All-Floating Coupled Data-Sparse Boundary and Interface-Concentrated Finite Element Tearing and Interconnecting Methods*

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Summary. Efficient and robust tearing and interconnecting solvers for large scale systems of coupled boundary and finite element domain decomposition equations are the main topic of this paper. In order to reduce the complexity of the finite element part from $O((H/h)^d)$ to $O((H/h)^{d-1})$, we use an interface-concentrated *hp* finite element approximation. The complexity of the boundary element part is reduced by data-sparse approximations of the boundary element matrices. Finally, we arrive at a parallel solver whose complexity behaves like $O((H/h)^{d-1})$ up to some polylogarithmic factor, where H, h, and d denote the usual scaling parameters of the subdomains, the minimal discretization parameter of the subdomain boundaries, and the spatial dimension, respectively.

1 Introduction

Domain Decomposition (DD) methods are nowadays not only used for constructing highly efficient parallel solvers for partial differential equations but also for coupling different physical fields, different meshes and different discretization techniques. Since Finite Element Methods (FEM) and Boundary Element Methods (BEM) exhibit certain complementary properties, it is sometimes very useful to couple these discretization techniques within some DD framework (see, e. g., [11] and the references cited there). The classical BEM has $O((H/h)^{d-1})$ unknowns (degree of freedom = DOF) and needs $O((H/h)^{2(d-1)})$ memory units for storing the dense BE system matrix whereas the standard FEM has $O((H/h)^d)$ DOF and needs $O((H/h)^d)$ memory units for storing the sparse FE stiffness matrix per subdomain. To reduce the memory demand for the BEM, various data sparse approximations of the dense

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BE matrices have been proposed during the last two decades. Let us here only mention the Fast Multipole Method (FMM) [30, 5], the panel clustering method [15], the $\mathcal{H}$-matrix technology [12, 13], and the Adaptive Cross Approximation (ACA) [1, 3], see also the book [29]. These methods replace the original dense BE matrix by a data-sparse approximation in such a way that the discretization error is not perturbed and the memory demand can be reduced from $O((H/h)^{2(d-1)})$ to $O((H/h)^{d-1})$ up to some polylogarithmic factor. The same is true for the matrix-by-vector multiplication that is the basic operation in an iteration process for solving the corresponding system of algebraic equations. Now, the FE parts have suddenly an higher complexity than the BE parts, at least, asymptotically. Under additional smoothness assumptions which are fulfilled in many practical applications (smooth coefficients and right-hand sides on the subdomains of the DD) it is possible to reduce the complexity of the FE parts from $O((H/h)^d)$ to $O((H/h)^{d-1})$. This reduction can be achieved by the boundary-concentrated FEM that was proposed by B.N. Khoromskij and J.M. Melenk in [17], see also [16, 9, 8, 4] and an earlier paper by H. Yserentant [37].

The Finite Element Tearing and Interconnecting (FETI) method, which was originally proposed by C. Farhat and F.-X. Roux in [10], and its modern versions FETI-DP and BDDC (see the monograph [36] and the references therein) are nowadays certainly the most popular DD methods, at least, in many engineering applications on large-scale parallel computers. The Boundary Element Tearing and Interconnecting (BETI) method was introduced in [23] as BE counterpart of the FETI methods. Coupled FETI-BETI methods were discussed in [24], see also [25] and the references therein. The tearing and interconnecting solvers exhibit only $O(1+\log(H/h))$ growth in the number of iterations. Furthermore, they do not suffer from coefficient jumps across the subdomain boundaries and they are highly parallel, but the total complexity is defined by the complexity of the matrix-by-vector multiplication and by the preconditioning operation.

In this paper, similar to [24], we use the tearing and interconnecting technique for solving coupled BE-FE domain decomposition equations, but we now use data-sparse approximations of the BE matrices involved and interface-concentrated FE approximations in some of the subdomains. Here, the interface is the union of the subdomain boundaries. The arithmetical cost and the memory demand of our All-Floating (AF) BETI-FETI solvers are basically proportional to the degrees of freedoms living on the skeleton of domain decomposition given. More precisely, the total complexity behaves like $O((H/h)^{d-1})$ up to a polylogarithmic factor in a parallel regime.

The rest of the paper is organized as follows: In Section 2, we introduce the symmetric coupled BE-FE discretization of a model potential equation in a non-overlapping DD framework, where we used data-sparse BE approximations in the BE subdomains and Interface-Concentrated (IC) FE approximations in the FE subdomains. Section 3 is devoted to the description of the AF-BETI-FETI solvers and to the construction of the preconditioners.

We state and prove our main result on the total complexity of our solver. Section 4 contains the results of our numerical experiments.

2 Symmetric BEM-FEM Coupling

2.1 Skeleton Variational Formulation

Let $\Omega \subset \mathbb{R}^d$ $(d = 2,3)$ be a bounded domain with a Lipschitz boundary $\Gamma = \partial\Omega$, and let us consider the Dirichlet boundary value problem (BVP) for the potential equations

$$-\mathrm{div}\,(a(x)\nabla u(x)) = f(x) \quad \text{for } x \in \Omega, \quad u(x) = g(x) \quad \text{for } x \in \Gamma \tag{1}$$

as our first model problem. We assume that the domain Ω is shape-regularly decomposed into p non-overlapping Lipschitz subdomains

$$\overline{\Omega} = \bigcup_{i=1}^{p} \overline{\Omega}_i, \Omega_i \cap \Omega_j = \emptyset \text{ for } i \neq j, \Gamma_i = \partial\Omega_i, \Gamma_{ij} = \Gamma_i \cap \Gamma_j, \Gamma_S = \bigcup_{i=1}^{p} \Gamma_i, \tag{2}$$

where we call Γ_S the skeleton of the domain decomposition. For the sake of simplicity, we assume that the boundary Γ and the subdomain boundaries Γ_i are polygonal ($d = 2$) or polyhedral ($d = 3$). We refer to the first q subdomains as BE subdomains, whereas the remaining $p - q$ subdomains are called FE subdomains. Thus, the extreme cases $q = 0$ and $q = p$ mean that there are no BE subdomains and no FE subdomains, respectively. We assume that the scalar coefficient function a is piecewise constant and the source function f is vanishing in the BE subdomains, i. e.,

$$a(x) = a_i > 0 \quad \text{and} \quad f(x) = 0 \quad \text{for } x \in \Omega_i, \quad i = 1, \ldots, q,$$

whereas the coefficient function $a(.) = a_i(.)$ and the source function $f(.) = f_i(.)$ are assumed to be analytic in the FE subdomains $\bar{\Omega}_i$, $i = q+1, \ldots, p$. Furthermore, we assume that the variation of the coefficient function $a_i(.)$ should be small, whereas the jumps across the subdomain boundaries can be very large. More precisely, there exist positive constants $\underline{a}_i$ and $\bar{a}_i$ such that

$$0 < \underline{a}_i \leq a(x) \leq \bar{a}_i \quad \forall x \in \overline{\Omega}_i$$

with a small ratio $\bar{a}_i/\underline{a}_i \geq 1$ for $i = q+1, \ldots, p$.
As a second typical model problem, we consider the potential equation (1) in $\mathbb{R}^d$ with an appropriate radiation condition for $|x| \to \infty$ instead of the Dirichlet boundary condition on Γ. In this case, we only add the exterior subdomain $\overline{\Omega}_0 := \mathbb{R}^d \setminus \Omega$ to the domain decomposition (2). In particular, we present numerical results for a simplified magnetic field problem given in the entire space $\mathbb{R}^2$ in Section 4. However, in this paper, we only analyse the case

of a bounded domain. The specialities of the analysis of the one-level BETI method and of the BETI-DP method for the case of unbounded domains are discussed in [28].

