SYMBOLIC SUMMATION FINDS OPTIMAL NESTED SUM REPRESENTATIONS

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Abstract. We consider the following problem: Given a nested sum expression, find a sum representation such that the nested depth is minimal. We obtain symbolic summation algorithms that solve this problem for sums defined, e.g., over hypergeometric, q-hypergeometric or mixed hypergeometric expressions.

1. Introduction

Karr's telescoping algorithm (Kar81; Kar85) based on his difference field theory of $\Pi\Sigma$-fields provides a general framework for symbolic summation. E.g., his algorithm, or a simplified version presented in (Sch05c), covers summation over hypergeometric terms (Gos78; Zei91), q-hypergeometric terms (PR97) or mixed hypergeometric terms (BP99). More generally, one can handle expressions in terms of indefinite nested sums or products. His algorithm is, in a sense, the discrete counterpart of Risch's integration algorithm (Ris69; Ris70).

As it turns out, the construction of the underlying $\Pi\Sigma$-field for a given sum expression can lead to sum representations with increased nested depth. Motivated by this observation, we derived a difference field theory for symbolic summation in (Sch05a; Sch07b) that refines Karr's $\Pi\Sigma$-fields. As a consequence, we obtained new algorithms that can express nested sums in the so called depth-optimal $\Pi\Sigma^*$-extensions. Reinterpreting these difference field expressions as sequences gives then sum representations with simplified nested depth.

In this article we show that the nested depth of such a simplified expression, produced by our algorithm from (Sch07b), is optimal in the ring of sequence. In order to derive this result, we exploit the fact (Sch07a) that a rather general subclass of $\Pi\Sigma^*$-extensions, the so called generalized d'Alembertian extensions, can be embedded in the ring of sequences, provided that the ground field can be embedded in the ring of sequences. This construction allows us to carry over results from our refined difference field theory (Sch07b) to the sequence setting. In particular, we can show that sum representations in a depth-optimal $\Pi\Sigma^*$-extension are always minimal in its depth.

From an applicational point of view our algorithms are able to produce d'Alembertian solutions (AP94), a subclass of Liouvillian solutions (HS99), of a given recurrence with minimal nested depth; for applications see, e.g., (Sch04; DPSW06; Sch07c; PS07; BBKS07; MS07). The presented algorithms are implemented in the summation package Sigma (Sch07c), that can be executed in the computer algebra system Mathematica.

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The general structure of this article is as follows. In Section 2 we introduce the problem to find optimal sum representations which we supplement by concrete examples. In Section 3 we define depth-optimal $\Pi\Sigma^*$-extensions and show how indefinite summation can be handled accordingly in such fields. After showing how generalized d'Alembertian extensions can be embedded in the ring of sequences in Section 4, we are ready to prove that our algorithms produce sum representations with optimal nested depth in Section 5. Applications are presented in Section 6.

2. The problem description in the ring of sequences

Let $\mathbb{N}$ be the non-negative integers, let $\mathbb{K}$ be a field[1] and consider the set of sequences $\mathbb{K}^{\mathbb{N}}$ with elements $(a_n)_{n=0}^{\infty} = \langle a_0, a_1, a_2, \dots \rangle$, $a_i \in \mathbb{K}$. With component-wise addition and multiplication we obtain a commutative ring; the field $\mathbb{K}$ can be naturally embedded by identifying $k \in \mathbb{K}$ with the sequence $\boldsymbol{k} := \langle k, k, k, \dots \rangle$. Given a sequence $\boldsymbol{a}$ and a nonnegative integer i, $\boldsymbol{a}(i)$ denotes the i-th entry in $\boldsymbol{a}$.

In order to turn the shift-operation

$$\mathcal{S} : \langle a_0, a_1, a_2, \dots \rangle \mapsto \langle a_1, a_2, a_3, \dots \rangle \tag{2.1}$$

to an automorphism, we follow the construction from (PWZ96, Sec. 8.2): We define an equivalence relation $\sim$ on $\mathbb{K}^{\mathbb{N}}$ with $(a_n)_{n=0}^{\infty} \sim (b_n)_{n=0}^{\infty}$ if there exists a $d \geq 0$ such that $a_k = b_k$ for all $k \geq d$. The equivalence classes form a ring which is denoted by $S(\mathbb{K})$; the elements of $S(\mathbb{K})$ will be denoted, as above, by sequence notation. Now it is immediate that $\mathcal{S} : S(\mathbb{K}) \to S(\mathbb{K})$ with (2.1) forms a ring automorphism.

In general, a *difference ring* (resp. difference field) $(\mathbb{A}, \sigma)$ is defined as a ring $\mathbb{A}$ (resp. field) with a ring automorphism (resp. field automorphism) $\sigma : \mathbb{A} \to \mathbb{A}$. The set of constants $\text{const}_\sigma\mathbb{A} = \{k \in \mathbb{A} \,|\, \sigma(k) = k\}$ forms a subring[2] (resp. subfield) of $\mathbb{A}$. We call $\text{const}_\sigma\mathbb{A}$ the *constant field* of $(\mathbb{A}, \sigma)$. A difference ring (resp. difference field) $(\mathbb{E}, \sigma)$ is a *difference ring extension* (resp. *difference field extension*) of a difference ring (resp. difference field) $(\mathbb{A}, \sigma')$ if $\mathbb{A}$ is a subring (resp. subfield) of $\mathbb{E}$ and $\sigma'(f) = \sigma(f)$ for all $f \in \mathbb{A}$; we call $(\mathbb{A}, \sigma')$ also a sub-difference ring (resp. field) of $(\mathbb{E}, \sigma)$. Since σ and σ' agree on $\mathbb{A}$, we do not distinguish them anymore.

Definition 2.1. The difference ring $(S(\mathbb{K}), \mathcal{S})$ is called the *ring of* $\mathbb{K}$*-sequences*, or in short, the *ring of sequences*.
$\boldsymbol{a} \in S(\mathbb{K})$ is called *rational* (resp. *hypergeometric*) if there is a rational function $p(n) \in \mathbb{K}(n)$ such that $\boldsymbol{a}(n) = p(n)$ (resp. $\frac{\boldsymbol{a}(n+1)}{\boldsymbol{a}(n)} = p(n)$) for all $n \geq r$ for some $r \in \mathbb{N}$. The *difference ring of all rational sequences* is denoted by $(R(\mathbb{K}), \mathcal{S})$.
Let $\mathbb{K} = \mathbb{K}'(q)$ be a transcendental extension. $\boldsymbol{a}$ is called *q-rational* (resp. *q-hypergeometric*) if there is a rational function $p(n) \in \mathbb{K}(n)$ s.t. $\boldsymbol{a}(n) = p(q^n)$ (resp. $\frac{\boldsymbol{a}(n+1)}{\boldsymbol{a}(n)} = p(q^n)$) for all $n \geq r$ for some $r \in \mathbb{N}$. The *difference ring of all q-rational sequences* is denoted by $(qR(\mathbb{K}), \mathcal{S})$.
Let $\mathbb{K} = \mathbb{K}'(q_1, \dots, q_l)$ be a transcendental extension. $\boldsymbol{a}$ is called *mixed rational* (resp. *mixed hypergeometric*, cf. (BP99)) if there is a rational function of the form $p(n, n_1, \dots, n_l) \in \mathbb{K}(n, n_1, \dots, n_l)$ such that $\boldsymbol{a}(n) = p(n, q_1^n, \dots, q_l^n)$ (resp. $\frac{\boldsymbol{a}(n+1)}{\boldsymbol{a}(n)} = p(n, q_1^n, \dots, q_l^n)$) for all $n \geq r$ for some $r \in \mathbb{N}$. The *difference ring of all mixed rational sequences* is denoted by $(qR_l(\mathbb{K}), \mathcal{S})$.

[1] All fields and rings are commutative and contain the rational numbers $\mathbb{Q}$.

[2] Subsequently, we assume that $\text{const}_\sigma\mathbb{A}$ is a field, which we usually denote by $\mathbb{K}$.

We introduce the sum ring over a subring $\mathbb{S}$ of $S(\mathbb{K})$; here we can choose, e.g., $\mathbb{S} = R(\mathbb{K})$ or $\mathbb{S} = qR_l(\mathbb{K})$.

Definition 2.2. Let $(\mathbb{S}, \mathcal{S})$ be a sub-difference ring of $(S(\mathbb{K}), \mathcal{S})$. The *sum ring over* $\mathbb{S}$, denoted by $(\Sigma(\mathbb{S}), \mathcal{S})$, is the smallest sub-difference ring of $(S(\mathbb{K}), \mathcal{S})$ s.t.

(1) $\mathbb{S} \subseteq \Sigma(\mathbb{S})$,
(2) $\boldsymbol{f} \in \Sigma(\mathbb{S})$ implies that $\boldsymbol{F} \in \Sigma(\mathbb{S})$ if $\mathcal{S}(\boldsymbol{F}) = \boldsymbol{F} + \boldsymbol{f}$.

In other words, $\Sigma(\mathbb{S})$ consists of all polynomial expressions of indefinite nested sums over $\mathbb{S}$. Note that the sum ring $\Sigma(R(\mathbb{K}))$ is a subring of Liouvillian sequences; see (HS99, Def. 3.3)

Example 2.3. Consider the sequences $\boldsymbol{a} = \langle A(n)\rangle_{n\geq 0}$ and $\boldsymbol{b} = \langle B(n)\rangle_{n\geq 0}$ with $A(n) = \sum_{k=1}^{n} k$ and $B(n) = \frac{n(n+1)}{2}$. Clearly, $\boldsymbol{a} = \boldsymbol{b} \in \Sigma(R(\mathbb{Q}))$.

Example 2.4. Subsequently, we denote the harmonic numbers of order $r > 0$ by $H_n^{(r)} = \sum_{k=1}^{n} 1/k^r$; we set $H_n = H_n^{(1)}$. Then $\langle H_n^{(r)}\rangle_{n\geq 0} \in \Sigma(R(\mathbb{Q}))$ for $r \in \mathbb{N}\setminus\{0\}$. Moreover, the sequences $\langle A(n)\rangle_{n\geq 0}, \langle W(n)\rangle_{n\geq 0}, \langle B(n)\rangle_{n\geq 0}$ represented by the indefinite nested sum expressions

$$A(n) = \sum_{r=1}^{n} \frac{\displaystyle\sum_{l=1}^{r} \frac{H_l^2 + H_l^{(2)}}{l} + \sum_{l=1}^{r} \frac{H_l}{l}}{r}, \quad W(n) = \sum_{r=1}^{n} \frac{\displaystyle\sum_{l=1}^{r} \frac{2\displaystyle\sum_{i=1}^{l} \frac{H_i}{i}}{l} + \sum_{l=1}^{r} \frac{H_l}{l}}{r},$$

$$B(n) = \frac{1}{12}\Big(H_n^4 + 2H_n^3 + 6(H_n+1)H_n^{(2)}H_n + 3\big(H_n^{(2)}\big)^2 + (8H_n+4)H_n^{(3)} + 6H_n^{(4)}\Big) \tag{2.2}$$

are also elements from $\Sigma(R(\mathbb{Q}))$. Note that for all $n \in \mathbb{N}$,

$$A(n) = W(n) = B(n),$$

i.e., $A(n), W(n), B(n)$ represent the same sequence. But, from the symbolic summation point of view, they differ in its representation depth.

