AN ALGORITHMIC APPROACH TO RAMANUJAN CONGRUENCES

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Abstract. In this paper we present an algorithm that takes as input a generating function of the form $\prod_{\delta|M}\prod_{n=1}^{\infty}(1-q^{\delta n})^{r_\delta}=\sum_{n=0}^{\infty}a(n)q^n$ and three positive integers m,t,p, and which returns true if $a(mn+t)\equiv 0 \pmod p, n\geq 0$, or false otherwise. Our method builds on work by Rademacher [12], Kolberg [6], Sturm [17], Eichhorn and Ono [3].

Keywords: partition congruences, number theoretic algorithm, modular forms

Introduction

Throughout this article M denotes a positive integer, and $r=(r_\delta)$ denotes a sequence of integers r_δ indexed by all positive integer divisors δ of M.

In this paper we present an algorithm that takes as input a generating function of the form $\prod_{\delta|M}\prod_{n=1}^{\infty}(1-q^{\delta n})^{r_\delta}=\sum_{n=0}^{\infty}a(n)q^n$ and three positive integers m,t,p, and which returns true if $a(mn+t)\equiv 0 \pmod p, n\geq 0$, or false otherwise. A similar algorithm for generating functions of the form $\prod_{n=1}^{\infty}(1-q^n)^{r_1}$ (i.e. the case $M=1$) has already been given in [3]. Our original plan was to implement that algorithm in order to prove some congruences from [1]. The algorithm we present here and the one in [3] both have in common that at the end one has to check that the congruence is true for the first coefficients up to a bound ν that the algorithm returns, and then to use the theorem of Sturm [17] to conclude that it is true for all coefficients. However we noticed that for our purpose the bound ν given in [3] was extremely high for some inputs. Encouraged by comments of Peter Paule we examined the problem in more detail. Finally our study resulted in a significant improvement of estimating the bound ν a priori. Our main tools to derive a better bound ν are the ones used by Rademacher [12], Newman [10]; Kolberg [6] was another major source of inspiration.

The organization of this paper is as follows: In section 1 we present the basic terminology. In section 2 we prepare some results needed to apply the theorem of Sturm. The main result, Theorem 2.13, can be viewed as a generalization of a theorem of R. Lewis [8]. In section 3 we estimate functions at different points; this is needed in order to prove they are indeed modular forms. In section 4 we show how to apply the theorem of Sturm in order to prove our desired congruence. In section 5 we conclude by giving some examples.

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1. Basic Terminology and Formulas

We use the notation $X \equiv_v Y$ if X and Y are congruent modulo v.

For integers m and n we let throughout $\gcd(m,n)$ denote the greatest common divisor of m and n which is always normalized to return positive values.

Let a be an integer relatively prime to 6, i.e. $\gcd(a,6)=1$. For such a one can easily show that $a^2-1 \equiv_{24} 0$. Similarly if $\gcd(a,3)=1$ then $a^2-1\equiv_3 0$, and finally, if $\gcd(a,2)=1$ then $a^2-1\equiv_8 0$. This facts will be used throughout the text.

For a positive integer N we define the following matrix groups:

$$M_2(\mathbb{Z})^* := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a,b,c,d \in \mathbb{Z}, ad-bc>0 \right\},$$
$$\Gamma := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z})^* \mid ad-bc=1 \right\},$$
$$\Gamma_\infty := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid c=0 \right\},$$
$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mid c \equiv_N 0 \right\}.$$

There is an explicit formula for the index (e.g. [15]):

$$[\Gamma:\Gamma_0(N)] = N\prod_{p\mid N}(1+p^{-1}). \tag{1}$$

Throughout we use the following conventions:

- $\mathbb{N}^*$ denotes the positive integers.
- $q := e^{2\pi i\tau}$.
- $\eta(\tau)$ denotes the Dedekind eta function for which

$$\eta(\tau) := q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n). \tag{2}$$

- $\mathbb{H} := \{x\in\mathbb{C} \mid \mathrm{Im}(x)>0\}$.
- $\mathbb{H}^* := \mathbb{H}\cup\mathbb{Q}\cup\{\infty\}$.
- $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z})^*$ acts on elements $\tau\in\mathbb{H}^*$ as $\gamma\tau := \frac{a\tau+b}{c\tau+d}$ because of the formula $\mathrm{Im}(\gamma\tau) = (ad-bc)\frac{\mathrm{Im}(\tau)}{|c\tau+d|^2}$ (e.g.[15]). Let $f(\tau)$ be a function of τ. We will later use that $f(\gamma_1(\gamma_2\tau)) = f((\gamma_1\gamma_2)\tau)$ where $\gamma_1,\gamma_2\in M_2(\mathbb{Z})^*$.
- $[x]_m$ denotes an element of $\mathbb{Z}_m$. (Note: $[x]_m=[y]_m$ iff $x\equiv_m y$.)

Definition 1.1. *Let $k\in\mathbb{Z}$. A modular form of weight k for a subgroup G of Γ is a function $f(\tau)$ defined on $\mathbb{H}^*$ such that:*

1. *$f(\tau)$ is holomorphic in $\mathbb{H}$;*
2. *$(c\tau+d)^{-k}f(\gamma\tau)=f(\tau)$ for all $\tau\in\mathbb{H}^*$ and all $\gamma\in G$;*
3. *for all $\gamma\in\Gamma$ the function $(c\tau+d)^{-k}f(\gamma\tau)$ has a Taylor series expansion in powers of $q^{\frac{1}{n}}$, n a positive integer, which converges in a nontrivial neighborhood of 0.*

Definition 1.2. *Let $a \in \mathbb{Z}$. For an odd integer $n > 0$ we define:*

- *If $n = 1$ then:*

$$\left(\frac{a}{n}\right) = \left(\frac{a}{1}\right) := 1. \tag{3}$$

- *If n is a prime p then:*

$$\left(\frac{a}{n}\right) = \left(\frac{a}{p}\right) := \begin{cases} 0 & \text{if } p \mid a \\ 1 & \text{if } a \text{ is a square modulo } p \\ -1 & \text{otherwise} \end{cases}. \tag{4}$$

- *If $p_1^{\alpha_1} \cdot \ldots \cdot p_k^{\alpha_k}$ is the prime factorization of n then:*

$$\left(\frac{a}{n}\right) := \left(\frac{a}{p_1}\right)^{\alpha_1} \cdot \ldots \cdot \left(\frac{a}{p_k}\right)^{\alpha_k}. \tag{5}$$

The symbol $\left(\frac{a}{n}\right)$ is called the Legendre-Jacobi symbol.

Lemma 1.3. *Let $n > 0$ be an odd integer, then the following relations hold:*

- *If a and b are integers then*

$$\left(\frac{a}{n}\right)\left(\frac{b}{n}\right) = \left(\frac{ab}{n}\right). \tag{6}$$

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$$\left(\frac{2}{n}\right) = (-1)^{\frac{n^2-1}{8}}. \tag{7}$$

- *If m is an odd integer then*

$$\left(\frac{m}{n}\right) = \left(\frac{n}{m}\right)(-1)^{\frac{m-1}{2}\frac{n-1}{2}}. \tag{8}$$

Proof. See [13], page 71. □

Definition 1.4. *We define $\epsilon : \{(a,b,c,d) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma\} \mapsto \mathbb{C}$ to be the unique mapping that satisfies*

$$\eta(\gamma\tau) = (-i(c\tau + d))^{\frac{1}{2}}\,\epsilon(a,b,c,d)\eta(\tau), \tag{9}$$

for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $\tau \in \mathbb{H}^*$.

Remark 1.5. This definition is meaningful because for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and $\tau \in \mathbb{H}^*$ we have $\eta^{24}(\gamma\tau) = (c\tau + b)^{12}\eta^{24}(\tau)$ (e.g. [14]). This also implies that $\epsilon^{24}(a,b,c,d) = 1$ for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

For $\gamma \in \Gamma$ with $\gcd(a,6) = 1, a > 0$ and $c > 0$ Newman [10] determined ϵ as

$$\epsilon(a,b,c,d) = \left(\frac{c}{a}\right) e^{-\frac{a\pi i}{12}(c-b-3)}. \tag{10}$$

Lemma 1.6 (Newman [10]). *Let $N \in \mathbb{N}^*$, $k \in \mathbb{Z}$ and $f : \mathbb{H}^* \mapsto \mathbb{C}$ a function such that for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ with $\gcd(a,6) = 1, a > 0, c > 0$ we have $f(\gamma\tau) = (c\tau + d)^{2k} f(\tau)$. Then for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ we have $f(\gamma\tau) = (c\tau + d)^{2k} f(\tau)$.*

Definition 1.7. *Given a positive integer m let $\varphi(t,\tau) : [0, m-1] \times \mathbb{H}^* \mapsto \mathbb{H}^*$ be a function with expansion $\varphi(t,\tau) = q^{-t} \sum_{n=0}^{\infty} a(n) q^n$. Let S_m be a complete set of non-equivalent representatives of the residue classes modulo m. For $\kappa \in \mathbb{N}$ with $\gcd(m,\kappa) = 1$ we define:*

$$M_{m,\kappa}(\varphi(t,\tau)) := \sum_{\lambda \in S_m} \varphi\left(t, \frac{\tau + \kappa\lambda}{m}\right). \tag{11}$$

In this paper we are always choosing

$$\kappa := \gcd(1 - m^2, 24). \tag{12}$$

With this choice clearly $\gcd(\kappa, m) = 1$.

Another property needed later is as follows:

Lemma 1.8. *Let κ be as defined in (12), then $6|\kappa m$.*

Proof. One can proceed by case distinction. For instance, if $2 \nmid m$ and $3|m$, then $m^2 - 1 \equiv_8 0$ because of $\gcd(m,2) = 1$. Hence by (12) we have $8|\kappa$, thus $6|\kappa m$. The other cases are similar.

□

Lemma 1.9. *Given positive integers m and κ, let $\varphi(t,\tau)$ be as in Definition 1.7. Then we have:*

$$M_{m,\kappa}(\varphi(t,\tau)) = m \sum_{n=0}^{\infty} a(mn + t) q^n. \tag{13}$$

Proof. By Definition 1.7 we have

$$\begin{aligned}
M_{m,\kappa}(\varphi(t,\tau)) = &\sum_{\lambda \in S_m} e^{-2\pi i t \frac{\tau+\kappa\lambda}{m}} \sum_{n=0}^{\infty} a(n) e^{2\pi i n \frac{\tau+\kappa\lambda}{m}} \\
= &\sum_{n=0}^{\infty} a(n) e^{-\frac{2\pi i \tau t}{m}} e^{\frac{2\pi i n \tau}{m}} \sum_{\lambda \in S_m} e^{2\pi i \lambda \frac{-\kappa t + \kappa n}{m}} \\
= &\sum_{\substack{n \geq 0 \\ n \equiv_m t}} m a(n) e^{-\frac{2\pi i \tau t}{m}} e^{\frac{2\pi i n \tau}{m}} \\
= &m \sum_{n=0}^{\infty} a(mn+t) q^{\frac{-t}{m}} q^{\frac{t}{m}} q^n \\
= &m \sum_{n=0}^{\infty} a(mn+t) q^n.
\end{aligned}$$

Note that the sum $\sum_{\lambda \in S_m} e^{2\pi i \lambda \frac{-\kappa t + \kappa n}{m}}$ equals m if $-\kappa t + \kappa n \equiv_m 0$. This is exactly the case when $n \equiv_m t$. For $n \not\equiv_m t$ the sum is 0. □

Definition 1.10. *Let $M \in \mathbb{N}^*$. By $R(M)$ we denote the set of all integer sequences (r_δ) indexed by all positive divisors δ of M.*

Definition 1.11. *For $m, M \in \mathbb{N}^*$, $t \in \mathbb{N}$ such that $0 \leq t \leq m-1$ and $r = (r_\delta) \in R(M)$, we define:*

$$f(\tau, r) := \prod_{\delta|M} \prod_{n=0}^{\infty} (1 - q^{\delta n})^{r_\delta} = \sum_{n=0}^{\infty} a(n) q^n, \tag{14}$$

and

$$g_{m,t}(\tau, r) := q^{\frac{24t + \sum_{\delta|M} \delta r_\delta}{24m}} \sum_{n=0}^{\infty} a(mn+t) q^n. \tag{15}$$

Lemma 1.12. *For $m, M \in \mathbb{N}^*$, $t \in \mathbb{N}$ such that $0 \leq t \leq m-1$ and $r = (r_\delta) \in R(M)$ we obtain the following representation:*

$$g_{m,t}(\tau, r) = \frac{1}{m} \sum_{\lambda=0}^{m-1} e^{\frac{2\pi i \kappa \lambda(-24t - \sum_{\delta|M} \delta r_\delta)}{24m}} \prod_{\delta|M} \eta^{r_\delta}\left(\frac{\delta(\tau + \kappa\lambda)}{m}\right). \tag{16}$$

Proof. Using (2) we see that $f(\tau, r) = q^{-\frac{\sum_{\delta|M} \delta r_\delta}{24}} \prod_{\delta|M} \eta^{r_\delta}(\delta\tau)$. Next applying $M_{m,\kappa}$ to $\varphi(t, \tau) := q^{-t} f(\tau, r)$, by Definition 1.7 we see that:

$$M_{m,\kappa}(\varphi(t,\tau)) = \sum_{\lambda=0}^{m-1} e^{2\pi i (\frac{\tau + \kappa\lambda}{m})(-t - \frac{\sum_{\delta|M} \delta r_\delta}{24})} \prod_{\delta|M} \eta^{r_\delta}\left(\frac{\delta(\tau+\kappa\lambda)}{m}\right)$$

$$= q^{\frac{-24t - \sum_{\delta|M} \delta r_\delta}{24m}} \sum_{\lambda=0}^{m-1} e^{\frac{2\pi i \kappa\lambda(-24t - \sum_{\delta|M} \delta r_\delta)}{24m}} \prod_{\delta|M} \eta^{r_\delta}\left(\frac{\delta(\tau+\kappa\lambda)}{m}\right).$$

Alternatively by Lemma 1.9 we obtain:

$$M_{m,\kappa}(\varphi(t,\tau)) = m \sum_{n=0}^{\infty} a(mn+t) q^n = m q^{-\frac{24t + \sum_{\delta|M} \delta r_\delta}{24m}} g_{m,t}(\tau, r).$$

Comparing the two expressions for $M_{m,\kappa}(\varphi(t,\tau))$ we obtain our assertion. □

The following lemma will be used at several occasions:

Lemma 1.13. *Given a real number k and maps $f : \mathbb{H}^* \mapsto \mathbb{C}$ and $g : \Gamma \times \mathbb{H}^* \mapsto \mathbb{C}$. Suppose for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ and for all $\tau \in \mathbb{H}^*$:*

$$(c\tau + d)^{-k} f(\gamma\tau) = g(\gamma, \tau).$$

Then for all $\xi = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_2(\mathbb{Z})^*$ and for all $\tau \in \mathbb{H}^*$:

$$\left(\frac{\gcd(A,C)}{AD - BC}(C\tau + D)\right)^{-k} f(\xi\tau)$$
$$= g\left(\begin{pmatrix} \frac{A}{\gcd(A,C)} & -y \\ \frac{C}{\gcd(A,C)} & x \end{pmatrix}, \frac{\gcd(A,C)\tau + Bx + Dy}{\frac{AD-BC}{\gcd(A,C)}}\right)$$

where the integers x and y are chosen such that $Ax + Cy = \gcd(A, C)$.