Following [25], we can easily see that the standard variational formulation of the BVP (1) in $H^1(\Omega)$ is equivalent to the following skeleton variational formulation: Given $g \in H^{1/2}(\Gamma)$, find $u \in H_g^{1/2}(\Gamma_S)$ such that

$$\sum_{i=1}^{p} \int_{\Gamma_i} (S_i u)(x) v(x) ds_x \; = \sum_{i=q+1}^{p} \int_{\Gamma_i} (N_i f)(x) v(x) ds_x \quad \forall v \in H_0^{1/2}(\Gamma_S), \tag{3}$$

where S_i and N_i denote the Steklov-Poincaré and the Newton potential operators, respectively. The manifold $H_g^{1/2}(\Gamma_S)$ and the corresponding subspace $H_0^{1/2}(\Gamma_S)$ are given by all functions from $H^{1/2}(\Gamma_S) := \{v|_{\Gamma_S} : v \in H^1(\Omega)\}$ with the trace g and 0 on Γ, respectively.

In order to discretize (3), we use the data-sparse BEM in the first q subdomains, whereas the IC-FEM is used for the remaining subdomains.

2.2 Data-Sparse Boundary Element Approximation

For the BE subdomains Ω_i, $i \leq q$, the local PDE reads as follows:

$$-a_i \Delta u(x) = 0 \quad \text{for } x \in \Omega_i\,.$$

The corresponding fundamental solution

$$U^*(x,\,y) := \begin{cases} -\dfrac{1}{2\pi} \log|x-y| & \text{for } d = 2\,, \\ \dfrac{1}{4\pi}\,\dfrac{1}{|x-y|} & \text{for } d = 3\,, \end{cases} \tag{4}$$

defines the usual boundary integral operators, namely the single layer potential operator $V_i : H^{-1/2}(\Gamma_i) \to H^{1/2}(\Gamma_i)$, the double layer potential operator $K_i : H^{1/2}(\Gamma_i) \to H^{1/2}(\Gamma_i)$, and the hypersingular integral operator $D_i : H^{1/2}(\Gamma_i) \to H^{-1/2}(\Gamma_i)$ by the relations

$$\begin{aligned} (V_i\, t)(x) &:= \int_{\Gamma_i} U^*(x,\,y)\, t(y)\, ds_y, \\ (K_i\, u)(x) &:= \int_{\Gamma_i} \frac{\partial}{\partial n_i(y)} U^*(x,\,y)\, u(y)\, ds_y, \text{ and} \\ (D_i\, u)(x) &:= -\frac{\partial}{\partial n_i(x)} \int_{\Gamma} \frac{\partial}{\partial n_i(y)} U^*(x,\,y)\, u(y)\, ds_y, \end{aligned} \tag{5}$$

respectively, where $x \in \Gamma_i$ and n_i denotes the outer unit normal vector on Γ_i. The adjoint double layer potential $K_i^\top : H^{-1/2}(\Gamma_i) \to H^{-1/2}(\Gamma_i)$ is given by $(K^\top t)(x) = \int_\Gamma \frac{\partial}{\partial n_i(x)} U^*(x,\,y)\, t(y)\, ds_y$. In two dimensions, we require the scaling assumption diam $\Omega_i < 1$ such that the single layer potential operator is coercive, see, e. g., [33]

We consider quasi-uniform triangulations $\mathcal{T}_i$ of the local boundaries Γ_i by line segments or affine triangles T with the mesh parameter h_i. We define the spaces of linear functions

$$\begin{aligned} V^h(\Gamma_i) &:= \{v \in H^{1/2}(\Gamma_i) : v|_T \text{ is affine linear on } T \quad \forall T \in \mathcal{T}_i\}, \\ V_0^h(\Gamma_i) &:= \{v \in V^h(\Gamma_i) : v|_\Gamma = 0\}\,. \end{aligned} \tag{6}$$

Moreover, we define the space of piecewise constant functions

$$Z_h(\Gamma_i) := \{v \in H^{-1/2}(\Gamma_i) : v|_T = \text{const} \quad \forall T \in \mathcal{T}_i\}\,. \tag{7}$$

With the standard bases $\{\varphi_k^{(i)}\}_k$ of $V^h(\Gamma_i)$ and $\{\psi_k^{(i)}\}$ of $Z_h(\Gamma_i)$, we define the boundary element matrices $\mathbf{V}_{C,i}$, $\mathbf{K}_{C,i}$, and $\mathbf{D}_{C,i}$ which correspond to the Galerkin approximations of the boundary integral operators V_i, K_i, and D_i respectively, and the mass matrix $\mathbf{M}_{C,i}$ by the Galerkin relations

$$\mathbf{V}_{C,i}[k,\,l] = \langle \psi_k,\, V_i\,\psi_l \rangle\,, \qquad \mathbf{K}_{C,i}[k,\,l] = \langle \psi_l,\, K_i\,\varphi_k \rangle\,, \tag{8}$$

$$\mathbf{D}_{C,i}[k,\,l] = \langle D_i\,\varphi_k,\, \varphi_k \rangle\,, \qquad \mathbf{M}_{C,i}[k,\,l] = \langle \psi_l,\, \varphi_k \rangle\,, \tag{9}$$

where $\langle\cdot,\,\cdot\rangle$ is the $H^{-1/2}$-$H^{1/2}$-duality pairing with the pivot space $L_2(\Gamma_i)$, see [24] for details. The index "C" stands for the coupling boundary Γ_C that coincides with Γ_S in the AF-BETI-FETI-case. The BE Schur complement

$$\mathbf{S}_{C,i}^{BE} := \alpha_i \mathbf{D}_{C,i} + \alpha_i \big(\tfrac{1}{2}\mathbf{M}_{C,i}^\top + \mathbf{K}_{C,i}^\top\big)\big[\mathbf{V}_{C,i}\big]^{-1}\big(\tfrac{1}{2}\mathbf{M}_{C,i} + \mathbf{K}_{C,i}\big)\,, \tag{10}$$

or, more precisely, the corresponding operator $S_{i,\,h} : V^h(\Gamma_i) \to V^h(\Gamma_i)^*$ appears as a symmetric BE approximation of the Steklov-Poincaré operator S_i. This is not the Galerkin approximation, but an outer approximation, see [25] for the approximation and spectral properties.

The BE matrices as defined above are fully populated. In order to save memory and to be able to perform fast matrix-by-vector operations, we replace the BE matrices $\mathbf{V}_{C,i}$, $\mathbf{K}_{C,i}$ and $\mathbf{D}_{C,i}$ by data-sparse approximations $\widetilde{\mathbf{V}}_{C,i}$, $\widetilde{\mathbf{K}}_{C,i}$ and $\widetilde{\mathbf{D}}_{C,i}$ without perturbing the approximation and the spectral properties. Such data-sparse approximation can be realized by different methods such as the FMM [30, 5], the panel clustering method [15], the $\mathcal{H}$-matrix technology [12, 13], and the ACA [1, 3], see also the book [29]. These methods allow us to reduce the memory demand and the arithmetical cost for one matrix-by-vector multiplication from $O((H/h)^{2(d-1)})$ to $O((H/h)^{d-1})$ up to some polylogarithmic factor.

