depth $<\mathfrak{d}$ which can be brought to an ordered $\Pi\Sigma^\delta$-extension $(\mathbb{H}_{r-1},\sigma)$ of $(\mathbb{G},\sigma)$ and which is $(\tilde{\boldsymbol{f}}_{\boldsymbol{r}},\mathfrak{d}-1,\mathbb{H}_{r-1})$–complete. By Proposition 23 we can adjoin the extensions t_i on top and get the ordered $\Pi\Sigma^\delta$-extension $(\mathbb{H}_{r-1}(t_1)\dots(t_e),\sigma)$ of $(\mathbb{G},\sigma)$. Now we are ready apply Corollary 54 ($\mathbb{H}_{r-1}$ is replaced by $\mathbb{H}$) and proceed with the reduction $r\to r-1$.

Corollary 54. *Let $\mathfrak{d}>0$ and let $(\mathbb{H}(t)(t_1)\dots(t_e),\sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{G},\sigma)$ with $\delta(t)\geq\mathfrak{d}$; if $e>0$, then* (23). *Let $(\mathbb{H}'(t)(t_1)\dots(t_e),\sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{H}(t)(t_1)\dots(t_e),\sigma)$ with extension depth $\leq\mathfrak{d}$. Let $r>0$ and $\boldsymbol{f_r}\in\mathbb{H}[t]_r^m$, and define $\tilde{\boldsymbol{f}}_{\boldsymbol{r}}$ and $\boldsymbol{f_{r-1}}$ as in* (19) *and* (21).

(1) $\sigma(t)-t\in\mathbb{F}$*: If* $(\mathbb{H},\sigma)$ *is* $(\tilde{\boldsymbol{f}}_{\boldsymbol{r}},\mathfrak{d}-1)$*–complete and in addition* $(\mathbb{H}'(t)(t_1)\dots(t_e),\sigma)$ *is* $(\boldsymbol{f}_{\boldsymbol{r-1}},\mathbb{H}'[t]_{r-1},\mathfrak{d})$*–complete, then* $(\mathbb{H}'(t)(t_1)\dots(t_e),\sigma)$ *is* $(\boldsymbol{f}_{\boldsymbol{r}},\mathbb{H}[t]_r,\mathfrak{d})$*–complete. If* $\tilde{B}_r$ *and* B_{r-1} *are bases of* $\mathrm{V}(\tilde{\boldsymbol{f}}_{\boldsymbol{r}},\mathbb{H})$ *and* $\mathrm{V}(\boldsymbol{f}_{\boldsymbol{r-1}},\mathbb{H}'[t]_{r-1})$*, respectively, we get a basis of* $\mathrm{V}(\boldsymbol{f}_{\boldsymbol{r}},\mathbb{H}'[t]_r)$ *following Remark 31.*

(2) $\alpha:=\frac{\sigma(t)}{t}\in\mathbb{F}$*: If* $(\mathbb{H}'(t)(t_1)\dots(t_e),\sigma)$ *is* $(\boldsymbol{f}_{\boldsymbol{r-1}},\mathbb{H}'[t]_{r-1},\mathfrak{d})$*-complete, then it follows that* $(\mathbb{H}'(t)(t_1)\dots(t_e),\sigma)$ *is* $(\boldsymbol{f}_{\boldsymbol{r}},\mathbb{H}'[t]_r,\mathfrak{d})$*–complete. If* $\tilde{B}_r$ *and* B_{r-1} *are bases of the solution spaces* $\mathrm{V}((\alpha^r,-1),\tilde{\boldsymbol{f}}_{\boldsymbol{r}},\mathbb{H})$ *and* $\mathrm{V}(\boldsymbol{f}_{\boldsymbol{r-1}},\mathbb{H}'[t]_{r-1})$*, respectively, we get a basis of* $\mathrm{V}(\boldsymbol{f}_{\boldsymbol{r}},\mathbb{H}'[t]_r)$ *following Remark 31.*

Proof. Let $(\mathbb{E},\sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{H}',\sigma)$ with extension depth $\mathfrak{d}$. Reorder it to $(\mathbb{H}'(x_1)\dots(x_l)(y_1)\dots(y_k)(t)(t_1)\dots(t_e),\sigma)$ with $\delta(x_i)<\mathfrak{d}$ for $1\leq i\leq l$ and $\delta(y_i)=\mathfrak{d}$ for $1\leq i\leq k$. Then by Corollary 39 $\mathrm{V}(\boldsymbol{f},\mathbb{E})=\mathrm{V}(\boldsymbol{f},\mathbb{F}(t_1)\dots(t_e))$. □

Summarizing, we obtain a reduction for $r=b,...,1$, which can be illustrated in Figure 3.

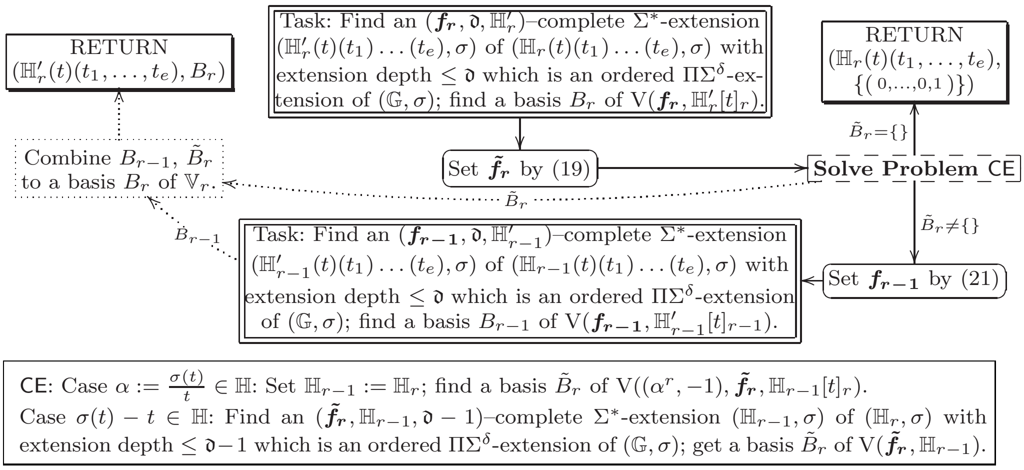

Fig. 3. The refined polynomial reduction.

Example 55. We continue the reduction from Example 53 with $r=2$.

$r=2$**:** Set $\boldsymbol{f_2}:=\boldsymbol{f}=(\frac{bq+s}{1+k+m})$. By (19) we get $\tilde{\boldsymbol{f}}_{\boldsymbol{2}}=(0)$. Clearly, $(\mathbb{H},\sigma)$ is $(\tilde{\boldsymbol{f}}_{\boldsymbol{2}},2)$–complete with the basis $\tilde{B}_2=\{(1,0),(0,1)\}$ of $\mathrm{V}(\tilde{\boldsymbol{f}}_{\boldsymbol{2}},\mathbb{H})$.

$r=1$**:** We get $\boldsymbol{f_1}=(\frac{bq+s}{1+k+m},-b^2q^2-2bqs)$ by (21) and $\tilde{\boldsymbol{f}}_{\boldsymbol{1}}=(\frac{1}{1+k+m},-2bq)$ by (19). We can construct the Σ^*-extension h with $\delta(h)=2$ which gives the ordered $\Pi\Sigma^\delta$-field $(\mathbb{K}(k)(q)(h)(b),\sigma)$ and which is $(\tilde{\boldsymbol{f}}_{\boldsymbol{1}},2)$–complete; a basis of $\mathrm{V}(\tilde{\boldsymbol{f}}_{\boldsymbol{1}},\mathbb{K}(k)(q)(h)(b))$ is $\tilde{B}_1=\{(1,0,h),(0,0,-1)\}$; see Example 57. This gives $\boldsymbol{f_0}=(-bqh,qp)$.

If $r=0$, we need a Σ^*-extension $(\mathbb{H}'_0(t)(t_1)\dots(t_e),\sigma)$ of $(\mathbb{H}_0(t_1)\dots(t_e),\sigma)$ with extension depth $\leq\mathfrak{d}$ which is an ordered $\Pi\Sigma^\delta$-extension of $(\mathbb{G},\sigma)$ and which is $(\boldsymbol{f_0},\mathfrak{d},\mathbb{H}'_0)$–complete.