We introduce the measure of depth that is defined for a given nested sum expression; for a formal description of such representations (like term algebras) we refer, e.g., to (NP97).

Definition 2.5. Let $(\mathbb{S}, \mathcal{S})$ be a sub-difference ring of $(S(\mathbb{K}), \mathcal{S})$ with a function $\delta : \mathbb{S} \to \mathbb{N}$. $\bar{F}(n)$ is a *sum expression over* $\mathbb{S}$, if $\bar{F}(n)$ is an expression in terms of indefinite nested sums over $\mathbb{S}$; by definition there is an $\boldsymbol{F} \in \Sigma(\mathbb{S})$ s.t. $\boldsymbol{F}(n) = \bar{F}(n)$ for all $n \geq r$ for some $r \in \mathbb{N}$. The *δ-depth* of $\bar{F}(n)$ is defined as follows.

(1) If no sum quantifiers occur in $\bar{F}(n)$, i.e., $\boldsymbol{F} \in \mathbb{S}$, the δ-depth of $\bar{F}(n)$ is $\delta(\boldsymbol{F})$.
(2) Let $\bar{f}(n)$ be a sum expression over $\mathbb{S}$ with δ-depth d. Then $\bar{F}(n) = \sum_{k=r}^{n} \bar{f}(k)$, for some $r \in \mathbb{N}$ properly chosen, has δ-depth $d+1$.
(3) Let $\bar{f}_1(n)$ and $\bar{f}_2(n)$ be sum expressions over $\mathbb{S}$ with δ-depths d_1 and d_2, respectively. Then $\bar{F}(n) = \bar{f}_1(n) + \bar{f}_2(n)$ or $\bar{F}(n) = \bar{f}_1(n)\bar{f}_2(n)$ have δ-depth $\max(d_1, d_2)$.

The *δ-depth* of $\boldsymbol{F} \in \Sigma(\mathbb{S})$ is the minimal δ-depth of a sum expression $\bar{F}(n)$ such that $\boldsymbol{F}(n) = \bar{F}(n)$ for all $n \geq r$ for some $r \in \mathbb{N}$. Such an $\bar{F}(n)$ with $\boldsymbol{F}(n) = \bar{F}(n)$ for all $n \geq r$ for some $r \in \mathbb{N}$ with minimal δ-depth is called *sum representation of* $\boldsymbol{F}$ *with optimal δ-depth.*

Remark. Note that by simple term rewriting sum expressions might cancel which then leads to better depth representations. We overcome this phenomenon later by representing such expressions in $\Pi\Sigma^*$-fields where the elements are in reduced form (the numerators and denominators are coprime); see Definition 3.6 below.

Example 2.6. Define $\delta : R(\mathbb{Q}) \to \mathbb{N}$ by

$$\delta(\boldsymbol{a}) = \begin{cases} 0, & \text{if } \boldsymbol{a} = \langle a, a, a, \dots \rangle \text{ for some } a \in \mathbb{Q} \\ 1, & \text{otherwise.} \end{cases} \tag{2.3}$$

In Example 2.3 the δ-depths of $A(n)$ is 2. Since $B(n)$ has no sum quantifier, the δ-depth of $B(n)$ is $\delta(\boldsymbol{b}) = 1$. Clearly, the δ-depth of $\boldsymbol{a}$ is 1 and $B(n)$ is a sum representation of $\boldsymbol{a}$ with optimal δ-depth.
In Example 2.4 the δ-depth of $A(n)$, $W(n)$ and $B(n)$ are 4, 5 and 2, respectively. Note that $B(n)$ has minimal δ-depth, i.e., the δ-depth of the sequence $\boldsymbol{a} = \langle A(n)\rangle_{n\geq 0} = \langle W(n)\rangle_{n\geq 0} = \langle B(n)\rangle_{n\geq 0}$ is 2 and $B(n)$ is a sum representation with optimal δ-depth.

Subsequently, we consider the following summation problem.

DOS: Depth Optimal Summation. *Given* a sub-difference ring $(\mathbb{S}, \mathcal{S})$ of $(S(\mathbb{K}), \mathcal{S})$ with a function $\delta : \mathbb{S} \to \mathbb{N}$ and *given* a sum expression over $\mathbb{S}$ that represents a sequence $\boldsymbol{a} \in \Sigma(\mathbb{S})$; *find* a sum representation of $\boldsymbol{a}$ with optimal δ-depth.

E.g., we shall derive algorithms if $(\mathbb{S}, \mathcal{S})$ is the rational or the mixed sequence ring over $\mathbb{K}$ with the canonic depth function

$$\delta(\boldsymbol{a}) = \begin{cases} 0, & \text{if } \boldsymbol{a} = \langle a, a, a, \dots \rangle \text{ for some } a \in \mathbb{K} \\ 1, & \text{otherwise.} \end{cases} \tag{2.4}$$

More generally, $(\mathbb{S}, \mathcal{S})$ can be a subring of $S(\mathbb{K})$ that contains, e.g., hypergeometric sequences or q-hypergeometric sequences; see Section 5 below.

3. Telescoping in refined difference fields

Loosely speaking, the telescoping problem can be formulated as follows:
Given a term expression $f(k)$ that represents a sequence $\boldsymbol{f} = \langle f(k)\rangle_{k\geq 0} \in S(\mathbb{K})$; *find* an "appropriate" term expression $g(k)$ such that

$$g(k+1) - g(k) = f(k+1) \tag{3.1}$$

holds for all $k \geq r$ for some $r \in \mathbb{N}$. Then summing this equation over k gives

$$g(n) - g(r) = \sum_{k=r+1}^{n} f(k).$$

To sum up, the sequence

$$\boldsymbol{g} = \langle 0, \dots, 0, g(r), g(r+1), g(r+2), \dots \rangle - \langle g(r)\rangle_{n\geq 0} \tag{3.2}$$

of the sum $\sum_{k=r+1}^{n} f(k)$ is represented by the "appropriate" term expression $g(n)$.

This task has been accomplished for the rational case ($\boldsymbol{f}, \boldsymbol{g} \in R(\mathbb{K})$), see (Abr71), for the q-rational case ($\boldsymbol{f}, \boldsymbol{g} \in qR(\mathbb{K})$), see e.g. (PS95), or the mixed rational case ($\boldsymbol{f}, \boldsymbol{g} \in qR_l(\mathbb{K})$), see (BP99). Moreover, there are algorithms for the (q–)hypergeometric and the mixed hypergeometric case, see (Gos78; PR97; BP99).

More generally, we consider the telescoping problem in $\Pi\Sigma^*$-fields in which one can represent expressions in terms of indefinite nested sums and products.

Definition 3.1. A difference field extension $(\mathbb{F}(t),\sigma)$ of $(\mathbb{F},\sigma)$ is called a $\Pi\Sigma^*$-*extension* if both difference fields share the same field of constants, t is transcendental over $\mathbb{F}$, and $\sigma(t)=t+a$ for some $a\in\mathbb{F}^*$ (a sum) or $\sigma(t)=a\,t$ for some $a\in\mathbb{F}^*$ (a product). If $\sigma(t)/t\in\mathbb{F}$ (resp. $\sigma(t)-t\in\mathbb{F}$), we call the extension also a Π-*extension* (resp. Σ^*-*extension*). In short, we say that $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ is a $\Pi\Sigma^*$-extension (resp. Π-extension, Σ^*-extension) of $(\mathbb{F},\sigma)$ if the extension is given by a tower of $\Pi\Sigma^*$-extensions (resp. Π-extensions, Σ^*-extensions). We call a $\Pi\Sigma^*$-extension $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ of $(\mathbb{F},\sigma)$ with $\sigma(t_i)=\alpha_i\,t_i+\beta_i$ *generalized d'Alembertian*, or in short *polynomial*, if $\alpha_i\in\mathbb{F}$ and $\beta_i\in\mathbb{F}[t_1,\dots,t_{i-1}]$ for all $1\leq i\leq e$. A $\Pi\Sigma^*$-*field* $(\mathbb{K}(t_1)\dots(t_e),\sigma)$ over $\mathbb{K}$ is a $\Pi\Sigma^*$-extension of $(\mathbb{K},\sigma)$ with constant field $\mathbb{K}$.

To be more precise, we consider the following subproblems:

R_1 (Representation 1): *Represent* $\boldsymbol{f}\in S(\mathbb{K})$, given as an expression $f(k)$ in terms of nested sums and products, in a $\Pi\Sigma^*$-field $(\mathbb{F},\sigma)$ with $f\in\mathbb{F}$.

T (Telescoping): Solve the telescoping problem (3.3) in $(\mathbb{F},\sigma)$: *Given* $f\in\mathbb{F}$; *find* a g in $\mathbb{F}$ or in an appropriate extension of $\mathbb{F}$ such that

$$\sigma(g)-g=f. \tag{3.3}$$

R_2 (Representation 2): *Reinterpret* the result g as an expression $g(k)$ in terms of nested sums and products such that (3.1) holds. This finally leads to the sum sequence (3.2) of $\sum_{k=r+1}^{n}f(k)$.

3.1. **A straightforward approach.** The following result from (Kar81) gives a first glance how these tasks can be treated.

Theorem 3.2. *Let $(\mathbb{F}(t),\sigma)$ be a difference field ext. of $(\mathbb{F},\sigma)$ with $\sigma(t)=a\,t+f$.*
(1) $(\mathbb{F}(t),\sigma)$ is a Σ^*-extension of $(\mathbb{F},\sigma)$ iff $a=1$ and there is no $g\in\mathbb{F}$ s.t. (3.3).
(2) $(\mathbb{F}(t),\sigma)$ is a Π-extension of $(\mathbb{F},\sigma)$ iff $t\neq 0$, $f=0$ and there are no $g\in\mathbb{F}^*$ and $m>0$ such that $\sigma(g)=a^m g$.

Loosely speaking, we can adjoin a sum formally by a Σ^*-extension if and only if there does not exist a solution of the telescoping problem.

Example 3.3. We start with the $\Pi\Sigma^*$-field $(\mathbb{Q}(k),\sigma)$ over $\mathbb{Q}$ with $\sigma(k)=k+1$. Now we consider the sum expressions of $A(k)$ in (2.2) (n replaced by k), say in the order

$$\begin{aligned}&\overset{(1)}{\to} H_k=\sum_{i=1}^{k}\frac{1}{i}\overset{(2)}{\to} H(k)=\sum_{i=1}^{k}\frac{H_i}{i}\overset{(3)}{\to} H_k^{(2)}=\sum_{i=1}^{k}\frac{1}{i^2}\\&\overset{(4)}{\to} a(k)=\sum_{i=1}^{k}\frac{H_i^2+H_i^{(2)}}{i}\overset{(5)}{\to} A(k),\end{aligned} \tag{3.4}$$

and represent them in terms of Σ^*-extensions following Theorem 3.2.1.

(1) Using, e.g., Gosper's algorithm (Gos78), Karr's algorithm (Kar81) or a simplified version of it presented in (Sch05c), we check that there is no $g\in\mathbb{Q}(k)$ with $\sigma(g)=g+\frac{1}{k+1}$. Hence, by Theorem 3.2.1 we adjoin H_k in form of the Σ^*-extension $(\mathbb{Q}(k)(h),\sigma)$ of $(\mathbb{Q}(k),\sigma)$ with $\sigma(h)=h+\frac{1}{k+1}$; note that the shift behavior $H_{k+1}=H_k+\frac{1}{k+1}$ is reflected by the automorphism σ.

