Convergence results for the Bayesian inversion theory*

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Abstract. In this paper we derive convergence results for regularized solutions of linear inverse problems obtained by the Bayesian approach in the Ky Fan metric. We show that the convergence rate is order optimal in finite dimensional spaces. Moreover, we prove that order optimal rates can be obtained for weighted Bayesian solutions when the dimension goes to infinity.

1. Introduction

We study the solution of the linear ill-posed problem

$$Tx = y \tag{1.1}$$

from noisy measurements of y, where $T \in L(\mathcal{X}, \mathcal{Y})$ and $\mathcal{X}$ and $\mathcal{Y}$ are Hilbert spaces. Opposed to the deterministic regularization theory (cf., e.g., [2]), where it is assumed that a bound for the noise is known, i.e., $\|y^\delta - y\| \le \delta$, we are interested in the case where the noise can be modelled by a normal random variable (cf., e.g., [3, 5]).

In a first step, we treat a finite dimensional version of equation (1.1) as it occurs when this problem is discretized (see Section 4), i.e., we deal with the solution of the problem

$$A\bar{x} = \bar{y}\,, \tag{1.2}$$

where $A \in \mathbb{R}^{m\times n}$ is a (usually ill-conditioned) matrix, $\bar{x} \in \mathbb{R}^n$, and $\bar{y} \in \mathbb{R}^m$.

In this paper, we use the Bayesian approach for obtaining an approximate solution of (1.2) (see [8] for a comprehensive introduction into the Bayesian inversion theory).

In the Bayesian framework all quantities included in the model of an inverse problem are treated as random variables. Even though the quantity of primary interest is assumed to be deterministic, all information available about it before performing the measurements is coded into the so-called *prior distribution*. Since we assume that the measurements are disturbed by an additive noise, we obtain the following linear model for the measurements

$$Y = A\,X + E\,,$$

where X, Y, and E are random variables from a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively.

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The Bayesian inversion theory is based on the Bayes formula. The solution of the inverse problem after performing the measurements is the *posterior distribution* of the random variables of interest. The Bayes formula describes how the prior information and the measurements have to be combined to give the posterior distribution; by this formula the posterior density is proportional to the product of the prior density and the *likelihood function* which is given by the model for the indirect measurements. Consequently, in the Bayesian approach not just a single regularized solution of (1.2) is obtained but instead a whole distribution is computed.

We examine the common case where all distributions are assumed to be normal, i.e., the prior distribution of X is given by $N(\bar{x}_0, \Gamma)$ and the noise model distribution of E is given by $N(0, \Sigma)$ with $\bar{x}_0 \in \mathbb{R}^n$ and positive definite symmetric matrices $\Gamma \in \mathbb{R}^{n\times n}$ and $\Sigma \in \mathbb{R}^{m\times m}$. Moreover, we assume that X and E are mutually independent. Then the posterior distribution μ_{post} of X conditioned on the data $\bar{y}^\sigma$ is again normal. It can be shown (cf. [8, Theorem 3.7]) that the posterior distribution is given by $N(\bar{x}_{\mathrm{post}}, \Gamma_{\mathrm{post}})$ with the posterior mean

$$\bar{x}_{\mathrm{post}} := (\Gamma^{-1} + A^T\Sigma^{-1}A)^{-1}(A^T\Sigma^{-1}\bar{y}^\sigma + \Gamma^{-1}\bar{x}_0) \tag{1.3}$$

and the posterior covariance matrix

$$\Gamma_{\mathrm{post}} := (\Gamma^{-1} + A^T\Sigma^{-1}A)^{-1}\,. \tag{1.4}$$

The data $\bar{y}^\sigma$ are a realization of the random variable $\bar{y} + E$, where $\bar{y}$ are the unknown exact data in equation (1.2). Thus, the posterior mean $\bar{x}_{\mathrm{post}}$ in (1.3) is also a realization of the random variable

$$X_{\mathrm{post}}(\omega) := (\Gamma^{-1} + A^T\Sigma^{-1}A)^{-1}(A^T\Sigma^{-1}(\bar{y} + E(\omega)) + \Gamma^{-1}\bar{x}_0) \tag{1.5}$$

and the posterior distribution μ_{post} is a realization of the random variable

$$M_{\mathrm{post}} : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathcal{M}(\mathbb{R}^n), \rho_{\mathrm{P}}), \qquad \omega \mapsto N(X_{\mathrm{post}}(\omega), \Gamma_{\mathrm{post}})\,, \tag{1.6}$$

where $\mathcal{M}(\mathbb{R}^n)$ is the set of all Borel measures in $\mathbb{R}^n$ and ρ_{P} is the Prokhorov metric in $\mathcal{M}(\mathbb{R}^n)$ (see Definition 2.1 below).

We are interested in where the random variables X_{post} and M_{post} converge to when the noise E tends to the zero random variable. These convergence issues were studied in [6, 7]. In this paper these results are generalized and improved. Moreover, we also consider the case when the dimensions m and n tend to infinity, i.e., when the finite dimensional problem (1.2) approaches some infinite dimensional problem (1.1).

The outline of the paper is as follows: in the next section we summarize some results about the Prokhorov and Ky Fan metrics necessary for our convergence analysis. In Section 3 we show that X_{post} converges to a special $\bar{x}_0$-minimum-norm-least-squares solution, $\bar{x}^\dagger$, and that M_{post} converges to a certain normal distribution with mean $\bar{x}^\dagger$. Finally, in Section 4 we present some convergence results for the case where m and n tend to infinity.

2. Preliminaries

As mentioned above, in the setup of this work, we treat the posterior distribution as a probability measure valued random variable. The set $\mathcal{M}(\mathbb{R}^n)$ of Borel measures in $\mathbb{R}^n$

is a metric space, when equipped with the *Prokhorov metric*, which is defined as follows (cf., e.g., [1]):

Definition 2.1. (Prokhorov metric). *Let μ_1 and μ_2 be Borel measures in a metric space $(\mathcal{X}, d_{\mathcal{X}})$. The distance between μ_1 and μ_2 in the* Prokhorov metric *is defined as*

$$\rho_{\mathrm{P}}(\mu_1, \mu_2) := \inf \{ \varepsilon > 0 : \ \mu_1(B) \le \mu_2(B^\varepsilon) + \varepsilon \ \ \forall B \in \mathcal{B}(\mathcal{X}) \} ,$$

where $\mathcal{B}(\mathcal{X})$ is the Borel σ-algebra in $\mathcal{X}$. The set B^ε is the ε-neighbourhood of B, i.e.,

$$B^\varepsilon := \{ x \in \mathcal{X} : \ \inf_{z \in B} d_{\mathcal{X}}(x, z) < \varepsilon \} .$$

For the special case $(\mathcal{M}(\mathbb{R}^n), \rho_{\mathrm{P}})$, the metric space $(\mathcal{X}, d_{\mathcal{X}})$ in Definition 2.1 is $\mathbb{R}^n$ equipped with some norm. Although all norms are equivalent in $\mathbb{R}^n$, it will turn out that it is appropriate to work with different norms. If Q is a positive definite symmetric matrix in $\mathbb{R}^{n \times n}$, we define the Q-norm as follows

$$\|\bar{x}\|_Q := \|Q^{-\frac{1}{2}} \bar{x}\| = (\bar{x}^T Q^{-1} \bar{x})^{\frac{1}{2}} , \tag{2.1}$$

where $\| \cdot \|$ denotes the Euclidean norm in $\mathbb{R}^n$.

