COMPUTING ONE OF VICTOR MOLL'S IRRESISTIBLE INTEGRALS WITH COMPUTER ALGEBRA

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ABSTRACT. We investigate a certain quartic integral from Victor Moll's book "Irresistible Integrals" and demonstrate how it can be solved by means of computer algebra methods, namely by using non-commutative Gröbner bases. We present recent implementations in the computer algebra systems SINGULAR and MATHEMATICA.

1. INTRODUCTION

The integral [1, (7.2.1)] which we deal with is

$$F(a,m) = \int_0^\infty \frac{1}{(x^4+2ax^2+1)^{m+1}}\,\mathrm{d}x. \tag{1}$$

From mathematical expert's view this integral might not look very challenging, and of course, Moll is able to compute its solution by hand. But nevertheless his computations are involved and need some quite special knowledge. From the software point of view both MAPLE and MATHEMATICA fail to evaluate (1) due to the presence of two parameters a, m (if they are set to concrete numbers the evaluation can be immediately done). We present computer algebra methods that allow to compute this integral in a purely automatic fashion with no expert's knowledge involved. The first approach is based on D-module theory whereas the second one follows Zeilberger's "holonomic systems approach". Our aim is to bring together these two directions since the underlying theoretical principles are identical. Moreover, we aim at a self-contained presentation of theory and algorithms.

2. PRELIMINARIES

Let $\mathbb{K}$ be a field. For the integration, we will need to deal with some special non-commutative algebras. It is common to define $\mathbb{K}$-algebras via generators and relations, especially if they have infinite dimension over $\mathbb{K}$. Let $\mathbf{X} = \{x_1,\dots,x_n\}$ be a finite set of symbols, then by $\mathbb{K}\langle\mathbf{X}\rangle$ one denotes a free associative algebra. Given a finite set $R = \{r_1(x),\dots,r_m(x)\} \subset \mathbb{K}\langle\mathbf{X}\rangle$, writing for an associative $\mathbb{K}$-algebra $A = \mathbb{K}\langle\mathbf{X} \mid R\rangle$ means $A \cong \mathbb{K}\langle\mathbf{X}\rangle/I_R$, where $I_R := \langle R\rangle$ is the two-sided ideal of $\mathbb{K}\langle\mathbf{X}\rangle$ generated by R. The elements of both R and I_R are often regarded as *relations* of A. This way of defining algebras has its roots in group theory, where a similar construction is performed. Since we are dealing with the algebras, which are in many sense close to commutative - in particular, each pair of variables is connected by some relation - we use shorter notation when writing the defining relations R. Namely, if we do not mention any relation between a pair of variables, these variables do commute.

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Given two algebras $A = \mathbb{K}\langle \mathbf{X} \rangle / I$ and $B = \mathbb{K}\langle \mathbf{Y} \rangle / J$, we identify $A \otimes_{\mathbb{K}} B$ with the algebra $\mathbb{K}\langle \mathbf{X}, \mathbf{Y} \mid I + J \rangle$, since in $A \otimes_{\mathbb{K}} B$ any element $a \otimes 1$ for $a \in A$ commutes with every element $1 \otimes b$ for $b \in B$.

In this article we deal with Weyl algebras, shift algebras and their tensor products over a field $\mathbb{K}$ of characteristic 0. Given a natural number $n \geq 1$ and a set of variables (also called *coordinates*) $\mathbf{X} = \{x_1, \ldots, x_n\}$, we construct first a commutative ring $R_n = \mathbb{K}[\mathbf{X}]$. We identify a polynomial $f \in R_n$ with the operator of multiplication by f. Given n natural operators $\partial_i := \partial_{x_i} = \frac{\partial}{\partial x_i}$ of partial differentiation with respect to the coordinate variable x_i, we define the algebra of linear partial differential operators with *polynomial* coefficients (also called the n-th Weyl algebra) to be

$$A_n := \mathbb{K}\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \mid \{\partial_j x_i = x_i \partial_j + \delta_{ij}{}^1 \ \forall 1 \leq i, j \leq n\} \rangle.$$