(2) With the algorithms from (Kar81) or (Sch05c) we show that there is no $g \in \mathbb{Q}(k)(h)$ with $\sigma(g) = g + \frac{\sigma(h)}{k+1}$. Thus we take the Σ^*-extension $(\mathbb{Q}(k)(h)(H), \sigma)$ of $(\mathbb{Q}(k)(h), \sigma)$ with $\sigma(H) = H + \frac{\sigma(h)}{k+1}$ and express $H(k)$ by H.
(3) With the algorithms from above, we find $g = 2H - h^2 \in \mathbb{Q}(k)(h)(H)$ with $\sigma(g) = g + \frac{1}{(k+1)^2}$, and represent[3] $H_k^{(2)}$ by g.
(4) We check algorithmically that there is no $g \in \mathbb{Q}(k)(h)(H)$ with $\sigma(g) = g + 2\frac{\sigma(H)}{k+1}$; thus we rephrase $a(k)$ as a in the Σ^*-extension $(\mathbb{Q}(k)(h)(H)(a), \sigma)$ of $(\mathbb{Q}(k)(h)(H), \sigma)$ with $\sigma(a) = a + 2\frac{\sigma(H)}{k+1}$.
(5) Finally, we fail to find a $g \in \mathbb{Q}(k)(h)(H)(a)$ such that $\sigma(g) = g + \frac{\sigma(H+a)}{k+1}$; thus we represent $A(k)$ with A in the Σ^*-extension $(\mathbb{Q}(k)(h)(H)(a)(A), \sigma)$ of $(\mathbb{Q}(k)(h)(H)(a), \sigma)$.

Reformulating A as a sum expression yields $W(k)$ (n replaced by k) from (2.2).

The product case works similarly; for more information and problematic cases we refer to (Sch05b). In this article, we consider mainly the following special case. Suppose that $\mathbb{K} = \mathbb{K}'(q_1, \dots, q_e)$ is a transcendental extension and consider the $\Pi\Sigma^*$-field $(\mathbb{K}(k), \sigma)$ with $\sigma(k) = k + 1$. Applying Thm. 3.2.2 iteratively shows that $(\mathbb{K}(k)(t_1) \dots (t_e), \sigma)$ is a Π-extension of $(\mathbb{K}(k), \sigma)$ with $\sigma(t_1) = q_i t_i$ for $1 \leq i \leq e$; this result is a direct consequence of (Sch07a, Thm. 6.1) and (Sch07b, Cor. 16)).

Definition 3.4. We call such a $\Pi\Sigma^*$-field $(\mathbb{K}(k)(t_1) \dots (t_e), \sigma)$ $(e \geq 0)$ introduced above a *mixed* $\Pi\Sigma^*$*-field over* $\mathbb{K}$ *with depth* 1.

We remark that the sums occurring in $W(n)$ pop up only in the numerator. Here the following result plays an important role.

Theorem 3.5 ((Sch07a))**.** *Let* $(\mathbb{F}(t_1) \dots (t_e), \sigma)$ *be a polynomial* $\Pi\Sigma^*$*-extension of* $(\mathbb{F}, \sigma)$*. For all* $g \in \mathbb{F}[t_1, \dots, t_e]$, $\sigma(g) - g \in \mathbb{F}[t_1, \dots, t_e]$ *if and only if* $g \in \mathbb{F}[t_1, \dots, t_e]$.

Namely, if, e.g., $A(n)$ consists only of sums that occur in the numerator, then by solving iteratively the telescoping problem, it is guaranteed that also the telescoping solutions will have only sums that occur in the numerators.

Note that the representation depth of $W(n)$ is reflected by the nested depth of the underlying difference field constructed in Example 3.3.

Definition 3.6. Let $(\mathbb{F}, \sigma)$ be a $\Pi\Sigma^*$-field over $\mathbb{K}$ with $\mathbb{F} := \mathbb{K}(t_1) \dots (t_e)$ where $\sigma(t_i) = a_i\, t_i$ or $\sigma(t_i) = t_i + a_i$ for $1 \leq i \leq e$. The *depth function for elements of* $\mathbb{F}$, $\delta_{\mathbb{K}} : \mathbb{F} \to \mathbb{N}$, is defined as follows.

(1) For any $g \in \mathbb{K}$, $\delta_{\mathbb{K}}(g) := 0$.
(2) If $\delta_{\mathbb{K}}$ is defined for $(\mathbb{K}(t_1) \dots (t_{i-1}), \sigma)$ with $i > 1$, we define $\delta_{\mathbb{K}}(t_i) := \delta_{\mathbb{K}}(a_i) + 1$; for $g = \frac{g_1}{g_2} \in \mathbb{K}(t_1) \dots (t_i)$, with $g_1, g_2 \in \mathbb{K}[t_1, \dots, t_i]$ coprime, we define
$$\delta_{\mathbb{K}}(g) := \max(\{\delta_{\mathbb{K}}(t_i)\mathbb{K} \,|\, t_i \text{ occurs in } g_1 \text{ or } g_2\} \cup \{0\}).$$

The *depth* of $(\mathbb{F}, \sigma)$ is $\max(\delta_{\mathbb{K}}(t_1), \dots, \delta_{\mathbb{K}}(t_e), 0)$. More generally, the *extension depth* of a $\Pi\Sigma^*$-extension $(\mathbb{F}(x_1) \dots (x_r), \sigma)$ of $(\mathbb{F}, \sigma)$ is $\max(\delta_{\mathbb{K}}(x_1), \dots, \delta_{\mathbb{K}}(x_r), 0)$.

Example 3.7. In Example 3.3 we have $\delta_{\mathbb{Q}}(k) = 1$, $\delta_{\mathbb{Q}}(h) = 2$, $\delta_{\mathbb{Q}}(H) = 3$, $\delta_{\mathbb{Q}}(a) = 4$, and $\delta_{\mathbb{Q}}(A) = 5$. The depth of $(\mathbb{Q}(k)(h)(H)(a)(A), \sigma)$ is 5.

[3]Note that there is no way to adjoin a Σ^*-extension h_2 of the desired type $\sigma(h_2) = h_2 + 1/(k+1)^2$, since otherwise $\sigma(g - h_2) = (g - h_2)$, i.e., $\text{const}_\sigma \mathbb{Q}(k)(h)(H)(h_2) \neq \mathbb{Q}$.

3.2. **A refined approach.** With the straightforward approach sketched in Example 3.3 we obtain an alternative sum representation $B(n)$ for $A(n)$ with larger depth. Motivated by such problematic situations, Karr's $\Pi\Sigma^*$-fields have been refined in the following way; see (Sch05a; Sch07b).

Definition 3.8. Let $(\mathbb{F},\sigma)$ be a $\Pi\Sigma^*$-field over $\mathbb{K}$. A difference field extension $(\mathbb{F}(s),\sigma)$ of $(\mathbb{F},\sigma)$ with $\sigma(s)=s+f$ is called *depth-optimal* Σ^*-extension, in short Σ^δ-extension, if there is no Σ^*-extension $(\mathbb{E},\sigma)$ of $(\mathbb{F},\sigma)$ with extension depth $\leq \delta_{\mathbb{K}}(f)$ such that there is $g\in\mathbb{E}$ as in (3.3). A $\Pi\Sigma^*$-extension $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ of $(\mathbb{F},\sigma)$ is depth-optimal, in short a $\Pi\Sigma^\delta$-extension, if all Σ^*-extensions are depth-optimal. A $\Pi\Sigma^\delta$-field consists of Π- and Σ^δ-extensions.

Note that a Σ^δ-extension is a Σ^*-extension by Theorem 3.2.1. Moreover, a $\Pi\Sigma^*$-field $(\mathbb{F},\sigma)$ with depth ≤ 2 and $k\in\mathbb{F}$ such that $\sigma(k)=k+1$ is always depth-optimal; see (Sch07b, Prop. 19). In particular, a mixed $\Pi\Sigma^*$-field with depth 1 is a $\Pi\Sigma^\delta$-field. Given any $\Pi\Sigma^\delta$-field, we obtain the following crucial property (Sch07b, Result 3).

Theorem 3.9. *Let $(\mathbb{F},\sigma)$ be a $\Pi\Sigma^\delta$-field over $\mathbb{K}$. Then for any $f,g\in\mathbb{F}$ such that* (3.3) *we have*

$$\delta_{\mathbb{K}}(f)\leq\delta_{\mathbb{K}}(g)\leq\delta_{\mathbb{K}}(f)+1. \tag{3.5}$$

In other words, in a given $\Pi\Sigma^\delta$-field we can guarantee that the depth of a telescoping solution is not bigger than the sum itself.

Example 3.10. We consider again the sum expressions in (3.4), but this time we use the refined algorithm presented in (Sch07b).

(1) As in Example 3.10 we compute the $\Pi\Sigma^\delta$-field $(\mathbb{Q}(k)(h),\sigma)$ and represent H_k with h. From this point on, our new algorithm works differently.
(2) Given $(\mathbb{Q}(k)(h),\sigma)$, we find the Σ^δ-extension $(\mathbb{Q}(k)(h)(h_2),\sigma)$ of $(\mathbb{Q}(k)(h),\sigma)$ with $\sigma(h_2)=h_2+\frac{1}{(k+1)^2}$ in which we find $H'=\frac{1}{2}(h^2+h_2)$ such that $\sigma(H')-H'=\frac{\sigma(h)}{k+1}$. Hence we represent $H(k)$ by H'.
(3) $H_k^{(2)}$ can be represented by h_2 in the already constructed $\Pi\Sigma^\delta$-field.
(4) Our algorithm finds the Σ^δ-extension $(\mathbb{Q}(k)(h)(h_2)(h_3),\sigma)$ of $(\mathbb{Q}(k)(h)(h_2),\sigma)$ with $\sigma(h_3)=h_3+\frac{1}{(k+1)^3}$ together with $a'=\frac{1}{3}(h^3+3hh_2+2h_3)$ such that $\sigma(a')-a'=\frac{\sigma(h^2+h_2)}{k+1}$; hence we rephrase $a(k)$ as a'.
(5) Finally, we find the Σ^δ-ext. $(\mathbb{Q}(k)(h)(h_2)(h_3)(h_4),\sigma)$ of $(\mathbb{Q}(k)(h)(h_2)(h_3),\sigma)$ with $\sigma(h_4)=h_4+\frac{1}{(k+1)^4}$ and get $A'=\frac{1}{12}(h^4+2h^3+6(h+1)h_2h+3h_2^2+(8h+4)h_3+6h_4)$ s.t. $\sigma(g)-g=\frac{\sigma(g_1+g_2)}{k+1}$; $A(k)$ is represented by A'.

Reinterpreting A' as a sum expression gives $B(k)$ (n replaced by k) from (2.2); see also Example 5.2.

To sum up, we can compute step by step a $\Pi\Sigma^\delta$-field in which we can represent nested sum expressions. To be more precise, we will exploit the following result.