The following estimate on the Prokhorov metric will be essential for our convergence analysis:

Proposition 2.2. *Let μ_1 and μ_2 be probability measures in $\mathbb{R}^n$ equipped with the Q-norm and μ_3 a probability measure in $\mathbb{R}^p$ equipped with the Euclidean norm, $1 \le p < n$, defined by*

$$\begin{aligned}
\mu_1(B) &:= \int_{\hat{B}} \pi_2(\xi_2)\, d\xi_2 , \quad B \in \mathcal{B}(\mathbb{R}^n), \quad \hat{B} := \{ \xi_2 \in \mathbb{R}^{n-p} : \ g(0, \xi_2) \in B \} , \\
\mu_2(B) &:= \int_{g^{-1}(B)} \pi_1(\xi_1) \pi_2(\xi_2)\, d(\xi_1, \xi_2) , \quad B \in \mathcal{B}(\mathbb{R}^n) , \\
\mu_3(B) &:= \int_B \pi_1(\xi_1)\, d\xi_1 , \quad B \in \mathcal{B}(\mathbb{R}^p) ,
\end{aligned}$$

where π_1 and π_2 are density functions in $\mathbb{R}^p$ and $\mathbb{R}^{n-p}$, respectively, and $g(\xi) := \tilde{x} + Q^{\frac{1}{2}} V \xi$, $\xi = (\xi_1, \xi_2)$, is an affine transformation with $\tilde{x} \in \mathbb{R}^n$ and a unitary matrix $V \in \mathbb{R}^{n \times n}$. Then it holds that

$$\rho_{\mathrm{P}}(\mu_1, \mu_2) = \rho_{\mathrm{P}}(\mu_3, \delta_0) = \rho(\mu_3) := \inf \{ \varepsilon > 0 : \ \mu_3(\mathbb{R}^p \backslash B_\varepsilon(0)) \le \varepsilon \} ,$$

where δ_0 is the point measure at $0 \in \mathbb{R}^p$ and $B_\varepsilon(0) := \{ \xi_1 \in \mathbb{R}^p : \ \|\xi_1\| < \varepsilon \}$.

Proof. Let $B \in \mathcal{B}(\mathbb{R}^n)$ be arbitrary but fixed. Then obviously

$$\bar{x} \in B^\varepsilon \Longleftrightarrow \bar{x} = g(\xi) \ \wedge \ \exists \eta \in g^{-1}(B) : \|g(\xi) - g(\eta)\|_Q = \|\xi - \eta\| < \varepsilon .$$

Thus, $B^\varepsilon = g((g^{-1}(B))^\varepsilon)$.

If $\varepsilon > 0$ is such that $\mu_3(\mathbb{R}^p \backslash B_\varepsilon(0)) \le \varepsilon$, then $1 \le \mu_3(B_\varepsilon(0)) + \varepsilon$ and hence, due to Fubini's Theorem and the fact that $\mu_1(B) \le 1$,

$$\begin{aligned}
\mu_1(B) &\le \mu_1(B) \mu_3(B_\varepsilon(0)) + \varepsilon = \mu_2(g(B_\varepsilon(0) \times \hat{B})) + \varepsilon \\
&\le \mu_2(g((g^{-1}(B))^\varepsilon)) + \varepsilon = \mu_2(B^\varepsilon) + \varepsilon .
\end{aligned}$$

Thus, due to Definition 2.1, $\rho_{\mathrm{P}}(\mu_1, \mu_2) \le \rho(\mu_3)$.

Let us now assume that $\mu_1(B) \le \mu_2(B^\varepsilon) + \varepsilon$ for all $B \in \mathcal{B}(\mathbb{R}^n)$. Then for the special set $B := g(\{0\} \times \mathbb{R}^{n-p})$ we obtain by Fubini's Theorem that

$$\begin{aligned} 1 &= \mu_1(B) \le \mu_2(B^\varepsilon) + \varepsilon = \mu_2(g((g^{-1}(B))^\varepsilon)) + \varepsilon \\ &= \mu_2(g(B_\varepsilon(0) \times \mathbb{R}^{n-p})) + \varepsilon = \mu_3(B_\varepsilon(0)) + \varepsilon\,. \end{aligned}$$

Therefore, $\mu_3(\mathbb{R}^p \backslash B_\varepsilon(0)) \le \varepsilon$ and hence $\rho_{\mathrm{P}}(\mu_1, \mu_2) \ge \rho(\mu_3)$. All together we have shown that $\rho_{\mathrm{P}}(\mu_1, \mu_2) = \rho(\mu_3)$.

Since it is obvious that $\delta_0(B) \le \mu_3(B^\varepsilon) + \varepsilon$ for all $B \in \mathcal{B}(\mathbb{R}^p)$ is equivalent to $1 \le \mu_3(B_\varepsilon(0)) + \varepsilon$, the last assertion $\rho_{\mathrm{P}}(\mu_3, \delta_0) = \rho(\mu_3)$ follows. ∎

We want to quantify the convergence in probability for $\mathcal{M}(\mathbb{R}^n)$-valued random variables. This can be achieved via the *Ky Fan metric* which measures distances between random variables on a metric space (cf., e.g., [1]):

Definition 2.3. (Ky Fan metric). *Let X_1 and X_2 be random variables in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in a metric space $(\mathcal{X}, d_{\mathcal{X}})$. The distance between X_1 and X_2 in the* Ky Fan metric *is defined as*

$$\rho_{\mathrm{K}}(X_1, X_2) := \inf \{\varepsilon > 0 :\ \mathbb{P}(d_{\mathcal{X}}(X_1(\omega), X_2(\omega)) > \varepsilon) < \varepsilon\}\,.$$

The following proposition shows that convergence rates are essentially preserved when they are lifted from a metric space to the space of random variables equipped with the Ky Fan metric. The result was proven in [6, Theorem 6]. For the convenience of the reader we include the proof.