Example 56. We continue Example 55 for the case $r=0$. By Ex. 50 we get the ordered $\Pi\Sigma^\delta$-field $(\mathbb{H}'(s),\sigma)$ with $\mathbb{H}'=\mathbb{K}(k)(q)(h)(b)(H)$ which is $(\boldsymbol{f_0},3,\mathbb{H})$–complete; a basis of $\mathrm{V}(\tilde{\boldsymbol{f}}_{\boldsymbol{0}},\mathbb{H}')$ is $\tilde{B}_0=\{(1,0,H),(0,0,1)\}$. Hence we obtain the bases $B_1=\{(1,0,sh+H),(0,0,1)\}$ of $\mathrm{V}(\boldsymbol{f_1},\mathbb{H}'(s))$ and $B_2=\{(1,sh+H),(0,1)\}$ of $\mathrm{V}(\boldsymbol{f_2},\mathbb{H}'(s))$.

Note that $\boldsymbol{f_0}$ is a vector in $\mathbb{H}_0$ which is a smaller field in the following sense: $(\mathbb{H}_0, \sigma)$ is a Σ^*-extension of $(\mathbb{H}, \sigma)$ with extension depth $< \mathfrak{d}$, but the extension t with $\delta(t) = \mathfrak{d}$ is eliminated (it pops up in the tower $(t)(t_1)\dots(t_e)$ above). Eventually, all extensions with depth $\mathfrak{d}$ are eliminated, and we get a difference field with depth $\mathfrak{d} - 1$; see Section 6.1.1.

Example 57. We are given the ordered $\Pi\Sigma^\delta$-field $(\mathbb{H}(b), \sigma)$ over $\mathbb{K}$ with $\mathbb{H} = \mathbb{K}(k)(q)$ and $\boldsymbol{f} = (\frac{1}{1+k+m}, -2bq)$. Following Figure 1 we get $\boldsymbol{p'} = \boldsymbol{f}$. The degree bound is 1 by (18). We start the reduction of Figure 3 with $r = 1$, set $\boldsymbol{f_1} := \boldsymbol{p'} = \boldsymbol{f}$, and get $\tilde{\boldsymbol{f_0}} = (0, -2q)$ by (19). A basis of $\mathrm{V}((\frac{1+m+K}{1+k}, -1), \tilde{\boldsymbol{f_0}}, \mathbb{H})$ is $\{(1, 0, 0)\}$. Hence $\boldsymbol{f_0} = (\frac{1}{1+k+m})$ by (21). Now we need an ordered $\Pi\Sigma^\delta$-field $(\mathbb{H}'(b), \sigma)$ which is a Σ^*-extension of $(\mathbb{H}(b), \sigma)$ with extension depth < 2 and which is $(\boldsymbol{f_0}, 2, \mathbb{H}')$–complete; note that b is eliminated. By Example 49 we get the $\Pi\Sigma^\delta$-field $(\mathbb{H}'(b), \sigma)$ with $\mathbb{H}' = \mathbb{K}(k)(q)(h)$ and the basis $B_0 = \{(1, h), (0, 1)\}$ of $\mathrm{V}(\boldsymbol{f_0}, \mathbb{H}')$. Completing the reduction, we get the basis $\{(1, 0, h), (0, 0, 1)\}$ of $\mathrm{V}(\boldsymbol{f}, \mathbb{H}'(b))$. By construction, $(\mathbb{H}'(b), \sigma)$ is $(\boldsymbol{f}, 2)$–complete.

6.2. Some refinements for Π*-extensions and polynomial extensions*

We sum up the construction from above: The derived Σ^δ-extensions are defined by entries of some vectors $\boldsymbol{f'}$ which occur within the reduction process; see Section 6.1.1. Internally, those vectors $\boldsymbol{f'}$ are determined by the reduction presented in Figure 1 (the rational reduction) and Figure 3 (the polynomial reduction). Exploiting additional properties in difference fields, we can predict how the $\boldsymbol{f'}$ and therefore the derived Σ^δ-extensions look like. The first result is needed in Lemma 65.

Corollary 58. *Let $(\mathbb{F}(y), \sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{G}, \sigma)$ with $\sigma(y)/y \in \mathbb{F}$ such that $(\mathbb{F}(y), \sigma)$ can be brought to an ordered $\Pi\Sigma^\delta$-extension of $(\mathbb{G}, \sigma)$; let $\boldsymbol{f} \in \mathbb{F}^n$ and $\mathfrak{d} \geq 0$. Then there is a Σ^*-extension of $(\mathbb{F}(y), \sigma)$ over $\mathbb{F}$ with extension depth $\leq \mathfrak{d}$ which can be brought to an ordered $\Pi\Sigma^\delta$-extension of $(\mathbb{G}, \sigma)$ and which is $(\boldsymbol{f}, \mathfrak{d}, \mathbb{F})$–complete.*

Proof. If $\delta(y) \geq \mathfrak{d}$, the corollary follows by Theorem 46 and Proposition 17.2. Let $\delta(y) < \mathfrak{d}$. We refine the inductive proof in Section 6. Suppose that the reduction holds for $\mathfrak{d} - 1$. As in Section 6 we assume that $(\mathbb{F}(t_1)\dots(t_e), \sigma)$ is a $\Pi\Sigma^*$-extension of $(\mathbb{G}, \sigma)$; if $e > 0$, then (23). Now reorder $(\mathbb{F}(y), \sigma)$ to an ordered $\Pi\Sigma^\delta$-extension $(\mathbb{F}', \sigma)$ of $(\mathbb{G}, \sigma)$. If $\delta(\mathbb{F}') < \mathfrak{d}$, we construct a Σ^*-extension over $\mathbb{F}$ as required; see Section 6.1.1. If $\delta(\mathbb{F}') \geq \mathfrak{d}$, set $\mathbb{F} = \mathbb{H}(t)$ with $\delta(t) \geq \mathfrak{d}$ as in Section 6.1.2 with $\sigma(t) = \alpha\, t + \beta$; note that $t \neq y$, since $\delta(y) < \mathfrak{d}$. Then by Corollary 52 $\boldsymbol{p'} \in \mathbb{H}[t]^m$. Define b by (18) and set $r := b$, $\mathbb{H}_r := \mathbb{H}$. Now we apply the reduction as given in Figure 3. Since $\tilde{\boldsymbol{f_r}}$ is free of y, we can apply the induction assumption: we can take –as required– a Σ^*-extension $(\mathbb{H}_{r-1}, \sigma)$ of $(\mathbb{H}_r, \sigma)$ where the new Σ^*-extensions do not depend on y. Note that $\mathrm{V}((\alpha^r, -1), \tilde{\boldsymbol{f_r}}, \mathbb{H}_r)$ is free of y by Proposition 17.2 (if t is a Σ^*-extension) or Lemma 38 (if t is a Π-extension). Hence $\boldsymbol{f_{r-1}}$ is free of y. Suppose we reach the base case (after at most r steps) with $\boldsymbol{f_0} \in \mathbb{H}_0$, free of y, where $(\mathbb{H}_0(t)(t_1)\dots(t_e), \sigma)$ is a Σ^*-extension of $(\mathbb{H}(t)(t_1)\dots(t_e), \sigma)$ which is free of y. By the reduction of the extensions with depth $\delta(t)$, the corollary follows. $\square$

Corollary 60 can be shown completely analogously by using the following Lemma 59.

Lemma 59 (([32],Thm. 2.5))**.** *Let $(\mathbb{H}(t_1)\dots(t_e), \sigma)$ be a polynomial $\Pi\Sigma^*$-extension of $(\mathbb{H}, \sigma)$. For all $g \in \mathbb{H}[t_1, \dots, t_e]$, $\sigma(g) - g \in \mathbb{H}[t_1, \dots, t_e]$ iff $g \in \mathbb{H}[t_1, \dots, t_e]$.*

Corollary 60. *Let $(\mathbb{H}(t_1)\dots(t_e),\sigma)$ be an ordered $\Pi\Sigma^{\delta}$-extension of $(\mathbb{G},\sigma)$ with $\delta(\mathbb{H})=d-1$ and $\delta(t_1)\geq d$ such that the Σ^{δ}-extension $(\mathbb{H}(t_1)\dots(t_e),\sigma)$ of $(\mathbb{H},\sigma)$ is polynomial; let $\boldsymbol{f}\in\mathbb{H}[t_1,\dots,t_e]^n$ and $\mathfrak{d}\geq 0$.*

(1) *Then there is Σ^*-extension $(\mathbb{H}'(s_1)\dots(s_r)(t_1)\dots(t_e),\sigma)$ of $(\mathbb{H}(t_1)\dots(t_e),\sigma)$ with $\delta(s_i)\geq d$ for $1\leq i\leq r$ and $\delta(\mathbb{H}')=d-1$ which can be brought to an ordered $\Pi\Sigma^{\delta}$-extension of $(\mathbb{G},\sigma)$ and which is $(\boldsymbol{f},\mathfrak{d})$–complete.*

(2) *In particular, the extension $(\mathbb{H}'(s_1)\dots(s_r)(t_1)\dots(t_e),\sigma)$ of $(\mathbb{H}',\sigma)$ is polynomial.*

(3) *We have $\mathrm{V}(\boldsymbol{f},\mathbb{H}'(s_1)\dots(s_r)(t_1)\dots(t_e))\subseteq\mathbb{K}^n\times\mathbb{H}'[s_1,\dots,s_r][t_1,\dots,t_e]$.*

(4) *It can be constructed explicitly, if $(\mathbb{G},\sigma)$ is σ-computable.*

Example 61. Consider the polynomial $\Pi\Sigma^{\delta}$-extension $(\mathbb{K}(k)(q)(b)(s),\sigma)$ of $(\mathbb{K}(k),\sigma)$ with (8) and let $f\in\mathbb{K}(k)[q,b,s]$. By Corollary 60 our construction will always yield a Σ^*-extension $(\mathbb{E},\sigma)$ of $(\mathbb{K}(k)(q)(b)(s),\sigma)$ with $\mathbb{E}=\mathbb{K}(k)(q)(b)(s)(s_1)\dots(s_r)$ where $(\mathbb{E},\sigma)$ is a polynomial $\Pi\Sigma^*$-extension of $(\mathbb{K}(k),\sigma)$; in particular, any solution g of (4) is in $\mathbb{K}(k)[q,b,s][s_1,\dots,s_r]$.