Note, that the action of an operator on a function from an appropriate function space will be denoted by $\bullet$, while $\cdot$ will be used for multiplication in operator algebras. Thus,

$$\partial_{x_i} \bullet f(x_1, \ldots, x_n) := \frac{\partial f(x_1, \ldots, x_n)}{\partial x_i}.$$

To each coordinate x_i we can also associate a partial shift operator s_i, which acts on a function $f(x_1, \ldots, x_i, \ldots, x_n)$ as

$$s_i \bullet f(x_1, \ldots, x_i, \ldots, x_n) := f(x_1, \ldots, x_i + 1, \ldots, x_n).$$

Given n such operators, we define the algebra of linear partial shift operators with *polynomial* coefficients (also called the n-th shift algebra) to be

$$S_n := \mathbb{K}\langle x_1, \ldots, x_n, s_1, \ldots, s_n \mid \{s_j x_i = x_i s_j + \delta_{ij} s_j \ \forall 1 \leq i, j \leq n\} \rangle.$$

Both A_n and S_n share many nice properties, for instance

- $\{x_1^{\alpha_1} \ldots x_n^{\alpha_n} \partial_1^{\beta_1} \ldots \partial_n^{\beta_n} \mid \alpha_i, \beta_i \in \mathbb{N}_0\}$ is a $\mathbb{K}$-basis for A_n,
- $\{x_1^{\alpha_1} \ldots x_n^{\alpha_n} s_1^{\beta_1} \ldots s_n^{\beta_n} \mid \alpha_i, \beta_i \in \mathbb{N}_0\}$ is a $\mathbb{K}$-basis for S_n,
- A_n and S_n are Noetherian domains (in particular, every module is finitely generated and there are no zero divisors),
- for any $i, j \in \mathbb{N}$, $A_{i+j} \cong A_i \otimes_{\mathbb{K}} A_j$ and $S_{i+j} \cong S_i \otimes_{\mathbb{K}} S_j$,
- there is a Gröbner basis theory for both types of algebras, very close to the theory in the commutative case, see e.g. [11, 8].

Picking some nice $\mathbb{K}$-basis for an algebra as above, we call these basis elements *monomials*. As one can see, the monomials are in one-to-one correspondence with their *exponent vectors*, say, $(\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n) \in \mathbb{N}^{2n}$. Hence, we can define a *monomial ordering* on A_n as follows (the cases of S_m and $A_n \otimes_{\mathbb{K}} S_m$ are completely analogous, see e.g. [8])

Definition 1. *A monomial ordering on A_n is a total ordering $\prec$ on the set of monomials, which satisfies for all $\alpha = (\alpha_x, \alpha_\partial), \beta = (\beta_x, \beta_\partial), \gamma = (\gamma_x, \gamma_\partial) \in \mathbb{N}^{2n}$*
(1) $\alpha \prec \beta \Rightarrow x^{\alpha_x} \partial^{\alpha_\partial} \prec x^{\beta_x} \partial^{\beta_\partial}$ *and*
(2) $x^{\alpha_x} \partial^{\alpha_\partial} \prec x^{\beta_x} \partial^{\beta_\partial} \ \Rightarrow \ x^{\alpha_x + \gamma_x} \partial^{\alpha_\partial + \gamma_\partial} \prec x^{\beta_x + \gamma_x} \partial^{\beta_\partial + \gamma_\partial}$.

Since every polynomial $f \in A_n$ can be uniquely written as a sum of monomials times coefficients, we call the highest monomial of f with respect to a given ordering the *leading monomial* of f. We denote the latter by $\mathrm{lm}(f)$.

Note that there is another requirement we need to be fulfilled in our class of algebras, namely $1 \prec x_i, \partial_j, s_k \ \ \forall\ i, j, k$, that is the monomial ordering is a *well-ordering.*

We say that $x^{\alpha_x} \partial^{\alpha_\partial}$ *divides* $x^{\beta_x} \partial^{\beta_\partial}$, if $\alpha_i \leq \beta_i$ for all i in the range. Note, that this just means, that there exist $\gamma \in \mathbb{N}^{2n}$ and $r \in A_n$, such that $x^{\beta_x} \partial^{\beta_\partial} = x^{\alpha_x} \partial^{\alpha_\partial} \cdot x^{\gamma_x} \partial^{\gamma_\partial} + r$ with $r = 0$ or $\mathrm{lm}(r) \prec x^{\alpha_x} \partial^{\alpha_\partial}$.