Proof. Define

$$\gamma := \begin{pmatrix} \frac{A}{\gcd(A,C)} & -y \\ \frac{C}{\gcd(A,C)} & x \end{pmatrix} \text{ and } \gamma' := \begin{pmatrix} \gcd(A,C) & Bx+Dy \\ 0 & \frac{AD-BC}{\gcd(A,C)} \end{pmatrix}.$$

Then the statement follows from the relation $\xi = \gamma\gamma'$ and by

$$f(\xi\tau) = f(\gamma(\gamma'\tau)) = \left(\frac{C}{\gcd(A,C)}(\gamma'\tau)+x\right)^k g(\gamma,\gamma'\tau).$$

□

2. The function $g_{m,t}(\tau,r)$ under modular substitutions

Throughout this section we will assume that $\gcd(a,6)=1, a>0$ and $c>0$ so that (10) will always apply and $a^2 \equiv_{24} 1$. For this reason it will be convenient to introduce the following notation:

$$\Gamma_0(N)^* := \{\gamma \in \Gamma_0(N) | a>0, c>0, \gcd(a,6)=1\}. \tag{17}$$

Because M and $r=(r_\delta)$ are assumed as fixed we will write $g_{m,t}(\tau) := g_{m,t}(\tau,r)$ and $f(\tau) := f(\tau,r)$ throughout.

We are interested in deriving a formula for $g_{m,t}(\gamma\tau)$ with $\gamma \in \Gamma_0(N)^*$ where N is an integer such that for every prime p with $p|m$ we have also $p|N$, i.e.,

$$p|m \text{ implies } p|N, \tag{18}$$

and such that for every $\delta|M$ with $r_\delta \neq 0$ we have $\delta|mN$, i.e.,

$$\delta|M \text{ implies } \delta|mN; \tag{19}$$

and some additional properties which we will specify later. For our purpose it is convenient to define the following set:

Definition 2.1. *We define*

$$\Delta := \left\{ (m,M,N,(r_\delta)) \in (\mathbb{N}^*)^3 \times R(M) \mid \begin{array}{l} m,M,N \textit{ and } (r_\delta) \textit{ satisfy} \\ \textit{the conditions } (18) \textit{ and } (19). \end{array} \right\}.$$

Lemma 2.2. *Let* $(m,M,N,(r_\delta)) \in \Delta$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)^*$ *and* λ *a nonnegative integer. Then:*

(i) *There exist integers* x *and* y *such that*

$$(a+\kappa\lambda c)x + mcy = 1 \tag{20}$$

and where $y := y_0(m\kappa c)^3$ *for some integer* y_0.

(ii) *There exists an integer* a' *satisfying* $a'a \equiv_{24c} 1$.

Let x, y *and* a' *be as in (i) and (ii). Then for*

$$\mu := \lambda dx + \frac{bx - ba'm^2}{\kappa} \tag{21}$$

the following statements hold:

(iii) *For ϵ as in Definition 1.4, $\tau \in \mathbb{H}^*$ and $\delta | M$ with $r_\delta \neq 0$ we have*

$$\begin{aligned}&\eta\left(\frac{\delta(\gamma\tau+\kappa\lambda)}{m}\right)\\ &= (-i(c\tau+d))^{\frac{1}{2}}\epsilon(a+\kappa\lambda c, -\delta y, \frac{mc}{\delta}, x)\eta\left(\frac{\delta(\tau+\kappa\mu)}{m}\right)e^{\frac{2\pi i abm\delta}{24}},\end{aligned} \tag{22}$$

and

$$\epsilon(a+\kappa\lambda c, -\delta y, \frac{mc}{\delta}, x) = \left(\frac{mc\delta}{a+\kappa\lambda c}\right)e^{-\frac{(a+\kappa\lambda c)\pi i}{12}(mc/\delta-3)}. \tag{23}$$

(iv) *The value μ is an integer, and if λ runs through a complete set of representatives of residue classes modulo m then so does μ; i.e., $\lambda \mapsto \mu$ is a bijection of $\mathbb{Z}_m$.*

(v)

$$\lambda \equiv_c \mu a^2 - ab\frac{1-m^2}{\kappa}. \tag{24}$$

Proof. We prove each part of Lemma 2.2 separately.

(i). We know that the equation

$$(a+\kappa\lambda c)x + mcy_0(m\kappa c)^3 = 1 \tag{25}$$

has integer solutions x and y_0 iff

$$\gcd(a+\kappa\lambda c, mc(m\kappa c)^3) = 1. \tag{26}$$

To prove (26) it suffices to prove $\gcd(a+\kappa\lambda c, m) = 1$ and $\gcd(a+\kappa\lambda c, \kappa c) = 1$. We have that

$$\gcd(a+\kappa\lambda c, \kappa c) = \gcd(a, \kappa c).$$

But $\gcd(a, c) = 1$ because of $ad - bc = 1$, and $\gcd(a, \kappa) = 1$ because of $\gcd(a, 6) = 1$ by assumption and κ being a divisor of 24. Next we see that $\gcd(a+\kappa\lambda c, c) = 1$ implies $\gcd(a+\kappa\lambda c, N) = 1$ because $N|c$. But $\gcd(a+\kappa\lambda c, N) = 1$ implies $\gcd(a+\kappa\lambda c, m) = 1$ by (18). This proves (26).

Note: Because of $y = y_0(m\kappa c)^3$ Lemma 1.8 gives

$$y \equiv_{24} 0. \tag{27}$$

(ii). The assumptions $\gcd(a, 6) = 1$ and $\gcd(a, c) = 1$ imply that $\gcd(a, 24c) = 1$, which is equivalent to the existence of an integer a' such that $a'a \equiv_{24c} 1$.

(22). For ϵ as in Definition 1.4 let

$$C := (-i(c\tau+d))^{\frac{1}{2}}\,\epsilon(a+\kappa\lambda c, -\delta y, \frac{mc}{\delta}, x). \tag{28}$$

For this part of the proof we exploit the relation:

$$\eta\left(\frac{\delta((a+\kappa\lambda c)\tau+b+\kappa\lambda d)}{mc\tau+md}\right) = C\eta\left(\frac{\delta\tau+\delta(b+\kappa\lambda d)x+md\delta y}{m}\right) \tag{29}$$

which is valid under the assumption that $(a+\kappa\lambda c)x + \delta y\frac{mc}{\delta} = 1$.

Note that $\frac{mc}{\delta}$ is a positive integer ($N|c$ and, by (19), $\delta | mN$), and that $(a+\kappa\lambda c)x + \delta y\frac{mc}{\delta} = 1$ because of (20).

Relation (29) is proven by applying Lemma 1.13 with $f(\tau) = \eta(\tau)$, $k = 1/2$,

$$g(\gamma, \tau) = (-i)^{\frac{1}{2}} \epsilon(a_0, b_0, c_0, d_0)\eta(\tau), \gamma = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \in \Gamma$$

and $\xi = \begin{pmatrix} \delta(a + \kappa\lambda c) & \delta(b + \kappa\lambda d) \\ mc & md \end{pmatrix}$.

We will also need that for all integers j we have as a trivial consequence of (2):

$$\eta(\tau + j) = \eta(\tau)e^{\frac{2\pi ij}{24}}. \tag{30}$$

Consequently,

$$\begin{aligned}
\eta\left(\frac{\delta(\gamma\tau + \kappa\lambda)}{m}\right) &= \eta\left(\frac{\delta((a + \kappa\lambda c)\tau + b + \kappa\lambda d)}{mc\tau + md}\right) && \text{(by substituting for } \gamma)\\
&= C\eta\left(\frac{\delta\tau + \delta(b + \kappa\lambda d)x + md\delta y}{m}\right) && \text{(by (29))}\\
&= C\eta\left(\frac{\delta\tau + \delta(b + \kappa\lambda d)x}{m}\right) && \text{(by (30) and (27))}\\
&= C\eta\left(\frac{\delta\tau + \delta(b + \kappa\lambda d)x - \delta ba'm^2}{m} + \delta ba'm\right) && \\
&= C\eta\left(\frac{\delta\tau + \delta(b + \kappa\lambda d)x - \delta ba'm^2}{m}\right)e^{\frac{2\pi i\delta a'bm}{24}} && \text{(by (30))}\\
&= C\eta\left(\frac{\delta(\tau + \kappa\mu)}{m}\right)e^{\frac{2\pi i\delta a'bm}{24}} && \text{(by (21))}\\
&= C\eta\left(\frac{\delta(\tau + \kappa\mu)}{m}\right)e^{\frac{2\pi i\delta abm}{24}} && \text{(because of } a' \equiv_{24} a).
\end{aligned}$$

In the last line we used fact *(ii)*, namely $aa' \equiv_{24c} 1$. This together with $a^2 \equiv_{24} 1$ implies that $a \equiv_{24} a'$ because of uniqueness of the inverse.

(23). First note that

$$\gcd(a + \kappa\lambda c, 6) = 1, \tag{31}$$

because of $\kappa c \equiv_6 0$ by Lemma 1.8 and (18) together with $N|c$.

We have that

$$
\begin{aligned}
&\epsilon(a+\kappa\lambda c, -\delta y, \frac{mc}{\delta}, x) \\
&= \left(\frac{mc/\delta}{a+\kappa\lambda c}\right) e^{-\frac{(a+\kappa\lambda c)\pi i}{12}(mc/\delta+\delta y-3)} && \text{(by (10) and (31))} \\
&= \left(\frac{mc/\delta}{a+\kappa\lambda c}\right) e^{-\frac{(a+\kappa\lambda c)\pi i}{12}(mc/\delta-3)} && \text{(by (27))} \\
&= \left(\frac{mc/\delta}{a+\kappa\lambda c}\right) \left(\frac{\delta^2}{a+\kappa\lambda c}\right) e^{-\frac{(a+\kappa\lambda c)\pi i}{12}(mc/\delta-3)} && \text{(see below)} \\
&= \left(\frac{mc\delta}{a+\kappa\lambda c}\right) e^{-\frac{(a+\kappa\lambda c)\pi i}{12}(mc/\delta-3)} && \text{(by (6)).}
\end{aligned}
$$

The third equality is shown as follows. If $\gcd(a+\kappa\lambda c, \delta) = 1$ then Definition 1.2 implies that $\left(\frac{\delta^2}{a+\kappa\lambda c}\right) = 1$. To prove relative primeness we see by (18) and (19) that each prime p dividing δ also divides N and consequently also c. So $\gcd(a+\kappa\lambda c, p) = \gcd(a, p)$. But since $p|c$ and $\gcd(a, c) = 1$ by $ad - bc = 1$, we conclude that $\gcd(a+\kappa\lambda c, \delta) = 1$.

(iv). In order to prove that μ is an integer we need to show that $bx - ba'm^2 \equiv_\kappa 0$. By (25) we obtain $ax \equiv_\kappa 1$. We also know by *(ii)* that $aa' \equiv_{24c} 1$. Because of $\kappa|24$ by (12), we have that $aa' \equiv_\kappa 1$. From this follows that $x \equiv_\kappa (a'a)x \equiv_\kappa a'(ax) \equiv_\kappa a'$. Consequently,

$$bx - ba'm^2 \equiv_\kappa bx - bxm^2 \equiv_\kappa bx(1-m^2) \equiv_\kappa 0,$$

using $\kappa|(1-m^2)$ from (12).

Next we show that the mapping $\lambda \mapsto \mu$ is a bijection of $\mathbb{Z}_m$ by providing an inverse using the observation that:

$$\mu - \frac{bx - ba'm^2}{\kappa} \equiv_m \lambda dx \text{ implies } \lambda \equiv_m (xd)^{-1}(\mu - \frac{bx - ba'm^2}{\kappa}).$$

The only non-trivial step is to show that d and x are indeed invertible modulo m. First of all, x is invertible modulo m because of (25). Because of $ad - bc = 1$ we have that $\gcd(c, d) = 1$, and since $N|c$ we have that $\gcd(N, d) = 1$. By (18) we get that $\gcd(m, d) = 1$ which shows that also d is invertible modulo m.

(v). By (25) we have that $ax \equiv_{\kappa c} 1$. From $aa' \equiv_{24c} 1$ and $\kappa|24$ we conclude that $aa' \equiv_{\kappa c} 1$ which implies $x \equiv_{\kappa c} a'$ by uniqueness of the inverse.

Because of the relation $ad - bc = 1$ we have that $ad \equiv_c 1$. From $ax \equiv_{\kappa c} 1$ it follows that $ax \equiv_c 1$ which implies $d \equiv_c x$ by uniqueness of the inverse.

Next we will show the validity of

$$\mu \equiv_c \lambda d^2 + bd\frac{1-m^2}{\kappa} \tag{32}$$

by the following chain of arguments starting with the Definition (21):

$$\kappa\mu \equiv_{\kappa c} \kappa\lambda dx + bx - ba'm^2 \equiv_{\kappa c} \kappa\lambda dx + bx - bxm^2 \equiv_{\kappa c} \kappa(\lambda dx + bx\frac{1-m^2}{\kappa})$$

which implies that

$$\mu \equiv_c \lambda dx + bx\frac{1-m^2}{\kappa} \equiv_c \lambda d^2 + bd\frac{1-m^2}{\kappa}.$$

We thus have proven (32). By multiplying the last congruence with a^2, we obtain:

$$\mu a^2 - ba\frac{1-m^2}{\kappa} \equiv_c \lambda.$$

We have again used that the inverse of d is a modulo c.

□

In order to arrive at our main result, Theorem 2.13, we need to introduce some additional assertions, Lemmas 2.3 to 2.10.