Proposition 2.4. *Let X_1, X_2 and Y_1, Y_2 be random variables on metric spaces $(\mathcal{X}, d_{\mathcal{X}})$ and $(\mathcal{Y}, d_{\mathcal{Y}})$, respectively. Let*

$$d_{\mathcal{X}}(X_1(\omega), X_2(\omega)) \le \Phi(d_{\mathcal{Y}}(Y_1(\omega), Y_2(\omega))) \tag{2.2}$$

for almost all $\omega \in \Omega$, where Φ is a monotonically increasing right-continuous function. Then

$$\rho_{\mathrm{K}}(X_1, X_2) \le \max\{\rho_{\mathrm{K}}(Y_1, Y_2), \Phi(\rho_{\mathrm{K}}(Y_1, Y_2))\}\,.$$

Proof. For an arbitrary $\varepsilon > 0$, (2.2) and the monotonicity of Φ imply that

$$\begin{aligned} \mathbb{P}\Big(d_{\mathcal{X}}(X_1(\omega), X_2(\omega)) > \Phi(\varepsilon)\Big) &\le \mathbb{P}\Big(\Phi(d_{\mathcal{Y}}(Y_1(\omega), Y_2(\omega))) > \Phi(\varepsilon)\Big) \\ &\le \mathbb{P}\Big(d_{\mathcal{Y}}(Y_1(\omega), Y_2(\omega))) > \varepsilon\Big)\,. \end{aligned}$$

Hence, by Definition 2.3,

$$\mathbb{P}\Big(d_{\mathcal{X}}(X_1(\omega), X_2(\omega)) > \max\{\varepsilon, \Phi(\varepsilon)\}\Big) < \max\{\varepsilon, \Phi(\varepsilon)\}$$

for any $\varepsilon > \rho_{\mathrm{K}}(Y_1, Y_2)$. The assertion now follows with the right-continuity of Φ. ∎

In the following proposition, we give an upper bound for the Ky Fan distance between a normal random variable and its mean. This result is an improvement of [6, Lemma 7].

Proposition 2.5. *Let $\bar{y} \in \mathbb{R}^m$ and $\Sigma \in \mathbb{R}^{m\times m}$ be a positive definite symmetric matrix. Let Y be a random variable with values in $\mathbb{R}^m$ and with distribution $N(\bar{y}, \Sigma)$. Then it holds in $(\mathbb{R}^m, \|\cdot\|_Q)$ that*

$$\rho_{\mathrm{K}}(Y, \bar{y}) = \rho(N(0,\Sigma)) := \inf\{\varepsilon > 0 :\ N(0,\Sigma)(\mathbb{R}^m \backslash B_\varepsilon(0)) \le \varepsilon\}\,, \tag{2.3}$$

where $B_\varepsilon(0) := \{\xi \in \mathbb{R}^m :\ \|\xi\|_Q < \varepsilon\}$. Moreover,

$$\rho(N(0,\Sigma)) \le \min\left\{1, \Big(2\lambda_{\max}(m - \ln^-(\lambda_{\max} a_m))\Big)^{\frac{1}{2}}\right\}\,, \tag{2.4}$$

where $\lambda_{\max}$ is the largest eigenvalue of $\Sigma^{\frac{1}{2}} Q^{-1} \Sigma^{\frac{1}{2}}$,

$$a_m := 2\pi m^2 \left(\tfrac{e}{2}\right)^m\,, \tag{2.5}$$

and $f^-(h) := \min\{0, f(h)\}$.

Proof. Formula (2.3) follows immediately from Definition 2.3. We will now show estimate (2.4).

Using spherical coordinates we obtain that

$$\begin{aligned} N(0,\Sigma)(\mathbb{R}^m\backslash B_\varepsilon(0)) &= (2\pi)^{-\frac{m}{2}} |\Sigma|^{-\frac{1}{2}} \int_{\|\xi\|_Q \ge \varepsilon} e^{-\frac{1}{2}\xi^T\Sigma^{-1}\xi}\, d\xi \\ &\le 2\,\Gamma(\tfrac{m}{2})^{-1} \int_{\varepsilon(2\lambda_{\max})^{-\frac{1}{2}}}^{\infty} t^{m-1} e^{-t^2}\, dt\,. \end{aligned}$$

Here $|\Sigma|$ denotes the determinant of Σ. Hence, by (2.3),

$$\rho(N(0,\Sigma)) \le (2\lambda_{\max})^{\frac{1}{2}} s \tag{2.6}$$

for any s satisfying the inequality

$$I_m(s) := 2\int_s^\infty t^{m-1} e^{-t^2}\, dt = \int_{s^2}^\infty u^{\frac{m}{2}-1} e^{-u}\, du \le \Gamma(\tfrac{m}{2})\,(2\lambda_{\max})^{\frac{1}{2}} s\,.$$

Since $u^{\frac{m}{2}-1}e^{-\frac{u}{2}}$ is strictly monotonically decreasing if $u > \max\{0, m-2\}$, it holds that $I_m(s) \le 2s^{m-2}e^{-s^2}$ if $s^2 > \max\{0, m-2\}$. Together with the formula

$$\Gamma(\tfrac{m}{2}) = (\tfrac{m}{2})^{\frac{m-1}{2}} e^{-\frac{m}{2}} \sqrt{2\pi}\, r_m\,, \quad m \in \mathbb{N}\,, \qquad r_{2m} \searrow 1\,, \quad r_{2m+1} \searrow 1\,, \tag{2.7}$$

which will be shown below, we obtain that (2.6) holds for any s satisfying

$$s^{m-3}e^{-s^2} \le (\tfrac{m}{2})^{\frac{m-1}{2}} e^{-\frac{m}{2}} (\pi\lambda_{\max})^{\frac{1}{2}} \quad \text{and} \quad s^2 > \max\{0, m-2\}\,. \tag{2.8}$$

We will now show that $s^2 = m - \ln^-(\lambda_{\max} a_m)$ satisfies these inequalities, where a_m is as in (2.5).