[1] δ_{ij} denotes the Kronecker symbol

Definition 2. *Let $\prec$ be a monomial ordering on A_n and $G \subset A_n$ a finite set of polynomials. Let I be a left ideal, generated by G. G is called a left Gröbner basis of I if and only if for any $f \in I \smallsetminus \{0\}$ there exists $g \in G$ satisfying* $\mathrm{lm}(g) \mid \mathrm{lm}(f)$.

Given a finite set of generators of a left ideal L, there is *Buchberger's algorithm* for computing a left Gröbner basis of L (see e.g. [11, 8]).

Let $M_n := R_n \setminus \{0\}$, then M_n is a multiplicatively closed subset of both A_n and S_n. Hence, using the algebraic formalism of "localization" and the fact that M_n is an Ore set, we can pass from A_n (resp. S_n) to its "Ore localization", that is an algebra $(A_n)_{M_n}$ (resp. $(S_n)_{M_n}$). In the language of systems of operator equations $(A_n)_{M_n}$ (resp. $(S_n)_{M_n}$) stays for the algebra of linear partial differential (resp. shift) operators with *rational* coefficients. The algebras with rational coefficients appear very often in practical applications. They - as well as the Weyl and the shift algebra - are special cases of Ore algebras. We refer to [10, 4, 3, 8] for more details on these algebras, their properties as well as computational aspects and Gröbner bases.

3. Integration with D-modules

Define $f := f(a,x) = x^4 + 2ax^2 + 1 \in \mathbb{K}[x,a]$, then we have to integrate the function $f^{-(m+1)}$ with respect to x.

D-module theory stands for "the theory of differential modules" and encompasses systems of linear partial differential equations with polynomial and rational coefficients. One of the most important algorithms, obtained with D-module theory (see [11] and references therein for the full picture) is the algorithm for computing the s-parametric annihilator of $f \in \mathbb{K}[x_1,\dots,x_n]$ for a symbolic s. That is, it is possible to compute a set of operators $\{P \in D[s] : P \bullet f^s = 0\} =: \mathrm{Ann}_{D[s]} f^s$, which is indeed a left ideal in the algebra $D[s] := A_n[s] = A_n \otimes_{\mathbb{K}} \mathbb{K}[s]$ (for historical reasons D stands for some n-th Weyl algebra). Additionally, there is an algorithm for computing $\mathrm{Ann}_D f^\lambda$ for any $\lambda \in \mathbb{C}$, which uses the previously mentioned one.

In the case of the integral (1), the polynomial f is in $\mathbb{K}[x,a] = R_2$. Then $D = A_2 = \mathbb{K}\langle x, a, \partial_x, \partial_a | \partial_x x = x\partial_x + 1, \partial_a a = a\partial_a + 1\rangle$ is the 2nd Weyl algebra and $D[s] = A_2 \otimes_{\mathbb{K}} \mathbb{K}[s]$. First, we are going to compute the left ideal $L := \mathrm{Ann}_{D[s]} f^s \subset A_2 \otimes_{\mathbb{K}} \mathbb{K}[s]$ for $s := -(m+1)$ being symbolic. L corresponds to the system of linear partial differential equations in operators $\partial_x, \partial_a, s$ with coefficients in $\mathbb{K}[x,a]$, which has f^s as a solution. That is $\forall h \in L,\ h \bullet f^s = 0$.