Replacing the BE matrices in $\mathbf{S}_{C,i}^{BE}$ by their data-sparse representations, we arrive at an approximate BE Schur complement

$$\widetilde{\mathbf{S}}_{C,i}^{BE} := \alpha_i \widetilde{\mathbf{D}}_{C,i} + \alpha_i \big(\tfrac{1}{2}\mathbf{M}_{C,i}^\top + \widetilde{\mathbf{K}}_{C,i}^\top\big)\big[\widetilde{\mathbf{V}}_{C,i}\big]^{-1}\big(\tfrac{1}{2}\mathbf{M}_{C,i} + \widetilde{\mathbf{K}}_{C,i}\big)\,, \tag{11}$$

It is also possible to approximate $\mathbf{S}_{C,i}^{BE}$ by a data-sparse approximation $\widetilde{\mathbf{S}}_{C,i}^{BE}$ replacing $\mathbf{V}_{C,i}^{-1}$ by $\widetilde{\mathbf{V}_{C,i}^{-1}}$ or by an $\mathcal{H}-LU$ factorization of $\widetilde{\mathbf{V}}_{C,i}$, [2], see also [14]. However, in our BETI-FETI-3 solver we make not use of data-sparse approximations of the BE Schur-complement, but only of the BE matrices $\mathbf{V}_{C,i}$, $\mathbf{K}_{C,i}$, and $\mathbf{D}_{C,i}$, cf. Section 3.

2.3 Interface-Concentrated Finite Element Approximation

The boundary-concentrated finite element method was introduced in [17], see also [16] for solvers and [8] for some error analysis. This technique was used in [4] for developing fast DD solvers. In the following we summarize the main aspects of the interface concentrated FEM in our context.

We consided γ-shape-regular triangulations $\mathcal{T}_i$ of the FEM subdomains Ω_i, $i = q+1, \ldots, p$, into affine triangles or tetrahedra K, i. e., each element K is the image $F_K(\widehat{K})$ of a reference simplex $\widehat{K}$ satisfying the inequality

$$h_K^{-1} \|F_K'\|_{L_\infty(K)} + h_K \|(F_K')^{-1}\|_{L_\infty(K)} \leq \gamma \qquad \forall K \in \mathcal{T}_i \,, \tag{12}$$

where $h_K := \operatorname{diam} K$ denotes the local mesh size. We assume that the resulting triangulation of the local boundary Γ_i is quasi uniform and denote the minimal mesh size on the boundary by h_i. We assume that the triangulation of Ω_i forms a geometric mesh. A mesh is called a geometric mesh if there exist positive constants c_1 and c_2 such that all elements $K \in \mathcal{T}_i$ fulfill the following properties:

1. if $\overline{K} \cap \Gamma_i \neq \emptyset$, then $h \leq h_k \leq c_2\, h$,
2. if $\overline{K} \cap \Gamma_i = \emptyset$, then $c_1 \inf_{x \in K} \operatorname{dist}(x,\, \Gamma_i) \;\leq\; h_K \;\leq\; c_2 \sup_{x\in K} \operatorname{dist}(x,\, \Gamma_i)$.

Figure 1 provides an example of a geometric mesh that is used in our numerical experiments presented in Section 4. In order to define the finite element spaces on Ω_i we associate to each element $K \in \mathcal{T}_i$ a polynomial degree $p_K \in \mathbb{N}$ and collect these numbers in the polynomial degree vector $\mathbf{p}^{(i)} := \{p_K : K \in \mathcal{T}_i\}$. Furthermore, for each edge e and face f of $\mathcal{T}_i$ we define

$$\begin{aligned} p_e &:= \min\{p_K : e \text{ is an edge of } K \in \mathcal{T}_i\}\,, \\ p_f &:= \min\{p_K : f \text{ is a face of } K \in \mathcal{T}_i\}\,, \end{aligned} \tag{13}$$

respectively. For an element K, we collect the degrees corresponding to K in a vector $\mathbf{p}_K$ defined by

$$\mathbf{p}_K := (p_K,\, \{p_e\}_{e:\text{edge of } K},\, \{p_f\}_{f:\text{face of } K})\,, \tag{14}$$

where, in two dimensions, we have to drop the face degrees.

For our geometric mesh $\mathcal{T}_i$ we assume that the vectors $\mathbf{p}^{(i)}$ are linear degree vectors, that means, for a fixed parameter $\alpha_i > 0$, called the slope of the linear degree vector $\mathbf{p}^{(i)}$, we choose the p_K for each element $K \in \mathcal{T}_i$ such that

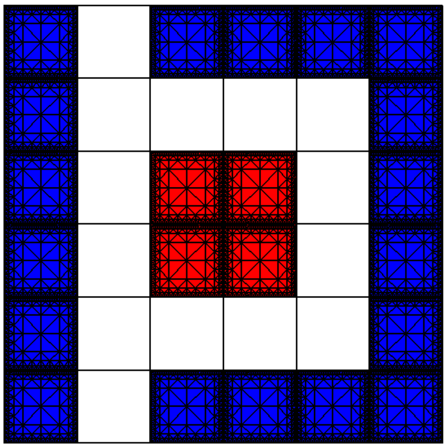

Fig. 1. Geometric meshes in the FE subdomains for our numerical example (magnetic valve) from Section 4.

$$1 + \alpha_i \, c_1 \, \log\big(\frac{h_K}{h_i}\big) \le p_K \le 1 + \alpha_i \, c_2 \, \log\big(\frac{h_K}{h_i}\big) \,,$$

for some positive constants c_1 and c_2. In practical applications, one usually chooses $\alpha_i = 1$. In particular, we have degree 1 on the elements touching the boundary.

The finite element spaces $V^h(\Omega_i)$ and $V_0^h(\Omega_i)$ are defined as follows:

$$\begin{aligned} V^h(\Omega_i) &:= \{v \in H^1(\Omega_i) : v \circ F_K \in \mathcal{P}_{\mathbf{p}_K}(\widehat{K}) \quad \forall K \in \mathcal{T}_i\} \,, \\ V_0^h(\Omega_i) &:= \{v \in V^h(\Omega_i) : v|_\Gamma = 0\} \,, \\ V^h(\Gamma_i) &:= \{v|_{\Gamma_i} : v \in V^h(\Omega_i)\} \,, \\ V_0^h(\Gamma_i) &:= \{v|_{\Gamma_i} : v \in V_0^h(\Omega_i)\} \,, \end{aligned} \tag{15}$$

where $\mathcal{P}_{\mathbf{p}_K}(\widehat{K})$ is the space of polynomials on the reference simplex $\widehat{K}$ with the polynomial degrees on the element, edges and faces given according to $\mathbf{p}_K$.