Let us first consider the case $\lambda_{\max} a_m \ge 1$. Then $s^2 = m$ and

$$m^{\frac{m-3}{2}} e^{-m} \le (\tfrac{m}{2})^{\frac{m-1}{2}} e^{-\frac{m}{2}} (\pi\lambda_{\max})^{\frac{1}{2}} \quad \Longleftrightarrow \quad \lambda_{\max} a_m \ge 1\,.$$

If $\lambda_{\max} a_m < 1$, then $s^2 = m - \ln(\lambda_{\max} a_m) > m$ and the left inequality in (2.8) is equivalent to

$$\Big(m - \ln(\lambda_{\max} a_m)\Big)^{\frac{m-3}{2}} e^{-m} \lambda_{\max} a_m \le (\tfrac{m}{2})^{\frac{m-1}{2}} e^{-\frac{m}{2}} (\pi\lambda_{\max})^{\frac{1}{2}}\,,$$

which together with (2.5) is equivalent to

$$\Big(1-\tfrac{1}{m}\ln(\lambda_{\max}a_m)\Big)^{\frac{m-3}{2}}(\lambda_{\max}a_m)^{\frac12}\le 1\,.$$

However, this inequality holds true, since $\Big(1-\tfrac{1}{m}\ln(h)\Big)^{\frac{m-3}{2}}h^{\frac12}$ is strictly monotonically increasing on the interval $(0,1]$.

Due to (2.6) and (2.8), this together with the fact that $N(0,\Sigma)(\mathbb{R}^m\backslash B_\varepsilon(0))<1$ proves assertion (2.4).

It remains to be shown that (2.7) holds. Actually, it is an immediate consequence of Stirling's formula (see, e.g., [4])

$$n! = n^n e^{-n}\sqrt{n}\,c_n\,,\quad n\in\mathbb{N}\,,\qquad \lim_{n\to\infty}c_n=\sqrt{2\pi}\,. \tag{2.9}$$

It is well-known that

$$\Gamma(\tfrac{m}{2}) = \begin{cases} (\frac{m}{2}-1)!\,, & m \text{ even}\,,\\ \dfrac{(m-1)!\sqrt{\pi}}{2^{m-1}(\frac{m-1}{2})!}\,, & m \text{ odd}\,,\end{cases}\qquad m\in\mathbb{N}\,.$$

This together with (2.9) implies that the formula in (2.7) is valid with

$$r_m := \begin{cases} (1-\frac{2}{m})^{\frac{m-1}{2}}\frac{e}{\sqrt{2\pi}}\,c_{\frac{m}{2}-1}\,, & m \text{ even}\,,\\ (1-\frac{1}{m})^{\frac{m-1}{2}}\sqrt{e}\,c_{m-1}/c_{\frac{m-1}{2}}\,, & m \text{ odd}\,,\end{cases}\qquad m\ge 3\,,\qquad \lim_{m\to\infty}r_m=1\,.$$

The monotonicity result in (2.7) is shown as follows: using $r_m=\Gamma(\frac{m}{2})(\frac{m}{2})^{-\frac{m-1}{2}}e^{\frac{m}{2}}/\sqrt{2\pi}$ and the fact that $\Gamma(x+1)=x\Gamma(x)$, we obtain that

$$r_{m+2}\le r_m \qquad\Longleftrightarrow\qquad e\le(1+\tfrac{2}{m})^{\frac{m+1}{2}}\,.$$

However, the latter inequality is easy to show. ∎

3. Convergence analysis for the finite dimensional setting

In this section, we show, where the random variables X_{post} and M_{post} (see (1.5) and (1.6), respectively) converge to when the noise E tends to the zero random variable. Note that $X_{\text{post}}(\omega)$ is the mean of the normal distribution $M_{\text{post}}(\omega)$. To be able to control how the noise tends to the zero random variable we assume that $\Sigma=\sigma^2\hat{\Sigma}$, where the largest eigenvalue of $\hat{\Sigma}$ is 1.

It will turn out that X_{post} converges to $\bar{x}^\dagger$, the $\bar{x}_0$-minimum-norm-least-squares solution (cf. [2]) defined as follows: $\bar{x}^\dagger$ minimizes the residual $\|A\bar{x}-\bar{y}\|_{\hat{\Sigma}}$ and among all minimizers it then minimizes $\|\bar{x}-\bar{x}_0\|_\Gamma$ (see (2.1) for the definition of the norms). Thus, the appropriate spaces for our convergence analysis are $(\mathcal{X},d_{\mathcal{X}})=(\mathbb{R}^n,\|\cdot\|_\Gamma)$ and $(\mathcal{Y},d_{\mathcal{Y}})=(\mathbb{R}^m,\|\cdot\|_{\hat{\Sigma}})$.

Let us assume in the following that λ_i, $1\le i\le p\le\min\{m,n\}$, are the positive singular values of the matrix $\hat{\Sigma}^{-\frac12}A\Gamma^{\frac12}$ and that v_i are orthonormal eigenvectors of $\Gamma^{\frac12}A^T\hat{\Sigma}^{-1}A\Gamma^{\frac12}$ in $(\mathbb{R}^n,\|\cdot\|)$, i.e.,

$$\Gamma^{\frac12}A^T\hat{\Sigma}^{-1}A\Gamma^{\frac12}v_i=\lambda_i^2v_i\,,\qquad \lambda_1\ge\ldots\ge\lambda_p>\lambda_{p+1}=\ldots=\lambda_n=0\,. \tag{3.1}$$

Then $\bar{x}^\dagger$ may be expressed as

$$\bar{x}^\dagger = \sum_{i=1}^{p} \lambda_i^{-2}(v_i^T \Gamma^{\frac{1}{2}} A^T \hat{\Sigma}^{-1} \bar{y}) \Gamma^{\frac{1}{2}} v_i + \sum_{i=p+1}^{n} (v_i^T \Gamma^{-\frac{1}{2}} x_0) \Gamma^{\frac{1}{2}} v_i \,. \tag{3.2}$$

Using the notations $V_1 := (v_1, \ldots, v_p)$, $V_2 := (v_{p+1}, \ldots, v_n)$, and $V := (V_1, V_2)$, and the splitting $\xi = (\xi_1, \xi_2)$ with $\xi_1 \in \mathbb{R}^p$ and $\xi_2 \in \mathbb{R}^{n-p}$, we will show that M_{post} converges to the normal distribution $\mu_{\bar{x}^\dagger} = N(\bar{x}^\dagger, \Gamma^{\frac{1}{2}} V_2 V_2^T \Gamma^{\frac{1}{2}})$, i.e., for all $B \in \mathcal{B}(\mathbb{R}^n)$

$$\mu_{\bar{x}^\dagger}(B) = (2\pi)^{-\frac{n-p}{2}} \int_{\hat{B}} e^{-\frac{1}{2}\xi_2^T \xi_2} \, d\xi_2 \,, \tag{3.3}$$

where $\hat{B} := \{\xi_2 \in \mathbb{R}^{n-p} : g(0, \xi_2) \in B\}$ and $g(\xi) := \bar{x}^\dagger + \Gamma^{\frac{1}{2}} V \xi$. Note that, in case $\mathcal{N}(A) = \{0\}$ (which is equivalent to $n = p$), this normal distribution is a point measure, i.e., $\mu_{\bar{x}^\dagger}(B) = 1$ if $\bar{x}^\dagger \in B$ and $\mu_{\bar{x}^\dagger}(B) = 0$ if $\bar{x}^\dagger \notin B$, usually denoted by $\delta_{\bar{x}^\dagger}$.