Lemma 2.3. *Let l, j be integers and C, a, s non-negative integers such that:*

(1) *the relation $p|l$ implies $p|C$ for any prime p;*
(2) $\gcd(a, l) = 1$*;*
(3) $l = 2^s j$ *where j is odd;*
(4) *a is odd and C is even.*

Then for any non-negative integer λ:

$$\left(\frac{l}{a+\lambda C}\right) = \left(\frac{l}{a}\right)(-1)^{\frac{\lambda C(j-1)}{4}}(-1)^{s\frac{2a\lambda C+\lambda^2C^2}{8}}. \tag{33}$$

Proof. By a similar reasoning as in the proof of (23) we see that $\gcd(a+\lambda C, l) = 1$ for all integers λ.

Next we can write $j = j_1 j_2$ where j_1 is squarefree and j_2 is a square. Clearly $j_1|C$ by assumption. Then:

$$\begin{aligned}
\left(\frac{j}{a+\lambda C}\right) &= \left(\frac{j_1}{a+\lambda C}\right)\left(\frac{j_2}{a+\lambda C}\right) && \text{(by (6))}\\
&= \left(\frac{j_1}{a+\lambda C}\right) && \text{(because of } \gcd(a+\lambda C, j) = 1)\\
&= (-1)^{\frac{a+\lambda C-1}{2}\frac{j_1-1}{2}}\left(\frac{a+\lambda C}{j_1}\right) && \text{(by (8))}\\
&= (-1)^{\frac{a+\lambda C-1}{2}\frac{j_1 j_2-1}{2}}\left(\frac{a+\lambda C}{j_1}\right) && \text{(because of } j_2 \equiv_4 1)\\
&= (-1)^{\frac{\lambda C(j-1)}{4}}(-1)^{\frac{a-1}{2}\frac{j_1-1}{2}}\left(\frac{a}{j_1}\right) && \text{(because of } a+\lambda C \equiv_{j_1} a)\\
&= (-1)^{\frac{\lambda C(j-1)}{4}}\left(\frac{j_1}{a}\right) && \text{(by (8))}\\
&= (-1)^{\frac{\lambda C(j-1)}{4}}\left(\frac{j}{a}\right) && \text{(by (6) and because of } \left(\frac{j_2}{a}\right) = 1).
\end{aligned}$$

Summarizing, we have proven:

$$\left(\frac{j}{a+\lambda C}\right) = (-1)^{\frac{\lambda C(j-1)}{4}}\left(\frac{j}{a}\right). \tag{34}$$

Next,

$$\left(\frac{2}{a+\lambda C}\right) = (-1)^{\frac{2a\lambda C+\lambda^2C^2}{8}}\left(\frac{2}{a}\right) \tag{35}$$

is easily seen by

$$\begin{aligned}\left(\frac{2}{a+\lambda C}\right) &= (-1)^{\frac{(a+\lambda C)^2-1}{8}} && \text{(by (7))}\\ &= (-1)^{\frac{2a\lambda C+\lambda^2C^2}{8}}\left(\frac{2}{a}\right) && \text{(by (7)).}\end{aligned}$$

The following derivation concludes the proof:

$$\begin{aligned}\left(\frac{l}{a+\lambda C}\right) &= \left(\frac{2}{a+\lambda C}\right)^s\left(\frac{j}{a+\lambda C}\right) && \text{(by (6))}\\ &= \left(\frac{2}{a}\right)^s\left(\frac{j}{a}\right)(-1)^{\frac{\lambda C(j-1)}{4}}(-1)^{s\frac{2a\lambda C+\lambda^2C^2}{8}} && \text{(by (34) and (35))}\\ &= \left(\frac{l}{a}\right)(-1)^{\frac{\lambda C(j-1)}{4}}(-1)^{s\frac{2a\lambda C+\lambda^2C^2}{8}} && \text{(by (6)).}\end{aligned}$$

□

In order to make the next lemmas more readable we need to introduce some helpful definitions:

Definition 2.4. *A tuple $(m, M, N, (r_\delta)) \in \Delta$ is said to be κ-proper, if*

$$\kappa N \sum_{\delta|M} r_\delta \frac{mN}{\delta} \equiv_{24} 0, \tag{36}$$

and

$$\kappa N \sum_{\delta|M} r_\delta \equiv_8 0, \tag{37}$$

for $\kappa = \gcd(1-m^2, 24)$.

Definition 2.5. *For $(m, M, N, (r_\delta)) \in \Delta$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)^*$ and λ a non-negative integer we define:*

$$\beta(\gamma,\lambda) := e^{\sum_{\delta|M}\frac{2\pi i r_\delta \delta a m b}{24}} \prod_{\delta|M}\left(\frac{mc\delta}{a+\kappa\lambda c}\right)^{|r_\delta|} e^{-\frac{(a+\kappa\lambda c)\pi i}{12}\sum_{\delta|M} r_\delta(mc/\delta-3)}. \tag{38}$$

Definition 2.6. *For M a positive integer and $(r_\delta) \in R(M)$ let $\pi(M, (r_\delta)) := (s, j)$ where s is a non-negative integer and j an odd integer uniquely determined by $\prod_{\delta|M} \delta^{|r_\delta|} = 2^s j$.*

Lemma 2.7. *Let $(m, M, N, (r_\delta)) \in \Delta$ be κ-proper, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)^*$, $(s, j) := \pi(M, (r_\delta))$. Then for λ a non-negative integer the following relations hold:*

$$\beta(\gamma,\lambda) = \prod_{\delta|M}\left(\frac{mc\delta}{a+\kappa\lambda c}\right)^{|r_\delta|} e^{-\frac{\pi i a}{12}\left(\sum_{\delta|M}\frac{mc}{\delta}r_\delta - \sum_{\delta|N} r_\delta \delta m b - 3\sum_{\delta|M} r_\delta\right)}, \tag{39}$$

and

$$\beta(\gamma,\lambda) = \begin{cases} \beta(\gamma,0) & ,\ \text{if } \kappa c \equiv_8 0 \\ \beta(\gamma,0)(-1)^{\frac{\kappa\lambda c(j-1)}{4}}(-1)^{s\frac{2a\kappa\lambda c+\kappa^2\lambda^2c^2}{8}} & ,\ \text{if } \sum_{\delta|M} r_\delta \equiv_2 0 \end{cases}. \tag{40}$$

Proof. (39)*:* From its definition $\beta(\gamma,\lambda)$ can be rewritten as

$$=\prod_{\delta|M}\left(\frac{mc\delta}{a+\kappa\lambda c}\right)^{|r_\delta|} e^{-\frac{\pi i a}{12}(\sum_{\delta|M} r_\delta\frac{mc}{\delta}-\sum_{\delta|M} mb\delta r_\delta-3\sum_{\delta|M} r_\delta)}$$
$$\cdot e^{-\frac{\pi i\kappa\lambda c}{12}(\sum_{\delta|M} r_\delta\frac{mc}{\delta}-3\sum_{\delta|M} r_\delta)}.$$

Because of $N|c$, (36) and (37) we can conclude that $\sum_{\delta|M} r_\delta c\kappa\frac{mc}{\delta}\equiv_{24} 0$ and $\kappa c\sum_{\delta|M} r_\delta\equiv_8 0$. Hence

$$e^{-\frac{\pi i\kappa\lambda c}{12}(\sum_{\delta|M} r_\delta\frac{mc}{\delta}-3\sum_{\delta|M} r_\delta)}=1. \tag{41}$$

(40)*:* Condition (37) implies that either $\sum_{\delta|M} r_\delta\equiv_2 0$ or $\kappa N\equiv_8 0$. From (39) by Lemma 2.3 we see that if $\kappa c\equiv_8 0$ then $\beta(\gamma,\lambda)=\beta(\gamma,0),\lambda\geq 0$.

If $\sum_{\delta|M} r_\delta\equiv_2 0$ we have

$$\begin{aligned}\prod_{\delta|M}\left(\frac{\delta mc}{a+\kappa\lambda c}\right)^{|r_\delta|} &=\left(\frac{\prod_{\delta|M}\delta^{|r_\delta|}}{a+\kappa\lambda c}\right) && \text{(by (6))}\\ &=\left(\frac{\prod_{\delta|M}\delta^{|r_\delta|}}{a}\right)(-1)^{\frac{\kappa\lambda c(j-1)}{4}}(-1)^{s\frac{2a\kappa\lambda c+\kappa^2\lambda^2c^2}{8}} && \text{(by Lemma 2.3)}\\ &=\prod_{\delta|M}\left(\frac{\delta mc}{a}\right)^{|r_\delta|}(-1)^{\frac{\kappa\lambda c(j-1)}{4}}(-1)^{s\frac{2a\kappa\lambda c+\kappa^2\lambda^2c^2}{8}} && \text{(by (6)).}\end{aligned}$$

In view of (39) this implies that

$$\beta(\gamma,\lambda)=\beta(\gamma,0)(-1)^{\frac{\kappa\lambda c(j-1)}{4}}(-1)^{s\frac{2a\kappa\lambda c+\kappa^2\lambda^2c^2}{8}}. \tag{42}$$

Note that in order to apply Lemma 2.3 above we need to verify that $p|\prod_{\delta|M}\delta^{|r_\delta|}$ implies $p|\kappa c$ and that $\gcd(a,\prod_{\delta|M}\delta^{|r_\delta|})=1$. This follows from (18) and (19) together with $\gcd(a,c)=1$ because of $ad-bc=1$.

□

Lemma 2.8. *Let $(m,M,N,(r_\delta))\in\Delta$ be κ-proper, $\gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma_0(N)^*$ and t an integer with $0\leq t\leq m-1$ such that the relation*

$$\frac{24m}{\gcd(\kappa(-24t-\sum_{\delta|M}\delta r_\delta),24m)}\mid N, \tag{43}$$

holds, then for $\tau\in\mathbb{H}^*$ we have that

$$\begin{aligned} g_{m,t}(\gamma\tau)=&(-i(c\tau+d))^{\frac{\sum_{\delta|M} r_\delta}{2}}e^{\frac{2\pi i ab(1-m^2)(24t+\sum_{\delta|M}\delta r_\delta)}{24m}}\\ &\cdot\frac{1}{m}\sum_{\lambda=0}^{m-1}\beta(\gamma,\lambda)e^{\frac{2\pi i\kappa\mu a^2(-24t-\sum_{\delta|M}\delta r_\delta)}{24m}}\prod_{\delta|M}\eta^{r_\delta}\left(\frac{\delta(\tau+\kappa\mu)}{m}\right),\end{aligned} \tag{44}$$

where μ is defined as in (21)*.*

Proof. Given two integers λ,λ' such that $\lambda\equiv_c\lambda'$, relation (43) implies

$$\lambda\equiv_{\frac{24m}{\gcd(24m,-24t-\sum_{\delta|M}\delta r_\delta)}}\lambda',$$

consequently

$$e^{\frac{2\pi i\lambda\kappa(-24t-\sum_{\delta|M}\delta r_\delta)}{24m}}=e^{\frac{2\pi i\lambda'\kappa(-24t-\sum_{\delta|M}\delta r_\delta)}{24m}}.$$

Therefore by *(v)* in Lemma 2.2 we conclude that:

$$e^{\frac{2\pi i\lambda\kappa(-24t-\sum_{\delta|M}\delta r_\delta)}{24m}} = e^{\frac{2\pi i\kappa(\mu a^2-\frac{ab(1-m^2)}{\kappa})(-24t-\sum_{\delta|M}\delta r_\delta)}{24m}}. \tag{45}$$

Hence,

$$g_{m,t}(\gamma\tau) = \frac{1}{m}\sum_{\lambda=0}^{m-1} e^{\frac{2\pi i\kappa\lambda(-24t-\sum_{\delta|M}\delta r_\delta)}{24m}} \prod_{\delta|M}\eta^{r_\delta}\left(\frac{\delta(\gamma\tau+\kappa\lambda)}{m}\right)$$

(by (16))

$$= (-i(c\tau+d))^{\frac{\sum_{\delta|M} r_\delta}{2}} \cdot \frac{1}{m}\sum_{\lambda=0}^{m-1} \beta(\gamma,\lambda) e^{\frac{2\pi i\kappa\lambda(-24t-\sum_{\delta|M}\delta r_\delta)}{24m}} \prod_{\delta|M}\eta^{r_\delta}\left(\frac{\delta(\tau+\kappa\mu)}{m}\right)$$

(by (22), (23) and (38))

$$= (-i(c\tau+d))^{\frac{\sum_{\delta|M} r_\delta}{2}} e^{\frac{2\pi i ab(1-m^2)(24t+\sum_{\delta|M}\delta r_\delta)}{24m}} \cdot \frac{1}{m}\sum_{\lambda=0}^{m-1} \beta(\gamma,\lambda) e^{\frac{2\pi i\kappa\mu a^2(-24t-\sum_{\delta|M}\delta r_\delta)}{24m}} \prod_{\delta|M}\eta^{r_\delta}\left(\frac{\delta(\tau+\kappa\mu)}{m}\right)$$

(by (45)).

□

Lemma 2.9. *Let $(m, M, N, (r_\delta)) \in \Delta$ be κ-proper, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)^*$, and t an integer with $0 \le t \le m-1$ such that (43) holds. Let t' be the unique integer satisfying $0 \le t' \le m-1$ and $t' \equiv_m ta^2 + \frac{a^2-1}{24}\sum_{\delta|M}\delta r_\delta$. Assume that $\kappa N \equiv_8 0$, then for $\tau \in \mathbb{H}^*$ we have that*

$$g_{m,t}(\gamma\tau) = \beta(\gamma, 0)\,(-i(c\tau+d))^{\frac{\sum_{\delta|M} r_\delta}{2}} e^{\frac{2\pi i ab(1-m^2)(24t+\sum_{\delta|M}\delta r_\delta)}{24m}} g_{m,t'}(\tau). \tag{46}$$

Proof.

$$g_{m,t}(\gamma\tau) = \beta(\gamma, 0)\,(-i(c\tau+d))^{\frac{\sum_{\delta|M} r_\delta}{2}} e^{\frac{2\pi i ab(1-m^2)(24t+\sum_{\delta|M}\delta r_\delta)}{24m}} \cdot \frac{1}{m}\sum_{\lambda=0}^{m-1} e^{\frac{2\pi i\kappa\mu(-24t'-\sum_{\delta|M}\delta r_\delta)}{24m}} \prod_{\delta|M}\eta^{r_\delta}\left(\frac{\delta(\tau+\kappa\mu)}{m}\right)$$

(by (44) and because $\beta(\gamma, 0) = \beta(\gamma, \lambda), \lambda \in \mathbb{Z}$ by (40))

$$= \beta(\gamma, 0)\,(-i(c\tau+d))^{\frac{\sum_{\delta|M} r_\delta}{2}} e^{\frac{2\pi i ab(1-m^2)(24t+\sum_{\delta|M}\delta r_\delta)}{24m}} g_{m,t'}(\tau)$$

(by (16) and *(iv)* in Lemma 2.2).