For each of the FE subdomains Ω_i, $i = q+1, \ldots, p$, we denote the local FE stiffness matrix corresponding to the bilinear form

$$a_i(u, v) = \int_{\Omega_i} a_i(x) \, \nabla u(x) \cdot \nabla v(x) \, dx$$

by $\mathbf{K}_i$ and group it with respect to the degrees of freedom on the (coupling) boundary (subscript C) and the remaining (interior) ones (subscript I),

$$\mathbf{K}_i = \begin{pmatrix} \mathbf{K}_{II}^{(i)} & (\mathbf{K}_{CI}^{(i)})^\top \\ \mathbf{K}_{CI}^{(i)} & \mathbf{K}_{CC}^{(i)} \end{pmatrix} . \tag{16}$$

We can now define the Schur complement matrix and the Newton potential matrix by

$$\mathbf{S}_{C,i}^{FE} := \mathbf{K}_{CC}^{(i)} - \mathbf{K}_{CI}^{(i)} \big[\mathbf{K}_{II}^{(i)}\big]^{-1} (\mathbf{K}_{CI}^{(i)})^\top \text{ and} \tag{17}$$

$$\mathbf{N}_{C,i}^{FE} := \Big[\mathbf{I}_C \,\Big|\, -\mathbf{K}_{CI}^{(i)} \big[\mathbf{K}_{II}^{(i)}\big]^{-1}\Big], \tag{18}$$

respectively, i. e., $\mathbf{N}_{C,i}^{FE}\,\mathbf{f} = \mathbf{f}_C - \mathbf{K}_{CI}^{(i)}[\mathbf{K}_{II}^{(i)}]^{-1}\mathbf{f}_I$.

It was shown in [17] that the boundary-concentrated FEM yields the same discretization error estimates as the standard FEM with linear finite elements on a uniformly refined mesh, but with an amount of unknowns (DOF) that is proportional to the number of grid points on the boundary, i. e., proportional to $(H_i/h_i)^{d-1}$ in our case. The same complexity holds for storage needed for the FE stiffness matrices $\mathbf{K}_i$ and for the cost for one matrix-by-vector multiplication.

2.4 Coupled System

Using now the data-sparse BE approximation $\widetilde{\mathbf{S}}_{C,i}^{BE}$ of the Steklov-Poincaré S_i in the BE subdomains and the IC FE approximations $\mathbf{S}_{C,i}^{FE}$ of S_i and $\mathbf{N}_{C,i}^{FE}$ of N_i and taking into account the Dirichlet boundary conditions (we assume for simplicity that g is a piecewise linear function), we can immediately derive the following linear system from the skeleton variational formulation (3):

$$\sum_{i=1}^{q} \mathbf{R}_{i,0}^\top \widetilde{\mathbf{S}}_{C,i}^{BE} \mathbf{R}_i \mathbf{u}_C + \sum_{i=q+1}^{p} \mathbf{R}_{i,0}^\top \mathbf{S}_{C,i}^{FE} \mathbf{R}_i \mathbf{u}_C = \sum_{i=q+1}^{p} \mathbf{R}_{i,0}^\top \mathbf{N}_i^{FE} (\mathbf{f}_{C,i}^\top, \mathbf{f}_{I,i}^\top)^\top \tag{19}$$

together with the Dirichlet condition $\mathbf{R}_D \mathbf{u}_C = \mathbf{g}_D$, where the vector $\mathbf{u}_C$ contains all nodal parameters living on the skeleton Γ_S including the (Dirichlet) boundary Γ, the restriction operator (sometimes also called subdomain connectivity matrix) $\mathbf{R}_i$ maps some vector $\mathbf{u}_C$ to $\mathbf{u}_{C,i}$, $\mathbf{R}_D$ maps some vector $\mathbf{u}_C$ to the vector containing only the values in the Dirichlet nodal points, the vector $\mathbf{g}_D$ contains the given Dirichlet data, and, finally, $\mathbf{R}_{i,0}$ coincides with $\mathbf{R}_i$ with exception of the columns the column index of which belongs to Dirichlet nodes. These columns have only zero entries. After homogenization of the Dirichlet boundary conditions $\mathbf{R}_D \mathbf{u}_C = \mathbf{g}_D$, system (19) is nothing but the primal coupled BE-FE Schur complement system. The article [4] discusses asymptotically optimal iterative methods for solving the primal coupled IC-FE (i. e., $q = 0$) Schur complement system or, more precisely, the rolled up system.

We mention that for the solution of the coupled data-sparse boundary and interface-concentrated finite element system (19) the same discretization error estimates hold as for the standard coupled BE-FE system [25].

In the next section, we apply the (all-floating) tearing and interconnecting technique for solving (19) following the BETI-FETI solvers proposed in [24] and [25] for standard coupled BE-FE systems.

3 All-Floating BETI – FETI Methods

3.1 Coupled BETI – FETI systems

Following [10] (see also [36]) and taking the all-floating idea into account [26, 27] (see also [7] for a closely related approach called total FETI), we tear the global skeleton potential vector $\mathbf{u}_C$ on all subdomain boundaries Γ_i including the Dirichlet parts by introducing the individual local unknowns $\mathbf{u}_{C,i} = \mathbf{R}_i \mathbf{u}_C$. The global continuity of the potentials on $\Gamma_i \setminus \Gamma$ and the Dirichlet boundary conditions on $\Gamma_D = \Gamma$ are now enforced by the constraints

$$\sum_{i=1}^{p} \mathbf{B}_{C,i} \mathbf{u}_{C,i} = \mathbf{g} \tag{20}$$

interconnecting the local potential vectors across the subdomain boundaries and the nodal parameters on the boundary Γ_D with the corresponding Dirichlet data. Each row of the matrix $\mathbf{B} = (\mathbf{B}_{C,1}, \ldots, \mathbf{B}_{C,p})$ is connected with a pair of matching nodes across the subdomain boundaries or with a Dirichlet node. The entries of the former rows are 1 and -1 for the indices corresponding to the matching nodes on the interface $\Gamma_S \setminus \Gamma_D$ and 0 otherwise, whereas an entry corresponding to a Dirichlet node on Γ_D is 1 and again 0 otherwise. The entries of the vector $\mathbf{g}$ coincides with the entries of $\mathbf{g}_D$ at the Dirichlet nodes and are 0 otherwise. Therefore, (20) implies that the corresponding finite element functions $u_{i,h}$ are continuous across the interface Γ_C, i. e., $u_{i,h} = u_{j,h}$ on $\Gamma_i \cap \Gamma_j \neq \emptyset$, and coincides with g on $\Gamma_i \cap \Gamma_D$. We assume here that the number of constraints at some matching node is equal to the number of matching subdomains minus one. This method resulting in a minimal number of Lagrange multipliers is called non–redundant, see, e. g., [18] or [36] for the use of redundant constraints. The latter case is even more often used in the implementation than the non–redundant.