Theorem 3.1. *Let X_{post} and M_{post} be the random variables defined in* (1.5) *and* (1.6), *respectively. Then the following estimates hold:*

$$\begin{aligned} \rho_{\text{K}}(X_{\text{post}}, \bar{x}^\dagger) \;&\leq\; \max\Big\{\rho_{\text{K}}(E, 0),\, \frac{\sigma^2}{\sigma^2 + \lambda_p^2} \|(I - P)(\bar{x}^\dagger - \bar{x}_0)\|_\Gamma \\ &\qquad + \frac{\max\{\sigma, \lambda_p\}}{\sigma^2 + \max^2\{\sigma, \lambda_p\}} \,\rho_{\text{K}}(E, 0)\Big\} \\ &=\; O\Big(\sigma\sqrt{1 + |\ln \sigma|}\Big)\,, \end{aligned} \tag{3.4}$$

where P denotes the orthogonal projector onto $\mathcal{N}(A)$ and

$$\rho_{\text{K}}(E, 0) \leq \sigma \left(2(m - \ln^-(\sigma^2 a_m))\right)^{\frac{1}{2}} = O\Big(\sigma\sqrt{1 + |\ln \sigma|}\Big) \tag{3.5}$$

with a_m as in (2.5). *Moreover,*

$$\begin{aligned} \rho_{\text{K}}(M_{\text{post}}, \mu_{\bar{x}^\dagger}) \;&\leq\; \max\Big\{\rho_{\text{K}}(E, 0),\, \frac{\sigma^2}{\sigma^2 + \lambda_p^2} \|(I - P)(\bar{x}^\dagger - \bar{x}_0)\|_\Gamma \\ &\qquad + \left(\frac{2\sigma^2}{\sigma^2 + \lambda_p^2}\Big(p - \ln^-\Big(\frac{\sigma^2 a_p}{\sigma^2 + \lambda_p^2}\Big)\Big)\right)^{\frac{1}{2}} \\ &\qquad + \frac{\max\{\sigma, \lambda_p\}}{\sigma^2 + \max^2\{\sigma, \lambda_p\}} \,\rho_{\text{K}}(E, 0)\Big\} \\ &=\; O\Big(\sigma\sqrt{1 + |\ln \sigma|}\Big) \end{aligned} \tag{3.6}$$

with $\mu_{\bar{x}^\dagger}$ as in (3.3) *and a_p as in* (2.5) *(with $m = p$).*

Proof. Using (1.5), (3.1), and (3.2) we obtain the estimate

$$\begin{aligned} &\|X_{\text{post}}(\omega) - \bar{x}^\dagger\|_\Gamma \\ &\quad\leq \left(\sum_{i=1}^{p} \frac{\sigma^4}{(\sigma^2 + \lambda_i^2)^2} (v_i^T \Gamma^{-\frac{1}{2}}(\bar{x}^\dagger - \bar{x}_0))^2\right)^{\frac{1}{2}} + \sup_{1 \leq i \leq p} \frac{\lambda_i}{\sigma^2 + \lambda_i^2} \|E(\omega)\|_{\hat{\Sigma}} \\ &\quad\leq \frac{\sigma^2}{\sigma^2 + \lambda_p^2} \|(I - P)(\bar{x}^\dagger - \bar{x}_0)\|_\Gamma + \frac{\max\{\sigma, \lambda_p\}}{\sigma^2 + \max^2\{\sigma, \lambda_p\}} \|E(\omega)\|_{\hat{\Sigma}} \,. \end{aligned} \tag{3.7}$$

This together with Proposition 2.4 and Proposition 2.5 (with $\bar{y} = 0$, $Q = \hat{\Sigma}$, and $\lambda_{\max} = \sigma^2$) yields the assertions (3.4) and (3.5).

We will now show that estimate (3.6) holds. Noting that $\rho_{\mathrm{P}}(M_{\mathrm{post}}(\omega), N(\bar{x}^\dagger, \Gamma_{\mathrm{post}})) \le \|X_{\mathrm{post}}(\omega) - \bar{x}^\dagger\|_\Gamma$ the triangle inequality yields that

$$\rho_{\mathrm{P}}(M_{\mathrm{post}}(\omega), \mu_{\bar{x}^\dagger}) \le \|X_{\mathrm{post}}(\omega) - \bar{x}^\dagger\|_\Gamma + \rho_{\mathrm{P}}(N(\bar{x}^\dagger, \Gamma_{\mathrm{post}}), \mu_{\bar{x}^\dagger})\,.$$

Note that for any $B \in \mathcal{B}(\mathbb{R}^n)$

$$\begin{aligned} N(\bar{x}^\dagger, \Gamma_{\mathrm{post}})(B) &= (2\pi)^{-\frac{n}{2}} |\Gamma_{\mathrm{post}}|^{-\frac{1}{2}} \int_B e^{-\frac{1}{2}(\bar{x}-\bar{x}^\dagger)^T \Gamma_{\mathrm{post}}^{-1} (\bar{x}-\bar{x}^\dagger)}\, d\bar{x} \\ &= \int_{g^{-1}(B)} (2\pi)^{-\frac{p}{2}} |F|^{\frac{1}{2}} e^{-\frac{1}{2}\xi_1^T F \xi_1} \cdot (2\pi)^{-\frac{n-p}{2}} e^{-\frac{1}{2}\xi_2^T \xi_2}\, d(\xi_1, \xi_2) \end{aligned}$$

with $F := I + V_1^T \Gamma^{\frac{1}{2}} A^T \Sigma^{-1} A \Gamma^{\frac{1}{2}} V_1 = \operatorname{diag}(1 + \sigma^{-2}\lambda_i^2)$. Now (3.3), Proposition 2.2 (with $\tilde{x} = \bar{x}^\dagger$ and $Q = \Gamma$), and Proposition 2.5 (with $m = p$, $\bar{y} = 0$, $\Sigma = F^{-1}$, $Q = I$, and $\lambda_{\max} = \frac{\sigma^2}{\sigma^2 + \lambda_p^2}$) imply that

$$\rho_{\mathrm{P}}(N(\bar{x}^\dagger, \Gamma_{\mathrm{post}}), \mu_{\bar{x}^\dagger}) = \rho(N(0, F^{-1})) \le \left(\frac{2\sigma^2}{\sigma^2 + \lambda_p^2} \Big(p - \ln^- \Big(\frac{\sigma^2 a_p}{\sigma^2 + \lambda_p^2} \Big) \Big) \right)^{\frac{1}{2}}.$$

This together with (3.7) and Proposition 2.4 proves assertion (3.6). ∎

As expected for regularization in finite dimensional spaces we obtain convergence rates with the same order as the noise.