Theorem 3.11. *Let $(\mathbb{F},\sigma)$ be a $\Pi\Sigma^\delta$-field over $\mathbb{K}$ and $f\in\mathbb{F}$.*

(1) There is a Σ^δ-extension $(\mathbb{E},\sigma)$ of $(\mathbb{F},\sigma)$ in which we have $g\in\mathbb{E}$ such that (3.3)*; $(\mathbb{E},\sigma)$ and g can be given explicitly.*
(2) In particular, suppose that $(\mathbb{F},\sigma)$ with $\mathbb{F}=\mathbb{G}(x_1,\dots,x_r)$ is a polynomial $\Pi\Sigma^\delta$-extension of $(\mathbb{G},\sigma)$ where $(\mathbb{G},\sigma)$ is a mixed $\Pi\Sigma^\delta$-field with depth 1*. If $f\in\mathbb{G}[x_1\dots,x_r]$, then $(\mathbb{E},\sigma)$ from part (1) can be given as a polynomial $\Pi\Sigma^\delta$-extension of $(\mathbb{G},\sigma)$; if $\mathbb{E}=\mathbb{F}(t_1,\dots,t_e)$, then $g\in\mathbb{G}[x_1,\dots,x_r][t_1,\dots,t_e]$.*

Proof. The first follows by (Sch07b, Result 1). The second part follows by Result 2, Lemma 44, and Corollary 60 of (Sch07b). □

4. Embedding of polynomial $\Pi\Sigma^*$-extensions in $S(\mathbb{K})$

In Examples 3.3 and 3.10 we illustrated how subtasks R$_1$, T, and R$_2$ can be handled in $\Pi\Sigma^*$-fields. In the following we will make steps R$_1$ and R$_2$ more precise.

Namely, following (Sch07a) we will embed the polynomial ring $\mathbb{F}[t_1,\dots,t_e]$ of a generalized d'Alembertian extension $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ of $(\mathbb{F},\sigma)$ with constant field $\mathbb{K}$ in the ring of sequences $(S(\mathbb{K}),\mathcal{S})$, provided that $(\mathbb{F},\sigma)$ can be embedded in $(S(\mathbb{K}),\mathcal{S})$. Then by constructing the monomorphism accordingly, R$_1$ is accomplished by $\tau^{-1}(\boldsymbol{f})$ and R$_2$ is carried out by $\tau(g)$; see Example 5.2 below.

A *difference ring homomorphism* $\tau:\mathbb{A}_1\to\mathbb{A}_2$ between difference rings $(\mathbb{A}_1,\sigma_1)$ and $(\mathbb{A}_2,\sigma_2)$ is a ring homomorphism with the additional property that $\tau(\sigma_1(f))=\sigma_2(\tau(f))$ for all $f\in\mathbb{A}_1$. If $(\mathbb{A}_1,\sigma)$ and $(\mathbb{A}_2,\sigma)$ are difference ring extensions of $(\mathbb{G},\sigma)$ and $\tau(g)=g$ for all $g\in\mathbb{G}$, we call τ also a $\mathbb{G}$-homomorphism. If τ is injective, we call τ a difference ring monomorphism (resp. a $\mathbb{G}$-monomorphism).

More precisely, we will construct a difference ring monomorphism of the form $\tau:\mathbb{F}[t_1,\dots,t_e]\to S(\mathbb{K})$ such that the constants $k\in\mathbb{K}$ are mapped to

$$\boldsymbol{k}=\langle k,k,\dots\rangle.$$

We will call such a difference ring monomorphism also a $\mathbb{K}$-*embedding*.

Let $(\mathbb{A},\sigma)$ be a difference ring with constant field $\mathbb{K}$ and let $\tau:\mathbb{A}\to S(\mathbb{K})$ be a $\mathbb{K}$-embedding. Then by definition of τ the following holds: For all $c\in\mathbb{K}$ there is a $d\in\mathbb{N}$ such that

$$\forall i\geq d:\ (\tau(c))(i)=c; \tag{4.1}$$

for all $f,g\in\mathbb{A}$ there is a $d\in\mathbb{N}$ such that

$$\forall i\geq d:\ (\tau(f\,g))(i)=(\tau(f))(i)\,(\tau(g))(i), \tag{4.2}$$

$$\forall i\geq d:\ (\tau(f+g))(i)=(\tau(f))(i)+(\tau(g))(i); \tag{4.3}$$

and for all $f\in\mathbb{A}$ and $j\in\mathbb{Z}$ there is a $d\in\mathbb{N}$ such that

$$\forall i\geq d\ (\tau(\sigma^j(f)))(i)=(\tau(f))(i+j). \tag{4.4}$$

To take into account the constructive aspects, we introduce the following functions.

Definition 4.1. Let $(\mathbb{A},\sigma)$ be a difference ring and let $\tau:\mathbb{A}\to S(\mathbb{K})$ be a $\mathbb{K}$-embedding. τ is called *operation-bounded* by $L:\mathbb{A}\to\mathbb{N}$ if for all $f\in\mathbb{A}$ and $j\in\mathbb{Z}$ with $d=d(f,j):=L(f)+\max(0,-j)$ we have (4.4) and for all $f,g\in\mathbb{A}$ with $d=d(f,g):=\max(L(f),L(g))$ we have (4.2) and (4.3); such a function is also called *o-function* for τ.

Example 4.2. Given the $\Pi\Sigma^\delta$-field $(\mathbb{K}(k),\sigma)$ over $\mathbb{K}$ with $\sigma(k)=k+1$, we construct a $\mathbb{K}$-embedding $(\mathbb{K}(k),\sigma)$ into the ring of sequences $(S(\mathbb{K}),\mathcal{S})$. We start with the $\mathbb{K}$-embedding $\tau_0:\mathbb{K}\to S(\mathbb{K})$ where $\tau_0(c)=\langle c,c,c,\dots\rangle$ for all $c\in\mathbb{K}$; for the o-function we can set $L_0(c)=0$ for all $c\in\mathbb{K}$. Now we define the difference ring homomorphism $\tau_1:\mathbb{K}(k)\to S(\mathbb{K})$ with $\tau_1(\frac{p}{q})=\langle F(k)\rangle_{k\geq0}$, p and q are coprime polynomials, such that

$$F(k)=\begin{cases}0 & q(k)=0\\ \frac{p(k)}{q(k)} & q(k)\neq0.\end{cases}$$

For the o-function $L_1(\frac{p(k)}{q(k)})$ we take the maximal non-negative integer l such that $q(k+l) \neq 0$ for all $k \in \mathbb{N}$. Note: since $p(k), q(k)$ have only finitely many roots, $\tau_1(\frac{p}{q}) = \mathbf{0}$ if and only if $\frac{p(k)}{q(k)} = 0$. Hence τ_1 is injective.
Since $\tau_1(\mathbb{K}(k)) = R(\mathbb{K})$, τ_1 forms an isomorphism between $\mathbb{K}(k)$ and $R(\mathbb{K})$.

Summarizing, the $\Pi\Sigma^\delta$-field $(\mathbb{K}(k), \sigma)$ with $\sigma(k) = k+1$ can be embedded into $(S(\mathbb{K}), \mathcal{S})$. More generally, if $(\mathbb{F}, \sigma)$ is a mixed $\Pi\Sigma^\delta$-field with depth 1, then there is a $\mathbb{K}$-embedding $\tau : \mathbb{F} \to S(\mathbb{K})$ and an o-function L, which can be given explicitly; for more details see (Sch07a) which relies on (BP99). As a consequence, we obtain a difference ring isomorphism between $qR_l(\mathbb{K})$ and mixed $\Pi\Sigma^\delta$-fields with depth 1.

Example 4.3. Take $(\mathbb{K}(k), \sigma)$ and τ_1 from Example 4.2 and consider the Σ^δ-extension $(\mathbb{K}(k)(h), \sigma)$ of $(\mathbb{K}(k), \sigma)$ with $\sigma(h) = h + \frac{1}{k+1}$. We define the difference ring homomorphism $\tau_2 : \mathbb{K}(k)[h] \to S(\mathbb{K})$ with $\tau_2(h) = \langle H_n \rangle_{n \geq 0}$ and

$$\tau_2(\sum_{i=0}^{d} f_i h^i) = \sum_{i=0}^{d} \tau_1(f_i)\tau_2(h)^i.$$

As o-function we can take $L_2(\sum_{i=0}^{d} f_i h^i) = \max(L_1(f_i) | 0 \leq i \leq d)$. Note that τ_2 is injective. Namely, suppose otherwise. Then we can take $f = \sum_{i=0}^{d} f_i h^i \in \mathbb{K}(k)[h] \setminus \{0\}$ with $\deg(f) = d$ minimal such that $\tau_2(f) = \mathbf{0}$. Since τ_1 is injective, $f \notin \mathbb{K}(k)$. Define

$$g := \sigma(f_d) f - f_d \sigma(f) \in \mathbb{K}(k)[h].$$

Note that $\deg(g) < d$ by construction. Moreover,

$$\tau_2(g) = \tau_1(\sigma(f_d))\tau_2(f) - \tau_1(f_d)\tau_2(\sigma(f)).$$

Since $\tau_2(f) = \mathbf{0}$ by assumption and $\tau_2(\sigma(f)) = \mathcal{S}(\tau_2(f)) = \mathcal{S}(\mathbf{0}) = \mathbf{0}$, it follows $\tau_2(g) = \mathbf{0}$. By the minimality of $\deg(f)$, $g = 0$, i.e., $\sigma(f_d) f - f_d \sigma(f) = 0$, or equivalently, $\frac{\sigma(f)}{f} = \frac{\sigma(f_d)}{f_d} \in \mathbb{K}(k)$. As $f \notin \mathbb{K}(k)$, this contradicts (Kar81, Theorem 4).

Example 4.4. Take $(\mathbb{K}(k), \sigma)$ and τ_1 from Example 4.2 and consider the Π-extension $(\mathbb{K}(k)(b), \sigma)$ of $(\mathbb{K}(k), \sigma)$ with $\sigma(b) = \frac{k+1}{2(2k+1)} b$. Since

$$\binom{2(k+1)}{k}^{-1} = \frac{k+1}{2(2k+1)} \binom{2k}{k}^{-1},$$

it easy to see that $\tau_2 : \mathbb{K}(k)[b] \to S(\mathbb{K})$ with $\tau_2(b) = \langle \binom{2n}{n}^{-1} \rangle_{n \geq 0}$ and

$$\tau_2(\sum_{i=0}^{d} f_i b^i) = \sum_{i=0}^{d} \tau_1(f_i)\tau_2(b)^i$$

forms a difference ring homomorphism; note that $\tau_2(b)$ has no zero entries by construction. We define the o-function by $L_2(\sum_{i=0}^{d} f_i b^i) = \max(L_1(f_i) | 0 \leq i \leq d))$. Suppose that τ_2 is not injective. Then take $f = \sum_{i=0}^{d} f_i b^i \in \mathbb{K}(k)[b] \setminus \{0\}$ with $\deg(f) = d$ minimal such that $\tau_2(f) = \mathbf{0}$. Similarly to Example 4.3, we can conclude that $\frac{\sigma(f)}{f} = \frac{\sigma(f_d)}{f_d} \left(\frac{k+1}{2(2k+1)}\right)^d \in \mathbb{K}(k)$. By (Kar81, Theorem 4), $f = u\, b^m$ where $m > 0$ and $u \in \mathbb{K}(k)^*$. Thus, $\mathbf{0} = \tau_2(f) = \tau_1(u)\tau_2(b)^m$. Since $\tau_1(u) \neq \mathbf{0}$ (τ_1 is injective and $u \neq 0$), $\tau_2(b)$ has infinitely many zeros; a contradiction.