6.3. Algorithmic considerations: An optimal algorithm

The building blocks from above can be summarized to Algorithm 1.

Algorithm 1 `FindDepthCompleteExt`$(\boldsymbol{f},\mathfrak{d},\mathbb{F},\mathbb{F}(t_1)\dots(t_e))$

In: An ordered $\Pi\Sigma^{\delta}$-extension $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ of a σ-computable $(\mathbb{G},\sigma)$, $\boldsymbol{f}\in\mathbb{F}^n$, $\mathfrak{d}\geq 0$.

Out: A Σ^*-extension $(\mathbb{F}'(t_1)\dots(t_e),\sigma)$ of $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ s.t. $(\mathbb{F}'(t_1)\dots(t_e),\sigma)$ is an ordered $\Pi\Sigma^{\delta}$-extension of $(\mathbb{G},\sigma)$ and is $(\boldsymbol{f},\mathfrak{d},\mathbb{F}')$–complete; a basis of $\mathrm{V}(\boldsymbol{f},\mathbb{F}')$.

1 IF $\mathfrak{d}=0$ or [a] ($\mathfrak{d}=1$ and $\mathrm{const}_{\sigma}\mathbb{G}=\mathbb{G}$ and $\sigma(k)=k+1$ for some $k\in\mathbb{F}$) THEN
2 Compute a basis B of $\mathrm{V}(\boldsymbol{f},\mathbb{F})$; RETURN $(B,\mathbb{F}(t_1)\dots(t_e))$. FI
3 IF $e\geq 1$ and $\delta(t_e)>\mathfrak{d}$ THEN (*Simplification I*)
4 Let $r\geq 0$ be minimal such that $\delta(t_r)=\mathfrak{d}$.
5 Execute $(B,(\mathbb{F}'(t_1)\dots(t_r),\sigma))$=`FindDepthCompleteExt`$(\boldsymbol{f},\mathfrak{d},\mathbb{F},\mathbb{F}(t_1)\dots(t_r))$.
6 RETURN $(B,\mathbb{F}'(t_1)\dots(t_e))$ FI
7 IF $\delta(\mathbb{F})<\mathfrak{d}$ THEN $\mathfrak{d}:=\delta(\mathbb{F})+1$ (*Simplification II*)
8 Let [b] $(B,\mathbb{H}(t_1)\dots(t_e))$=`FindDepthCompleteExt`$(\boldsymbol{f},\mathfrak{d}-1,\mathbb{F}(t_1)\dots(t_e),\mathbb{F}(t_1)\dots(t_e))$
9 Construct a Σ^*-extension $(\mathbb{H}'(t_1)\dots(t_e),\sigma)$ of $(\mathbb{H}(t_1)\dots(t_e),\sigma)$ and a basis B' of $\mathrm{V}(\boldsymbol{f},\mathbb{H}')$ as in Sec. 6.1.1. Return $(B',\mathbb{H}'(t_1)\dots(t_e))$ FI

Reduction phase: Let $\mathbb{F}:=\mathbb{H}(t)$ with $\delta(t)\geq\delta(\mathbb{H})\geq\mathfrak{d}$; if $e>0$, then (23).

10 Follow the rational reduction as in Figure 1. Let R be a basis of $\mathrm{V}(\boldsymbol{r},\mathbb{H}(t)_{(\mathrm{r})})$.
11 IF $R=\{\}$, THEN RETURN $(\{(0,\dots,0,1)\},\mathbb{F}(t_1)\dots(t_e))$ FI
12 Take $\boldsymbol{p'}\in\mathbb{H}[t]^m$, b by (17) and (18). Apply the polynomial reduction [c] from Figure 3 for $r=b,\dots,1$. If the reduction stops earlier, return the corresponding result. Otherwise, take $\boldsymbol{f_0}$ with the computed ordered $\Pi\Sigma^{\delta}$-extension $(\mathbb{H}_0(t)(t_1)\dots(t_e),\sigma)$ of $(\mathbb{G},\sigma)$.
13 Execute $(B_0,\mathbb{H}'(t)(t_1)\dots(t_e)$:=`FindDepthCompleteExt`$(\boldsymbol{f_0},\mathfrak{d},\mathbb{H}_0,\mathbb{H}_0(t)(t_1)\dots(t_e))$.
14 Finish the reductions from Figures 3 and 1; let B' be the basis of $\mathrm{V}(\boldsymbol{f},\mathbb{H}'(t))$.
15 RETURN $(B',\mathbb{H}'(t)(t_1)\dots(t_e))$.

[a] We are in the base case or we apply Lemma 44.

[b] For an operative improvement towards an optimal algorithm see Lemma 62.

[c] If $\sigma(t)-t\in\mathbb{H}$ and $r>0$, CE is solved by $(B_r,\mathbb{H}_{r-1})$:=`FindDepthCompleteExt`$(\boldsymbol{f}_r,\mathfrak{d}-1,\mathbb{H}_r,\mathbb{H}_r)$.

Now suppose that we remove lines 8 and 9 and return instead $(B,(\mathbb{F}(t_1)\dots(t_e),\sigma))$

where B is a basis of $\mathrm{V}(\boldsymbol{f}, \mathbb{F})$. Then this modified version of Algorithm 1 boils down to the recursive reduction presented in Section 5; see Remark 33. In other words, the execution of lines 8 and 9 is the heart of our new algorithm. In the sequel, we will optimize this part further. For this task we will refine the reduction phase (lines 10–14) as follows.

Let $(\mathbb{F}(t_1)\dots(t_e), \sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{F}, \sigma)$ over $\mathbb{F}$ with $\mathfrak{d} = \delta(t_i)$ for $1 \leq i \leq e$ and $\delta(\mathbb{F}) < \mathfrak{d}$; moreover let $V = \{1 \leq i \leq e | \sigma(t_i) - t_i \in \mathbb{F}\}$ and let $\boldsymbol{f'} \in \mathbb{F}(t_1, \dots, t_e)$. Note: by executing `FindDepthCompleteExt`$(\boldsymbol{f'}, \mathfrak{d}, \mathbb{F}, \mathbb{F}(t_1)\dots(t_e)))$ it calls itself with depth $\mathfrak{d}$ in line 13; for all other recursive calls (in line 12, see footnote) we use depth $< \mathfrak{d}$. Finally, if we enter the completion phase (lines 8 and 9) with depth $\mathfrak{d}$ we can assume that $\boldsymbol{f}$ contains all the elements $\sigma(t_i) - t_i$ for $i \in V$. This follows by Lemmata 35 and 29. Given such a refined reduction, we can simplify the completion phase as follows.