In order to compute a system I of such equations for the function $F(a,s)$, we use Theorem 5.5.1 of [11], which states the following: let J be the right ideal of A_2, generated by all partial differential operators, corresponding to variables, with respect to which we perform integration (in our case this is just ∂_x, but the Theorem, as well as the whole approach, which goes back to Takayama [13, 12], holds for the multiple variable case too). Then

$$I = (L + J) \cap \big(\mathbb{K}\langle a, \partial_a \mid \partial_a a = a\partial_a + 1\rangle \otimes_{\mathbb{K}} \mathbb{K}[s]\big),$$

where the latter algebra is a natural $\mathbb{K}$-subalgebra of $A_2 \otimes_{\mathbb{K}} \mathbb{K}[s]$.

In general the sum of a left and of a right ideals carries no left or right structure. However, in the setting we work with a right ideal is very special one and, as we can see, there is a structure of left ideal on the intersection I of the sum of ideals above with a subalgebra.

We work with the special $\mathbb{K}$-basis of the algebra $A_2 \otimes_{\mathbb{K}} \mathbb{K}[s]$, namely $\{\partial_x^\alpha x^\beta a^\gamma \partial_a^\delta s^\epsilon \mid \alpha, \beta, \gamma, \delta, \epsilon \in \mathbb{N}_0\}$. In particular, each monomial of any polynomial in a left Gröbner basis of $L = \mathrm{Ann}_{D[s]} f^s$ is presented in this form. Moreover, we compute a left Gröbner basis G of L with respect to an ordering which eliminates x, ∂_x, i.e., any monomial containing ∂_x or x is bigger than one, which does not contain both of them.

Instead of summing L with the right ideal J (generated by ∂_x), we perform the right reduction of G with respect to J, what amounts to just skipping any monomial of every polynomial of G, if it is of the form $\partial_x^\alpha x^\beta a^\gamma \partial_a^\delta s^\epsilon$, where $\alpha \geq 1$. We may throw such a monomial away, because it belongs to the ideal J. After such a procedure we get a new set of polynomials G', where ∂_x does not appear. Since we used elimination ordering for both x and ∂_x for G and, moreover, monomials containing ∂_x are not present in G', it remains to pick those elements of G', which do not contain x. These elements then belong to the algebra $\mathbb{K}\langle a, \partial_a \mid \partial_a a = a\partial_a + 1\rangle[s]$ and, according to the Theorem 5.5.1 of [11], they generate the left ideal I we are looking for.

Now we illustrate the computation for the integral (1) with the computer algebra system Singular:Plural [5, 6]. This system has a library for computations with algebraic D-modules `dmod.lib` [9], which we are going to use.

```
LIB "dmod.lib";       // load the library for D-modules
ring r = 0,(a,x),dp;  // define a commutative ring
poly f = x^4 + 2*a*x^2 + 1;
def A = Sannfs(f); // A is a ring with the result object
                   // in it
setring A;
```

In the ring `A`, which stays for $D[s]$ (see above), there is an object called `LD` of the type `ideal`, which is the s-parametric annihilator ideal $L = \operatorname{Ann}_{D[s]} f^s$ as before. Its Gröbner basis consists of four operators

$$
\begin{aligned}
&2x^2\partial_a + 2a\partial_a - x\partial_x\ ,\\
&x^3\partial_x - 2a^2\partial_a + ax\partial_x - 4x^2 s + 2\partial_a\ ,\\
&4a^2\partial_a^2 - x^2\partial_x^2 - 8a\partial_a s + 4a\partial_a - 4\partial_a^2 + 4x\partial_x s - x\partial_x\ ,\\
&2a^2x\partial_a + ax^2\partial_x - 4axs - 2x\partial_a + \partial_x.
\end{aligned}
$$

Now, we change the order of variables into $\partial_x, x, a, \partial_a, s$; adjust the non-commutative relations respectively; set the monomial ordering, eliminating ∂_x, x and compute the left Gröbner basis of the ideal L, mapped from the ring `A`.

```
ring rr = 0, (Dx, x, a, Da, s), (a(1,1),dp);
matrix @D[5][5];
@D[1,2] = -1; @D[3,4] = 1;
def RR = nc_algebra(1,@D);
setring RR; // a new non-commutative ring
map M = A, a, x, Da, Dx, s; // map from A to RR using names
ideal LD = M(LD); // the image of LD in the new ring
LD = groebner(LD); // left Groebner basis of LD
```