More generally, we arrive at the following result; see (Sch07a) for a detailed proof.

Lemma 4.5. *Let $(\mathbb{F}(t_1)\dots(t_e)(t),\sigma)$ be a polynomial $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$ with $\mathbb{K}:=\text{const}_\sigma\mathbb{F}$ and $\sigma(t)=\alpha\,t+\beta$. Let $\tau:\mathbb{F}[t_1]\dots[t_e]\to S(\mathbb{K})$ be a $\mathbb{K}$-embedding.*

(1) Then there is a $\mathbb{K}$-embedding $\tau':\mathbb{F}[t_1]\dots[t_e][t]\to S(\mathbb{K})$ with $\tau'(f)=\tau(f)$ for all $f\in\mathbb{F}[t_1,\dots,t_e]$ and

$$(\tau'(t))(k)=\begin{cases} c\displaystyle\prod_{i=r}^{k}(\tau(\alpha))(i-1) & \text{if } \sigma(t)=\alpha\,t\\ \displaystyle\sum_{i=r}^{k}(\tau(\beta))(i-1)+c & \text{if } \sigma(t)=t+\beta,\end{cases}\tag{4.5}$$

for some $r\in\mathbb{N}$ and $c\in\mathbb{K}$; we require $c\neq 0$, if $\beta=0$.

(2) If there is an o-function L for τ, there is an o-function L' for τ'; if L is given explicitly, the lower bound r in (4.5) and L' can be given explicitly.

Applying Lemma 4.5 iteratively, gives the following theorem.

Theorem 4.6. *Let $(\mathbb{F}(x_1)\dots(x_r)(t_1)\dots(t_e),\sigma)$ be a polynomial $\Pi\Sigma^*$-extension of $(\mathbb{F},\sigma)$ with $\mathbb{K}:=\text{const}_\sigma\mathbb{F}$. If $\tau:\mathbb{A}\to S(\mathbb{K})$ with $\mathbb{A}=\mathbb{F}[x_1,\dots,x_r]$ is a $\mathbb{K}$-embedding with o-function L, there is a $\mathbb{K}$-embedding $\tau':\mathbb{A}[t_1,\dots,t_e]\to S(\mathbb{K})$ with an o-function L' such that $\tau'(f)=\tau(f)$ for all $f\in\mathbb{A}$. If $(\mathbb{F},\sigma)$ is a mixed $\Pi\Sigma^\delta$-field with depth 1, τ' and L' can be given explicitly.*

Example 4.7. E.g., take the $\Pi\Sigma^\delta$-field $(\mathbb{F},\sigma)$ with $\mathbb{F}=\mathbb{K}(k)$ and $\sigma(k)=k+1$ together with the $\mathbb{K}$-embedding $\tau:\mathbb{F}\to S(\mathbb{K})$ as carried out in Example 4.2. Moreover, take a Π-extension $(\mathbb{F}(x_1)\dots(x_r),\sigma)$ of $(\mathbb{F},\sigma)$ such that $\frac{\sigma(x_i)}{x_i}\in\mathbb{F}$ for $1\leq i\leq r$. Then we can extend the $\mathbb{K}$-embedding τ to $\tau':\mathbb{F}[x_1,\dots,x_e]\to S(\mathbb{K})$ such that for all $1\leq i\leq e$,

$$(\tau(x_i))(n)=c\prod_{k=r}^{n}\alpha_i(k)$$

for some $r\in\mathbb{N}$, $c\in\mathbb{K}^*$ and $\alpha(k)\in\mathbb{K}(k)$ such that $\alpha(k)=\tau(\frac{\sigma(x_i)}{x_i})(k)$ for all $k\geq r$. In other words, we can model a finite set of hypergeometric sequences which do not contain algebraic relations. Similarly, we are in the position to handle q-hypergeometric sequences or mixed hypergeometric sequences.

Example 4.8. E.g., take the $\Pi\Sigma^\delta$-field $(\mathbb{Q}(k),\sigma)$ with $\sigma(k)=k+1$ together with the $\mathbb{Q}$-embedding $\tau:\mathbb{Q}(k)\to S(\mathbb{Q})$ as carried out in Ex. 4.2 ($\mathbb{K}=\mathbb{Q}$). Moreover, consider the $\Pi\Sigma^\delta$-field $(\mathbb{Q}(k)(h)(h_1)(h_2)(h_3)(h_4),\sigma)$ from Ex. 3.10. Then we obtain the $\mathbb{Q}$-embedding $\tau':\mathbb{Q}(k)[h,h_2,h_3,h_4]\to S(\mathbb{Q})$ such that $\tau'(f)=\tau(f)$ for all $f\in\mathbb{Q}(k)$ and such that $\tau'(h)=\langle H_n\rangle_{n\geq 0}$ and $\tau'(h_i)=\langle H_n^{(i)}\rangle_{n\geq 0}$ for $2\leq i\leq 4$.

Remark 4.9. In (Sch07a) we constructed this $\mathbb{K}$-embedding $\tau:\mathbb{F}[t_1,\dots,t_e]\to S(\mathbb{K})$ in order to carry over the algebraic independence of $\Pi\Sigma^*$-extensions into $(S(\mathbb{K}),\mathcal{S})$. In particular, if we set $\mathbb{S}:=\tau(\mathbb{F}[t_1,\dots,t_e])$, then the sub-difference ring $(\mathbb{S},\mathcal{S})$ of $(S(\mathbb{K}),\mathcal{S})$ and $(\mathbb{F}[t_1,\dots,t_e],\sigma)$ are isomorphic. This result has direct implications in symbolic summation: E.g., the summation principles telescoping, creative telescoping, parameterized telescoping, or recurrence solving in $\mathbb{S}$ and in $\mathbb{F}[t_1,\dots,t_e]$ are exactly the same thing; for more details of these principles we refer to (PWZ96) for the (q–)hypergeometric case and to (Sch07c) for the general $\Pi\Sigma^*$-field case.

In the next section we shall use this $\mathbb{K}$-embedding construction τ in order to carry over the depth-behavior (see Theorem 3.9) to $(S(\mathbb{K}),\mathcal{S})$. As a consequence, we can then solve problem DOS.

5. Finding optimal nested sum representations

Suppose that we have represented our basic summation objects in terms of indefinite nested sums and products in a well chosen $\Pi\Sigma^\delta$-field. More precisely, suppose that we are given a mixed $\Pi\Sigma^\delta$-field $(\mathbb{F}, \sigma)$ over $\mathbb{K}$ with depth 1 and a polynomial $\Pi\Sigma^\delta$-extension $(\mathbb{F}(x_1)\dots(x_r), \sigma)$ of $(\mathbb{F}, \sigma)$. Furthermore, we assume that we are given a $\mathbb{K}$-embedding

$$\tau : \mathbb{F}[x_1, \dots, x_r] \to S(\mathbb{K}) \tag{5.1}$$

which is constructed explicitly by iterative application of Lemma 4.5. This leads directly to the sub-difference ring $(\mathbb{S}, \mathcal{S})$ of $(S(\mathbb{K}), \mathcal{S})$ with

$$\mathbb{S} := \tau(\mathbb{F}[x_1, \dots, x_r]).$$

The following remarks are crucial. From the algorithmic point of view, the elements of $\mathbb{S}$ are given by $a \in \mathbb{F}[x_1, \dots, x_r]$ and we can link them to $S(\mathbb{K})$ with $\tau(a)$. From the symbolic summation point of view, we represent the sequences from $\mathbb{S}$ by indefinite nested sum and product expressions that we find by applying the $\mathbb{K}$-embedding τ to a. In a nutshell, we have full control on $\mathbb{S}$ or $\mathbb{F}[t_1, \dots, t_e]$, and can switch between the nested sum-product representation and the $\Pi\Sigma^\delta$-field representation in an algorithmic fashion. In this way we can model rational-sequences (see Example 4.2), q-rational sequences, or mixed rational sequences. Moreover, hypergeometric sequences, q-hypergeometric sequences, or mixed hypergeometric sequences can be handled which are transcendental to each other; see, e.g., Example 4.4 or, more generally, Example 4.7 for the hypergeometric case. Furthermore, sum expressions over such sequences can be represented accordingly in $\mathbb{S}$; see, e.g., Example 4.3 and Example 4.8.

Now define

$$\delta : \begin{cases} \mathbb{S} & \to & \mathbb{N} \\ f & \mapsto & \delta_{\mathbb{K}}(\tau^{-1}(f)); \end{cases} \tag{5.2}$$

note that δ gives the nested depth of the sequences as they are described in the underlying $\Pi\Sigma^\delta$-field. Then we solve problem DOS for such a given $\mathbb{S}$ as follows.

A solution to DOS. *Given*[a] $\boldsymbol{A} \in \Sigma(\mathbb{S})$ in terms of a sum expression $A(n)$ over $\mathbb{S}$. *Find* a Σ^δ-extension $D := (\mathbb{F}(x_1)\dots(x_r)(s_1)\dots(s_u), \sigma)$ of $(\mathbb{F}(x_1)\dots(x_r), \sigma)$ such that D is a polynomial extension of $(\mathbb{F}, \sigma)$ and *find* a $\mathbb{K}$-embedding

$$\tau' : \underbrace{\mathbb{F}[x_1, \dots, x_r][s_1, \dots, s_u]}_{=:\mathbb{A}} \to S(\mathbb{K}) \tag{5.3}$$

by iterative application of Lemma 4.5 (starting with (5.1)) with the following properties: We obtain an $a \in \mathbb{A}$ such that $\tau'(a) = \boldsymbol{A}$ and such that $d := \delta_{\mathbb{K}}(a)$ is the δ-depth of $\boldsymbol{A}$.

[a] As mentioned above, we assume that we have full control on the elements of $\mathbb{S}$ in form of nested sum-product expressions or equivalently on the corresponding elements of $\mathbb{F}[t_1, \dots, t_e]$; the sums over $\mathbb{S}$ from $\Sigma(\mathbb{S})$ are given in form of nested sums that are specified by the user.

More precisely, in Theorem 5.1 we will show that a sequence $\boldsymbol{A} \in \Sigma(\mathbb{S})$ can be represented in a $\Pi\Sigma^\delta$-field algorithmically just as required in our solution to DOS presented above. Finally, in Theorem 5.5 we prove that any such representation in a $\Pi\Sigma^\delta$-field, in particular, our constructed one, has optimal δ-depth d. Reinterpreting those elements as nested sums and products using (5.3) gives then a sum representation of $\boldsymbol{A}$ with optimal δ-depth d.