Lemma 62. *Let $(\mathbb{E}, \sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{F}, \sigma)$ over $\mathbb{F}$ with $\mathbb{E} = \mathbb{F}(t_1)\dots(t_e)$ such that $\mathfrak{d} = \delta(t_i)$ for $1 \leq i \leq e$ and $\delta(\mathbb{F}) < \mathfrak{d}$; let $\mathbb{K} = \mathrm{const}_\sigma\mathbb{F}$ and $V = \{1 \leq i \leq e | \sigma(t_i) - t_i \in \mathbb{F}\}$. Let $\boldsymbol{f} \in \mathbb{F}^n$ where the the last $|V|$ entries are $\sigma(t_i) - t_i \in \mathbb{F}$ for $i \in V$. Then:*

(1) *If $(\mathbb{F}, \sigma)$ is $(\boldsymbol{f}, \mathfrak{d} - 1)$–complete, then $(\mathbb{E}, \sigma)$ is $(\boldsymbol{f}, \mathfrak{d} - 1)$–complete.*

(2) *If a basis of $\mathrm{V}(\boldsymbol{f}, \mathbb{F})$ is given, by row-operations in $\mathbb{K}^n \times \mathbb{F}$ over $\mathbb{K}$ one can construct a Σ^*-extension $(\mathbb{E}(s_1)\dots(s_r), \sigma)$ of $(\mathbb{E}, \sigma)$ s.t. $(\mathbb{E}(s_1)\dots(s_r), \sigma)$ is $(\boldsymbol{f}, \mathbb{F}(s_1)\dots(s_r), \mathfrak{d})$–complete; a basis of $\mathrm{V}(\boldsymbol{f}, \mathbb{F}(s_1)\dots(s_r))$ can be extracted with no extra cost.*

Proof. **(1)** Suppose that $(\mathbb{E}, \sigma)$ is not $(\boldsymbol{f}, \mathfrak{d} - 1)$–complete. Then there is a Σ^*-extension $(\mathbb{E}(x_1)\dots(x_r), \sigma)$ of $(\mathbb{F}, \sigma)$ with $g \in \mathbb{E}(x_1)\dots(x_r) \setminus \mathbb{E}$ and $\boldsymbol{c} \in \mathbb{K}^n$ such that $\sigma(g) - g = \boldsymbol{cf}$. By Proposition 17, $g = \sum_{i \in V} d_i t_i + w$ for some $d_i \in \mathbb{K}$ and $w \in \mathbb{F}(x_1)\dots(x_r) \setminus \mathbb{F}$. Thus there is $\boldsymbol{e} \in \mathbb{K}^n$ such that $\sigma(w) - w = \boldsymbol{ef}$ and therefore $(\mathbb{F}, \sigma)$ is not $(\boldsymbol{f}, \mathfrak{d} - 1)$–complete.
(2) Suppose that the entries $\sigma(t_i) - t_i$ with $i \in V$ occur in the last $u = |V|$ entries of $\boldsymbol{f}$; in particular suppose that they are sorted in the order as the corresponding extensions t_i occur in $\mathbb{E}$. Take a basis of $\mathrm{V}(\boldsymbol{f}, \mathbb{F})$ and apply row operations such that one gets a basis $B = \{(c_{i1}, \dots, c_{in}, g_i), 1 \leq i \leq m\} \,\dot{\cup}\, \{(0, \dots, 0, 1)\}$ where $\boldsymbol{C} = (c_{ij})$ is in reduced form. If $\boldsymbol{C}$ is the identity matrix, $(\mathbb{E}, \sigma)$ is $(\boldsymbol{f}, \mathbb{F}, \mathfrak{d})$–complete by Lemma 47.2 and we are done.

Otherwise, let $T \neq \{\}$ be the set of all integers $1 \leq k \leq n - u$ such that the kth column does not have a corner element in $\boldsymbol{C}$. Suppose that $T = \{j_1, \dots, j_r\}$ with $n - u \geq j_1 > j_2 > \dots > j_r \geq 1$. Now consider the difference field $(\mathbb{E}(s_1)\dots(s_r), \sigma)$ extension of $(\mathbb{E}, \sigma)$ where $\mathbb{E}(s_1)\dots(s_r)$ is a rational function field and for all $1 \leq k \leq r$ we have $\sigma(s_k) - s_k = f_{j_k}$. We prove that this is a Σ^*-extension which is $(\boldsymbol{f}, \mathbb{F}, \mathfrak{d})$–complete.

Let $0 \leq k \leq r$ be minimal such that s_k is not a Σ^*-extension. Then there is a $g \in \mathbb{E}(s_1)\dots(s_{k-1})$ with $\sigma(g) - g = f_{j_k}$. Hence, by Proposition 17, $g = \sum_{i=1}^{k-1} c_i s_i + \sum_{i \in V} d_i t_i + w$ with $w \in \mathbb{F}, c_i, d_i \in \mathbb{K}$. Hence $f_{j_k} - \sum_{i=1}^{k-1} c_i f_{j_i} - \sum_{i=n-u}^{n} d'_i f_i = \sigma(w) - w$ for some $d'_i \in \mathbb{K}$; thus $\boldsymbol{b} = (0, \dots, 0, 1, \dots) \in \mathrm{V}(\boldsymbol{f}, \mathbb{F})$ where 1 is at the j_kth position. Since $\boldsymbol{C}$ is in row reduced form and the j_kth position has no corner entry, we cannot generate $\boldsymbol{b}$; a contradiction that B is a basis. Hence $(\mathbb{E}(s_1)\dots(s_r), \sigma)$ is a Σ^*-extension of $(\mathbb{E}, \sigma)$. We get besides B the solutions $B' = \{(0, \dots, 1, \dots, 0, s_k) | 1 \leq k \leq r\}$ where the 1 is at the j_kth position and $B'' = \{(0, \dots, 1, \dots, 0, t_k) | 1 \leq k \leq e\}$ where the 1 is at the $(n - e + k)$th position. Since the $n + 1$ elements in $B \,\dot{\cup}\, B' \,\dot{\cup}\, B''$ are lin. independent, $(\mathbb{E}(s_1)\dots(s_r), \sigma)$ is $(\boldsymbol{f}, \mathbb{F}, \mathfrak{d})$–complete by Lemma 47. Clearly, $B \,\dot{\cup}\, B'$ is a basis of $\mathrm{V}(\boldsymbol{f}, \mathbb{F}(s_1)\dots(s_r))$. $\square$

Crucial improvements. The first consequence is that we can replace line 8 by executing the function call $(B, \mathbb{H})$=`FindDepthCompleteExt`$(\boldsymbol{f}, \mathfrak{d} - 1, \mathbb{F}, \mathbb{F})$. Then by Prop. 23 we get an ordered $\Pi\Sigma^\delta$-extension $(\mathbb{H}(t_1)\dots(t_e), \sigma)$ of $(\mathbb{G}, \sigma)$. Moreover, by Lemma 62.1

$(\mathbb{H}(t_1)\dots(t_e),\sigma)$ is $(\boldsymbol{f},\mathfrak{d}-1)$–complete, as needed in Lemma 48.

Finally, by Lemma 62.2 we simplify the construction of step 9 as follows: By analyzing the basis B of $\mathrm{V}(\boldsymbol{f},\mathbb{H})$, that has been computed already in step 8, we get a Σ^*-extension of $(\mathbb{H}(t_1)\dots(t_r),\sigma)$ of $(\mathbb{G},\sigma)$ which can be brought to an ordered $\Pi\Sigma^\delta$-extension $(\mathbb{H}'(t_1)\dots(t_r),\sigma)$ and which is $(\boldsymbol{f},\mathbb{H}',\mathfrak{d})$–complete; we extract a basis of $\mathrm{V}(\boldsymbol{f},\mathbb{H}')$.

Example 63. In Example 50 we claim that $(\mathbb{F}(s),\sigma)$ with $\mathbb{F}=\mathbb{K}(k)(q)(h)(b)$ is $(\boldsymbol{f},2)$–complete where $\boldsymbol{f}=(-bqh,bq)$. Note that $\sigma(s)-s=bq$. Hence by Lemma 62.1 it suffices to show that $(\mathbb{F},\sigma)$ is $(\boldsymbol{f},2)$–complete. With our algorithm this can be easily checked; during this check we get the basis $\{(0,0,1)\}$ of $\mathrm{V}(\boldsymbol{f},\mathbb{F})$. Following the proof of Lemma 62.2, $(\mathbb{F}(s)(H),\sigma)$ is a Σ^*-extension of $(\mathbb{F}(s),\sigma)$ with $\sigma(H)=H-bqh$ and we get the basis $\{(0,0,1),(1,0,H)\}$ of $\mathrm{V}(\boldsymbol{f},\mathbb{F}(s)(H))$. By Lemma 48 H is depth-optimal.

We emphasize that the modified algorithm differs from the reduction presented in Section 5 (which is similar to Karr's algorithm) by just analyzing the sub-results and by inserting extensions if necessary.

In a nutshell, choosing an appropriate $\Pi\Sigma^\delta$-extension manually and solving problem PT *with the recursive algorithm from Section 5 (or Karr's algorithm) is as expansive as executing our new algorithm which finds such an appropriate $\Pi\Sigma^\delta$-extension and which outputs the corresponding solution to problem* PT*.*

7. Proving the main results (from Section 3)

We need the following preparation to prove Result 2.