At this stage we have to perform the addition of the left ideal L with the right ideal J, generated by ∂_x and intersect the result with the subalgebra $\mathbb{K}\langle a, \partial_a \mid \partial_a a = a\partial_a + 1\rangle[s]$. We go along the lines, described above.

```
ideal DD = Dx ;
ideal J = rightNF(LD,DD); // reduce with Dx from the right
ideal NJ = nselect(J,1,2); // see below
NJ = groebner(NJ); // left Groebner basis of NJ
```

We achieve these operations by computing the right normal forms of generators of left Gröbner basis of `LD` with respect to ∂_x. Invoking `nselect` command we select those generators, which do not include the variables from 1 to 2, that is ∂_x and x. As we can see, the ideal called `NJ`, which stay for I as above, is a principal ideal indeed. It is generated by the polynomial

$$4a^2\partial_a^2 - 4\partial_a^2 - 8a\partial_a s + 4a\partial_a - 4s - 1.$$

Depending on the monomial ordering used, sometimes an invertible element might appear as a factor.

Now we substitute s by $-m-1$ and rewrite some terms, giving back the answer: the integral $F(a,m)$ is annihilated by the left principal ideal of the algebra $\mathbb{K}\langle a, \partial_a \mid \partial_a a = a\partial_a + 1\rangle[m]$, which is generated by the operator

$$4(a-1)(a+1)\partial_a^2 + 4a(2m+3)\partial_a + (4m+3).$$

Of course, it is not yet a final answer, but an important part of it. In the next sections we show how we come to a closed form for the integral.

4. Holonomic systems and ∂-finite functions

We will now demonstrate how the symbolic evaluation of integrals like (1) can be performed in a different, more general framework, following D. Zeilberger's "holonomic systems approach" [14]. This theory was extended by F. Chyzak [2, 3, 4] who introduced the concept of ∂-finite functions and proposed Ore algebras to describe them. Moreover he implemented the underlying algorithms in the Maple package Mgfun.

For the construction of an Ore algebra, one starts with a commutative algebra like $\mathbb{K}[\mathbf{X}]$ or $\mathbb{K}(\mathbf{X})$ and adds one or several Ore extensions. These extensions introduce operators that necessarily commute with each other but usually do not commute with the variables $\mathbf{X}$. This setting is quite general (see e.g. [10]) and here we consider only special operators, namely the partial derivatives ∂_x, ∂_a and the shift s_m. For example, the Ore algebra that we will use here is $\mathbb{O} = \mathbb{K}(x,a,m)[\partial_x; 1, \partial_x][\partial_a; 1, \partial_a][s_m; s_m, 0]$. This algebra can also be realized as an Ore localization $(A_2 \otimes_{\mathbb{K}} S_1)_B$ where $A_2 = \mathbb{K}\langle x, a, \partial_x, \partial_a \mid \partial_x x = x\partial_x + 1, \partial_a a = a\partial_a + 1\rangle$, $S_1 = \mathbb{K}\langle m, s_m \mid s_m m = m s_m + s_m\rangle$, and B is the multiplicatively closed set $\mathbb{K}[x,a,m] \setminus \{0\} \subset A_2 \otimes_{\mathbb{K}} S_1$.

A function f is called ∂-finite w.r.t. a rational Ore algebra $\mathbb{K}(\mathbf{X})[\mathbf{P}; ., .]$ if the $\mathbb{K}(\mathbf{X})$-vector space spanned by all $(\mathbf{X^m P^n}) \bullet f$ is finite-dimensional over $\mathbb{K}(\mathbf{X})$. The following example will clarify this definition.