Theorem 5.1. *Let $(\mathbb{F}(x_1)\dots(x_r),\sigma)$ be a $\Pi\Sigma^\delta$-field over $\mathbb{K}$ and (5.1) be a $\mathbb{K}$-embedding as stated above; set $\mathbb{S}:=\tau(\mathbb{F}[x_1,\dots,x_r])$, and define (5.2). Let $\boldsymbol{A}\in\Sigma(\mathbb{S})$ be given by a sum expression $A(n)$ over $\mathbb{S}$ with δ-depth d. Then there is a Σ^δ-extension $D:=(\mathbb{F}(x_1)\dots(x_r)(s_1)\dots(s_u),\sigma)$ of $(\mathbb{F}(x_1)\dots(x_r),\sigma)$ such that D is a polynomial extension of $(\mathbb{F},\sigma)$ and there is a $\mathbb{K}$-embedding (5.3) with the following property: There is an $a\in\mathbb{A}$ such that $\tau(a)=\boldsymbol{A}$ and such that $\delta_{\mathbb{K}}(a)\leq d$. This extension, τ' and a can be given explicitly.*

Proof. If $d=0$, then $\boldsymbol{A}\in\mathbb{S}$ is given as an indefinite nested sum and product expression and we get $a:=\tau^{-1}(\boldsymbol{A})\in\mathbb{F}[x_1,\dots,x_r]$ as required; by assumption, this construction can be given explicitly.
Now suppose that the statement holds for sum expressions with δ-depth $\leq d$, and let $\boldsymbol{A}\in\Sigma(\mathbb{S})$ be given by a sum expression $A(n)$ with δ-depth $d{+}1$. By assumption take a Σ^δ-extension $D:=(\mathbb{F}(x_1)\dots(x_r)(s_1)\dots(s_u),\sigma)$ of $(\mathbb{F}(x_1)\dots(x_r),\sigma)$ such that D is a polynomial extension of $(\mathbb{F},\sigma)$ and let $\tau:\mathbb{F}[x_1,\dots,x_r][s_1,\dots,s_u]\to S(\mathbb{K})$ be a $\mathbb{K}$-embedding with the following property: For any expression $F(n)$ in $A(n)$ with δ-depth $\leq d$ we are given explicitly $f\in\mathbb{F}[x_1,\dots,x_r][s_1,\dots,s_u]$ such that $\tau(f)=\boldsymbol{F}$ with $\boldsymbol{F}(n)=F(n)$ all $n\geq r$ for some $r\in\mathbb{N}$ and such that $\delta_{\mathbb{K}}(f)\leq d$. Since $A(n)$ has δ-depth $d+1$, there pops up a sum of the form $B(n)=\sum_{k=r+1}^{n}F(k)$ where $F(k)$ has δ-depth d. Take $f\in\mathbb{F}[x_1,\dots,x_r][s_1,\dots,s_u]$ with $(\tau(f))(k)=F(k+1)$ for all $k\geq L(f)$. Then by Theorem 3.11 we can take a Σ^δ-extension $D:=(\mathbb{F}(x_1)\dots(x_r)(s_1)\dots(s_u)(t_1)\dots(t_e),\sigma)$ of $(\mathbb{F}(x_1)\dots(x_r)(s_1)\dots(s_u),\sigma)$ such that D a polynomial extension of $(\mathbb{F},\sigma)$ and in which we have $g\in\mathbb{A}$ with $\mathbb{A}:=\mathbb{F}[x_1,\dots,x_r][s_1,\dots,s_u][t_1,\dots,t_e]$ such that (3.3). Moreover, we can extend the embedding to $\tau:\mathbb{A}\to S(\mathbb{K})$ explicitly and obtain a corresponding o-function L' by Theorem 4.6. Take $l:=\max(r,L(f),L'(g))+1$ and define $c:=\sum_{k=r+1}^{l}F(k)-(\tau(g))(l-1)$. Then for all $n\geq l$,

$$\begin{aligned}(\tau(g+c))(n)&=(\tau(\sigma^{-1}(g)+\sigma^{-1}(f)))(n)+c=(\tau(g))(n-1)+(\tau(f))(n-1)+c\\&=(\tau(g))(n-1)+F(n)+c=\dots=(\tau(g))(l-1)+\sum_{k=l}^{n}F(k)+c=B(n).\end{aligned}$$

Since $\delta_{\mathbb{K}}(g)\leq\delta_{\mathbb{K}}(f)+1$ by Thm. 3.9, $\delta_{\mathbb{K}}(g+c)=\delta_{\mathbb{K}}(g)\leq\delta_{\mathbb{K}}(f)+1\leq d+1$. Note that the Σ^δ-extension with g and the $\mathbb{K}$-embedding τ can be given explicitly; moreover, we can give L' and therefore l and c explicitly. We proceed for all sums with δ-depth $d+1$ in $A(n)$ and complete the induction step. □

We remark that this translation mechanism presented in Theorem 5.1 is implemented in the summation package Sigma (Sch07c).

Example 5.2. Consider the $\Pi\Sigma^\delta$-field $(\mathbb{Q}(k)(h)(h_2)(h_3)(h_4),\sigma)$ from Example 3.10 and the $\mathbb{Q}$-embedding $\tau':\mathbb{Q}(k)[h,h_2,h_3,h_4]\to S(\mathbb{Q})$ from Example 4.8. By construction, we can link the sums in (3.4) with $h,H',h_2,a',A'\in\mathbb{Q}(k)[h,h_2,h_3,h_4]$ from Example 3.10 as follows:

$$\begin{aligned}&\langle H_n\rangle_{n\geq0}=\tau'(h),\quad\langle H(n)\rangle_{n\geq0}=\tau'(H'),\quad\langle H_n^{(2)}\rangle_{n\geq0}=\tau'(h_2),\\&\langle a(n)\rangle_{n\geq0}=\tau'(a')\quad,\langle A(n)\rangle_{n\geq0}=\tau'(A').\end{aligned}\tag{5.4}$$

The δ-depths of $H_k,H(k),H_k^{(2)},a(k),A(k)$ are $2,3,2,3,4$, respectively; δ is given by (2.3). The corresponding depths in the $\Pi\Sigma^\delta$-field, $\delta_{\mathbb{Q}}(h)=\delta_{\mathbb{Q}}(H')=\delta_{\mathbb{Q}}(h_2)=\delta_{\mathbb{Q}}(a')=\delta_{\mathbb{Q}}(A')=2$, are the same or have been improved.

Note that we rely on the fact that all our sums are represented in $\Pi\Sigma^\delta$-fields; for general $\Pi\Sigma^*$-fields, the depth might be bigger than the δ-depth, see Example 3.3.

Lemma 5.3. *Let $(\mathbb{F}(x_1)\dots(x_r),\sigma)$ be a polynomial $\Pi\Sigma^\delta$-extension of $(\mathbb{F},\sigma)$ and let $(\mathbb{F},\sigma)$ be a $\Pi\Sigma^\delta$-field over $\mathbb{K}$ with a $\mathbb{K}$-embedding (5.1); let $(\mathbb{F}(t_1)\dots(t_e),\sigma)$ be a polynomial Σ^*-extension of $(\mathbb{F},\sigma)$ with a $\mathbb{K}$-embedding $\rho:\mathbb{F}[t_1,\dots,t_e]\to S(\mathbb{K})$. There is a Σ^*-extension $D:=(\mathbb{F}(x_1)\dots(x_r)(y_1)\dots(y_l),\sigma)$ of $(\mathbb{F}(x_1)\dots(x_r),\sigma)$ such that D is a polynomial Σ^*-extension of $(\mathbb{F},\sigma)$ with the following properties:*

(1) There is an $\mathbb{F}$-monomorphism $\lambda:\mathbb{F}(t_1,\dots,t_e)\to\mathbb{F}(x_1,\dots,x_r)(y_1,\dots,y_l)$ such that for all $a\in\mathbb{F}(t_1,\dots,t_e)$,

$$\delta_{\mathbb{K}}(\lambda(a))\leq\delta_{\mathbb{K}}(a);\tag{5.5}$$

and such that for all $a\in\mathbb{F}[t_1,\dots,t_e]$,

$$\lambda(a)\in\mathbb{F}[x_1,\dots,x_r][y_1,\dots,y_l].\tag{5.6}$$

(2) There is a $\mathbb{K}$-embedding $\tau':\mathbb{F}[x_1,\dots,x_r][y_1,\dots,y_l]\to S(\mathbb{K})$ such that for all $a\in\mathbb{F}[t_1,\dots,t_e]$,

$$\tau'(\lambda(a))=\rho(a).\tag{5.7}$$

Proof. The base case $e=0$ holds with $\lambda(a)=a$ for all $a\in\mathbb{F}$ and $\tau':=\rho$. Suppose the lemma holds for e extensions $(\mathbb{H},\sigma)$ with $\mathbb{H}=\mathbb{F}(t_1)\dots(t_e)$ and let $(\mathbb{D},\sigma)$ with $\mathbb{D}:=\mathbb{F}(x_1)\dots(x_r)(y_1)\dots(y_r)$, τ, ρ and λ as stated above; set $\mathbb{S}=\mathbb{F}[x_1,\dots,x_r,y_1,\dots,y_l]$. Now let $(\mathbb{H}(t),\sigma)$ be a Σ^*-extension of $(\mathbb{H},\sigma)$ with $f:=\sigma(t)-t\in\mathbb{F}[t_1\dots,t_e]$, and take a $\mathbb{K}$-embedding $\rho':\mathbb{F}[t_1,\dots,t_e][t]\to S(\mathbb{K})$ such that $\rho'(a)=\rho(a)$ for all $a\in\mathbb{F}[t_1\dots,t_e]$.
Case 1: If there is no $g\in\mathbb{D}$ such that

$$\sigma(g)-g=\lambda(f),\tag{5.8}$$

we can take the Σ^*-extension $(\mathbb{D}(y),\sigma)$ of $(\mathbb{D},\sigma)$ with $\sigma(y)=y+\lambda(f)$ by Theorem 3.2.1 and we can define an $\mathbb{F}$-monomorphism $\lambda':\mathbb{H}(t)\to\mathbb{S}(y)$ s.t. $\lambda'(a)=\lambda(a)$ for all $a\in\mathbb{H}$ and such that $\tau'(t)=y$. By construction, $\delta_{\mathbb{K}}(y)=\delta_{\mathbb{K}}(\lambda(f))+1$. Since

$$\delta_{\mathbb{K}}(\lambda(f))+1\leq\delta_{\mathbb{K}}(f)+1=\delta_{\mathbb{K}}(t),\tag{5.9}$$

$\delta_{\mathbb{K}}(\lambda(a))\leq\delta_{\mathbb{K}}(a)$ for all $a\in\mathbb{H}(t)$. Moreover, since $\lambda(f)\in\mathbb{S}$, it follows that $(\mathbb{S}(y),\sigma)$ is a polynomial extension of $(\mathbb{F},\sigma)$. Moreover, for all $a\in\mathbb{F}[t_1,\dots,t_e,t]$, $\lambda(a)\in\mathbb{S}[y]$. This proves part (1).
Now we define the $\mathbb{K}$-embedding $\tau':\mathbb{S}[y]\to S(\mathbb{K})$ by $\tau'(a)=\tau(a)$ for all $a\in\mathbb{S}$ and $\tau'(y)$ as in the right hand side of (4.5) where $\beta=\lambda(f)$ and $c=(\rho'(t))(r-1)$; for some $r\in\mathbb{N}$ properly chosen. Then for all $k\geq r$ (r is chosen big enough),

$$\begin{aligned}(\tau'(\lambda'(t)))(k)=(\tau'(y))(k)&=\sum_{i=r}^{k}(\tau(\lambda(f)))(i-1)+(\rho'(t))(r-1)\\&=\sum_{i=r}^{k}(\rho(f))(i-1)+(\rho'(t))(r-1)=\sum_{i=r+1}^{k}(\rho(f))(i-1)+h(r)\end{aligned}$$

with $h(r) = (\rho(f))(r-1) + (\rho'(t))(r-1) = (\rho'(f+t))(r-1) = (\rho'(\sigma(t)))(r-1) = (\rho'(t))(r)$. Applying this reduction $k-r+1$ times shows that

$$\begin{aligned}(\tau'(\lambda'(t)))(k) &= \sum_{i=r}^{k}(\rho(f))(i-1) + \rho'(t)(r-1)\\ &= \sum_{i=r+1}^{k}(\rho(f))(i-1) + \rho'(t)(r) = \dots = (\rho'(t))(k).\end{aligned}$$

Hence $\tau'(\lambda'(t)) = \rho'(t)$, and thus $\tau'(\lambda'(a)) = \rho'(a)$ for all $a \in \mathbb{F}[t_1 \dots, t_e][t]$.