4. Convergence analysis for the infinite dimensional setting

In this section, we turn back to the infinite dimensional equation (1.1) formulated between real separable Hilbert spaces. As mentioned in the introduction, this equation is usually discretized leading to a finite dimensional problem (1.2). One possibility to discretize equation (1.1) is via projection. We choose the following approach (see [2, Section 5.2]):

Let $\{\mathcal{Y}_n\}$ be a sequence of finite dimensional subspaces of $\overline{\mathcal{R}(T)}$ with the following approximation property

$$\lim_{n\to\infty} \|(I - Q_n)y\| = 0 \qquad \text{for all} \quad y \in \overline{\mathcal{R}(T)}\,, \tag{4.1}$$

where Q_n is the orthogonal projector of $\mathcal{Y}$ onto $\mathcal{Y}_n$. This condition is especially satisfied if $\mathcal{Y}_n \subset \mathcal{Y}_{n+1}$ for all $n \in \mathbb{N}$ and if the union of $\mathcal{Y}_n$ is dense in $\overline{\mathcal{R}(T)}$. Instead of (1.1) we now want to solve

$$T_n x = Q_n y\,, \qquad T_n := Q_n T\,. \tag{4.2}$$

As in the finite dimensional approach we assume that we have a prior distribution for X. Now X is a random variable from a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to the Hilbert space $\mathcal{X}$ with Gaussian distribution $N(x_0, \Gamma)$, where $x_0 \in \mathcal{X}$ and Γ is a positive definite self adjoint trace class operator (see, e.g., [9]). We will look for stable solutions of (4.2) in the space $\mathcal{X}_n := \Gamma T^* \mathcal{Y}_n$, where T^* denotes the adjoint of $T : \mathcal{X} \to \mathcal{Y}$.

Assuming that $\mathcal{Y}_n$ has dimension n and that $\{\varphi_i^n\}_{1\le i\le n}$, is a basis for $\mathcal{Y}_n$, problem (4.2) is equivalent to

$$A_n\bar{x} = \bar{y} \qquad \text{with} \qquad A_n := G_n^{-1}H_n\,, \tag{4.3}$$

where

$$x = \sum_{i=1}^{n}\bar{x}_i\Gamma T^*\varphi_i^n\,, \quad Q_n y = \sum_{i=1}^{n}\bar{y}_i\varphi_i^n\,, \quad H_n := [\langle\, \Gamma T^*\varphi_i^n, T^*\varphi_j^n \,\rangle]\,, \quad G_n := [\langle\, \varphi_i^n, \varphi_j^n \,\rangle]\,.$$

Note that $H_n = [\langle\, T^{\#}\varphi_i^n, T^{\#}\varphi_j^n \,\rangle_\Gamma]$, where $T^{\#} = \Gamma T^*$ is the adjoint of $T : \mathcal{X}_\Gamma \to \mathcal{Y}$ and $\mathcal{X}_\Gamma := \mathcal{D}(\Gamma^{-\frac{1}{2}}) \subset \mathcal{X}$ is the Hilbert space with inner product

$$\langle\, x_1, x_2 \,\rangle_\Gamma := \langle\, \Gamma^{-\frac{1}{2}}x_1, \Gamma^{-\frac{1}{2}}x_2 \,\rangle\,.$$

Moreover, for $x \in \mathcal{X}_n$ and $y \in \mathcal{Y}$ it holds that

$$\|x\|_\Gamma^2 = \bar{x}^T H_n \bar{x} \qquad \text{and} \qquad \|Q_n y\|^2 = \bar{y}^T G_n \bar{y}\,. \tag{4.4}$$

Finally, we need an error model. It is known from the deterministic regularization approach that for the finite dimensional equation (4.2) only noise in $\mathcal{Y}_n$ matters. Therefore, we can choose a random noise model, where the random variable E_n has its values in $\mathcal{Y}_n$. We assume in this section that the distribution of the noise E_n is given by $N(0, \sigma^2 Q_n)$ and as always that E_n is independent of X. Note that the values of E_n belong to $\mathcal{Y}_n$ with probability 1 (cf., e.g., [10]).

From the prior distribution of X on $\mathcal{X}$ we can derive a prior distribution for $\bar{X} := H_n^{-1}[\langle\, X, T^*\varphi_i^n \,\rangle]$, the random variable corresponding to $\bar{x} \in \mathbb{R}^n$: it is well known (see, e.g., [9]) that $[\langle\, X, T^*\varphi_i^n \,\rangle]$ is normally distributed with mean $[\langle\, x_0, T^*\varphi_i^n \,\rangle]$ and covariance matrix H_n. Then the proper prior distribution for $\bar{X}$ is given by $N(\bar{x}_0, H_n^{-1})$ with $\bar{x}_0 = H_n^{-1}[\langle\, x_0, T^*\varphi_i^n \,\rangle]$. Note that

$$P_n x_0 = \sum_{i=1}^{n}\bar{x}_{0,i}\Gamma T^*\varphi_i^n$$

if $x_0 \in \mathcal{X}_\Gamma$, where P_n is the orthogonal projector from $\mathcal{X}_\Gamma$ onto $\mathcal{X}_n \subset \mathcal{X}_\Gamma$. In a similar way we see that the corresponding distribution of the noise in $\mathbb{R}^n$ is given by $N(0, \sigma^2 G_n^{-1})$.

The Bayesian approach now yields that the posterior distribution μ_{post} for the finite dimensional problem (4.3) is given by $N(\bar{x}_{\text{post}}, \Gamma_{\text{post}})$ with (see (1.3) and (1.4))

$$\bar{x}_{\text{post}} = (\sigma^2 G_n + H_n)^{-1}(G_n\bar{y}^\sigma + \sigma^2 G_n \bar{x}_0) \quad \text{and} \quad \Gamma_{\text{post}} = (G_n + \sigma^{-2}H_n)^{-1}G_n H_n^{-1}\,.$$

Obviously, there is an appropriate element $x_{\text{post},n} = \sum_{i=1}^n \bar{x}_{\text{post},i}\Gamma T^*\varphi_i^n$ in $\mathcal{X}_n$ given by

$$x_{\text{post},n} = (\sigma^2\Gamma^{-1} + T_n^*T_n)^{-1}(T_n^* y_n^\sigma + \sigma^2\Gamma^{-1}P_n x_0)\,, \tag{4.5}$$

where y_n^σ is a realization of the random variable $Q_n y + E_n$ such that $Q_n y_n^\sigma = \sum_{i=1}^n \bar{y}_i^\sigma \varphi_i^n$.