□

Lemma 2.10. *Let $(m, M, N, (r_\delta)) \in \Delta$ be κ-proper, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)^*$, $(s, j) := \pi(M, (r_\delta))$ and t an integer with $0 \le t \le m-1$ such that (43) holds. Assume further that $\sum_{\delta|M} r_\delta \equiv_2 0$ and $2|m$.*

(i) *If* $s \equiv_2 0$ *let* t' *be the unique integer satisfying* $t' \equiv_m ta^2 + \frac{a^2-1}{24}\sum_{\delta|M}\delta r_\delta - \frac{3mca^2(j-1)}{24}$ *and* $0 \le t' \le m-1$*. Then for* $\tau \in \mathbb{H}^*$ *we have that*

(47)
$$g_{m,t}(\gamma\tau) = (-1)^{\frac{abc(1-m^2)(j-1)}{4}}\beta(\gamma,0) \cdot (-i(c\tau+d))^{\frac{\sum_{\delta|M} r_\delta}{2}} e^{\frac{2\pi i ab(1-m^2)(24t+\sum_{\delta|M}\delta r_\delta)}{24m}} g_{m,t'}(\tau).$$

(ii) *If* $\kappa c \equiv_4 0$ *let* t' *be the unique integer satisfying* $t' \equiv_m -\frac{3mcsa^2}{24} + ta^2 + \frac{a^2-1}{24}\sum_{\delta|M}\delta r_\delta$ *and* $0 \le t' \le m-1$*. Then for* $\tau \in \mathbb{H}^*$ *we have that*

(48)
$$g_{m,t}(\gamma\tau) = (-1)^{\frac{sa^2bc(1-m^2)}{4}}\beta(\gamma,0) \cdot (-i(c\tau+d))^{\frac{\sum_{\delta|M} r_\delta}{2}} e^{\frac{2\pi i ab(1-m^2)(24t+\sum_{\delta|M}\delta r_\delta)}{24m}} g_{m,t'}(\tau).$$

Proof. (i):

$$g_{m,t}(\gamma\tau) = (-i(c\tau+d))^{\frac{\sum_{\delta|M} r_\delta}{2}} e^{\frac{2\pi i ab(1-m^2)(24t+\sum_{\delta|M}\delta r_\delta)}{24m}}\beta(\gamma,0) \cdot \frac{1}{m}\sum_{\lambda=0}^{m-1}(-1)^{\frac{\kappa\lambda c(j-1)}{4}} e^{\frac{2\pi i\kappa\mu a^2(-24t-\sum_{\delta|M}\delta r_\delta)}{24m}}\prod_{\delta|M}\eta^{r_\delta}\left(\frac{\delta(\tau+\kappa\mu)}{m}\right)$$

(by (44) and (42), together with $2|m$ which implies $2|c$ because of (18))

$$=(-1)^{\frac{abc(1-m^2)(j-1)}{4}}\beta(\gamma,0)\,(-i(c\tau+d))^{\frac{\sum_{\delta|M} r_\delta}{2}} e^{\frac{2\pi i ab(1-m^2)(24t+\sum_{\delta|M}\delta r_\delta)}{24m}} \cdot \frac{1}{m}\sum_{\lambda=0}^{m-1}(-1)^{\frac{\kappa\mu a^2 c(j-1)}{4}} e^{\frac{2\pi i\kappa\mu a^2(-24t-\sum_{\delta|M}\delta r_\delta)}{24m}}\prod_{\delta|M}\eta^{r_\delta}\left(\frac{\delta(\tau+\kappa\mu)}{m}\right)$$

(by (24) and $c \equiv_2 0$)

$$=(-1)^{\frac{abc(1-m^2)(j-1)}{4}}\beta(\gamma,0)\,(-i(c\tau+d))^{\frac{\sum_{\delta|M} r_\delta}{2}} e^{\frac{2\pi i ab(1-m^2)(24t+\sum_{\delta|M}\delta r_\delta)}{24m}} \cdot \frac{1}{m}\sum_{\lambda=0}^{m-1} e^{\frac{2\pi i\kappa\mu a^2(3mc(j-1)-24t-\sum_{\delta|M}\delta r_\delta)}{24m}}\prod_{\delta|M}\eta^{r_\delta}\left(\frac{\delta(\tau+\kappa\mu)}{m}\right)$$
$$=(-1)^{\frac{abc(1-m^2)(j-1)}{4}}\beta(\gamma,0)\,(-i(c\tau+d))^{\frac{\sum_{\delta|M} r_\delta}{2}} e^{\frac{2\pi i ab(1-m^2)(24t+\sum_{\delta|M}\delta r_\delta)}{24m}} \cdot \frac{1}{m}\sum_{\lambda=0}^{m-1} e^{\frac{2\pi i\kappa\mu(-24t'-\sum_{\delta|M}\delta r_\delta)}{24m}}\prod_{\delta|M}\eta^{r_\delta}\left(\frac{\delta(\tau+\kappa\mu)}{m}\right)$$

(by substituting for t')

$$=(-1)^{\frac{abc(1-m^2)(j-1)}{4}}\beta(\gamma,0)\,(-i(c\tau+d))^{\frac{\sum_{\delta|M} r_\delta}{2}} e^{\frac{2\pi i ab(1-m^2)(24t+\sum_{\delta|M}\delta r_\delta)}{24m}} g_{m,t'}(\tau)$$

(by (16) and *(iv)* in Lemma 2.2).

(ii):

$$g_{m,t}(\gamma\tau) = (-i(c\tau+d))^{\frac{\sum_{\delta|M} r_\delta}{2}} e^{\frac{2\pi i ab(1-m^2)(24t+\sum_{\delta|M}\delta r_\delta)}{24m}} \beta(\gamma,0) \cdot \frac{1}{m}\sum_{\lambda=0}^{m-1}(-1)^{\frac{sa\kappa\lambda c}{4}} e^{\frac{2\pi i\kappa\mu a^2(-24t-\sum_{\delta|M}\delta r_\delta)}{24m}} \prod_{\delta|M}\eta^{r_\delta}\left(\frac{\delta(\tau+\kappa\mu)}{m}\right)$$

(by (44) and (42))

$$=(-1)^{\frac{sa^2bc(1-m^2)}{4}}\beta(\gamma,0)\,(-i(c\tau+d))^{\frac{\sum_{\delta|M} r_\delta}{2}} e^{\frac{2\pi i ab(1-m^2)(24t+\sum_{\delta|M}\delta r_\delta)}{24m}} \cdot \frac{1}{m}\sum_{\lambda=0}^{m-1}(-1)^{\frac{\kappa\mu a^3 cs}{4}} e^{\frac{2\pi i\kappa\mu a^2(-24t-\sum_{\delta|M}\delta r_\delta)}{24m}} \prod_{\delta|M}\eta^{r_\delta}\left(\frac{\delta(\tau+\kappa\mu)}{m}\right)$$

(by (24) and $c \equiv_2 0$))

$$=(-1)^{\frac{sa^2bc(1-m^2)}{4}}\beta(\gamma,0)\,(-i(c\tau+d))^{\frac{\sum_{\delta|M} r_\delta}{2}} e^{\frac{2\pi i ab(1-m^2)(24t+\sum_{\delta|M}\delta r_\delta)}{24m}} \cdot \frac{1}{m}\sum_{\lambda=0}^{m-1} e^{\frac{2\pi i\kappa\mu(-24t'-\sum_{\delta|M}\delta r_\delta)}{24m}} \prod_{\delta|M}\eta^{r_\delta}\left(\frac{\delta(\tau+\kappa\mu)}{m}\right)$$

(by substituting for t')

$$=(-1)^{\frac{sa^2bc(1-m^2)}{4}}\beta(\gamma,0)\,(-i(c\tau+d))^{\frac{\sum_{\delta|M} r_\delta}{2}} e^{\frac{2\pi i ab(1-m^2)(24t+\sum_{\delta|M}\delta r_\delta)}{24m}} g_{m,t'}(\tau)$$

(by (16) and *(iv)* in Lemma 2.2).

□

Note that if $2 \nmid m$ then $\kappa N \equiv_8 0$ and Lemma 2.9 applies. If $2|m$ and $\kappa N \not\equiv_8 0$ then the Lemma 2.10 applies.

Let $(m, M, N, (r_\delta)) \in \Delta$ and s, j integers such that $\pi(M, (r_\delta)) = (s, j)$. In the next theorem we will also assume that:

$$\kappa N \equiv_4 0 \text{ or } s \equiv_2 0, \tag{49}$$

and

$$2 \nmid N \text{ or } 8|N(1-j) \text{ or } 8|Ns. \tag{50}$$

Definition 2.11. *We define*

$$\Delta^* := \{ \textit{ all tuples } (m, N, N, t, (r_\delta)) \textit{ with properties as listed in } (51)\} :$$

$$\begin{aligned} &(m, M, N, (r_\delta)) \in \Delta \textit{ is } \kappa\textit{-proper, } t \in \mathbb{N}, 0 \le t \le m-1; \\ &\textit{in addition } (43), (49) \textit{ and } (50) \textit{ hold.} \end{aligned} \tag{51}$$

Definition 2.12. *Let $m, M, N \in \mathbb{N}^*$ and $(r_\delta) \in R(M)$. Define the operation $\odot$: $\Gamma_0(N)^* \times \{0, \dots, m-1\} \mapsto \{0, \dots, m-1\}$, $(\gamma, t) \mapsto \gamma \odot t$, where for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the image $\gamma \odot t$ is uniquely defined by the relation*

$$\gamma \odot t \equiv_m ta^2 + \frac{a^2-1}{24} \sum_{\delta|M} \delta r_\delta. \tag{52}$$

Finally we arrive at the main theorem of this section which can be viewed as a generalization of a theorem of R. Lewis; see Remark 2.14 below.

Theorem 2.13. *Let $(m, M, N, t, (r_\delta) = r) \in \Delta^*$, $g_{m,t}(\tau, r)$ be as in Definition 1.11, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)^*$, and β as in Definition 2.5. Then for all $\tau \in \mathbb{H}^*$ we have that*

$$g_{m,t}(\gamma\tau, r) = \beta(\gamma, 0) \left(-i(c\tau + d)\right)^{\frac{\sum_{\delta|M} r_\delta}{2}} e^{2\pi i \frac{ab(1-m^2)(24t + \sum_{\delta|M} \delta r_\delta)}{24m}} \cdot g_{m,\gamma\odot t}(\tau, r). \tag{53}$$

Proof. If $\kappa N \equiv_8 0$ then (53) follows from Lemma 2.9. If $\kappa N \not\equiv_8 0$ then $2|m$, and because of (49), either *(i)* or *(ii)* in Lemma 2.10 apply. When *(i)* applies we have $\frac{3mca^2(j-1)}{24} \equiv_m 0$ because of (50), and consequently $t' = \gamma \odot t$. Also because of (50) we have that $(-1)^{\frac{abc(1-m^2)(j-1)}{4}}$ in (47) is equal to 1 for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)^*$ and (53) holds. When *(ii)* applies, the proof is analogous. $\square$

Remark 2.14. Theorem 2.13 extends Theorem 1 in [8] which covers products of the form $\prod_{n=1}^{\infty}(1-q^n)^{r_1}$ where r_1 is a fixed integer.

3. Formulas for $g_{m,t}(\gamma\tau)$ when $\gamma \in \Gamma$

Usually $g_{m,t}(\tau) = g_{m,t}(\tau, r)$ as defined in Definition 1.11 is not a modular form. But if we choose a sequence $(a_\delta) \in R(N)$ properly, we can always make sure that $\left(\prod_{t' \in P(t)} g_{m,t'}(\tau)\right) \left(\prod_{\delta|N} \eta^{a_\delta}(\delta\tau)\right)$ (with $P(t)$ as in (66)) is a modular form. To prove this we need some formulas for $\prod_{\delta|N} \eta^{a_\delta}(\delta(\gamma\tau))$ and for $g_{m,t}(\gamma\tau)$ that are valid for all γ in Γ, in order to check condition *(3)* in Definition 1.1 of a modular form. This is done in the Lemmas 3.1 to 3.6 below.

Recall that $\kappa = \gcd(1-m^2, 24)$.

Lemma 3.1. *Let $(m, M, N, (r_\delta)) \in \Delta$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. For $\delta|M$ with $\delta > 0$ and λ an integer let $x(\delta, \lambda)$ and $y(\delta, \lambda)$ be any fixed solutions to the equation $\delta(a + \kappa\lambda c) \cdot x(\delta, \lambda) + mc \cdot y(\delta, \lambda) = \gcd(\delta(a + \kappa\lambda c), mc)$. Further define*

$$w(\delta, \lambda, \gamma) := \frac{\gcd(\delta(a + \kappa\lambda c), mc)\tau + \delta(b + \kappa\lambda d)x(\delta, \lambda) + mdy(\delta, \lambda)}{\frac{\delta m}{\gcd(\delta(a+\kappa\lambda c), mc)}}. \tag{54}$$

Then there exists a map $C : \Gamma \mapsto \mathbb{C}$ such that for all $\gamma \in \Gamma$ and $\tau \in \mathbb{H}^*$ the following relation holds:

$$\prod_{\delta|M} \eta^{r_\delta} \left(\frac{\delta(\gamma\tau + \kappa\lambda)}{m}\right) = C(\gamma)(c\tau + d)^{\frac{1}{2}\sum_{\delta|M} r_\delta} \prod_{\delta|M} \eta^{r_\delta} \left(w(\delta, \lambda, \gamma)\right). \tag{55}$$

In addition, there exist mappings $C' : \Gamma \mapsto \mathbb{C}$ and $\mu : \mathbb{Z} \mapsto \mathbb{Z}$ such that for all $\gamma \in \Gamma_0(N)$ and $\tau \in \mathbb{H}^*$ the following relation holds:

$$\prod_{\delta|M} \eta^{r_\delta}\left(\frac{\delta(\gamma\tau+\kappa\lambda)}{m}\right) = C'(\gamma)(c\tau+d)^{\frac{1}{2}\sum_{\delta|M} r_\delta} \prod_{\delta|M} \eta^{r_\delta}\left(\frac{\delta(\tau+\kappa\mu(\lambda))}{m}\right), \tag{56}$$

where μ is chosen such that $[\lambda]_m \mapsto [\mu(\lambda)]_m$ is a bijection of $\mathbb{Z}_m$.