Case 2: Otherwise, if there is a $g \in \mathbb{D}$ s.t. (5.8), then $g \in \mathbb{S}$ by Theorem 3.5. In particular, $\delta_{\mathbb{K}}(g) \leq \delta_{\mathbb{K}}(\lambda(f)) + 1$ by Theorem 3.9. With (5.9), it follows that

$$\delta_{\mathbb{K}}(g) \leq \delta_{\mathbb{K}}(t). \tag{5.10}$$

Since $(\lambda(\mathbb{H})(g), \sigma)$ is a difference field (it is a sub-difference field of $(\mathbb{D}, \sigma)$), g is transcendental over $\lambda(\mathbb{H})$ by Theorem 3.2.1. In particular, we can define the $\mathbb{F}$-monomorphism $\lambda' : \mathbb{H}(t) \to \mathbb{D}$ with $\lambda'(a) = \lambda(a)$ for all $a \in \mathbb{H}$ and $\lambda'(t) = g + \rho(g) - \tau(g)$. Since $g \in \mathbb{S}$, $\lambda'(t) \in \mathbb{S}$, and therefore $\lambda(a) \in \mathbb{S}$ for all $a \in \mathbb{F}[t_1, \dots, t_e][t]$. With (5.10) and our induction assumption it follows that $\delta_{\mathbb{K}}(\tau'(a)) \leq \delta_{\mathbb{K}}(a)$ for all $a \in \mathbb{H}(t)$. This proofs part (1). Note that $\tau(\lambda'(t)) = \tau(g) + \rho(g) - \tau(g) = \rho(g)$ by construction. Hence, $\tau(\lambda'(a)) = \rho(a)$ for all $a \in \mathbb{F}[t_1 \dots, t_e][t]$. This proves part (2) and completes the induction step. □

Theorem 5.4. *Let $(\mathbb{F}(x_1)\dots(x_r), \sigma)$ be a polynomial $\Pi\Sigma^\delta$-extension of $(\mathbb{F}, \sigma)$ and let $(\mathbb{F}, \sigma)$ be a $\Pi\Sigma^\delta$-field over $\mathbb{K}$ with a $\mathbb{K}$-embedding (5.1). Then for any polynomial Σ^*-extension $(\mathbb{F}(t_1)\dots(t_e), \sigma)$ of $(\mathbb{F}, \sigma)$ with a $\mathbb{K}$-embedding $\rho : \mathbb{F}[t_1, \dots, t_e] \to S(\mathbb{K})$ and any $\boldsymbol{s} \in \tau(\mathbb{F}[x_1, \dots, x_r]) \cap \rho(\mathbb{F}[t_1, \dots, t_e])$ we have*

$$\delta_{\mathbb{K}}(\tau^{-1}(\boldsymbol{s})) \leq \delta_{\mathbb{K}}(\rho^{-1}(\boldsymbol{s})).$$

Proof. Take a Σ^*-extension $(\mathbb{F}(x_1)\dots(x_r)(y_1)\dots(y_l), \sigma)$ of $(\mathbb{F}(x_1)\dots(x_r), \sigma)$, an $\mathbb{F}$-monomorphism $\lambda : \mathbb{F}(t_1)\dots(t_e) \to \mathbb{F}(x_1)\dots(x_r)(y_1)\dots(y_l)$ as in (5.5) and (5.6) for all $a \in \mathbb{F}[t_1, \dots, t_e]$ and a $\mathbb{K}$-embedding $\tau' : \mathbb{F}[x_1, \dots, x_r, y_1, \dots, y_l] \to S(\mathbb{K})$ as in (5.7) for all $a \in \mathbb{F}[t_1, \dots, t_e]$; this is possible by Lemma 5.3. Now take $\boldsymbol{s} \in \tau(\mathbb{F}[x_1, \dots, x_r]) \cap \rho(\mathbb{F}[t_1, \dots, t_e])$ and define $f := \tau^{-1}(\boldsymbol{s}) \in \mathbb{F}[x_1, \dots, x_r]$ and $g := \rho^{-1}(\boldsymbol{s}) \in \mathbb{F}[t_1 \dots, t_e]$. Take $g' := \lambda(g) \in \mathbb{F}[x_1, \dots, x_r, y_1, \dots, y_l]$ with $\delta_{\mathbb{K}}(g') \leq \delta_{\mathbb{K}}(g)$. Since $\tau'(g') = \rho(g) = \boldsymbol{s}$ and $\tau'(f) = \boldsymbol{s}$, and since τ' is injective, $g' = f$. Thus, $\delta_{\mathbb{K}}(f) = \delta_{\mathbb{K}}(g') \leq \delta_{\mathbb{K}}(g)$. □

Theorem 5.5. *Let $(\mathbb{F}(x_1)\dots(x_r), \sigma)$ be a $\Pi\Sigma^\delta$-field with a $\mathbb{K}$-embedding (5.1) as stated above; set $\mathbb{S} := \tau(\mathbb{F}[x_1, \dots, x_r])$, and define (5.2). Let $\boldsymbol{A} \in \Sigma(\mathbb{S})$ and suppose that we are given a Σ^δ-extension $D := (\mathbb{F}(x_1)\dots(x_r)(y_1)\dots(y_u), \sigma)$ of $(\mathbb{F}(x_1)\dots(x_r), \sigma)$ s.t. D is a polynomial extension of $(\mathbb{F}, \sigma)$ and that we are given a $\mathbb{K}$-embedding (5.3). If $a \in \mathbb{A}$ with $\tau(a) = \boldsymbol{A}$, $\delta_{\mathbb{K}}(a)$ is the δ-depth of $\boldsymbol{A}$.*

Proof. Take a sum representation of $\boldsymbol{A}$ with optimal δ-depth d. By Theorem 5.1 we can take a Σ^*-extension $D := (\mathbb{F}(x_1)\dots(x_r)(s_1)\dots(s_u), \sigma)$ of $(\mathbb{F}(x_1)\dots(x_r), \sigma)$ s.t. D is a polynomial extension of $(\mathbb{F}, \sigma)$ and we can assume that there is a $\mathbb{K}$-embedding $\rho : \mathbb{F}[x_1, \dots, x_r][s_1, \dots, s_u] \to S(\mathbb{K})$ with $a' \in \mathbb{F}[x_1, \dots, x_r][s_1, \dots, s_u]$ s.t. $\rho(a') = \boldsymbol{A}$ and $\delta_{\mathbb{K}}(a') \leq d$. By Theorem 5.4, it follows that $\delta_{\mathbb{K}}(a) = \delta_{\mathbb{K}}(a')$. Since a and a' give sum representations of $\boldsymbol{A}$ with δ-depth $\leq d$, and since d is minimal, $\delta_{\mathbb{K}}(a) = d$. □

Example 5.6. Since the elements $h, H', h_2, a', A' \in \mathbb{Q}(k)[h, h_2, h_3, h_4]$ with depth 2 from Example 5.2 are given in a $\Pi\Sigma^\delta$-field, the δ-depths of the sequences (5.4) are 2; here $\delta : R(\mathbb{Q}) \to \mathbb{N}$ is given by (2.3). Reinterpreting h, H', h_2, a', A' in (5.4) as sum expressions leads to the representations

$$H_k = \sum_{i=1}^{k} \frac{1}{i}, \qquad H(k) = \frac{1}{2}(H_k^2 + H_k^{(2)}), \quad H_k^{(2)} = \sum_{i=1}^{k} \frac{1}{i^2},$$
$$a(k) = \frac{1}{3}\left(H_k^3 + 3H_k^{(2)} H_k + 2H_k^{(3)}\right), \quad A(k) = B(k)$$

on the right sides with optimal δ-depth 2; δ is given by (2.3), and the sum expressions $H(k), a(k), A(k), B(k)$ are given in (3.4) and (2.2).

6. Application: Simplification of d'Alembertian solutions

The d'Alembertian solutions (AP94; Sch01), a subclass of Liouvillian solutions (HS99), of a given recurrence relation are computed by factorizing the recurrence into linear right hand factors as much as possible. Given this factorization, one can read of the d'Alembertian solutions which are of the form

$$h(n) \sum_{k_1=c_1}^{n} b_1(k_1) \sum_{k_2=c_2}^{k_2} b_2(k_2) \cdots \sum_{k_s=c_s}^{k_{s-1}} b_s(k_s) \tag{6.1}$$

for lower bounds $c_1, \dots, c_s \in \mathbb{N}$; here the $b_i(k_i)$ and $h(n)$ are given by the objects form the coefficients of the recurrence or by products over such elements. Note that such solutions can be represented in $\Pi\Sigma^\delta$-fields if the occurring products can be rephrased accordingly in Π-extensions. Then applying our refined algorithms to such solutions (6.1), we can find sum representations with minimal nested depth. Typical examples can be found, e.g., in (DPSW06; OS06; KS06; Sch07c; MS07).

In the following we present two examples with detailed computation steps that have been provided by the summation package Sigma (Sch07c).