We want to find Ore operators in $\mathbb{O}$ that annihilate the integrand $g(x,a,m) = 1/(x^4+2ax^2+1)^{m+1}$. First observe that g is hyperexponential in x and a, i.e., $\frac{\partial_x \bullet g}{g}$ and $\frac{\partial_a \bullet g}{g}$ are rational functions in x and a respectively, e.g.,

$$\frac{\partial_x \bullet g(x,a,m)}{g(x,a,m)} = \frac{(-m-1)\left(4x^3+4ax\right)}{x^4+2ax^2+1}.$$

Moreover g is hypergeometric in m which means that $\frac{s_m \bullet g}{g} = \frac{g(x,a,m+1)}{g(x,a,m)}$ is a rational function in m. Hence we can compute first order annihilating operators for $g(x,a,m)$ in $\operatorname{Ann}_{\mathbb{O}} g = \{R \in \mathbb{O} \mid R \bullet g = 0\}$. Note that we use the term "annihilator" for any ideal of annihilating operators.

```
g = 1/(x^4+2*a*x^2+1)^(m+1);
ann = Annihilator[g, {S[m], Der[a], Der[x]}]
```

$$\begin{aligned}\{&(x^4+2ax^2+1)\partial_x + 4mx^3 + 4x^3 + 4ax + 4amx,\\ &(x^4+2ax^2+1)\partial_a + 2mx^2 + 2x^2,\\ &(x^4+2ax^2+1)s_m - 1\}\end{aligned}$$

An easy check ensures that these polynomials indeed constitute a Gröbner basis of the left ideal they generate. Moreover all leading monomials have degree 1; hence the corresponding ideal is a left maximal ideal and $\dim_{\mathbb{K}(x,a,m)} \mathbb{O}/\operatorname{Ann}_{\mathbb{O}} g = 1$, so g is indeed ∂-finite w.r.t. $\mathbb{O}$.

In order to perform the integration w.r.t. x, we are interested in finding operators in $\operatorname{Ann}_{\mathbb{O}} g$ of the following special form:

$$P(a, m, \partial_a, s_m) + \partial_x Q(x, a, m, \partial_x, \partial_a, s_m),$$

since

$$\begin{aligned} 0 &= \int_0^\infty \big(P(a, m, \partial_a, s_m) + \partial_x Q(x, a, m, \partial_x, \partial_a, s_m)\big) \bullet g(x, a, m)\,\mathrm{d}x \\ &= P \bullet F(a, m) + \Big[Q \bullet g(x, a, m)\Big]_{x=0}^{x=\infty}. \end{aligned} \tag{2}$$

For this purpose we will use Takayama's algorithm [13, 12]. It is designed in a way that it computes P (the part one is mainly interested in) without computing Q. Informally spoken, one first divides out the right ideal generated by ∂_x and then eliminates x by performing a Gröbner basis computation over a module. To this aim we have to compute in the Ore algebra $\mathbb{K}(a, m)[x][\partial_x; 1, \partial_x][\partial_a; 1, \partial_a][s_m; s_m, 0]$ because otherwise we were not able to eliminate x. More details on Takayama's algorithm were given in the previous section.

The fact that Q is not considered at all leads to the prerequisite that the integral must have natural boundaries: An integral $\int_u^v h(x, \dots)\mathrm{d}x$ is said to have natural boundaries if $[R \bullet h]_{x=u}^{x=v} = 0$ for all operators R in the respective algebra. In particular, the inhomogeneous part in (2) will vanish. If the integral does not have natural boundaries, we can end up with an inhomogeneous equation.

If we now look at the integral (1) we see that unfortunately it does not have natural boundaries, e.g.,

$$\Big[1 \bullet g(x, a, m)\Big]_{x=0}^{x=\infty} = -1.$$

We nevertheless can apply Takayama's algorithm, but we have to use an extended version where also Q is computed. Such an extension is included in [7].

```
Takayama[ann, {x}, OreAlgebra[x, Der[x], S[m], Der[a]],
         Extended -> True]
```

$$\begin{aligned} &\{\{(-4m-4)s_m + 2a\partial_a + (4m+3), \\ &\quad (4a^2-4)\partial_a^2 + (8ma+12a)\partial_a + (4m+3)\}, \\ &\ \{x, (-4m-4)xs_m + 2ax\partial_a + x\}\}. \end{aligned} \tag{3}$$