Proof. (55)*:* Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. Apply Lemma 1.13 with k set to $\frac{1}{2}$, $f(\tau)$ to $\eta(\tau)$, $g(\gamma,\tau)$ to $(-i)^{\frac{1}{2}}\epsilon(a,b,c,d)\eta(\tau)$ and ξ to $\begin{pmatrix} \frac{\delta(a+\kappa\lambda c)}{mc} & \frac{\delta(b+\kappa\lambda d)}{md} \end{pmatrix}$; then for all $\delta|M$ with $\delta > 0$ the following relation holds:

$$\begin{aligned}&\left(\frac{\gcd(\delta(a+\kappa\lambda c), mc)}{\delta m} m(c\tau+d)\right)^{-\frac{1}{2}} \eta\left(\frac{\delta((a+\kappa\lambda c)\tau+b+\kappa\lambda d)}{m(c\tau+d)}\right)\\ =&(-i)^{\frac{1}{2}}\epsilon\left(\frac{\delta(a+\kappa\lambda c)}{\gcd(\delta(a+\kappa\lambda c), mc)}, -y(\delta,\lambda), \frac{mc}{\gcd(\delta(a+\kappa\lambda c), mc)}, x(\delta,\lambda)\right)\eta(w(\delta,\lambda,\gamma)).\end{aligned}$$

Taking the product over $\delta|M$ on both sides and using that

$$\eta\left(\frac{\delta(\gamma\tau+\kappa\lambda)}{m}\right) = \eta\left(\frac{\delta((a+\kappa\lambda c)\tau+b+\kappa\lambda d)}{m(c\tau+d)}\right)$$

proves (55).

(56)*:* In order to prove (56) we first will prove that $\gcd(\delta(a+\kappa\lambda c), mc) = \delta$ if $N|c$. By (19) we see that $\delta|mc$ hence $\gcd(\delta(a+\kappa\lambda c), mc) = \delta\gcd(a+\kappa\lambda c, \frac{mc}{\delta})$. Also since $\gcd(a+\kappa\lambda c, c) = 1$ because of $ad-bc = 1$, and $\gcd(a+\kappa\lambda c, m) = 1$ because of (18), we can conclude that $\gcd(a+\kappa\lambda c, \frac{mc}{\delta}) = 1$. Next, for $\lambda \in \mathbb{Z}$ let $x_0(\lambda)$ and $y_0(\lambda)$ be any solutions to the equation $(a+\kappa\lambda c)x_0(\lambda)+mcy_0(\lambda) = 1$. Then we can define $x(\delta,\lambda) := x_0(\lambda)$ and $y(\delta,\lambda) := \delta y_0(\lambda)$ because of $\gcd(\delta(a+\kappa\lambda c), mc) = \delta$. Consequently,

$$\eta(w(\delta,\lambda,\gamma)) = \eta\left(\frac{\delta\tau+\delta(b+\kappa\lambda d)x_0(\lambda)}{m} + \delta d y_0(\lambda)\right). \tag{57}$$

Next, let X and Y be integers such that $\kappa X + mY = 1$. Such integers clearly exist by (12). Define $\mu(\lambda) := (b+\kappa\lambda d)Xx_0(\lambda)$. Then

$$\begin{aligned}\eta\left(\frac{\delta(\tau+\kappa\mu(\lambda))}{m}\right) &= \eta\left(\frac{\delta(\tau+\kappa(b+\kappa\lambda d)Xx_0(\lambda))}{m}\right)\\ &=\eta\left(\frac{\delta(\tau+(b+\kappa\lambda d)x_0(\lambda))}{m} - Y(b+\kappa\lambda d)x_0(\lambda)\right).\end{aligned} \tag{58}$$

This shows that

$$\eta(w(\delta,\lambda,\gamma)) = \epsilon\eta\left(\frac{\delta(\tau+\kappa\mu(\lambda))}{m}\right)$$

for some 24-th root of unity ϵ because of (30) and by (57) and (58). It only remains to show that μ is a bijection of $\mathbb{Z}_m$. Note that $x_0(\lambda)$ is invertible modulo m because of $(a+\kappa\lambda c)x_0(\lambda)+mcy_0(\lambda) = 1$ implying $(\mu(\lambda)X^{-1}x_0(\lambda)^{-1} - b)\kappa^{-1}d^{-1} \equiv_m \lambda$. Note that d is invertible modulo m because of $\gcd(c,d) = 1$ which by (18) implies $\gcd(m,d) = 1$. □

Remark 3.2. Note that (56) is very similar to (22) in Lemma 2.2 but here we lifted the restriction $\gcd(a,6) = 1$, $a > 0$, $c > 0$.

Lemma 3.3. *Let $\gamma_0 \in \Gamma$, $(m, M, N, (r_\delta)) \in \Delta$, $t \in \mathbb{Z}$ with $0 \leq t \leq m-1$, and define the mappings $p : \Gamma \times [0, \dots, m-1] \mapsto \mathbb{Q}$ and $p : \Gamma \mapsto \mathbb{Q}$ by*

$$p(\gamma, \lambda) := \frac{1}{24} \sum_{\delta | M} r_\delta \frac{\gcd^2(\delta(a + \kappa\lambda c), mc)}{\delta m}, \tag{59}$$

and

$$p(\gamma) := \min_{\lambda \in \{0, \dots, m-1\}} p(\gamma, \lambda). \tag{60}$$

Then for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)\gamma_0\Gamma_\infty$ there exists a positive integer k and a Taylor series $h(q)$ in powers of $q^{\frac{1}{k}}$ such that for $\tau \in \mathbb{H}^*$ we have

$$(c\tau + d)^{-\frac{1}{2}\sum_{\delta|M} r_\delta} g_{m,t}(\gamma\tau) = h(q) q^{p(\gamma_0)}. \tag{61}$$

Proof. We write $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \gamma_N \gamma_0 \gamma_\infty$ where $\gamma_N = \begin{pmatrix} a_N & b_N \\ c_N & d_N \end{pmatrix} \in \Gamma_0(N)$, $\gamma_\infty = \begin{pmatrix} 1 & b_\infty \\ 0 & 1 \end{pmatrix} \in \Gamma_\infty$ and $\gamma_0 = \begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix} \in \Gamma$. Then

$$g_{m,t}(\gamma\tau) = \frac{1}{m} \sum_{\lambda=0}^{m-1} C_1(\lambda) \prod_{\delta|M} \eta^{r_\delta}\left(\frac{\delta(\gamma\tau + \kappa\lambda)}{m}\right)$$

(by (16)) with suitably chosen $C_1 : \{0, \dots, m-1\} \mapsto \mathbb{C}$)

$$= (c_N(\gamma_0\gamma_\infty\tau) + d_N)^{\frac{1}{2}\sum_{\delta|M} r_\delta} \cdot \frac{1}{m} \sum_{\mu=0}^{m-1} C_2(\mu(\lambda)) \prod_{\delta|M} \eta^{r_\delta}\left(\frac{\delta(\gamma_0\gamma_\infty\tau + \kappa\mu(\lambda))}{m}\right)$$

(by (56) with suitably chosen $C_2 : \{0, \dots, m-1\} \mapsto \mathbb{C}$)

$$= ((c_N(\gamma_0\gamma_\infty\tau) + d_N)(c_0(\gamma_\infty\tau) + d_0))^{\frac{1}{2}\sum_{\delta|M} r_\delta} \cdot \frac{1}{m} \sum_{\mu=0}^{m-1} C_3(\mu(\lambda)) \prod_{\delta|M} \eta^{r_\delta}\left(\frac{\gcd^2(\delta(a_0 + \kappa\mu(\lambda)c_0), mc)\tau + C_4(\mu(\lambda))}{\delta m}\right)$$

(by (55) with suitably chosen $C_3 : \{0, \dots, m-1\} \mapsto \mathbb{C}$ and $C_4 : \{0, \dots, m-1\} \mapsto \mathbb{C}$)

$$= (c\tau + d)^{\frac{1}{2}\sum_{\delta|M} r_\delta} \cdot \frac{1}{m} \sum_{\mu=0}^{m-1} C_3(\mu(\lambda)) \prod_{\delta|M} \eta^{r_\delta}\left(\frac{\gcd^2(\delta(a_0 + \kappa\mu(\lambda)c_0), mc)\tau + C_4(\mu(\lambda))}{\delta m}\right)$$

(because of $\begin{pmatrix} a_N & b_N \\ c_N & d_N \end{pmatrix}\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}\begin{pmatrix} 1 & b_\infty \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$)

$$= (c\tau + d)^{\frac{1}{2}\sum_{\delta|M} r_\delta} \sum_{\mu(\lambda)=0}^{m-1} C_3(\mu(\lambda)) q^{p(\gamma_0, \mu(\lambda))} h(\mu(\lambda), q)$$

(where for each $\mu(\lambda)$, $h(\mu(\lambda),q)$ is a Taylor series in powers of $q^{24p(\gamma_0,\mu(\lambda))}$ by (2))

$$=(c\tau+d)^{\frac{1}{2}\sum_{\delta|M} r_\delta} q^{p(\gamma_0)} h(q)$$

(with $h(q) := q^{p(\gamma_0)} \sum_{\mu=0}^{m-1} C_3(\mu(\lambda)) q^{p(\gamma_0,\mu(\lambda))-p(\gamma_0)} h(\mu(\lambda),q)$).

$\square$

Lemma 3.4. *Let $N \in \mathbb{N}^*$, $(a_\delta) \in R(N)$, $f(\tau) := \prod_{\delta|N} \eta^{a_\delta}(\delta\tau)$, and define the mapping $p^* : \Gamma \mapsto \mathbb{C}$ by $p^*\left(\begin{pmatrix} a_0 & b_0 \\ c_0 & d_0 \end{pmatrix}\right) := \frac{1}{24}\sum_{\delta|N} \frac{a_\delta \gcd^2(\delta,c_0)}{\delta}$. Then for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ there exists an integer k and a Taylor series $h^*(q)$ in powers of $q^{\frac{1}{k}}$ such that*

$$(c\tau+d)^{-\frac{1}{2}\sum_{\delta|N} a_\delta} f(\gamma\tau) = h^*(q) q^{p^*(\gamma)}. \tag{62}$$

Furthermore, for $\gamma_1 \in \Gamma$ and $\gamma_2 \in \Gamma_0(N)\gamma_1\Gamma_\infty$ we have $p^*(\gamma_1) = p^*(\gamma_2)$.

Proof. Let $w_\delta := \gcd(\delta a, c)\frac{\gcd(\delta a,c)\tau+\delta b x_\delta + d y_\delta}{\delta}$ where $x_\delta, y_\delta \in \mathbb{Z}$ such that $a\delta x_\delta + c y_\delta = \gcd(a\delta, c)$ for any fixed $\delta|N$ with $\delta > 0$. Then

$$\begin{aligned}(c\tau+d)^{-\frac{1}{2}\sum_{\delta|N} a_\delta} \prod_{\delta|N} \eta^{a_\delta}(\delta\gamma\tau) =&(c\tau+d)^{-\frac{1}{2}\sum_{\delta|N} a_\delta} \prod_{\delta|N} \eta^{a_\delta}\left(\frac{\frac{\delta a}{\gcd(\delta a,c)} w_\delta - y_\delta}{\frac{c}{\gcd(\delta a,c)} w_\delta + x_\delta}\right)\\ =&C \prod_{\delta|N} \eta^{a_\delta}(w_\delta)\end{aligned}$$

(by (9)) with suitably chosen $C \in \mathbb{C}$)

$$=C q^{p^*(\gamma)} \prod_{\delta|N} h^*(\delta,q)$$

(by (2) for some Taylor series $h^*(\delta,q)$ where $\delta|N$ (with constant term 1)). This proves (62).

To prove the remaining part of Lemma 3.4 let $\gamma_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\gamma_2 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Because of $\gamma_2 \in \Gamma_0(N)\gamma_1\Gamma_\infty$ we have that $\gamma_2 = \gamma_N\gamma_1\gamma_\infty$ for some $\gamma_N = \begin{pmatrix} a' & b' \\ c'N & d' \end{pmatrix} \in \Gamma_0(N)$ and $\gamma_\infty = \begin{pmatrix} 1 & b_\infty \\ 0 & 1 \end{pmatrix} \in \Gamma_\infty$. This shows that $C = ac'N + d'c$ and clearly $\gcd(d', c'N) = 1$ because of $a'd' - c'Nd' = 1$. For $\delta|N$ this implies that $\gcd(\delta, C) = \gcd(\delta, ac'\delta\frac{N}{\delta} + d'c) = \gcd(\delta, d'c) = \gcd(\delta, c)$. By this we have shown that the sums $p^*(\gamma_1)$ and $p^*(\gamma_2)$ have the same summands which proves that they are identical. $\square$

Theorem 3.5. *Let $(m, M, N, (r_\delta)) \in \Delta$, $t \in \mathbb{Z}$ with $0 \le t \le m-1$, p be as in Lemma 3.3, (a_δ) and p^* be as in Lemma 3.4, and $\gamma_0 \in \Gamma$. Then for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)\gamma_0\Gamma_\infty$ the expression*

$$q^{-(p(\gamma_0)+p^*(\gamma_0))}(c\tau+d)^{-\frac{1}{2}\sum_{\delta|M} r_\delta - \frac{1}{2}\sum_{\delta|N} a_\delta} g_{m,t}(\gamma\tau)\prod_{\delta|N}\eta^{a_\delta}(\delta(\gamma\tau)) \tag{63}$$

finds a representation as a Taylor series in powers of $q^{\frac{1}{k}}$ for some positive integer k.