By introducing Lagrange multipliers $\lambda \in \mathbb{R}^L$, the linear system (19) is obviously equivalent to the following extended saddle point system

$$\begin{pmatrix} \mathbf{S}_{C,1} & & & \mathbf{B}_{C,1}^\top \\ & \ddots & & \vdots \\ & & \mathbf{S}_{C,p} & \mathbf{B}_{C,s}^\top \\ \mathbf{B}_{C,1} & \ldots & \mathbf{B}_{C,p} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{u}_{C,1} \\ \vdots \\ \mathbf{u}_{C,p} \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathbf{b}_{C,1} \\ \vdots \\ \mathbf{b}_{C,p} \\ \mathbf{g} \end{pmatrix}, \tag{21}$$

with the singular Schur complements

$$\mathbf{S}_{C,i} = \begin{cases} \widetilde{\mathbf{S}}_{C,i}^{BE} & \text{for } i = 1, \ldots, q, \\ \mathbf{S}_{C,i}^{FE} & \text{for } i = q+1, \ldots, p, \end{cases} \tag{22}$$

with $(\ker(\mathbf{S}_{C,i}) = \operatorname{span}\{\mathbf{1}_{C,i}\}$ for all $i = 1, \ldots, p)$, and the corresponding right-hand sides

$$\mathbf{b}_{C,i} = \begin{cases} \mathbf{0} & \text{for } i = 1, \ldots, q, \\ \mathbf{f}_{C,i} - \mathbf{K}_{CI,i}\mathbf{K}_{II,i}^{-1}\mathbf{f}_{I,i} & \text{for } i = q+1, \ldots, p, \end{cases} \tag{23}$$

It is clear that there exists a unique solution of system (21). We note that in the redundant case the Lagrange parameter λ is only unique up to $\ker \mathbf{B_C}^\top$. However, the solution $\mathbf{u}_C$ is unique, and the redundancy does not result in any implementational difficulties, cf. [36] or [28].

Following [25], we can transform saddle point system (21) to the equivalent saddle point system (BETI-FETI-2 system)

$$\begin{pmatrix} \overline{\mathbf{S}}_{C,1} & & & \overline{\mathbf{B}}_{C,1}^\top \\ & \ddots & & \vdots \\ & & \overline{\mathbf{S}}_{C,p} & \overline{\mathbf{B}}_{C,p}^\top \\ \overline{\mathbf{B}}_{C,1} & \ldots & \overline{\mathbf{B}}_{C,p} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{v}_{C,1} \\ \vdots \\ \mathbf{v}_{C,p} \\ \lambda_0 \end{pmatrix} = \begin{pmatrix} \mathbf{d}_{C,1} \\ \vdots \\ \mathbf{d}_{C,p} \\ \mathbf{d} \end{pmatrix}, \tag{24}$$

with symmetric and positive definite (SPD) diagonal blocks

$$\overline{\mathbf{S}}_{C,i} = \mathbf{S}_{C,i} + \beta_i \mathbf{1}_{C,i}\mathbf{1}_{C,i}^\top, \quad i = 1, \ldots, p, \tag{25}$$

and with the block matrices

$$\overline{\mathbf{B}}_{C,i} = \mathbf{L}_0^\top \mathbf{P}^\top \mathbf{B}_{C,i}, \quad i = 1, \ldots, p, \tag{26}$$

where β_i is some positive regularization parameter. The matrix $\mathbf{P}$ denotes the well-studied FETI projector $\mathbf{I} - \mathbf{Q}\mathbf{G}(\mathbf{G}^\top\mathbf{Q}\mathbf{G})^{-1}\mathbf{G}^\top$, where

$$\mathbf{G} = (\mathbf{B}_{C,1}\mathbf{1}_{C,1}, \ldots, \mathbf{B}_{C,p}\mathbf{1}_{C,p})$$

and $\mathbf{Q}$ is a diagonal matrix which takes care of the coefficient jumps (see, e. g., [36]). We only mention that, for the FE subdomains Ω_i, $i = q+1, \ldots, p$, where the coefficient $a(\cdot)$ is slightly varying, we use the upper bounds $\overline{a}_i$ for the entries in Q. The matrix $\mathbf{L}_0$ is defined by the representation of λ in the form

$$\lambda = \mathbf{L}_0\lambda_0 + \lambda_e \tag{27}$$

with known $\lambda_e = \mathbf{Q}\mathbf{G}(\mathbf{G}^\top\mathbf{Q}\mathbf{G})^{-1}\mathbf{e}$ fulfilling the constraints $\mathbf{G}^\top\lambda_e = \mathbf{e}$, and unknown $\mathbf{L}_0\lambda_0 \in \ker \mathbf{G}^\top$, i. e., $\mathbf{G}^\top\mathbf{L}_0\lambda_0 = \underline{0}$. The right-hand side of BETI-FETI-2 system (24) is defined by the relations $\mathbf{d}_{C,i} = \mathbf{b}_{C,i} - \mathbf{B}_{C,i}^\top\lambda_e$ for $i = 1, \ldots, p$ and $\mathbf{d} = \mathbf{L}_0^\top\mathbf{P}^\top\mathbf{g}$.

Once the vectors $\mathbf{v}_{C,1}, \ldots, \mathbf{v}_{C,p}$ and λ_0 are defined from (24), we get λ from (27) and $\mathbf{u}_{C,1}, \ldots, \mathbf{u}_{C,p}$ from the relations

$$\mathbf{u}_{C,i} = \mathbf{v}_{C,i} + \gamma_i\mathbf{1}_{C,i} \quad i = 1, \ldots, p, \tag{28}$$

with $\gamma = (\gamma_i)_{i=1,\ldots,p} = (\mathbf{G}^\top\mathbf{Q}\mathbf{G})^{-1}\mathbf{G}^\top\mathbf{Q}(\mathbf{g} - \mathbf{B}_{C,1}\mathbf{v}_{C,1} - \ldots - \mathbf{B}_{C,p}\mathbf{v}_{C,p})$.

On the one hand, eliminating the unknowns $\mathbf{v}_{C,1},\dots,\mathbf{v}_{C,p}$ from the BETI-FETI-2 system (24), we arrive at the SPD BETI-FETI Schur complement system (BETI-FETI-1 system)

$$\mathbf{L}_0^\top\mathbf{P}^\top\mathbf{F}\mathbf{P}\mathbf{L}_0\lambda_0 = \mathbf{L}_0^\top\mathbf{P}^\top\mathbf{b} \tag{29}$$

with the BETI-FETI matrix $\mathbf{F}$ and the right-hand side $\mathbf{b}$

$$\mathbf{F} = \sum_{i=1}^{p}\mathbf{B}_{C,i}\overline{\mathbf{S}}_{C,i}^{-1}\mathbf{B}_{C,i}^\top \quad \text{and} \quad \mathbf{b} = \sum_{i=1}^{p}\mathbf{B}_{C,i}\overline{\mathbf{S}}_{C,i}^{-1}\mathbf{d}_{C,i} - \mathbf{g}, \tag{30}$$

respectively. In the redundant case the Lagrange parameter λ_0 is only unique up to $\ker\mathbf{B_C}^\top$.