The induced posterior distribution $\mu_{\text{post},n}$ with values in $\mathcal{X}$ is then given by $N(x_{\text{post},n}, \Gamma_{\text{post},n})$ with $\Gamma_{\text{post},n} := (\Gamma^{-1} + \sigma^{-2}T_n^*T_n)^{-1} = \Gamma^{\frac{1}{2}}(I + \sigma^{-2}\Gamma^{\frac{1}{2}}T_n^*T_n\Gamma^{\frac{1}{2}})^{-1}\Gamma^{\frac{1}{2}}$. Note that $\Gamma_{\text{post},n}$ is a trace class operator.

The question is now, where the corresponding random variables $X_{\text{post},n}$ and $M_{\text{post},n}$ (compare (1.5) and (1.6)) converge to as σ goes to 0 and as n tends to infinity.

For this convergence analysis we need an estimate for $\rho_{\mathrm{K}}(E_n, 0)$. To obtain a qualitatively nice estimate that does not involve too many parameters we assume that the following condition

$$\sigma n c^n \geq 1 \quad \text{for some} \quad c > 1 \tag{4.6}$$

holds. Then Proposition 2.5 (with $m = n$, $\bar{y} = 0$, $\Sigma = \sigma^2 G_n^{-1}$, and $Q = G_n^{-1}$) together with Definition 2.3, (2.1), and (4.4) implies that

$$\rho_{\mathrm{K}}(E_n, 0) = O(\sigma n^{\frac{1}{2}}) \,. \tag{4.7}$$

Checking the proof of Proposition 2.5 one can show that this estimate is sharp if $\sigma n \to 0$. Obviously σ has to go faster to zero than $n^{\frac{1}{2}}$ tends to infinity so that $\rho_{\mathrm{K}}(E_n, 0)$ will converge to zero. This reflects the fact that E_n is a projection of the white noise which is not a Gaussian distribution in $\mathcal{Y}$. On the other hand, condition (4.6) means that σ is not allowed to go too fast to zero as n tends to infinity, however, it might still tend to zero exponentially fast.

From Tikhonov regularization one knows that the regularization parameter is not allowed to go to zero too fast compared to the noise level (see, e.g., [2]). In view of (4.7) we can not expect that $x_{\mathrm{post},n}$ as defined in (4.5) will converge. Therefore, we consider the following *weighted Bayesian approach*:

Viewing the random variable X in the space $\mathcal{X}$ equipped with the weighted norm $\alpha^{\frac{1}{2}}\sigma^{-1}\|\cdot\|_{\mathcal{X}}$ implies that the appropriate prior distribution for $\bar{X}$ in $\mathbb{R}^n$ is given by $N(\bar{x}_0, \sigma^2\alpha^{-1}H_n^{-1})$. Now the standard Bayesian approach in $\mathbb{R}^n$ yields a posterior distribution $\mu^\alpha_{\mathrm{post}} = N(\bar{x}^\alpha_{\mathrm{post}}, \Gamma^\alpha_{\mathrm{post}})$ with

$$\bar{x}^\alpha_{\mathrm{post}} = (\alpha G_n + H_n)^{-1}(G_n\bar{y}^\sigma + \alpha G_n \bar{x}_0) \quad \text{and} \quad \Gamma^\alpha_{\mathrm{post}} = \sigma^2(\alpha G_n + H_n)^{-1} G_n H_n^{-1} \,.$$

The induced posterior distribution with values in $\mathcal{X}$ is then given by $N(x^\alpha_{\mathrm{post},n}, \Gamma^\alpha_{\mathrm{post},n})$ with

$$x^\alpha_{\mathrm{post},n} = (\alpha\Gamma^{-1} + T_n^*T_n)^{-1}(T_n^* y_n^\sigma + \alpha\Gamma^{-1}P_n x_0) \quad \text{and} \quad \Gamma^\alpha_{\mathrm{post},n} := \sigma^2(\alpha\Gamma^{-1} + T_n^*T_n)^{-1} \,.$$

For a convergence analysis in $\mathcal{X}$ one would have to use results from regularization in Hilbert scales (see [2, Section 8.5]), i.e., one would need an assumption like $\|Tx\| \sim \|\Gamma^{-\frac{a}{2}}x\|$ for some $a > 0$. We will not pursue this approach here, but we rather derive results on convergence and convergence rates in $\mathcal{X}_\Gamma$.

The appropriate induced posterior distribution $\mu^\alpha_{\mathrm{post},n}$ in $\mathcal{X}_{n,\Gamma}$, the space $\mathcal{X}_n$ equipped with the $\mathcal{X}_\Gamma$-norm, is given by $N(x^\alpha_{\mathrm{post},n}, \tilde{\Gamma}^\alpha_{\mathrm{post},n})$ with

$$\tilde{\Gamma}^\alpha_{\mathrm{post},n} := \sigma^2(\alpha I + T^\# Q_n T)^{-1} \,.$$

Note that $\mu^\alpha_{\mathrm{post},n}$ is not a Gaussian distribution in $\mathcal{X}_\Gamma$, since $\tilde{\Gamma}^\alpha_{\mathrm{post},n}$ is not a trace class operator in $\mathcal{X}_\Gamma$.

As in the finite dimensional case the weighted posterior mean $x^\alpha_{\mathrm{post},n}$ and the weighted posterior distribution $\mu^\alpha_{\mathrm{post},n}$ are realizations of random variables $X^\alpha_{\mathrm{post},n}$ and $M^\alpha_{\mathrm{post},n}$, respectively, i.e.,

$$\begin{aligned} X^\alpha_{\mathrm{post},n}(\omega) &:= (\alpha\Gamma^{-1} + T_n^*T_n)^{-1}(T_n^*(Q_n y + E_n(\omega)) + \alpha\Gamma^{-1}P_n x_0) \\ &= (\alpha I + T^\# Q_n T)^{-1}(T^\# Q_n(y + E_n(\omega)) + \alpha P_n x_0) \,, \end{aligned} \tag{4.8}$$

$$M^{\alpha}_{\mathrm{post},n} : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathcal{M}(\mathcal{X}_{n,\Gamma}), \rho_{\mathrm{P}}), \qquad \omega \mapsto N(X^{\alpha}_{\mathrm{post},n}(\omega), \tilde{\Gamma}^{\alpha}_{\mathrm{post},n}) . \tag{4.9}$$

In the next theorem, we present convergence rates for $\rho_{\mathrm{K}}(X^{\alpha}_{\mathrm{post},n}, x^\dagger)$ and $\rho_{\mathrm{K}}(M^{\alpha}_{\mathrm{post},n}, \delta_{x_n^\dagger})$, where $x^\dagger = T^\dagger y$, $T^\dagger$ is the Moore-Penrose inverse of $T : \mathcal{X}_\Gamma \to \mathcal{Y}$, and $x_n^\dagger = T_n^\dagger y$. Note that $x_n^\dagger = P_n x^\dagger$ (cf., e.g., [2, Theorem 3.24]) and that $\delta_{x_n^\dagger}$ is a point measure in $\mathcal{X}_{n,\Gamma}$.