Lemma 64. *Let $(\mathbb{F}(y),\sigma)$ be a Π-extension of $(\mathbb{F},\sigma)$ which can be brought to an ordered $\Pi\Sigma^\delta$-extension of $(\mathbb{G},\sigma)$. Let $f\in\mathbb{F}$ and let $(\mathbb{E},\sigma)$ be a Σ^*-extension of $(\mathbb{F},\sigma)$ with extension depth $\mathfrak{d}$ and $g\in\mathbb{E}$ such that (4). Then there is a Σ^*-extension $(\mathbb{S},\sigma)$ of $(\mathbb{F},\sigma)$ with extension depth $\leq\mathfrak{d}$ and $g'\in\mathbb{S}$ such that $\sigma(g')-g'=f$ and $\delta(g')\leq\delta(g)$.*

Proof. Write $\mathbb{E}=\mathbb{F}(y)(s_1)\dots(s_e)$ with $\mathfrak{d}=\max_i\delta(s_i)$. Since we can bring $(\mathbb{F}(y),\sigma)$ to an ordered $\Pi\Sigma^\delta$-extension of $(\mathbb{G},\sigma)$, we can apply Corollary 58: There is a Σ^*-extension $(\mathbb{S},\sigma)$ of $(\mathbb{F}(y),\sigma)$ over $\mathbb{F}$ with $\mathbb{S}=\mathbb{F}(y)(x_1)\dots(x_r)$ which can be brought to an ordered $\Pi\Sigma^\delta$-extension of $(\mathbb{G},\sigma)$ and in which we have $g'\in\mathbb{S}$ s.t. $\sigma(g')-g'=f$; by Proposition 17.2, $g'\in\mathbb{F}(x_1)\dots(x_r)$. By Theorem 25 we can take a Σ^*-extension $(\mathbb{S}',\sigma)$ of $(\mathbb{S},\sigma)$ and an $\mathbb{F}(y)$-monomorphism $\tau:\mathbb{E}\to\mathbb{S}'$ s.t. (12) for all $a\in\mathbb{E}$. Note that $\sigma(\tau(g))-\tau(g)=f=\sigma(g')-g'$. Since $\tau(g),g'\in\mathbb{S}'$, $\tau(g)=g'+c$ for some $c\in\mathrm{const}_\sigma\mathbb{G}$. Therefore, $\delta(g)\geq\delta(\tau(g))=\delta(g')$. Since $\delta(\tau(s_i))\leq\delta(s_i)\leq\mathfrak{d}$ for $1\leq i\leq r$, $g'\in\mathbb{F}(x_i|\delta(x_i)\leq\mathfrak{d})=:\mathbb{S}''$. By construction, $(\mathbb{S}'',\sigma)$ is a Σ^*-extension of $(\mathbb{F},\sigma)$ with extension depth $\leq\mathfrak{d}$. □

Lemma 65. *Let $(\mathbb{F}(x)(y),\sigma)$ be a $\Pi\Sigma^\delta$-extension of $(\mathbb{F},\sigma)$ and suppose that $(\mathbb{F}(x),\sigma)$ and $(\mathbb{F}(y),\sigma)$ can be brought to ordered $\Pi\Sigma^\delta$-extensions of $(\mathbb{G},\sigma)$. Then the $\Pi\Sigma^*$-extension $(\mathbb{F}(y)(x),\sigma)$ of $(\mathbb{F},\sigma)$ is depth-optimal.*

Proof. First we show that $(\mathbb{F}(y),\sigma)$ is a $\Pi\Sigma^\delta$-extension of $(\mathbb{F},\sigma)$. If y is a Π-extension, we are done. Otherwise, let y be a Σ^*-extension with $\sigma(y)=y+f$ which is not depth-optimal. Hence, we can take a Σ^*-extension $(\mathbb{F}(s_1)\dots(s_e),\sigma)$ of $(\mathbb{F},\sigma)$ with extension depth $\leq\delta(f)$ and $g\in\mathbb{F}(s_1)\dots(s_e)$ such that (4). There are two cases.

Case 1a: x is a Π-extension. By Corollary 16 $(\mathbb{F}(s_1)\dots(s_e)(x),\sigma)$ is a Π-extension of $(\mathbb{F}(s_1)\dots(s_e),\sigma)$. By reordering, $(\mathbb{F}(x)(s_1)\dots(x_e),\sigma)$ is a Σ^*-extension of $(\mathbb{F}(x),\sigma)$. Consequently, $(\mathbb{F}(x)(y),\sigma)$ is not a Σ^δ-extension of $(\mathbb{F}(x),\sigma)$, a contradiction.
Case 1b: x is a Σ^*-extension. Bring $(\mathbb{F}(x),\sigma)$ to an ordered $\Pi\Sigma^\delta$-extension $(\mathbb{S},\sigma)$ of $(\mathbb{G},\sigma)$. Hence, by Thm. 25 there is a Σ^*-extension $(\mathbb{E},\sigma)$ of $(\mathbb{S},\sigma)$ with extension depth $\leq \delta(f)$ and an $\mathbb{F}$-monomorphism $\tau:\mathbb{F}(s_1)\dots(s_e)\to\mathbb{E}$ with $\sigma(\tau(g))-\tau(g)=f$. Since $\mathbb{S}=\mathbb{F}(x)$ (as fields), $(\mathbb{F}(x)(y),\sigma)$ is not a Σ^δ-extension of $(\mathbb{F}(x),\sigma)$; a contradiction.

Second, we show that $(\mathbb{F}(y)(x),\sigma)$ is a $\Pi\Sigma^\delta$-extension of $(\mathbb{F}(y),\sigma)$. If x is a Π-extension, we are done. Otherwise, let x be a Σ^*-extension. If y is a Σ^*-extension and $\delta(y)\geq\delta(x)$, the statement follows by Lemma 22 and by Proposition 24. What remains to consider are the cases that y is a Π-extension or that y is a Σ^*-extension with $\delta(y)<\delta(x)$. Now suppose that $(\mathbb{F}(y)(x),\sigma)$ is a Σ^*-extension of $(\mathbb{F}(y),\sigma)$ with $\sigma(x)=x+f$ which is not depth-optimal. Hence, we can take a Σ^*-extension $(\mathbb{F}(y)(s_1)\dots(s_e),\sigma)$ of $(\mathbb{F}(y),\sigma)$ with extension depth $\leq\delta(f)$ and $g\in\mathbb{F}(y)(s_1)\dots(s_e)$ such that (4).
Case 2a: y is a Π-extension. By Lemma 64, there is a Σ^*-extension $(\mathbb{S},\sigma)$ of $(\mathbb{F},\sigma)$ with extension depth $\leq\delta(f)$ and $g'\in\mathbb{S}$ such that $\sigma(g')-g'=f$; hence $(\mathbb{F}(x),\sigma)$ is not a Σ^δ-extension of $(\mathbb{F},\sigma)$, a contradiction.
Case 2b: y is a Σ^*-extension with $\delta(y)>\delta(x)$. Reorder $(\mathbb{F}(y),\sigma)$ to a $\Pi\Sigma^\delta$-extension of $(\mathbb{G},\sigma)$. By Theorem 21 $\delta(g)\leq\delta(f)+1$. Since $\delta(f)+1=\delta(x)<\delta(y)$, g is free of y. Hence, $(\mathbb{F}(s_1)\dots(s_e),\sigma)$ is a Σ^*-extension of $(\mathbb{F},\sigma)$ with $g\in\mathbb{F}(s_1)\dots(s_e)$ such that (4), and therefore $(\mathbb{F}(x),\sigma)$ is not a Σ^δ-extension of $(\mathbb{F},\sigma)$; a contradiction. $\square$

• **Result 2.** If $e=0,1$ nothing has to be shown. Let $(\mathbb{G}(t_1)\dots(t_e)(x),\sigma)$ be a $\Pi\Sigma^\delta$-extension of $(\mathbb{G},\sigma)$ with $e\geq 1$ and suppose the theorem holds for $e\geq 1$ extension. Choose any possible reordering. If x stays on top, by the induction assumption all extensions below are depth-optimal. x remains depth-optimal, since the field below has not changed. This shows this case. Otherwise, suppose that t_i for some $1\leq i\leq e$ is on top. Then we can reorder our field to the $\Pi\Sigma^*$-extension $(\mathbb{H}(t_i)(x),\sigma)$ of $(\mathbb{G},\sigma)$ with $\mathbb{H}:=\mathbb{G}(t_1)\dots(t_{i-1})(t_{i+1})\dots(t_e)$. By the induction assumption we can bring $(\mathbb{H}(t_i),\sigma)$ and $(\mathbb{H}(x),\sigma)$ to ordered $\Pi\Sigma^\delta$-extensions of $(\mathbb{G},\sigma)$. Thus, we can apply Lemma 65 and get the $\Pi\Sigma^\delta$-extension $(\mathbb{H}(x)(t_i),\sigma)$ of $(\mathbb{G},\sigma)$. By the induction assumption we can bring the extensions in $\mathbb{H}$ to the desired order without change the $\Pi\Sigma^\delta$-property.