On the other hand, unfolding the Schur complements $\mathbf{S}_{C,i}$ on the block diagonal of the BETI-FETI-2 system (24), we get the larger saddle point system

$$\begin{pmatrix} \overline{\mathbf{K}}_1 & & & \overline{\mathbf{B}}_1^\top \\ & \ddots & & \vdots \\ & & \overline{\mathbf{K}}_p & \overline{\mathbf{B}}_p^\top \\ \overline{\mathbf{B}}_1 & \dots & \overline{\mathbf{B}}_p & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_p \\ \lambda_0 \end{pmatrix} = \begin{pmatrix} \mathbf{f}_1 \\ \vdots \\ \mathbf{f}_p \\ \mathbf{d} \end{pmatrix}, \tag{31}$$

that is also called BETI-FETI-3 system, with the regular, symmetric, but indefinite BE (subdomain) matrices

$$\overline{\mathbf{K}}_i = \alpha_i \begin{pmatrix} \widetilde{\mathbf{V}}_{C,i} & -\big(\frac{1}{2}\mathbf{M}_{C,i}^\top + \widetilde{\mathbf{K}}_{C,i}^\top\big) \\ -\big(\frac{1}{2}\mathbf{M}_{C,i}^\top + \widetilde{\mathbf{K}}_{C,i}^\top\big) & -(\widetilde{\mathbf{D}}_{C,i} + \beta_i\mathbf{1}_{C,i}\mathbf{1}_{C,i}^\top) \end{pmatrix} \tag{32}$$

for $i = 1,\dots,q$ and the SPD FE (subdomain) stiffness matrices

$$\overline{\mathbf{K}}_i = \begin{pmatrix} \mathbf{K}_{II}^{(i)} & (\mathbf{K}_{CI}^{(i)})^\top \\ \mathbf{K}_{CI}^{(i)} & \overline{\mathbf{K}}_{CC}^{(i)} \end{pmatrix}, \text{ with } \overline{\mathbf{K}}_{CC}^{(i)} = \mathbf{K}_{CC}^{(i)} + \beta_i\mathbf{1}_{C,i}\mathbf{1}_{C,i}^\top, \tag{33}$$

for $i = q+1,\dots,p$. The matrices $\overline{\mathbf{B}}_i$ are nothing but the extension to interior indices by zero columns, i. e., $\overline{\mathbf{B}}_i = (\overline{\mathbf{B}}_{C,i}\ 0)$ for $i = 1,\dots,p$. The unknown vectors $\mathbf{u}_i$ and the given right-hand side vectors $\mathbf{f}_i$ are defined as follows:

$$\mathbf{u}_i = \begin{pmatrix} \mathbf{t}_{C,i} \\ \mathbf{v}_{C,i} \end{pmatrix}, \quad \mathbf{f}_i = \begin{pmatrix} \mathbf{0} \\ -\mathbf{B}_{C,i}^\top\lambda_e \end{pmatrix} \quad \text{for } i = 1,\dots,q,$$

and

$$\mathbf{u}_i = \begin{pmatrix} \mathbf{u}_{I,i} \\ \mathbf{v}_{C,i} \end{pmatrix}, \quad \mathbf{f}_i = \begin{pmatrix} \mathbf{f}_{I,i} \\ \mathbf{f}_{C,i} - \mathbf{B}_{C,i}^\top\lambda_e \end{pmatrix} \quad \text{for } i = q+1,\dots,p.$$

If $q = 0$ (i. e., no BE subdomains), then system (31) is an one-fold saddle point system that was treated in [4]. In the case of a pure BETI-3 system (i. e., $q = p$), we get a two-fold saddle point system that was studied in [19]. In the next subsection we focus on the mixed case of the BETI-FETI-3 system (31).

3.2 BETI–FETI Preconditioners and Solvers

The system matrix $\mathbf{L}_0^\top \mathbf{P}^\top \mathbf{F}\mathbf{P}\mathbf{L}_0$ of the BETI-FETI-1 system (29) is SPD. The Preconditioned Conjugate Gradient (PCG) method is a very efficient solver for ill-conditioned SPD systems provided that an appropriate preconditioner is available. If we start the PCG iteration with λ_e, then it is clear that we can perform the PCG iteration on the subspace range $\mathbf{P} = \ker \mathbf{G}^\top = \operatorname{range} \mathbf{L}_0$, see Algorithm 1 in [23] or Fig. 6.4 in [36]. Thus, we need a good preconditioner $\mathbf{C}$ that is spectrally equivalent to $\mathbf{F}$ on the subspace $\ker \mathbf{G}^\top$. At least, two efficient BETI-FETI preconditioners $\mathbf{C}$ are available:

- the scaled Dirichlet BETI-FETI preconditioner (see, e. g., [36]):
$$\mathbf{C}^{-1} = \mathbf{A}\overline{\mathbf{S}}_C\mathbf{A}^\top, \tag{34}$$
with $\overline{\mathbf{S}}_C = \operatorname{diag}(\overline{\mathbf{S}}_{C,i})$, see (25), and
- the scaled data-sparse hypersingular preconditioner [23]:
$$\mathbf{C}^{-1} = \mathbf{A}\overline{\mathbf{D}}_C\mathbf{A}^\top, \tag{35}$$
with $\overline{\mathbf{D}}_C = \operatorname{diag}(\overline{\mathbf{D}}_{C,i}) = \operatorname{diag}(\widetilde{\mathbf{D}}_{C,i} + \beta_i \mathbf{1}_{C,i}\mathbf{1}_{C,i}^\top)$.

In the redundant case the matrix $\mathbf{A} = (\mathbf{W}_{C,1}\mathbf{B}_{C,1}, \ldots, \mathbf{W}_{C,p}\mathbf{B}_{C,p})$ is defined by the interconnecting matrix $\mathbf{B} = \operatorname{diag}(\mathbf{B}_{C,i})$ and by the scaling or weighting matrix $\mathbf{W} = \operatorname{diag}(\mathbf{W}_{C,i})$ with appropriate diagonal matrices $\mathbf{W}_{C,i}$ whose entries depend on the coefficients, see [18] and [36] for further details. For the non-redundant case, a similar matrix $\mathbf{A}$ can be constructed [18, 36], see also [26] for implementational details. We mention that the scaled data-sparse hypersingular preconditioner (35) uses the data-sparse hypersingular blocks $\overline{\mathbf{D}}_{C,i}$ not only in the BE subdomains but also in the FE subdomains.

Theorem 1. *Let $\mathbf{C}$ be one of the BETI-FETI preconditioners defined by (34) or (35). Then there exist positive constants $\underline{c}$ and $\overline{c}$ such that the spectral inequalities*
$$\underline{c}\,(\mathbf{C}\underline{\lambda}, \underline{\lambda}) \le (\mathbf{F}\underline{\lambda}, \underline{\lambda}) \le \overline{c}\,(1 + \log(H/h))^2\,(\mathbf{C}\underline{\lambda}, \underline{\lambda}) \tag{36}$$
hold for all $\underline{\lambda} \in \ker \mathbf{G}^\top$, where the constant $\underline{c}$ and $\overline{c}$ are independent of h_i, H_i, and the coefficient jumps across the interfaces, and $H/h = \max H_i/h_i$.

Proof. The matrices $\overline{\mathbf{S}}_{C,i}$ and $\overline{\mathbf{D}}_{C,i}$ are spectrally equivalent to an auxiliary Schur complement $\overline{\mathbf{S}}_{C,i}^{auxFE}$ arising from a standard finite element discretization with linear triangular or tetrahedral elements on an auxiliary quasi-uniform triangular or tetrahedral mesh generated from the given quasi-uniform boundary mesh on $\partial\Omega_i$ with the average mesh size h_i in both the FE- and BE-subdomains, see [23] and [4]. The spectral inequalities (36) were proved for $\overline{\mathbf{S}}_{C,i}^{auxFE}$ (see [18] or [36] for the standard FETI method, see [28] for details on the all-floating method), they are also valid if we replace $\overline{\mathbf{S}}_{C,i}^{auxFE}$ in $\mathbf{F}$ and $\mathbf{C}^{-1}$ by the spectrally equivalent matrices $\overline{\mathbf{S}}_{C,i}$ or $\overline{\mathbf{D}}_{C,i}$. This completes the proof of the theorem.