Proof. By Lemmas 3.3 and 3.4, for each $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)\gamma_0\Gamma_\infty$ there exists a positive integer k, and Taylor series $h(q)$ and $h^*(q)$ in powers of $q^{\frac{1}{k}}$ such that

$$(c\tau+d)^{-\frac{1}{2}(\sum_{\delta|M} r_\delta + \sum_{\delta|N} a_\delta)} g_{m,t}(\gamma\tau)\prod_{\delta|N}\eta^{a_\delta}(\delta(\gamma\tau)) = h(q)h^*(q)q^{p(\gamma_0)+p^*(\gamma_0)}. \tag{64}$$

□

Lemma 3.6. *Let $F : \mathbb{H}^* \mapsto \mathbb{C}$ be a mapping, k an integer, and l a positive integer. Assume that for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ there exists a positive integer n and a Taylor series $h(\gamma, q)$ in powers of $q^{\frac{1}{n}}$ such that for all $\tau \in \mathbb{H}^*$ the relation $(c\tau+d)^{-k}F(\gamma\tau) = h(\gamma, q)$ holds. Then for all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ there exists a positive integer n' and a Taylor series $h^*(\gamma, q)$ in powers of $q^{\frac{1}{n'}}$ such that for all $\tau \in \mathbb{H}^*$ the relation $(c\tau+d)^{-k}F(l(\gamma\tau)) = h^*(\gamma, q)$ holds.*

Proof. We apply Lemma 1.13 with $f(\tau) = F(\tau)$, $g(\gamma, \tau) = h(\gamma, q)$, $\xi = \begin{pmatrix} al & bl \\ c & d \end{pmatrix}$, $g := \gcd(al, c)$, and x, y some integers such that $alx + cy = g$. As a consequence we have that

$$\left(\frac{g}{l}(c\tau+d)\right)^{-k} f(l(\gamma\tau)) = h^*\left(\begin{pmatrix} \frac{al}{g} & -y \\ \frac{c}{g} & x \end{pmatrix}, q^{\frac{g^2}{l}} e^{\frac{2\pi i g}{l}(blx+dy)}\right). \tag{65}$$

Choosing $n' = \frac{g^2}{l}n$ and

$$h^*(\gamma, q) = (g/l)^k h\left(\begin{pmatrix} \frac{al}{g} & -y \\ \frac{c}{g} & x \end{pmatrix}, q^{\frac{g^2}{l}} e^{\frac{2\pi i g}{l}(blx+dy)}\right)$$

concludes the proof. □

Definition 3.7. *We define*

$$\mathbb{Z}_n^* := \{[x]_n \in \mathbb{Z}_n | \gcd(x, n) = 1\},$$

and

$$\mathbb{S}_n := \{y^2 | y \in \mathbb{Z}_n^*\}.$$

Lemma 3.8. *For all integers $w \ge 2$ we have $24\sum_{s\in\mathbb{S}_w} s = [0]_w$. If $\gcd(w, 6) = 1$ then $\sum_{s\in\mathbb{S}_w} s = [0]_w$.*

Proof. If $\gcd(w,6) = 1$ then $[2^2]_w \in \mathbb{S}_w$, which implies that $[2^2]_w \sum_{s\in\mathbb{S}_w} s = \sum_{s\in\mathbb{S}_w} s$. This is because multiplication by an element of $\mathbb{S}_w$ just permutes the summands. Consequently $[2^2-1]_w \sum_{s\in\mathbb{S}_w} s = [0]_w$, but 2^2-1 is invertible modulo w and we can conclude that $\sum_{s\in\mathbb{S}_w} s = [0]_w$. If we assume that $w = 2^s3^t$ then $[5^2-1]_w \sum_{s\in\mathbb{S}_w} s = [0]_w$. Next consider a general $w = 2^s3^tu, \gcd(u,6) = 1$. We have a ring isomorphism $\phi : \mathbb{Z}_{2^s3^tu} \mapsto \mathbb{Z}_{2^s3^t} \times \mathbb{Z}_u$ given by $\phi([x]_{2^s3^tu}) = ([x]_{2^s3^t}, [x]_u)$. Obviously,

$$\begin{aligned}\phi([24]_w \sum_{s\in\mathbb{S}_w} s) = \phi([24]_w) \sum_{\substack{s \in \mathbb{S}_u,\\ s' \in \mathbb{S}_{2^s3^t}}} (s,s')\\ =&\phi([24]_w)([|\mathbb{S}_{2^s3^t}|]_u \sum_{s\in\mathbb{S}_u} s, [|\mathbb{S}_u|]_{2^s3^t} \sum_{s\in\mathbb{S}_{2^s3^t}} s)\\ =&([24|\mathbb{S}_u|]_{2^s3^t} \sum_{s\in\mathbb{S}_{2^s3^t}} s, [24|\mathbb{S}_{2^s3^t}|]_u \sum_{s\in\mathbb{S}_u} s) = ([0]_{2^s3^t}, [0]_u).\end{aligned}$$

Since ϕ is an isomorphism its kernel is $\{0\}$, which proves the lemma. ☐

Definition 3.9. *For $m, M \in \mathbb{N}^*$, $(r_\delta) \in R(M)$ and $t \in \mathbb{N}$ with $0 \le t \le m-1$ we define the map $\overline{\odot} : \mathbb{S}_{24m} \times \{0,\dots,m-1\} \mapsto \{0,\dots,m-1\}$ where the image $[s]_{24m}\overline{\odot}t$ is uniquely determined by the relation $[s]_{24m}\overline{\odot}t \equiv_m ts + \frac{s-1}{24}\sum_{\delta|M}\delta r_\delta$. We also define*

$$P(t) := \{[s]_{24m}\overline{\odot}t | [s]_{24m} \in \mathbb{S}_{24m}\}. \tag{66}$$

Lemma 3.10. *Let m, t, M, N be positive integers with $0 \le t \le m-1$ such that* (18) *holds. Let $(r_\delta) \in R(M)$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)^*$, and $\odot$ as in* (52)*. Moreover, define*

$$w := \frac{24m}{\gcd(\kappa(24t + \sum_{\delta|M} r_\delta), 24m)}.$$

Then the following statements hold:

(i) $\gamma \odot t = [a^2]_{24m}\overline{\odot}t$.
(ii) $[x]_{24m}\overline{\odot}t = [y]_{24m}\overline{\odot}t$ *iff* $x \equiv_w y$ *for all* $x, y \in \mathbb{Z}$.
(iii)

$$P(t) = \{\gamma \odot t | \gamma \in \Gamma_0(N)^*\}. \tag{67}$$

(iv) *For* $[s]_{24m} \in \mathbb{S}_{24m}$ *we have*

$$P(t) = \{[s]_{24m}\overline{\odot}t' | t' \in P(t)\}. \tag{68}$$

(v) $\chi := \prod_{t'\in P(t)} e^{2\pi i \frac{ab(1-m^2)(24t' + \sum_{\delta|M}\delta r_\delta)}{24m}}$ *is a 24-th root of unity.*

Proof. (i): If $\gamma \in \Gamma_0(N)^*$ then $\gcd(a,6) = 1$. By (18) and because of $\gcd(a,N) = 1$ we also have that $\gcd(a,m) = 1$, hence $\gcd(a,24m) = 1$. This means that $[a^2]_{24m} \in \mathbb{S}_{24m}$. *(ii):* Assume that $[s_1]_{24m}\overline{\odot}t = [s_2]_{24m}\overline{\odot}t$ for $[s_1]_{24m}, [s_2]_{24m} \in \mathbb{S}_{24m}$. Then

$$\kappa(s_1t + \frac{s_1-1}{24}\sum_{\delta|M}\delta r_\delta) \equiv_m \kappa(s_2t + \frac{s_2-1}{24}\sum_{\delta|M}\delta r_\delta)$$

because of $\gcd(\kappa,m) = 1$. Consequently,

$$\kappa(24t + \sum_{\delta|M}\delta r_\delta)(s_1 - s_2) \equiv_{24m} 0$$

and thus,

$$s_1 - s_2 \equiv_w 0.$$

(iii): By *(i)* we have

(69)

$$\{\gamma \odot t | \gamma \in \Gamma_0(N)^*\}$$
$$=\{[a^2]_{24m}\overline{\odot}t | \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)^*\} \subseteq \{[s]_{24m}\overline{\odot}t | [s]_{24m} \in \mathbb{S}_{24m}\}.$$

To show the other inclusion let $[s]_{24m} \in \mathbb{S}_{24m}$. By definition there exists an $[a]_{24m} \in \mathbb{Z}^*_{24m}$ such that $[s]_{24m} = [a^2]_{24m}$. Because $\gcd(a, 24) = 1$ we have $\gcd(a, 6) = 1$. We want to show that there exists a λ such that $\gcd(a + 24\lambda m, N) = 1$ because then there exist integers x and y such that

$$\begin{pmatrix} a + 24\lambda m & -y \\ N & x \end{pmatrix} \odot t = [s]_{24m}\overline{\odot}t$$

and the other inclusion is shown. It is sufficient to show that for each prime p with $p|N$ there exists an integer λ_p s.t. $\gcd(a + 24\lambda_p m, p) = 1$ because then by Chinese remaindering there exists a λ s.t. for all $p|N$ we have that $\lambda \equiv_p \lambda_p$. If p is such that $p|N$ and $p|24m$ then we simply choose $\lambda_p = 0$. If $p|N$ and $p \nmid 24m$ and $p|a$ then choose $\lambda_p = 1$, if $p \nmid a$ choose $\lambda_p = 0$.

(iv): We have to show that given $[s]_{24m} \in \mathbb{S}_{24m}$, the mapping $[s]_{24m}\overline{\odot}t : P(t) \mapsto P(t)$ is a bijection. This is clear because the inverse is $[s]^{-1}_{24m}\overline{\odot}t$.

(v): Let S be a subset of $\mathbb{S}_{24m}$ such that for $[r_1]_{24m}, [r_2]_{24m} \in S$ we have $r_1 \not\equiv_w r_2$, and such that for all $[s]_{24m} \in \mathbb{S}_{24m}$ there exists $[r]_{24m} \in S$ with $r \equiv_w s$. Then by *(ii)*:

$$P(t) = \{ts + \frac{s-1}{24}\sum_{\delta|M} \delta r_\delta | [s]_{24m} \in S\}.$$

It is straight-forward to prove that the set S gives a complete set of representatives of $\mathbb{S}_w$. Next note that

$$\begin{aligned}
\chi = & \prod_{t' \in P(t)} e^{2\pi i \frac{ab(1-m^2)(24t' + \sum_{\delta|M} \delta r_\delta)}{24m}} \\
= & \prod_{[s]_w \in \mathbb{S}_w} e^{2\pi i \frac{sab(1-m^2)(24t + \sum_{\delta|M} \delta r_\delta)}{24m}} \\
= & e^{\frac{2\pi i ab(1-m^2)(24t + \sum_{\delta|M} \delta r_\delta)}{24m} \sum_{[s]_w \mathbb{S}_w} s}.
\end{aligned}$$

Since $\kappa|(1 - m^2)$ and $24\sum_{s \in \mathbb{S}_w} s \equiv_w 0$, by Lemma 3.8 we conclude that χ is a 24-th root of unity.

□

4. Proving Congruences by Sturm's Theorem

4.1. **Proof Strategy.** Let M be a positive integer and $r = (r_\delta) \in R(M)$. Let $f(\tau, r) = \prod_{\delta|M} \prod_{n=1}^{\infty} (1 - q^{\delta n})^{r_\delta} = \sum_{n=0}^{\infty} a(n) q^n$ be as in Definition 1.11. Let m and u be positive integers and t an integer satisfying $0 \leq t \leq m - 1$. We want to prove or disprove the conjecture $a(mn+t) \equiv_u 0, n \geq 0$. It is convenient to introduce the following definition:

Definition 4.1. *For u a positive integer and $c(\tau) := \sum_{n=0}^{\infty} c(n)q^n$ a power series we define* $\mathrm{ord}_u(c(\tau)) := \inf\{n \mid u \nmid c(n)\}$*; we write $c(\tau) \equiv_u 0$ if* $\mathrm{ord}_u(c(\tau)) = \infty$.

First note that if $c_1(\tau)$ and $c_2(\tau)$ are power series in q and if p is a prime number then the relation $c_1(\tau)c_2(\tau) \equiv_p 0$ implies either $c_1(\tau) \equiv_p 0$ or $c_2(\tau) \equiv_p 0$.

Let u' be a divisor of u and assume that we already know that $u'|a(mn+t)$ for all $n \geq 0$. If we can prove that $\frac{a(mn+t)}{u'} \equiv_p 0$ for some prime p dividing u/u', clearly it follows by induction that $a(mn+t) \equiv_u 0, n \geq 0$. In other words, our aim is to prove

$$\frac{1}{u'} \sum_{n=0} a(mn+t)q^n \equiv_p 0,$$

which is equivalent to proving

$$\left(\frac{1}{u'} \sum_{n=0}^{\infty} a(mn+t)q^n\right)^{24} \equiv_p 0,$$

which in turn is equivalent to proving

$$H(\tau) := \left(\frac{1}{u'} \sum_{n=0}^{\infty} a(mn+t)q^n\right)^{24} h_1(\tau) \equiv_p 0, \tag{70}$$

where $h_1(\tau)$ is a power series in q with $h_1(\tau) \not\equiv_p 0$. We will choose $h_1(\tau)$ in such a way that $H(\tau)$ becomes a modular form of weight k for some subgroup G of Γ and some positive integer k. Then by Theorem 4.2 below it is sufficient to show that $\mathrm{ord}_p(H(\tau)) > \frac{k}{12}[\Gamma : G]$ in order to conclude that $H(\tau) \equiv_p 0$ and hence $a(mn+t)/u' \equiv_p 0, n \geq 0$. In order to derive a bound for $\mathrm{ord}_p(H(\tau))$ we will use that for given power series $c_1(\tau)$ and $c_2(\tau)$ with $\mathrm{ord}_p(c_1(\tau)) \geq b_1$ for some $b_1 \in \mathbb{N}$ and $\mathrm{ord}_p(c_2(\tau)) \geq b_2$ for some $b_2 \in \mathbb{N}$ then $\mathrm{ord}_p(c_1(\tau)c_2(\tau)) \geq b_1 + b_2$.

We will consider two types of congruences:

Type 1: $a(mn+t) \equiv_u 0, n \geq 0$;

Type 2: $a(mn+t') \equiv_u 0, t' \in P(t), n \geq 0$.

Obviously congruences of Type 2 are special cases of congruences of Type 1 but we have observed that one can be "$|P(t)|$ times faster in practical computations" when considering congruences of Type 2. At the current stage this observation relies on experimental data and is not yet proved;for a comparison see Example 5.2.

Before entering a detailed discussion of how to prove congruences of Type 1 and 2 we recall a theorem of J. Sturm.

Theorem 4.2 (Sturm [17])**.** *Let k be an integer and $c(\tau) = \sum_{n=0}^{\infty} c(n)q^n$ a modular form of weight k for a subgroup G of Γ. Assume that* $\mathrm{ord}_u(c(\tau)) > \frac{k}{12}[\Gamma : G]$ *then* $c(\tau) \equiv_u 0$.

For setting up the lemmas in the next two subsections we have collected valuable ideas from [16, p. 134, Cor. 9.1.4], attributed to Buzzard.