• **Result 1.** This follows by Theorem 40, Corollary 60 and Result 2. In particular, $(\mathbb{E},\sigma)$ and $g\in\mathbb{E}$ can be computed as follows.

1. Reorder $(\mathbb{F},\sigma)$ to an ordered $\Pi\Sigma^*$-extension of $(\mathbb{G},\sigma)$.
2. Execute $(B,\mathbb{E})$:=`FindDepthCompleteExt`$((f),\delta(f)+1,\mathbb{F},\mathbb{F})$ and extract g from B s.t. (4).

• **Result 3.** This is a direct consequence of Theorem 21 and Result 2.

• **Result 4.** This follows by Theorem 25 and Result 2.

• **Result 5.** This is implied by the following more general statement: there is a Σ^*-extension $(\mathbb{S},\sigma)$ of $(\mathbb{E},\sigma)$ and a $\Pi\Sigma^\delta$-extension $(\mathbb{D},\sigma)$ of $(\mathbb{F},\sigma)$ with an $\mathbb{F}$-isomorphism $\tau:\mathbb{S}\to\mathbb{D}$ as in (12) for all $a\in\mathbb{E}$; we can assume that $\mathbb{E}$ is ordered.
We prove this result by induction on the number of extensions in $\mathbb{E}$. For $\mathbb{E}=\mathbb{F}$, take $\mathbb{D}:=\mathbb{F}$ and $\mathbb{S}=:\mathbb{F}$ with $\tau=\mathrm{id}_{\mathbb{F}}$. Now suppose we have shown the result for $\mathbb{E}=\mathbb{F}(t_1)\dots(t_{e-1})$ with $e\geq 1$. I.e., we are given a $\Pi\Sigma^\delta$-extension $(\mathbb{D},\sigma)$ of $(\mathbb{F},\sigma)$, a Σ^*-extension $(\mathbb{S},\sigma)$ of $(\mathbb{E},\sigma)$ with $\mathbb{S}=\mathbb{E}(s_1)\dots(s_u)$ and an $\mathbb{F}$-isomorphism $\tau:\mathbb{S}\to\mathbb{D}$ as in (12) for all $a\in\mathbb{E}$. Let $(\mathbb{E}(t),\sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{E},\sigma)$ with $\delta(t)\geq\delta(\mathbb{E})$.

Case 1: Suppose that t is a Π-extension with $\sigma(t) = \alpha t$. Then by Corollary 16 we can construct the Π-extension $(\mathbb{S}(t), \sigma)$ of $(\mathbb{S}, \sigma)$. Moreover, by Proposition 18.2 we can construct the Π-extension $(\mathbb{D}(x), \sigma)$ of $(\mathbb{D}, \sigma)$ with $\sigma(x) = \tau(\alpha)x$ and can extend the $\mathbb{F}$-isomorphism τ to $\tau : \mathbb{S}(t) \to \mathbb{D}(x)$ with $\tau(t) = x$. By reordering, we get the Σ^*-extension $(\mathbb{E}(t)(s_1)\dots(s_u), \sigma)$ of $(\mathbb{E}(t), \sigma)$ with the $\mathbb{F}$-isomorphism $\tau : \mathbb{E}(t)(s_1)\dots(s_u) \to \mathbb{D}(x)$. As $\delta(\tau(\alpha)) \leq \delta(\alpha)$, it follows that $\delta(\tau(t)) \leq \delta(t)$. Hence (12) for all $a \in \mathbb{E}(t)$.
Case 2: Suppose that t is a Σ^*-extension with $\sigma(t) = t + \beta$. We consider two subcases
Case 2a: If there is a $g \in \mathbb{S}$ with $\sigma(g) - g = \beta$, let $j \geq 1$ be minimal such that $g \notin \mathbb{E}(s_1)\dots(s_{j-1})$. Then by Theorem 3.1 there is the Σ^*-extension $(\mathbb{E}(s_1)\dots(s_{j-1})(t), \sigma)$ of $(\mathbb{E}(s_1)\dots(s_{j-1}), \sigma)$ with $\sigma(t) = t + \beta$. Furthermore, there is an $\mathbb{E}(s_1)\dots(s_{j-1})$-isomorphism $\rho : \mathbb{E}(s_1)\dots(s_{j-1})(t) \to \mathbb{E}(s_1)\dots(s_{j-1})(s_j)$ with $\rho(t) = g$ by Prop. 18.1. By reordering we get the Σ^*-extension $(\mathbb{E}(t)(s_1)\dots(s_{j-1}), \sigma)$ of $(\mathbb{E}(t), \sigma)$. Now we can construct a Σ^*-extension $(\mathbb{S}', \sigma)$ of $(\mathbb{E}(t)(s_1)\dots(s_{j-1}), \sigma)$ with an $\mathbb{E}(t)(s_1)\dots(s_{j-1})$-isomorphism $\rho : \mathbb{S}' \to \mathbb{S}$ by Prop. 18.3. Hence we arrive at an $\mathbb{F}$-isomorphism $\tau' : \mathbb{S}' \to \mathbb{D}$ with $\tau' := \tau \circ \rho$. Finally, observe that for all $a \in \mathbb{E}$ we have $\tau'(a) = \tau(\rho(a)) = \tau(a)$ and $\tau'(t) = \tau(\rho(t)) = \tau(g)$. Since $\sigma(\tau(g)) - \tau(g) = \tau(\beta)$, $\delta(\tau(g)) \leq \delta(\tau(\beta)) + 1$ by Result 3. With $\delta(\tau(\beta)) + 1 \leq \delta(\beta) + 1 = \delta(t)$ it follows that $\delta(\tau'(t)) \leq \delta(t)$. Since $\delta(\tau'(a)) = \delta(\tau(a)) \leq \delta(a)$ for all $a \in \mathbb{E}$, we get $\delta(\tau'(a)) \leq \delta(a)$ for all $a \in \mathbb{E}(t)$.
Case 2b: Suppose that there is no $g \in \mathbb{S}$ with $\sigma(g) - g = \beta$. Then there is no $g \in \mathbb{D}$ with $\sigma(g) - g = \tau(\beta)$. By Result 1 there is a Σ^δ-extension $(\mathbb{D}(y_1)\dots(y_v), \sigma)$ of $(\mathbb{D}, \sigma)$ such that $\sigma(g) - g = \tau(\beta)$ for some $g \in \mathbb{D}(y_1)\dots(y_v) \setminus \mathbb{D}(y_1)\dots(y_{v-1})$. Moreover, by Proposition 18.3 it follows that there is a Σ^*-extension $(\mathbb{S}(x_1)\dots(x_{v-1}), \sigma)$ of $(\mathbb{S}, \sigma)$ and an $\mathbb{F}$-isomorphism $\tau' : \mathbb{S}(x_1)\dots(x_{v-1}) \to \mathbb{D}(y_1)\dots(y_{v-1})$ where $\tau'(a) = \tau(a)$ for all $a \in \mathbb{E}$. Furthermore, we can construct the Σ^*-extension $(\mathbb{S}(x_1)\dots(x_{v-1})(t), \sigma)$ of $(\mathbb{S}(x_1)\dots(x_{v-1}), \sigma)$ with $\sigma(t) = t+\beta$ by Proposition 17.1. Finally, we can construct the $\mathbb{F}$-isomorphism $\tau'' : \mathbb{S}(x_1)\dots(x_{v-1})(t) \to \mathbb{D}(y_1)\dots(y_{v-1})(y_v)$ with $\tau''(a) = \tau'(a)$ for all $a \in \mathbb{S}(x_1)\dots(x_{v-1})$ and $\tau''(t) = g$ by Proposition 18.1. By reordering of $\mathbb{S}(x_1)\dots(x_{v-1})(t)$ we obtain the Σ^*-extension $(\mathbb{S}', \sigma)$ of $(\mathbb{E}(t), \sigma)$ with $\mathbb{S}' = \mathbb{E}(t)(s_1)\dots(s_u)(x_1)\dots(x_{v-1})$. As above, $\delta(\tau''(t)) = \delta(g) \leq \delta(\tau(\beta)) + 1 \leq \delta(\beta) + 1 = \delta(t)$. Since $\tau''(a) = \tau'(a) = \tau(a)$ for all $a \in \mathbb{E}$, $\delta(\tau''(a)) \leq \delta(a)$ for all $a \in \mathbb{E}$. Thus $\delta(\tau''(a)) \leq \delta(a)$ for all $a \in \mathbb{E}(t)$.
Note that this construction can be given explicitly, if $(\mathbb{G}, \sigma)$ is σ-computable.