Theorem 1 immediately implies that we need at most $I(\varepsilon) = O((1 + \log(H/h)) \log \varepsilon^{-1})$ PCG iterations in order to reduce the initial error by the factor $\varepsilon \in (0,1)$. The complexity of the one PCG iteration depends on the complexity of the preconditioning operation $\mathbf{C}^{-1}\rho$ and the costs for the matrix-by-vector multiplication $\mathbf{F}\lambda$. If we use the scaled data-sparse hypersingular preconditioner (35), then the cost for the preconditioning operation is proportional to $(H/h)^{d-1}$ up to a polylogarithmic factor in a parallel regime. In principle, this complexity estimate can also be obtained for the other operations if we use special direct solvers (see [2] and [16] for BEM and IC-FEM, respectively) for the systems with the regularized Neumann matrices (32) and (33) arising in the matrix-by-vector multiplication $\mathbf{F}\lambda$, and for the systems with the system matrices $\widetilde{\mathbf{V}}_{C,i}$ and $\mathbf{K}_{II}^{(i)}$ arising in the scaled Dirichlet BETI-FETI preconditioner (34).

However, in the BETI-FETI-2 and in the BETI-FETI-3 cases, we avoid the inversions of the local Schur complements $\overline{\mathbf{S}}_{C,i}$, i. e., the solution of the corresponding local Neumann problems. The BETI-FETI-3 solver even avoids the multiplication with the local Schur complements $\overline{\mathbf{S}}_{C,i}$, i. e., the solution of the corresponding local Dirichlet problems.

Let us here only discuss the ingredients of the BETI-FETI-3 solver. The BETI-FETI-2 solver can directly benefit from these ingredients. In the case $q = p$ (only BE subdomains), system (31) is pure two-fold saddle point problem The BETI-3 solver was discussed and analyzed in [19]. In order to solve system (31) in the mixed case $1 < q < p$, we need preconditioners for the blocks $\widetilde{\mathbf{V}}_{C,i}$ $(i = 1,\ldots,q)$, $\overline{\mathbf{S}}_{C,i}$ $(i = 1,\ldots,q)$, $\overline{\mathbf{K}}_i$ $(i = q+1,\ldots,p)$, and for the BETI-FETI-1 matrix $\mathbf{F}$. We propose the following preconditioners for these blocks [19, 4]:

- Multigrid or multilevel preconditioners $\mathbf{C}_{C,i}^{V}$ for the data-sparse single layer potential matrices $\widetilde{\mathbf{V}}_{C,i}$ in the BE subdomain $(i = 1,\ldots,q)$ which are asymptotically optimal [22]. Algebraic version are also available [21, 32, 19].
- Opposite order preconditioners $\mathbf{C}_{C,i}^{S}$ for the BE Schur complement matrices $\overline{\mathbf{S}}_{C,i}$ $(i = 1,\ldots,q)$ which are also asymptotically optimal [34, 19].
- Asymptotically optimal preconditoners of the form

$$\mathbf{C_i} = \begin{pmatrix} \mathbf{I}_{I,i} & \mathbf{0} \\ -\mathbf{E}_{IC,i}^{\top} & \mathbf{I}_{C,i} \end{pmatrix} \begin{pmatrix} \mathbf{C}_{I,i} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_{C,i} \end{pmatrix} \begin{pmatrix} \mathbf{I}_{I,i} & -\mathbf{E}_{IC,i} \\ \mathbf{0} & \mathbf{I}_{C,i} \end{pmatrix} \tag{37}$$

  for the regularized IC-FE Neumann matrix $\overline{\mathbf{K}}_i$ in the FE subdomains $(i = q+1,\ldots,p)$ were proposed in [4], where $\mathbf{C}_{I,i}$ denotes the Additive Schwarz Preconditioner proposed in [4] for $\mathbf{K}_{II}^{(i)}$, $\mathbf{C}_{C,i}$ is an asymptotically optimal preconditioner for the local FE Schur complement matrices $\overline{\mathbf{S}}_{C,i}$, e. g., the BPX preconditioner [35], and $\mathbf{E}_{IC,i}$ is the matrix representation of the multilevel bounded extension operator proposed in [4].

- For $\mathcal{F}$ we propose to use the BETI-FETI preconditioner

$$\mathbf{C}^{-1} = (\mathbf{C}^F)^{-1} = \mathbf{A}\mathrm{diag}(\overline{\mathbf{D}}_{C,i})\mathbf{A}^\top \tag{38}$$

that uses the data-sparse hypersingular blocks $\overline{\mathbf{D}}_{C,i} = \widetilde{\mathbf{D}}_{C,i} + \beta_i \mathbf{1}_{C,i}\mathbf{1}_{C,i}^\top$ in the BE subdomains ($i = 1, \ldots, q$) and the inexact FE Schur complement

$$\overline{\mathbf{D}}_{C,i} = \overline{\mathbf{K}}_{CC}^{(i)} + \mathbf{E}_{IC,i}^\top \mathbf{K}_{II}^{(i)} \mathbf{E}_{IC,i} - \mathbf{K}_{CI}^{(i)} \mathbf{E}_{IC,i} - \mathbf{E}_{IC,i}^\top \mathbf{K}_{IC}^{(i)} \tag{39}$$

in the FE subdomains ($i = q+1, \ldots, p$). The inexact Schur complement (39) does not contain $(\mathbf{K}_{II}^{(i)})^{-1}$. The matrix-vector multiplication $\overline{\mathbf{D}}_{C,i}\overline{\mathbf{v}}_{C,i}$ costs $O((H_i/h_i)^{d-1})$ arithmetical operations and is, therefore, more efficient than the application of the hypersingular matrix performed by a Fast Multipole Method (FMM) where two additional logarithmic factors appear [19]. Since the inexact Schur complement (39) is spectrally equivalent to the exact Schur complement $\overline{\mathbf{S}}_{C,i}$ with spectral constants which are independent on h_i and H_i [4], the new BETI-FETI preconditioner $\mathbf{C}$ fulfills the spectral inequalities (36).

Now we can solve the BETI-FETI-3 system (31) by means of the generalized Bramble-Pasciak PCG studied in [19].

Theorem 2. *Let us assume that the BETI-FETI-3 system (31) is solved by means of the generalized Bramble-Pasciak PCG method with the appropriately scaled block preconditioners $\mathbf{C}_{C,i}^V$, $\mathbf{C}_{C,i}^S$, $\mathbf{C}_\mathbf{i}$ and $\mathbf{C}^F$ proposed above. Then not more than $I(\varepsilon) = O((1 + \log(H/h)) \log \varepsilon^{-1})$ iterations and $ops(\varepsilon) = O((H/h)^{d-1}(1+\log(H/h))^3 \log \varepsilon^{-1})$ arithmetical operations (provided that the FMM is used for the BE matrices) are required in order to reduce the initial error by the factor $\varepsilon \in (0, 1)$ in a parallel regime. The number of iterations $I(\varepsilon)$ is robust with respect to the jumps in the coefficients. Moreover, at most $O(H/h)(1 + \log(H/h))^2$ storage units are needed per subdomain (processor).*

Proof. The estimate for the number $I(\varepsilon)$ of iterations is a direct consequence of results given in [19] and of the spectral estimates for the block preconditioners. The complexity estimates for the arithmetical costs and the memory demand follow from the corresponding complexity estimates for the block matrices involved in the matrix-vector multiplications and in the preconditioning. This completes the proof of the theorem.