We are interested in the ordinary differential equation in a (the second operator). Note that it is the same as the result obtained with the first method. The corresponding Q is $(-4m-4)xs_m + 2ax\partial_a + x$. Now we verify that $[Q \bullet g]_{x=0}^{x=\infty}$ indeed vanishes although the integral does not have natural boundaries:

```
inhom = Simplify[ApplyOreOperator[%[[2,2]], g]]
```

$$x\left(x^4 + 2ax^2 + 1\right)^{-m-2}\left(-x^4 + 2ax^2 + 4m\left(ax^2 + 1\right) + 3\right)$$

```
inhom /. x -> 0
```

$$0$$

```
Limit[inhom, x -> Infinity, Assumptions -> m >= 0]
```

$$0$$

Hence, we derived in a purely automatic fashion an ordinary differential equation in a that is satisfied by the integral.

5. Closed form solution

Up to now we did not present a closed form solution of the integral, but only a differential equation in the parameter a:

$$(4m+3)F(a,m)+4a(2m+3)F'(a,m)+4\left(a^2-1\right)F''(a,m)=0. \tag{4}$$

For solving this differential equation we can use standard tools. Since it has order 2, we need the initial values $F(0,m)$ and $F'(0,m)$:

```
in0 = Integrate[g /. a -> 0, {x, 0, Infinity},
                Assumptions -> m >= 0]
```

$$\frac{\Gamma\left(\frac{5}{4}\right)\Gamma\left(m+\frac{3}{4}\right)}{\Gamma(m+1)}$$

```
in1 = Integrate[D[g, a] /. a -> 0, {x, 0, Infinity},
                Assumptions -> m >= 0]
```

$$-\frac{2\Gamma\left(\frac{7}{4}\right)\Gamma\left(m+\frac{5}{4}\right)}{3\Gamma(m+1)}$$

We solve (4) with Mathematica's command DSolve:

```
DSolve[{(4m+3)F[a] + 4a(2m+3)F'[a] + 4(a^2-1)F''[a] == 0,
       F[0] == in0, F'[0] == in1}, F[a], a]
```

After some simplification we end up with the final result:

$$F(a,m)=-\frac{(1+i)(-i)^m 2^{-m-1}\left(a^2-1\right)^{-\frac{m}{2}-\frac{1}{4}}\sqrt{\pi}Q_m^{\left(m+\frac{1}{2}\right)}(a)}{\Gamma(m+1)},$$

where $Q_\lambda^{(\mu)}(z)$ denotes the associated Legendre function of the second kind.

Note that we computed this solution completely automatically with no necessity of human insight to the specific problem. V. Moll as an expert in the field of integrals gives the following slightly simpler solution involving Jacobi polynomials:

$$F(a,m)=2^{-m-\frac{3}{2}}(a+1)^{-m-\frac{1}{2}}\pi P_m^{\left(m+\frac{1}{2},-m-\frac{1}{2}\right)}(a)$$

With our software [7] we can immediately prove the correctness of this solution:

```
Annihilator[Pi*JacobiP[m, m+1/2, -m-1/2, a]/2^(m+3/2)/
           (a+1)^(m+1/2), {Der[a], S[m]}]
```

$$\begin{aligned}\{&-4m+(-2a)\partial_a+(4m+4)s_m-3,\\ &4m+\left(4a^2-4\right)\partial_a^2+(8ma+12a)\partial_a+3\}\end{aligned}$$

Observe that this annihilator is exactly the same as (3). By comparing the initial values

$$F(0,0)=\frac{\pi}{2\sqrt{2}} \quad \text{and} \quad F'(0,0)=-\frac{\pi}{4\sqrt{2}}$$

we complete the proof.

6. Conclusion

We presented computer algebra methods for the automatic solution of a parametrized integral. We want to emphasize that these methods are applicable to a wide class of integration (and summation) problems. In particular the second method works for the large class of holonomic functions (including hypergeometric, hyperexponential, algebraic, and Special functions). As a more challenging example let's just mention the integral

$$\int_0^\infty \frac{1}{(x^4+ax^3+bx^2+cx+d)^m}\mathrm{d}x$$

which contains more parameters, but nevertheless can be tackled in an analogous way. The problem here is only that the resulting differential equations are so involved that the standard tools are not able to find a closed form solution.

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