• **Result 6.** The induction base $e = 0$ is obvious. Suppose Result 6 holds for $e \geq 0$ extensions, and consider a $\Pi\Sigma^*$-extension $(\mathbb{F}(t_1)\dots(t_{e+1}), \sigma)$ of $(\mathbb{F}, \sigma)$ with $g \in \mathbb{F}(t_1)\dots(t_{e+1})$ s.t. (4). Then by assumption there is a Σ^δ-extension $(\mathbb{F}(t_1)(s_1)\dots(s_r), \sigma)$ of $(\mathbb{F}(t_1), \sigma)$ with $g' \in \mathbb{F}(t_1)(s_1)\dots(s_r)$ such that $\delta(g') \leq \delta(g)$ and $\sigma(g') - g' = f$. Now we apply Result 5 (if t_1 is a Σ^*-extension) and Lemma 64 together with Result 5 (if t_1 is a Π-extension): It follows that there is a Σ^δ-extension $(\mathbb{S}, \sigma)$ of $(\mathbb{F}, \sigma)$ with $g'' \in \mathbb{S}$ s.t. $\delta(g'') \leq \delta(g')$ and $\sigma(g'') - g'' = f$. Since $\delta(g'') \leq \delta(g)$, we are done.

• **Result 7.** This is a direct consequence of Results 1 and 6.

• **Result 8.** Let $(\mathbb{E}, \sigma)$ be such a Σ^δ-extension of $(\mathbb{F}, \sigma)$ with $\mathbb{K} = \text{const}_\sigma\mathbb{F}$; take $g \in \mathbb{E}$ as in (4). **(1)** Let $(\mathbb{H}, \sigma)$ be a $\Pi\Sigma^*$-extension of $(\mathbb{F}, \sigma)$ with $g' \in \mathbb{H}$ s.t. $\sigma(g') - g' = f$. By Result 6 there is a Σ^δ-extension $(\mathbb{S}, \sigma)$ of $(\mathbb{F}, \sigma)$ with $\mathbb{S} = \mathbb{F}(x_1)\dots(x_r)$ and $h \in \mathbb{S}$ such that $\sigma(h) - h = f$ and $\delta(h) \leq \delta(g')$. By Result 4 we get a Σ^δ-extension $(\mathbb{E}', \sigma)$ of $(\mathbb{E}, \sigma)$ and an $\mathbb{F}$-monomorphism $\tau : \mathbb{S} \to \mathbb{E}'$ as in (12) for all $a \in \mathbb{S}$. Hence $\delta(\tau(h)) \leq \delta(h) \leq \delta(g')$. Since $\tau(h), g \in \mathbb{E}'$ and $\sigma(\tau(h)) - \tau(h) = f$, $\tau(h) = g + c$ for some $c \in \mathbb{K}$. Hence $\delta(\tau(h)) = \delta(g)$.
(2) Suppose in addition that $\delta(s_e) = \mathfrak{d}$ and $g \in \mathbb{E} \setminus \mathbb{F}(s_1, \dots, x_{e-1})$. By the above

considerations, $\delta(\tau(x_i)) \leq \delta(x_i)$ for $1 \leq i \leq r$ and $\tau(h) = g + c$ for some $c \in \mathbb{K}$. Hence there is an i with $1 \leq i \leq r$ s.t. s_e occurs in $\tau(x_i)$. Hence $\mathfrak{d} = \delta(s_e) \leq \delta(\tau(x_i)) \leq \delta(x_i)$.

• **Result 9.** By Theorem 42 we can take a Σ^δ-extension $(\mathbb{E}, \sigma)$ of $(\mathbb{F}, \sigma)$ which is $(\boldsymbol{f}, \mathfrak{d})$–complete. Now let $(\mathbb{H}, \sigma)$ be any $\Pi\Sigma^*$-extension of $(\mathbb{F}, \sigma)$ with extension depth $\leq \mathfrak{d}$ and $g \in \mathbb{H}$, $\boldsymbol{c} \in \mathbb{K}^n$ s.t. (3). Then by Results 1 and 8.1 we take a Σ^δ-extension $(\mathbb{S}, \sigma)$ of $(\mathbb{F}, \sigma)$ with $g' \in \mathbb{S}$ such that $\sigma(g') - g' = \boldsymbol{c}\boldsymbol{f} =: f$ and $\delta(g') \leq \delta(g)$. Moreover, by Result 4 we take a Σ^*-extension $(\mathbb{E}', \sigma)$ of $(\mathbb{E}, \sigma)$ with extension depth $\leq \mathfrak{d}$ and an $\mathbb{F}$-monomorphism $\tau : \mathbb{S} \to \mathbb{E}'$ s.t. $\delta(h) \leq \delta(g')$ for $h := \tau(g')$. Since $\sigma(h) - h = f$ and $(\mathbb{E}, \sigma)$ is $(\boldsymbol{f}, \mathfrak{d})$–complete, $h \in \mathbb{E}$; in particular, $\delta(h) \leq \delta(g)$. $(\mathbb{E}, \sigma)$ can be constructed explicitly:

Reorder $(\mathbb{F}, \sigma)$ to an ordered $\Pi\Sigma^*$-extension of $(\mathbb{G}, \sigma)$.
$(B, \mathbb{E})$:=`FindDepthCompleteExt`$(\boldsymbol{f}, \delta(\boldsymbol{f}), \mathbb{F}, \mathbb{F})$.

8. Applications from Particle Physics

We conclude our article by non-trivial applications from particle physics [4, 17]. For the computations we used the summation package Sigma [34] which contains in its inner core our new difference field theory.

8.1. *Finding recurrence relations with smaller order*

In massive higher order calculations of Feynman diagrams [4] the sum

$$A(N) = \sum_{i=1}^{\infty} \frac{B(N,i)}{i+N+2} S_1(i) S_1(N+i),$$

where $B(N,i) = \frac{\Gamma(N)\Gamma(i)}{\Gamma(N+i)}$ denotes the beta function [2, p. 5], arose. It turns out that our refined creative telescoping method produces –analogous to Example 14– a recurrence with minimal order:

$$\begin{aligned}(N+2)A(N) - (N+3)A(N+1) &= 2\tfrac{N^5+5N^4+21N^3+38N^2+28N+8}{N^4(N+1)^2(N+2)^2}\\ &+ 2\tfrac{(-1)^N}{N(N+2)}\Big(- \tfrac{(3N+4)(\zeta_2+2S_{-2}(N))}{(N+1)(N+2)} - 2\zeta_3 - 2S_{-3}(N) - 2\zeta_2 S_1(N) - 4S_{1,-2}(N)\Big)\\ &+ \tfrac{1}{N+1}(S_2(N) - \zeta_2) + \tfrac{N^6+8N^5+31N^4+66N^3+88N^2+64N+16}{N^3(N+1)^2(N+2)^3} S_1(N);\end{aligned}$$

note that standard creative telescoping produces a recurrence of order 4 only; see [4, p. 6]. Given this optimal recurrence of order 1, the closed form [6]

$$\begin{aligned}A(N) = &\tfrac{2(-1)^N}{N(N+1)(N+2)}\Big[2S_{-2,1}(N) - 3S_{-3}(N) - 2S_{-2}(N)S_1(N) - \zeta_2 S_1(N) - \zeta_3 - \tfrac{2S_{-2}(N)+\zeta_2}{N+1}\Big]\\ &- 2\tfrac{S_3(N)-\zeta_3}{N+2} - \tfrac{S_2(N)-\zeta_2}{N+2}S_1(N) + \tfrac{2+7N+7N^2+5N^3+N^4}{N^3(N+1)^3(N+2)}S_1(N) + 2\tfrac{2+7N+9N^2+4N^3+N^4}{N^4(N+1)^3(N+2)}\end{aligned}$$

can be read off immediately.
We remark that in this example the algebraic object $(-1)^N$ occurs which cannot be handled in a direct fashion in $\Pi\Sigma^*$-fields. As it turns out, our algorithmic framework can be slightly extended such that it works also in this case; the technical details are omitted here.

Similar examples for our refined creative telescoping method can be found, e.g., in [21, 16, 13, 18].

[6] ζ_k denotes the Riemann zeta function at k; e.g., $\zeta_2 = \pi^2/6$.