Remark 1. The results of Theorems 1 and 2 are proved for bounded domains Ω. In the case of an unbounded domain Ω, where the exterior subdomain $\overline{\Omega}_0$ has to be added to the domain decomposition (cf. Section 2.1), there are some specialities. A rigorous analysis of the unbounded case can be found in [28].

Remark 2. In general, 2D magnetic field problems lead to non-linear potential problems due to the non-linear dependence of the reluctivity ν from the induction. The Newton linearization results in potential problems of the form (1) where $a(\cdot)$ becomes a matrix-valued coefficient in the ferromagnetic materials. Typically, $a(\cdot)$ is smooth in the ferromagnetic subdomains, but shows large variation, especially along the material interfaces. In [20], we studied this case and proposed special scaling matricies $\mathbf{W}_{C,i}$ resulting in a robust behavior of the preconditioner with respect to these variations, at least in numerical experiments.

4 Numerical Experiments

A very simplified two-dimensional model of a magnetic valve (see Figure 2, left) leads to the potential equation $-\mathrm{div}\,(a\nabla u) = f$ in $\mathbb{R}^2$ with an appropriate radiation condition for $|x| \to \infty$. The reluctivity $a(.)$ is piecewise constant, but exhibits large jumps across the interface between ferromagnetic materials and air. More precisely, $a(.)$ is equal to ν_0, ν_0, $10^{-5} \cdot \nu_0$ and $10^{-3} \cdot \nu_0$ in the air, coil, armature and iron core, respectively, where $\nu_0 := \frac{1}{4\,\pi} \cdot 10^7$. The right-hand side f is concentrated in the coil and is nothing else but the current density. In our test case, we set $f = 2.3 \cdot 10^6$ in the coil and zero elsewhere.
The domain decomposition is shown in Figure 2 (right). Our computational domain is the entire space $\mathbb{R}^2$ where the exterior subdomain Ω_0 is an air subdomain and the interior subdomain Ω is decomposed into 14 air subdomain Ω_i $(i = 1, , \ldots, q = 14)$ and into 22 other subdomains Ω_i $(i = 15, , \ldots, p = 36)$ which are discretized by the IC-FEM. The triangulation of the FE subdomains by interface-concentrated (geometric) meshes is shown in Figure 1. All air subdomains including the exterior subdomain Ω_0 are discretized by the BEM. Figure 3 displays the potential field (left) and the norm of the B-field (right) in the FE subdomains and on the boundary of the BE subdomains. Using the representation formula, one could easily calculate both fields in any point of the BE subdomains.

The generation of the BE matrices is preformed by the software package *OSTBEM* [31]. Furthermore, we use some of the *hp*-FEM routines from the open source package *NGSolve* (`www.hpfem.jku.at`). On the finest level, we get polynomial basis function up to degree 7 in FE subdomains.

Table 1 provides some numerical features of our experiments. The first six columns give an overview over the DOF involved in the BETI-FETI methods. It can be seen from the second (also from the fifth) and the third columns that the local BEM DOF and the local IC-FEM DOF grow both like $O(H/h)$ whereas the standard FEM DOF grows like $O((H/h)^2)$ which is illustrated in the *fourth column* for comparison. The total number of Lagrange multipiers grows like h^{-1} (column 6). The logarithmic grow of the number of PCG iteration is confirmed by **column 7**. There is almost no change in the number of iteration if we would artificially change the jumps of the coefficients across the interfaces. We used the BETI-FETI-1 solver with the scaled Dirichlet BETI-FETI preconditioner (34). In the FE subdomains we used the direct solver *PARDISO* [6] whereas in the BE subdomains we used dense BE matrices and a *standard LU decomposition* using *LAPACK* for solving the local Neumann and Dirichlet problems. Of course, in order to reduce the complexity (CPU-time) in the BE subdomains one has to replace the dense matrices by data-sparse representations and the standard LU decomposition by an $\mathcal{H} - LU$ decomposition. In 3D, the use of data-sparse techniques is absolutely necessary. Moreover, at least for large-scale problems, we expect that the BETI-FETI-3 solver is more efficient than the BETI-FETI-1 and BETI-FETI-2 solvers (cf. also [19] for the pure BE case).

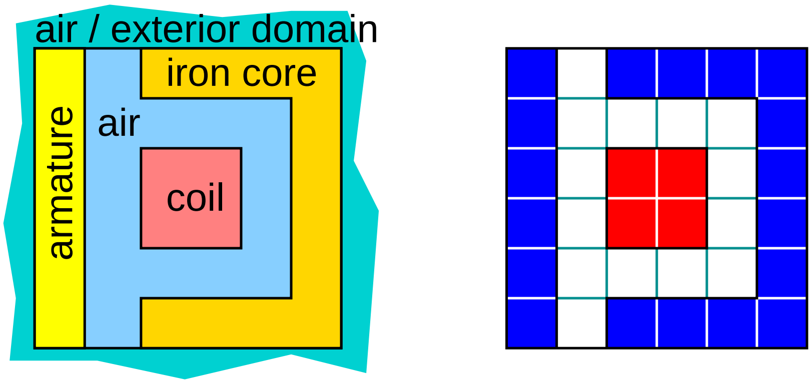

Fig. 2. The magnetic valve (left) and the domain decomposition (right).

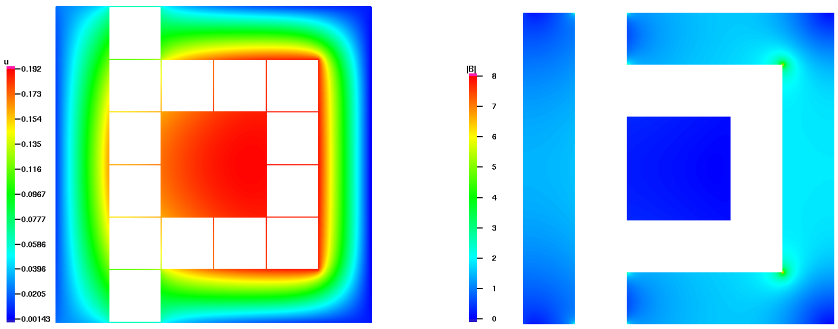

Fig. 3. The potential field (left) and the B-field (right).

Table 1. Numerical results BETI–IC-FETI-1: DOF, Number $I(\varepsilon)$ of PCG-iterations for $\varepsilon = 10^{-8}$, and CPU in seconds.

global DOF	local BEM DOF	local IC-FEM DOF	*cf. local FEM DOF*	external DOF	Lagrange DOF	PCG $I(\varepsilon)$	CPU [sec]
155	8	9	*9*	48	298	**9**	0.3
499	16	21	*25*	96	466	**10**	0.4
1715	32	61	*81*	192	802	**12**	0.9
6259	64	189	*289*	384	1474	**14**	2.8
23795	128	514	*1089*	768	2818	**16**	13.0
92659	256	1405	*4225*	1536	5506	**18**	63.9
365555	512	3373	*16641*	3072	11012	**20**	344.9
1452019	1024	7645	*66049*	6144	21634	**22**	1990.3

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