6.1. **An example from particle physics.** In massive higher order calculations of Feynman diagrams the evaluation of the sum

$$S(n) = \sum_{i=1}^{\infty} \frac{H_{i+n}^2}{i^2}$$

was needed; see (BBKS07). In order to accomplish this task, Sigma computes in a first step the recurrence relation

$$\begin{aligned}
&-(n+2)(n+1)^3\left(n^2+7n+16\right)S(n)\\
&\quad+(n+2)\left(5n^5+62n^4+318n^3+814n^2+1045n+540\right)S(n+1)\\
&\quad-2\left(5n^6+84n^5+603n^4+2354n^3+5270n^2+6430n+3350\right)S(n+2)\\
&\quad+2\left(5n^6+96n^5+783n^4+3478n^3+8906n^2+12530n+7610\right)S(n+3)\\
&\quad-(n+4)\left(5n^5+88n^4+630n^3+2318n^2+4453n+3642\right)S(n+4)\\
&\quad+(n+4)(n+5)^3\left(n^2+5n+10\right)S(n+5) = -\frac{4(n+7)}{(n+3)(n+4)}H_n\\
&\quad-\frac{2\left(2n^7+35n^6+235n^5+718n^4+824n^3-283n^2-869n+10\right)}{(n+1)(n+2)(n+3)^2(n+4)^2(n+5)}
\end{aligned}$$

by a generalized version of Zeilberger's creative telescoping (Zei91). Given this recurrence, Sigma computes the d'Alembertian solutions

$$A_1(n)=1,\quad A_2(n)=H_n,\quad A_3(n)=H_n^2,$$
$$A_4(n)=\sum_{i=2}^{n}\frac{\displaystyle\sum_{j=2}^{k}\frac{(2j-1)\sum_{k=1}^{j}\frac{1}{(2k-3)(2k-1)}}{(j-1)j}}{i},$$
$$A_5(n)=\sum_{i=3}^{n}\frac{\displaystyle\sum_{j=3}^{i}\frac{(2j-1)\displaystyle\sum_{k=3}^{j}\frac{2(k-2)(k-1)kH_k-(2k-1)\left(3k^2-6k+2\right)}{(k-2)(k-1)k(2k-3)(2k-1)}}{(j-1)j}}{i},$$
$$B(n)=\sum_{i=4}^{n}\frac{\displaystyle\sum_{j=4}^{i}\frac{(2j-1)\displaystyle\sum_{k=4}^{j}\frac{\displaystyle\sum_{l=4}^{k}\frac{(2l-3)\left(l^2-3l+6\right)\tilde{B}(l)}{(l-3)(l-2)(l-1)l}}{(2k-3)(2k-1)}}{(j-1)j}}{i}$$

where

$$\tilde{B}(l)=\sum_{r=3}^{l}-\frac{2\left(2r^6-27r^5+117r^4-254r^3+398r^2+2(r-3)(r-2)(r-1)(r+2)H_r r-446r+204\right)}{(r-2)(r-1)r(r^2-5r+10)(r^2-3r+6)}.$$

To be more precise, $\langle A_i(n)\rangle_{n\geq 0}\in\Sigma(R(\mathbb{Q}))$, $1\leq i\leq 5$, are the five linearly independent solutions of the homogeneous version of the recurrence and $\langle B(n)\rangle_{n\geq 0}\in\Sigma(R(\mathbb{Q}))$ is one particular solution of the recurrence itself. If we define $\delta: R(\mathbb{Q})\to\mathbb{N}$ by (2.3), then $A_1(n),\dots,A_5(n),B(n)$ have δ-depths $0,2,2,4,5,7$, respectively. As a consequence, we obtain the general solution

$$B(n)+c_1A_1(n)+c_2A_2(n)+c_3A_3(n)+c_4A_4(n)+c_5A_5(n)\tag{6.2}$$

for constants c_i. Checking initial values shows that we have to choose[4]

$$c_1=\tfrac{17}{10}\zeta_2^2,\ c_2=\tfrac{1}{12}(48\zeta_3-67),\ c_3=\tfrac{31}{12},\ c_4=\tfrac{1}{4}(23-8\zeta_2),\ c_5=-\tfrac{1}{2}$$

in order to match (6.2) with $S(n)$.

Finally, Sigma simplifies the derived expressions further and finds sum representations with minimal nested depth (see problem DOS). Namely, it computes the $\Pi\Sigma^{\delta}$-field $(\mathbb{Q}(k)(h)(h_2)(h_4)(H),\sigma)$ with $\sigma(k)=k+1$, $\sigma(h)=h+\frac{1}{k+1}$, $\sigma(h_2)=h_2+\frac{1}{(k+1)^2}$, $\sigma(h_4)=h_4+\frac{1}{(k+1)^4}$ and $\sigma(H)=H+\frac{\sigma(h)}{(k+1)^2}$ together with the $\mathbb{Q}$-embedding $\tau:\mathbb{Q}(k)[h,h_2,h_4,H]\to S(\mathbb{K})$ such that $\tau(f)=\tau_1(f)$ for all $f\in\mathbb{Q}(k)$ from Example 4.2 and such that $\tau(h)=\langle H_n\rangle_{n\geq 0}$, $\tau(h_2)=\langle H_n^{(2)}\rangle_{n\geq 0}$, $\tau(h_4)=\langle H_n^{(4)}\rangle_{n\geq 0}$ and

[4] ζ_k denotes the Riemann zeta function at k; e.g., $\zeta_2=\pi^2/6$.

$\tau(H) = \langle\sum_{k=1}^{n} \frac{H_k}{k^2}\rangle_{n\geq 0}$. Moreover, it finds

$$\begin{aligned}
&a_1 = 1, \quad a_2 = h, \quad a_3 = h^2, \quad a_4 = \frac{1}{2}\left(h_2 - h^2\right),\\
&a_5 = \frac{1}{2}\left(-h^2 + 2h_2\, h - h\right),\\
&b = \frac{1}{24}\left(h^2 - 48hH + 128h - 12h_2^2 + (12h - 69)\, h_2 - 12h_4\right)
\end{aligned}$$

such that $\tau(a_i) = \langle A_i(n)\rangle_{n\geq 0}$ for $1 \leq i \leq 5$ and $\tau(b) = \langle B(n)\rangle_{n\geq 0}$. As a consequence of Theorem 5.5, the expressions

$$\begin{aligned}
&A_1(n) = 1, \quad A_2(n) = H_n, \quad A_3(n) = H_n^2, \quad A_4(n) = \frac{1}{2}\left(H_n^{(2)} - H_n^2\right),\\
&A_5(n) = \frac{1}{2}\left(-H_n^2 + 2H_n^{(2)} H_n - H_n\right),\\
&B(n) = \tfrac{1}{24} H_n^2 - 2H_n \sum_{k=1}^{n} \frac{H_k}{k^2} + \tfrac{16}{3} H_n - \tfrac{1}{2}\left(H_n^{(2)}\right)^2 + \left(\tfrac{1}{2} H_n - \tfrac{69}{24}\right) H_n^{(2)} - \tfrac{1}{2} H_n^{(4)}
\end{aligned}$$

are sum representations with optimal δ-depths $0, 2, 2, 2, 2, 3$. To this end, we obtain (BBKS07, equ. 3.14):

$$\sum_{i=1}^{\infty} \frac{H_{i+n}^2}{i^2} = \frac{17}{10}\zeta_2^2 + 4H_n\zeta_3 + H_n^2\zeta_2 - H_n^{(2)}\zeta_2 - \frac{1}{2}\left(\left(H_n^{(2)}\right)^2 + H_n^{(4)}\right) - 2H_n \sum_{k=1}^{n} \frac{H_k}{k^2}.$$

6.2. **A nontrivial harmonic sum identity.** We look for an indefinite nested sum representation of the definite sum

$$S(n) = \sum_{k=0}^{n} \binom{n}{k}^2 H_k^2,$$

which is of similar type as in (DPSW06). First, Sigma finds with creative telescoping the recurrence relation

$$\begin{aligned}
&8(n+1)(2n+1)^3\left(64n^4 + 480n^3 + 1332n^2 + 1621n + 735\right) S(n) - 4(768n^8 + 8832n^7\\
&+ 43056n^6 + 115708n^5 + 186452n^4 + 183201n^3 + 106442n^2 + 33460n + 4533) S(n+1)\\
&+2(n+2)(384n^7 + 4224n^6 + 18968n^5 + 44610n^4 + 58679n^3 + 42775n^2 + 16084n + 2616) S(n+2)\\
&- (n+2)(n+3)^3\left(64n^4 + 224n^3 + 276n^2 + 141n + 30\right) S(n+3) =\\
&- 3\left(576n^6 + 4896n^5 + 16660n^4 + 28761n^3 + 26171n^2 + 11574n + 1854\right).
\end{aligned}$$

Solving the recurrence in terms of d'Alembertian solutions and checking initial values yield the sum representation

$$S(n) = \binom{2n}{n}\left(\frac{1}{2}A_1(n) - \frac{19}{28}A_2(n) + B(n)\right)$$

with

$$A_1(n) = \sum_{i=1}^{n} \frac{4i-3}{i(2i-1)},$$

$$A_2(n) = \sum_{i=2}^{n} \frac{(4i-3)\displaystyle\sum_{j=2}^{i} \frac{64j^4-288j^3+468j^2-323j+84}{(j-1)j(2j-3)(4j-7)(4j-3)}}{i(2i-1)},$$

$$B(n) = -\sum_{i=2}^{n} \frac{(4i-3)\displaystyle\sum_{j=2}^{n} \frac{\left(64j^4-288j^3+468j^2-323j+84\right)\tilde{B}(j)}{(j-1)j(2j-3)(4j-7)(4j-3)}}{i(2i-1)}$$

where

$$\tilde{B}(j) = \sum_{k=1}^{j} -\frac{3(2k-3)(2k-1)(4k-7)\left(576k^6-5472k^5+20980k^4-41559k^3+44882k^2-25113k+5760\right)}{k(64k^4-544k^3+1716k^2-2379k+1227)(64k^4-288k^3+468k^2-323k+84)\binom{2k}{k}}.$$

Now consider the $\Pi\Sigma^\delta$-field $(\mathbb{Q}(k)(b), \sigma)$ with $\sigma(k) = k+1$ and $\sigma(b) = \frac{k+1}{2(2k+1)}b$, and take the $\mathbb{Q}$-monomorphism $\tau_2 : \mathbb{Q}(k)[b] \to \sigma$ from Example 4.4 ($\mathbb{K} = \mathbb{Q}$). Then $(\mathbb{S}, \mathcal{S})$ with $\mathbb{S} := \tau_2(\mathbb{Q}(k)[b])$ is a sub-difference ring of $(S(\mathbb{Q}), \mathcal{S})$ such that

$$\langle A_1(n)\rangle_{n\geq 0},\ \langle A_2(n)\rangle_{n\geq 0},\ \langle B(n)\rangle_{n\geq 0} \in \Sigma(\mathbb{S}).$$

If we define $\delta : \mathbb{S} \to \mathbb{N}$ by (5.2), then $A_1(n)$, $A_2(n)$ and B have δ-depths $2, 3, 5$, respectively. Finally, we represent $A_1(n)$, $A_2(n)$ and B in a $\Pi\Sigma^\delta$-field and obtain the sum representations

$$\begin{aligned} A_1(n) =& 2\left(2H_n - H_{2n}\right), \\ A_2(n) =& 2\left(4H_n^2 + 4H_n + H_{2n}^2 + \left(-4H_n - 2\right)H_{2n} - H_{2n}^{(2)}\right), \\ B(n) =& \frac{3}{14}\left(44H_n^2 + 16H_n + 11H_{2n}^2 - \left(44H_n + 8\right)H_{2n} - 11H_{2n}^{(2)} + 14\sum_{i=1}^{n}\frac{1}{i^2\binom{2i}{i}}\right). \end{aligned} \tag{6.3}$$

Summarizing, we have solved problem DOS for the expressions $A_1(n)$, $A_2(n)$ and $B(n)$, i.e., in the right hand sides of (6.3) we obtained sum representations with optimal δ-depths $2, 2, 3$, respectively. Finally, this leads to the identity

$$\sum_{k=0}^{n}\binom{n}{k}^2 H_k^2 = \binom{2n}{n}\left(4H_n^2 - 4H_{2n}H_n + H_{2n}^2 - H_{2n}^{(2)} + 3\sum_{i=1}^{n}\frac{1}{i^2\binom{2n}{n}}\right).$$

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