8.2. Simplification of d'Alembertian solution

As worked out in [17] Sigma could reproduce the evaluation [7] of a Feynman diagram that occurred in [37] during the computation of the third-order QCD corrections to deep-inelastic scattering by photon exchange. More precisely, in Mellin space the related Feynman diagram could be expressed in terms of the recurrence

$$\begin{aligned}-N(N+1)^2(N+2)(3N+7)F(N)+(N+1)(N+2)^2(N+3)(3N+4)F(N+1)\\+N(N+1)(N+2)(N+3)(3N+7)F(N+2)\\-(N+1)(N+2)(N+3)(N+4)(3N+4)F(N+3)=f(N)\end{aligned}$$

with inhomogeneous part

$$\begin{aligned}f(N)=&\Big(1-(-1)^N\Big)\Big(\tfrac{6N^4+38N^3+81N^2+66N+14}{N(N+1)(N+2)}\big(24\zeta_3+16S_{-3}(N)\big)\\&+\tfrac{16(N+2)(N+3)(3N+4)}{(N+1)^4}+\tfrac{16\big(6N^7+56N^6+213N^5+429N^4+496N^3+339N^2+138N+28\big)}{N^2(N+1)^2(N+2)^2}S_{-2}(N)\Big)\\&-\tfrac{8\big(12N^7+115N^6+462N^5+1026N^4+1383N^3+1152N^2+552N+112\big)}{N^2(N+1)^2(N+2)^2}S_{-2}(N)\\&-\tfrac{16\big(9N^4+61N^3+144N^2+138N+42\big)\zeta_3}{N(N+1)(N+2)}-\tfrac{8\big(12N^4+81N^3+189N^2+180N+56\big)}{N(N+1)(N+2)}S_{-3}(N)\\&+\tfrac{8\big(3N^4+18N^3+30N^2+7N-12\big)}{(N+1)^2(N+2)^2}S_2(N)+\tfrac{8\big(N^2+9N+12\big)}{(N+1)(N+2)}S_3(N)\end{aligned}$$

and initial values in terms of ζ-values (which are not printed here). Sigma easily computes the general d'Alembertian solution

$$\begin{aligned}F(N)=c_1\frac{1}{N+1}+c_2\frac{(-1)^N}{N+1}+c_3\frac{(-1)^N NS_{-1}(N)-2}{N(N+1)}\\ \underbrace{-\frac{1}{N+1}\sum_{k=4}^{N}(-1)^k\underbrace{\sum_{j=4}^{k}\frac{(-1)^j(3j-2)}{(j-2)(j-1)j}\overbrace{\sum_{i=4}^{j}\frac{f(i-3)}{(3i-5)(3i-2)}}^{=A(j)}}_{=B(k)}}_{=C(N)}\end{aligned}\tag{24}$$

for constants c_1,c_2,c_3. Checking initial values shows that $c_1=\frac{41\zeta_3}{7}$, $c_2=\frac{1}{7}(53\zeta_3-70\zeta_5)$ and $c_3=-\frac{12\zeta_3}{7}$. Now the main task is to simplify (24) further. With, e.g., Karr's algorithm [11] the inner sum $A(j)$ can be eliminated and one gets a rather big expression for $A(j)$ in terms of single nested harmonic sums $S_i(j)$. In other words, we obtain an expression for (24) where the depth is reduced by one. To get a representation with optimal nested depth, we execute our refined algorithm. The result is an expression for $C(N)$ in terms of two nested sum expressions only:

$$C(N)=-\frac{(-1)^N B(N)}{2(N+1)}-\frac{1}{4(N+1)}\sum_{k=4}^{N}\frac{f(k-3)}{k-1)(k-2)}+\frac{(3N-2)(3N+1)}{4(N-1)N(N+1)}A(N).$$

Finally, splitting these sums by partial fraction decomposition, we get the solution [17]:

$$F(N)=\frac{2\Big(6\zeta_3+(-1)^N(6\zeta_3-5N^2\zeta_5)\Big)}{N^2(N+1)}-\frac{8(1+(-1)^N)S_{-5}(N)}{N+1}-\frac{4\Big(1+(-1)^N\Big)S_5(N)}{N+1}$$

[7] For the original computation [37] the package Summer [36] based on Form was used which is specialized to manipulate huge expressions in terms of harmonic sums.

$$
+ S_2(N)\Big(\frac{4S_3(N)}{N+1} - \frac{4\zeta_3}{N+1}\Big) + S_{-3}(N)\Big(\frac{8(1+(-1)^N)}{N^2(N+1)} - \frac{4(2+(-1)^N)S_{-2}(N)}{N+1}
- \frac{4S_2(N)}{N+1}\Big) + S_{-2}(N)\Big(\frac{-12\zeta_3N^3+(-1)^N(8-8N^3\zeta_3)+8}{N^3(N+1)} - \frac{(4+8(-1)^N)S_3(N)}{N+1}\Big)
+ \frac{8S_{-3,-2}(N)}{N+1} + \frac{(4-4(-1)^N)S_{-3,2}(N)}{N+1} + \frac{(4+12(-1)^N)S_{-2,3}(N)}{N+1} - \frac{8S_{2,3}(N)}{N+1}.
$$

For further examples how one can simplify d'Alembertian solutions [1] with our algorithms see, e.g., [21, 16, 13, 18]. We note that in the derived result no algebraic relations between the harmonic sums occur. In the next section we show how Sigma eliminates, or equivalently, finds such algebraic relations explicitly and efficiently.

8.3. *Finding algebraic relations of nested sums*

During the calculation of Feynman integrals harmonic sums arise frequently; see, for instance, [6, 36, 37, 4, 17] for further literature. In order to derive compact representations of such computations, one can use, e.g., results from [5] where all relations of harmonic sums are classified in general and tabulated up to nested depth 6. Alternatively, we illustrate how this task can be handled efficiently in the general $\Pi\Sigma^\delta$-field setting. Consider, e.g., the sums $S_{4,2}(N)$, $S_{2,4}(N)$, $S_{2,1,1,1,1}(N)$, $S_{1,2,1,1,1}(N)$, $S_{1,1,2,1,1}(N)$, $S_{1,1,1,2,1}(N)$, $S_{1,1,1,1,2}(N)$ which are algebraically independent – except the last one: here the relation

$$
\begin{aligned}
S_{1,1,1,1,2}(N) = \frac{1}{8}\Big(&2S_1(N)^6 + 7S_2(N)S_1(N)^4 + 4S_2(N)^2S_1(N)^2\\
&+ 8S_{1,1,1,2}(N)S_1(N) + 8S_{1,1,2,1}(N)S_1(N) + 8S_{1,2,1,1}(N)S_1(N)\\
&+ 8S_{2,1,1,1}(N)S_1(N) + S_2(N)^3 + 24S_{1,1,1}(N)^2 + 8S_{2,4}(N) + 8S_{4,2}(N)\\
&+ \big(-4S_1(N)^2 - 2S_2(N)\big)S_4(N) + \big(-16S_1(N)^3 - 24S_2(N)S_1(N)\big)S_{1,1,1}(N)\\
&- 8S_{1,1,1,2,1}(N) - 8S_{1,1,2,1,1}(N) - 8S_{1,2,1,1,1}(N) - 8S_{2,1,1,1,1}(N)\Big)
\end{aligned}
\tag{25}
$$

pops up. With the naive reduction from Section 5, Sigma finds (25) by representing the sums in a $\Pi\Sigma^*$-field $(\mathbb{Q}(t_1)\dots(t_{18}),\sigma)$ where the depths of the Σ^*-extensions $t_1,\dots,t_{18}$ are: $1,2,2,2,3,3,3,3,3,4,4,4,4,5,5,5,5,6$, respectively. With a standard notebook (2.16 GHz) we needed 772 seconds to construct this $\Pi\Sigma^*$-field in order to get the relation (25).

Applying our new algorithms, we can represent the harmonic sums in a $\Pi\Sigma^\delta$-field with again 18 extensions, but this time the depths are $1,2,2,2,2,2,2,3,3,3,3,3,3,3,3,3,3,3$, respectively. E.g., the extensions correspond to the sum representations (10) and

$$
\begin{aligned}
S_{2,1,1,1,1}(N) =& \frac{1}{24}\sum_{k=1}^{N}\frac{S_1(k)^4+6S_2(k)S_1(k)^2+8S_3(k)S_1(k)+3S_2(k)^2+6S_4(k)}{k^2},\\
S_{1,2,1,1,1}(N) =& \frac{1}{6}\Big(\sum_{k=1}^{N}\frac{-(kS_1(k)-1)\big(S_1(k)^3+3S_2(k)S_1(k)+2S_3(k)\big)}{k^3}\\
&\qquad + S_1(N)\sum_{k=1}^{N}\frac{S_1(k)^3+3S_2(k)S_1(k)+2S_3(k)}{k^2}\Big), \text{ etc;}
\end{aligned}
$$

note that the representation of $S_{2,4}(N)$ and $S_{4,2}(N)$ in the corresponding $\Pi\Sigma^\delta$-field has been carried out in details in Example 7. In total we needed 37 seconds (instead of 772 seconds) to construct the underlying $\Pi\Sigma^\delta$-field. Based on this optimal $\Pi\Sigma^\delta$-field representation, by backwards transformation the relation (25) can be found automatically.

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