Proving Congruences of Type 1.

Lemma 4.3. *Let $(m,M,N,t,(r_\delta)=r)\in\Delta^*$, $(a_\delta)\in R(N)$, n be the number of double cosets in $\Gamma_0(N)\backslash\Gamma/\Gamma_\infty$ and $\{\gamma_1,\dots,\gamma_n\}\subseteq\Gamma$ a complete set of representatives of the double cosets $\Gamma_0(N)\backslash\Gamma/\Gamma_\infty$. Assume that $p^*(\gamma_i)+|P(t)|p(\gamma_i)\geq 0$ for $1\leq i\leq n$ and with p and p^* as in the lemmas 3.3 and 3.4. Next define:*

$$\begin{aligned}\nu :=&\frac{1}{24}\left(\left(\sum_{\delta|N}a_\delta+|P(t)|\sum_{\delta|M}r_\delta\right)[\Gamma:\Gamma_0(N)]-\sum_{\delta|N}\delta a_\delta\right)\\&-\frac{1}{m}\sum_{t'\in P(t)}t'-\frac{|P(t)|}{24m}\sum_{\delta|M}\delta r_\delta.\end{aligned}$$

Then for $f(\tau,r)=\sum_{n=0}^\infty a(n)q^n$ and $g_{m,t}(\tau,r)$ as in Definition 1.11 the following statements hold:

(i) *$\{(\prod_{t'\in P(t)}g_{m,t'}(\tau))(\prod_{\delta|N}\eta^{a_\delta}(\delta\tau))\}^{24}$ is a modular form for the group $\Gamma_0(N)$ of weight $12\sum_{\delta|N}a_\delta+12|P(t)|\sum_{\delta|M}r_\delta$.*

(ii) *For any $u\in\mathbb{N}^*$ we have: If $\mathrm{ord}_u(\sum_{n=0}^\infty a(mn+t)q^n)>\nu$ then $\sum_{n=0}^\infty a(mn+t)q^n\equiv_u 0$.*

Proof. *(i):* Let $\gamma=\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\Gamma_0(N)^*$ and let χ be as in *(v)* in Lemma 3.10. Then:

$$\left(\prod_{t'\in P(t)}g_{m,t'}(\gamma\tau)\right)^{24}=(\beta(\gamma,0)\chi)^{24}(c\tau+d)^{12|P(t)|\sum_{\delta|M}r_\delta}\left(\prod_{t'\in P(t)}g_{m,[a^2]_{24m}\odot t'}(\tau)\right)^{24}$$

(by (53), and *(i)* in Lemma 3.10)

$$=(c\tau+d)^{12|P(t)|\sum_{\delta|M}r_\delta}\left(\prod_{t'\in P(t)}g_{m,t'}(\tau)\right)^{24}\tag{71}$$

(by *(iv)* and *(v)* in Lemma 3.10).

By (9) we get:

$$\left(\prod_{\delta|N}\eta^{a_\delta}\left(\frac{a(\delta\tau)+b\delta}{\frac{c}{\delta}(\delta\tau)+d}\right)\right)^{24}=(c\tau+d)^{12\sum_{\delta|N}a_\delta}\left(\prod_{\delta|N}\eta^{a_\delta}(\delta\tau)\right)^{24}.\tag{72}$$

By (71) and (72) we obtain:

$$\begin{aligned}&\left(\left(\prod_{t'\in P(t)}g_{m,t'}(\gamma\tau)\right)\left(\prod_{\delta|N}\eta^{a_\delta}(\delta(\gamma\tau))\right)\right)^{24}\\&=(c\tau+d)^{12\sum_{\delta|N}a_\delta+12|P(t)|\sum_{\delta|M}r_\delta}\left(\left(\prod_{t'\in P(t)}g_{m,t'}(\tau)\right)\left(\prod_{\delta|N}\eta^{a_\delta}(\delta\tau)\right)\right)^{24}\end{aligned}\tag{73}$$

We want to prove that

$$V(\tau) := \left(\left(\prod_{t'\in P(t)} g_{m,t'}(\tau)\right)\left(\prod_{\delta|N}\eta^{a_\delta}(\delta\tau)\right)\right)^{24} \tag{74}$$

is a modular form of weight $12\sum_{\delta|N}a_\delta + 12|P(t)|\sum_{\delta|M}r_\delta$ for the group $\Gamma_0(N)$. Clearly, condition (1) of Definition 1.1 is satisfied. Also condition (2) is satisfied because of (73) and because of Lemma 1.6. The only assertion left to verify is condition (3). Let $\gamma\in\Gamma_0(N)\gamma_i\Gamma_\infty, i\in\{1,\dots,n\}$, then by Lemmas 3.4 and 3.3 there exists a positive integer k such that $h_1(q),\dots,h_{|P(t)|}(q), h^*(q)$ are Taylor series in powers of $q^{\frac{1}{k}}$ such that:

$$(c\tau+d)^{-12\sum_{\delta|N}a_\delta+12|P(t)|\sum_{\delta|M}r_\delta}V(\gamma\tau) = q^{24p^*(\gamma_i)+24|P(t)|p(\gamma_i)}h^*(q)\prod_{j=1}^{|P(t)|}h_j(q). \tag{75}$$

But by assumption $p^*(\gamma_i)+|P(t)|p(\gamma_i)\geq 0$, so also condition (3) of Definition 1.1 is satisfied.

(ii): Assume that $a(mn+t)\equiv_{u'} 0$ for some integer u' that divides u. Let $l\in\mathbb{N}^*$ be such that

$$h_0(\tau) := \frac{1}{l}\left(\prod_{t'\in P(t), t'\neq t}\left(\sum_{n=0}^{\infty}a(mn+t')q^n\right)\right)^{24}\left(\prod_{n=1}^{\infty}\prod_{\delta|N}(1-q^{\delta n})^{a_\delta}\right)^{24}, \tag{76}$$

is a power series with integral coefficients such that for any prime p we have $h_0(\tau)\not\equiv_p 0$. Then $\frac{V(\tau)}{lu'^{24}}$ in (74) can be written as:

$$\begin{aligned}\frac{V(\tau)}{lu'^{24}} =& \frac{1}{lu'^{24}}q^{\sum_{t'\in P(t)}(24t'+\sum_{\delta|M}\delta r_\delta)/m}\left(\prod_{t'\in P(t)}\left(\sum_{n=0}^{\infty}a(mn+t')q^n\right)\right)^{24}\\ &\cdot q^{\sum_{\delta|N}\delta a_\delta}\left(\prod_{n=1}^{\infty}\prod_{\delta|N}(1-q^{\delta n})^{a_\delta}\right)^{24}\\ =&\left(\frac{1}{u'}\sum_{n=0}^{\infty}a(mn+t)q^n\right)^{24}q^{\frac{1}{m}\left(24\sum_{t'\in P(t)}t'+|P(t)|\sum_{\delta|M}\delta r_\delta\right)+\sum_{\delta|M}\delta a_\delta}h_0(\tau).\end{aligned}$$

If we choose $h_1(\tau) := q^{\frac{1}{m}\left(24\sum_{t'\in P(t)}t'+|P(t)|\sum_{\delta|M}\delta r_\delta\right)+\sum_{\delta|M}\delta a_\delta}h_0(\tau)$ then in order to prove $\frac{a(mn+t)}{u'}\equiv_p 0, n\geq 0$ for some prime p dividing u/u' we need to prove

$$\frac{V(\tau)}{lu'^{24}} = \left(\frac{1}{u'}\sum_{n=0}^{\infty}a(mn+t)q^n\right)^{24}h_1(\tau)\equiv_p 0, \tag{77}$$

which is exactly (70) above. From the above derivation we note that

$$\mathrm{ord}_p(h_1(\tau)) \geq \frac{1}{m}\left(24\sum_{t'\in P(t)}t' + |P(t)|\sum_{\delta|M}\delta r_\delta\right)+\sum_{\delta|M}\delta a_\delta \tag{78}$$

which is an integer because of $V(\tau+1)=V(\tau)$. Because of

$$\mathrm{ord}_p\left(\left(\frac{1}{u'}\sum_{n=0}^{\infty}a(mn+t)q^n\right)^{24}\right) > 24\nu,$$

by assumption, we have that

$$\mathrm{ord}_p\left(\frac{1}{lu'^{24}}V(\tau)\right) > \left(\sum_{\delta|N} a_\delta + |P(t)|\sum_{\delta|M} r_\delta\right)[\Gamma:\Gamma_0(N)]$$

because of (77), (78) and by substituting according to the definition of ν. Theorem 4.2 allows us to conclude that $\frac{1}{lu'^{24}}V(\tau) \equiv_p 0$ which implies that $\frac{a(mn+t)}{u'} \equiv_p 0, n \geq 0$.

□

Proving Congruences of Type 2.

Lemma 4.4. *Let u be a positive integer, $(m, M, N, t, (r_\delta)) \in \Delta^*$, $(a_\delta) \in R(N)$, n be the number of double cosets in $\Gamma_0(N)\backslash\Gamma/\Gamma_\infty$ and $\{\gamma_1, \ldots, \gamma_n\} \subseteq \Gamma$ a complete set of representatives of the double cosets $\Gamma_0(N)\backslash\Gamma/\Gamma_\infty$. Assume that $p(\gamma_i) + p^*(\gamma_i) \geq 0, i = 1, \ldots, n$, with p and p^* as in the Lemmas 3.3 and 3.4. Furthermore, let $l := \frac{24m}{\kappa}$, $t_{\min} := \min_{t' \in P(t)} t'$ and*

$$\nu := \frac{1}{24}\left(\left(\sum_{\delta|N} a_\delta + \sum_{\delta|M} r_\delta\right)[\Gamma:\Gamma_0(N)] - \sum_{\delta|N}\delta a_\delta\right) - \frac{1}{24m}\sum_{\delta|M}\delta r_\delta - \frac{t_{\min}}{m}.$$

Then

(i) $(\prod_{\delta|N}\eta^{a_\delta}(l\delta\tau)\sum_{t'\in P(t)} g_{m,t'}(l\tau))^{24}$ *is a modular form of weight* $12(\sum_{\delta|M} r_\delta + \sum_{\delta|N} a_\delta)$ *for the group* $\Gamma_0(lN)$.

(ii) *If* $\mathrm{ord}_u(\sum_{n=0}^{\infty} a(mn+t')q^n) > \nu$ *for all* $t' \in P(t)$ *then* $\sum_{n=0}^{\infty} a(mn+t')q^n \equiv_u 0$ *for all* $t' \in P(t)$.

Proof. *(i):* Clearly condition *(1)* of Definition 1.1 is satisfied.

In order to prove condition *(2)* we only need to consider $\gamma \in \Gamma_0(lN)^*$ because of Lemma 1.6. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(lN)^*$ then by Theorem 2.13 the following relation holds:

$$\begin{aligned} g_{m,t}(l(\gamma\tau)) =& \beta\left(\begin{pmatrix} a & lb \\ \frac{c}{l} & d \end{pmatrix}, 0\right)(-i(c\tau+d))^{\frac{\sum_{\delta|M} r_\delta}{2}} \\ & \cdot e^{2\pi i\frac{abl(1-m^2)(24t+\sum_{\delta|M}\delta r_\delta)}{24m}} \cdot g_{m,[a^2]_{24m}\overline{\odot}t}(l\tau) \\ =& \beta\left(\begin{pmatrix} a & lb \\ \frac{c}{l} & d \end{pmatrix}, 0\right)(-i(c\tau+d))^{\frac{\sum_{\delta|M} r_\delta}{2}} g_{m,[a^2]_{24m}\overline{\odot}t}(l\tau). \end{aligned} \tag{79}$$

By (79) and *(iv)*, *(i)* of Lemma 3.10 we obtain:

$$\left(\sum_{t'\in P(t)} g_{m,t'}(l(\gamma\tau))\right)^{24} = ((c\tau+d))^{12\sum_{\delta|M} r_\delta}\left(\sum_{t'\in P(t)} g_{m,t'}(l\tau)\right)^{24}. \tag{80}$$

By (80) and (72) we obtain:

$$\begin{aligned}(81)\qquad &\left(\prod_{\delta|N}\eta^{a_\delta}(\delta l(\gamma\tau))\sum_{t'\in P(t)}g_{m,t'}(l(\gamma\tau))\right)^{24}\\ &\qquad = ((c\tau+d))^{12(\sum_{\delta|N}a_\delta+\sum_{\delta|M}r_\delta)}\left(\prod_{\delta|N}\eta^{a_\delta}(\delta l\tau)\sum_{t'\in P(t)}g_{m,t'}(l\tau)\right)^{24};\end{aligned}$$

hence condition *(2)* of Definition 1.1 is satisfied.

In order to prove *(3)* in Definition 1.1 fix a $t'\in P(t)$, and a $\gamma\in\Gamma_0(N)\gamma_i\Gamma_\infty, i\in\{1,\dots,n\}$. Then by Lemmas 3.3 and 3.4 there exist positive integers k,k' and Taylor series $h(q),h^*(q)$ in powers of $q^{\frac{1}{k}}$ and $q^{\frac{1}{k'}}$, respectively, such that

$$(82)\quad (c\tau+d)^{-12(\sum_{\delta|N}a_\delta+\sum_{\delta|M}r_\delta)}g_{m,t'}(\gamma\tau)\prod_{\delta|M}\eta^{a_\delta}(\delta(\gamma\tau)) = h(q)h^*(q)q^{p(\gamma_i)+p^*(\gamma_i)}.$$

Because of the positivity of $p(\gamma_i)+p^*(\gamma_i)$, there exists an positive integer j such that $h(q)h^*(q)q^{p(\gamma_i)+p^*(\gamma_i)}$ is a Taylor series in powers of $q^{\frac{1}{j}}$. Summarizing, we have proven that for all $t'\in P(t)$ and all γ there exists a positive integer k and a Taylor series $h(\gamma,q)$ such that

$$(83)\qquad (c\tau+d)^{-12(\sum_{\delta|N}a_\delta+\sum_{\delta|M}r_\delta)}g_{m,t'}(\gamma\tau)\prod_{\delta|M}\eta^{a_\delta}(\delta(\gamma\tau)) = h(\gamma,q).$$

Then by Lemma 3.6 there exist positive integers $k'_{t'}, t'\in P(t)$ and Taylor series $h^*(t',\gamma,q), t'\in P(t)$ in powers of $q^{\frac{1}{k'_{t'}}}$ such that

$$(84)\qquad (c\tau+d)^{-12(\sum_{\delta|N}a_\delta+\sum_{\delta|M}r_\delta)}g_{m,t'}(l(\gamma\tau))\prod_{\delta|M}\eta^{a_\delta}(\delta l(\gamma\tau)) = h^*(t',\gamma,q).$$

This proves that

$$\begin{aligned}(85)\qquad &(c\tau+d)^{-12(\sum_{\delta|N}a_\delta+\sum_{\delta|M}r_\delta)}\left(\prod_{\delta|N}\eta^{a_\delta}(\delta l(\gamma\tau))\sum_{t'\in P(t)}g_{m,t'}(l(\gamma\tau)\right)^{24}\\ &\qquad\qquad = \left(\sum_{t'\in P(t)}h^*(t',\gamma,q)\right)^{24}.\end{aligned}$$

So we have proven condition *(3)* of Definition 1.1.