Theorem 4.1. *Let* (4.6) *hold, let $X^{\alpha}_{\mathrm{post},n}$ and $M^{\alpha}_{\mathrm{post},n}$ be the random variables defined in* (4.8) *and* (4.9), *respectively, and suppose that T is compact. Moreover, assume that $x^\dagger, x_0 \in \mathcal{X}_\Gamma$ and that*

$$(I-P)x_0 - x^\dagger = (T^{\#}T)^{\mu} v , \quad v \in \mathcal{N}(T)^{\perp} , \quad \mu \in [0,1] , \tag{4.10}$$

where P denotes the orthogonal projector from $\mathcal{X}_\Gamma$ onto $\mathcal{N}(T) \subset \mathcal{X}_\Gamma$.

If $\sigma n^{\frac{1}{2}} \to 0$ and if $\alpha \to 0$ and $\frac{\sigma^2 n}{\alpha} \to 0$ in case $\mu = 0$ and $\alpha \sim (\sigma n^{\frac{1}{2}})^{\frac{2}{2\mu+1}}$ in case $\mu > 0$ as $\sigma \to 0$ and $n \to \infty$, then it holds that

$$\rho_{\mathrm{K}}(X^{\alpha}_{\mathrm{post},n}, x^\dagger) = O\Big(\|(I-P_n)x^\dagger\|_\Gamma\Big) + \begin{cases} o(1) , & \mu = 0 , \\ O\Big(\gamma_n^{2\mu} + (\sigma n^{\frac{1}{2}})^{\frac{2\mu}{2\mu+1}}\Big) , & \mu > 0 , \end{cases} \tag{4.11}$$

and

$$\rho_{\mathrm{K}}(M^{\alpha}_{\mathrm{post},n}, \delta_{x_n^\dagger}) = \begin{cases} o(1) , & \mu = 0 , \\ O\Big(\gamma_n^{2\mu} + (\sigma n^{\frac{1}{2}})^{\frac{2\mu}{2\mu+1}}\Big) , & \mu > 0 , \end{cases} \tag{4.12}$$

where $\gamma_n := \|(I-Q_n)T\|_{\mathcal{X}_\Gamma, \mathcal{Y}}$.

Proof. Noting that $P_n x_0 = P_n(I-P)x_0$ and $Q_n T x_n^\dagger = Q_n y$, and using standard estimation techniques for Tikhonov regularized solutions (see [2, Section 5.2]), we obtain the following estimate

$$\begin{aligned} \|X^{\alpha}_{\mathrm{post},n}(\omega) - P_n x^\dagger\|_\Gamma &\le \|\alpha(\alpha I + T^{\#}Q_n T)^{-1} P_n(x_0 - x^\dagger)\|_\Gamma \\ &\quad + \|(\alpha I + T^{\#}Q_n T)^{-1} T^{\#} Q_n E_n(\omega)\|_\Gamma \\ &\le \|(I-P_n)((I-P)x_0 - x^\dagger)\|_\Gamma + \frac{1}{2\sqrt{\alpha}} \|E_n(\omega)\| \\ &\quad + \begin{cases} o(1) , & \mu = 0 , \\ O(\gamma_n^{2\mu} + \alpha^{\mu}) , & \mu > 0 . \end{cases} \end{aligned}$$

Now assertion (4.11) follows together with (4.7), Proposition 2.4, the choices for α, and the fact (see [2, Lemma 5.10]) that

$$\|(I-P_n)((I-P)x_0 - x^\dagger)\|_\Gamma = \begin{cases} o(1) , & \mu = 0 , \\ O(\gamma_n^{2\mu}) , & \mu > 0 . \end{cases}$$

The proof of estimate (4.12) is similar to the one of estimate (3.6) noting that as in the finite dimensional case

$$\rho_{\mathrm{P}}(M^{\alpha}_{\mathrm{post},n}(\omega), \delta_{x_n^\dagger}) \le \|X^{\alpha}_{\mathrm{post},n}(\omega) - P_n x^\dagger\|_\Gamma + \rho_{\mathrm{P}}(N(x_n^\dagger, \tilde{\Gamma}^{\alpha}_{\mathrm{post},n}), \delta_{x_n^\dagger})$$

and

$$\begin{aligned} \rho_{\mathrm{P}}(N(x_n^\dagger, \tilde{\Gamma}^{\alpha}_{\mathrm{post},n}), \delta_{x_n^\dagger}) &= \rho(N(0, \Gamma^{\alpha}_{\mathrm{post}})) = \rho(N(0, \mathrm{diag}^{-1}(\sigma^{-2}(\alpha + \lambda_i^2)))) \\ &\le \Big(\frac{2\sigma^2}{\alpha + \lambda_n^2}\big(n - \ln^{-}\big(\frac{\sigma^2 a_n}{\alpha + \lambda_n^2}\big)\big)\Big)^{\frac{1}{2}} , \end{aligned}$$

where $a_n := 2\pi n^2 \left(\frac{e}{2}\right)^n$ and λ_i, $1 \leq i \leq n$, are the singular values of the matrix $G_n^{-\frac{1}{2}} H_n^{\frac{1}{2}}$ or equivalently the singular values of $T_n : \mathcal{X}_\Gamma \to \mathcal{Y}$. Moreover, due to (4.6), we obtain that $\rho_{\mathrm{P}}(N(x_n^\dagger, \tilde{\Gamma}_{\mathrm{post},n}^\alpha), \delta_{x_n^\dagger}) = O(\alpha^{-\frac{1}{2}} \sigma n^{\frac{1}{2}})$. ∎

The rate in (4.11) is known to be order optimal. Note that $\gamma_n \to 0$, since T is compact.

The a priori parameter choice of α depends on the parameter μ in (4.10). This parameter is in general not known. For a posteriori parameter selection criterions that do not need this knowledge see [2].

The convergence results in Theorem 4.1 show that, although the rates in the finite dimensional case (see Theorem 3.1) are order optimal, one might obtain better estimates with a weighted approach, especially for large scale problems.

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