(ii): First we note that given positive integers u',ν' and a power series $c(\tau) := \sum_{n=0}^\infty c(n)q^n$ such that $\mathrm{ord}_{u'}(c(\tau)) > \nu'$ we have that $\mathrm{ord}_u\left(\sum_{n=0}^\infty c(n)q^{an+b}\right) > a\nu'+b$ for any positive integers a and b.

We have proven above that $V_2(\tau) := (\prod_{\delta|N}\eta^{a_\delta}(l\delta\tau)\sum_{t'\in P(t)}g_{m,t'}(l\tau))^{24}$ is a modular form of weight $12(\sum_{\delta|N}a_\delta+\sum_{\delta|M}r_\delta)$.

Let u' be a divisor of u and p a divisor of u/u'. Assume that $u'|a(mn+t')$ for $n\geq 0$ and $t'\in P(t)$. We have that

$$\frac{V_2(\tau)}{u'^{24}} = q^{\frac{24}{\kappa}\sum_{\delta|M}\delta r_\delta + l\sum_{\delta|N}\delta a_\delta + \frac{24^2}{\kappa}t_{\min}} \left(\sum_{t'\in P(t)}\sum_{n=0}^{\infty}\frac{a(mn+t')}{u'}q^{\frac{24}{\kappa}(mn+t'-t_{\min})}\right)^{24}$$
$$\cdot\left(\prod_{n=1}^{\infty}\prod_{\delta|N}(1-q^{l\delta n})^{a_\delta}\right)^{24}.$$

For this rewriting we have used the definition of $g_{m,t}(\tau)$, the definition of l and that $\eta(\tau)$ can be written as an infinite product according to (2). We observe that

$$\mathrm{ord}_p(V_2(\tau)) > \frac{24}{\kappa}\sum_{\delta|M}\delta r_\delta + l\sum_{\delta|N}\delta a_\delta + \frac{24^2}{\kappa}t_{\min} + \frac{24^2}{\kappa}m\nu, \tag{86}$$

by looking at the above rewriting of $\frac{V_2(\tau)}{u'^{24}}$ and using the assumption that

$$\mathrm{ord}_u\left(\sum_{n=0}^{\infty}a(mn+t')q^n\right) > \nu,$$

for $t' \in P(t)$. If we substitute for ν in (86) we obtain:

$$\mathrm{ord}_p(V_2(\tau)) > \left(\sum_{\delta|N}a_\delta + \sum_{\delta|M}r_\delta\right)[\Gamma:\Gamma_0(N)]l.$$

Next observe that $[\Gamma:\Gamma_0(N)]l = [\Gamma:\Gamma_0(Nl)]$ because there in no prime q such that $q|l$ and $q\nmid N$ together with (1). Next apply Theorem 4.2 and we obtain $\frac{V_2(\tau)}{u'^{24}} \equiv_p 0$. This completes the proof.

□

5. Examples

Example 5.1. The generating function for broken 2-diamonds according to Andrews and Paule [1] is given by

$$\prod_{n=1}^{\infty}\frac{(1-q^{2n})(1-q^{5n})}{(1-q^n)^3(1-q^{10n})} = \sum_{n=0}^{\infty}\Delta_2(n)q^n.$$

In this paper they are stating some conjectures about the congruence properties of this function such as

$$\Delta_2(10n+2) \equiv_2 0, n \geq 0, \tag{87}$$

and

$$\Delta_2(25n+14) \equiv_5 0, n \geq 0. \tag{88}$$

The first congruence (87) has been proven in [5] and the second (88) in [2]. Following our approach, alternative proofs can be provided as follows. Since Chan [2] also proved that $\Delta_2(25n+24) \equiv_5 0, n \geq 0$ we can consider this to be congruences of Type 2; i.e., we will apply Lemma 4.4. We observe that $(25, 10, 10, 14, (r_1, r_2, r_5, r_{10}) = (-3, 1, 1, -1)) \in \Delta^*$. A complete set of representatives of the double cosets $\Gamma_0(10)\backslash\Gamma/\Gamma_\infty$ is given by

$$\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \gamma_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \gamma_2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \gamma_3 = \begin{pmatrix} 1 & 0 \\ 5 & 1 \end{pmatrix}.$$

Also let $(a_1, a_2, a_5, a_{10}) = (73, -21, -15, 5) \in R(10)$. According to Lemma 4.4 we need to show that $p^*(\gamma_k) + p(\gamma_k) \geq 0, k = 0, 1, 2, 3$ which can be readily verified from the data below.

$$p^*(\gamma_0) = \frac{1}{24}(1^2\frac{73}{1} - 2^2\frac{21}{2} - 5^2\frac{15}{5} + 10^2\frac{5}{10}) = \frac{6}{24},$$

$$p^*(\gamma_1) = \frac{1}{24}(\frac{73}{1} - \frac{21}{2} - \frac{15}{5} + \frac{5}{10}) = \frac{60}{24},$$

$$p^*(\gamma_2) = \frac{1}{24}(1^2\frac{73}{1} - 2^2\frac{21}{2} - 1^2\frac{15}{5} + 2^2\frac{5}{10}) = \frac{30}{24},$$

$$p^*(\gamma_3) = \frac{1}{24}(1^2\frac{73}{1} - 1^2\frac{21}{2} - 5^2\frac{15}{5} + 5^2\frac{5}{10}) = 0,$$

$$p(\gamma_0) = \min_{\lambda\in\{0,\dots,24\}} \frac{1}{24}(-\gcd^2(1\cdot(1+24\lambda\cdot 0), 25\cdot 0)\frac{3}{1\cdot 25} + \gcd^2(2\cdot(1+24\lambda\cdot 0), 25\cdot 0)\frac{1}{2\cdot 25}$$

$$+\gcd^2(5\cdot(1+24\lambda\cdot 0), 25\cdot 0)\frac{1}{5\cdot 25} - \gcd^2(10\cdot(1+24\lambda\cdot 0), 25\cdot 0)\frac{1}{10\cdot 25}) = -\frac{1}{100},$$

$$p(\gamma_1) = \min_{\lambda\in\{0,\dots,24\}} \frac{1}{24}(-\gcd^2(1\cdot(0+24\lambda\cdot 1), 25\cdot 1)\frac{3}{1\cdot 25} + \gcd^2(2\cdot(0+24\lambda\cdot 1), 25\cdot 1)\frac{1}{2\cdot 25}$$

$$+\gcd^2(5\cdot(0+24\lambda\cdot 1), 25\cdot 1)\frac{1}{25\cdot 5} - \gcd^2(10\cdot(0+24\lambda\cdot 1), 25\cdot 1)\frac{1}{10\cdot 25}) = -\frac{5}{2},$$

$$p(\gamma_2) = \min_{\lambda\in\{0,\dots,24\}} \frac{1}{24}(-\gcd^2(1\cdot(1+24\lambda\cdot 2), 25\cdot 2)\frac{3}{1\cdot 25} + \gcd^2(2\cdot(1+24\lambda\cdot 2), 25\cdot 2)\frac{1}{2\cdot 25}$$

$$+\gcd^2(5\cdot(1+24\lambda\cdot 2), 25\cdot 2)\frac{1}{25\cdot 5} - \gcd^2(10\cdot(1+24\lambda\cdot 2), 25\cdot 2)\frac{1}{10\cdot 25}) = -\frac{5}{4},$$

$$p(\gamma_3) = \min_{\lambda\in\{0,\dots,24\}} \frac{1}{24}(-\gcd^2(1\cdot(1+24\lambda\cdot 5), 25\cdot 5)\frac{3}{1\cdot 25} + \gcd^2(2\cdot(1+24\lambda\cdot 5), 25\cdot 5)\frac{1}{2\cdot 25}$$

$$+\gcd^2(5\cdot(1+24\lambda\cdot 5), 25\cdot 5)\frac{1}{5\cdot 25} - \gcd^2(10\cdot(1+24\lambda\cdot 5), 25\cdot 5)\frac{1}{10\cdot 25}) = 0.$$

Further we have that $[\Gamma : \Gamma_0(10)] = 18$, $\sum_{\delta|10} a_\delta = 42$, $\sum_{\delta|10} r_\delta = -2$, $\sum_{\delta|10} \delta r_\delta = -6$ and $\sum_{\delta|10} \delta a_\delta = 6$ hence $\nu = \frac{1}{24}(40 \cdot 18 - 6) - \frac{1}{24\cdot 25} \cdot (-6) - \frac{14}{25} = 146/5 \approx 30$. Consequently by Lemma 4.4 *(ii)* we have that if $\Delta_2(25n + 14) \equiv_5 0$ and $\Delta_2(25n + 24) \equiv_5 0$ for $n = 0, \dots, 30$ then $\Delta_2(25n + 14) \equiv_5 \Delta_2(25n + 24) \equiv_5 0$ for all nonnegative n. Also note that by *(ii)* in Lemma 4.4 we have that

$$q^{144}\left(\sum_{n=0}^{\infty} \Delta_2(25n + 14)q^{25n+14} + \Delta_2(25n + 24)q^{25n+24}\right)^{24} \left(\prod_{n=1}^{\infty}\prod_{\delta|10}(1 - q^{25\delta n})^{a_\delta}\right)^{24},$$

is a modular form of weight 480 for the group $\Gamma_0(250)$.

Hirschhorn and Sellers [5] proved that $\Delta_2(10n + 6) \equiv_2 0, n \geq 0$. To prove (87) and Hirschhorn and Sellers result we can again apply Lemma 4.4. This time we have that $(10, 10, 40, 2, (r_1, r_2, r_5, r_{10}) = (-3, 1, 1, -1)) \in \Delta^*$. If we choose $(a_\delta) = (a_1, a_2, a_4, a_5, a_8, a_{10}, a_{20}, a_{40}) = (33, -15, 0, -6, 0, 3, 0, 0)$ then all conditions of Lemma 4.4 applies and we get that $\nu \geq 39$. Consequently, verification of $\Delta_2(10n + 2) \equiv_2 0$ and $\Delta_2(10n + 6) \equiv_2 0$ for $0 \leq n \leq 39$ implies that (87) is true for all $n \geq 0$.

Example 5.2. The generating function

$$\prod_{n=1}^{\infty}\frac{1}{(1-q^{3n})(1-q^n)^3}=\sum_{n=0}^{\infty}a(n)q^n$$

appears in [11]. Here Ono proves that the numbers $a(63n+j), j=22,40,49, n\geq 0$ are divisible by 7.

Ono uses Sturms criterion and needs to compute 148147 coefficients of a certain generating function.

In order to solve this problem we can again apply Lemma 4.4. We find that $(63,3,21,22,(r_1,r_3)=(-3,-1))\in\Delta^*$ and see that the Lemma applies with $(a_\delta)=(a_1,a_3,a_7,a_{21})=(240,-77,-33,11)$. We find that $\nu\geq 182$ hence we need to verify that $a(63n+22)\equiv_7 a(63n+40)\equiv_7 a(63n+49)\equiv 0$ for $0\leq n\leq 182$ in order to conclude that this congruences hold for all nonnegative n.

However Ono restates the problem by defining:

$$\sum_{n=0}^{\infty}b(n)q^n=\left(\prod_{n=1}^{\infty}\frac{(1-q^n)^{14}}{(1-q^{7n})^2}\right)\sum_{n=0}^{\infty}a(n)q^n.$$

He observes that $a(63n+j)\equiv_7 0, j=22,40,49$ is equivalent to $b(63n+j)\equiv_7 0, j=22,40,49$. This is clear since $\prod_{n=1}^{\infty}\frac{(1-q^n)^{14}}{(1-q^{7n})^2}\equiv_7 1$.

We can again apply Lemma 4.4 to this reformulated problem. With input $(63,21,21,22,(r_1,r_3,r_7,r_{21})=(11,-1,-2,0))\in\Delta^*$ we see that the lemma applies with $(a_\delta)=(a_1,a_3,a_7,a_{21})=(5,-1,0,0)$. This time we find that $\nu\geq 16$ which is a huge improvement. Because of $a(n)\equiv_7 b(n)$ for all nonnegative n we need to show $a(63n+22)\equiv_7 a(63n+40)\equiv_7 a(63n+49)\equiv 0$ for $0\leq n\leq 16$ in order to conclude that this congruences hold for all nonnegative n.

We can also prove the congruence $b(63n+22)\equiv_7 0$ with Lemma 4.3 and with the same input $(63,21,21,22,(r_1,r_3,r_7,r_{21})=(11,-1,-2,0))\in\Delta^*$. We see that all conditions of Lemma 4.3 are satisfied if we choose $(a_\delta)=(a_1,a_3,a_7,a_{21})=(15,-4,0,0)$, and we get that $\nu\geq 45$ (approximately 3 times higher in comparison to using Lemma 4.4). Hence we need to verify that $b(63n+22)\equiv_7 0$ for $0\leq n\leq 45$ in order for the congruence to be true for all nonnegative n. Also *(i)* in Lemma 4.3 gives us that

$$q^{45}\left(\prod_{t'\in\{22,40,49\}}\left(\sum_{n=0}^{\infty}b(63n+t')q^n\right)\right)^{24}\left(\prod_{n=1}^{\infty}\frac{(1-q^n)^{15}}{(1-q^{3n})^4}\right)^{24},$$

is a modular form of weight 420 for the group $\Gamma_0(21)$.

Example 5.3. In this example we are considering several generating functions and consider their congruence properties. Given a positive integer M we assume a generating function to be of the form $\prod_{\delta|M}\prod_{n=1}^{\infty}(1-q^{\delta n})^{r_\delta}=\sum_{n=0}^{\infty}a(n)q^n$, and we abbreviate such a generating function by $\prod\delta^{r_\delta}$. In the table below the second column describes the generating function that we are considering. In columns 3, 4 and 5 we specify the integers m, t and p for which we wish to prove that $a(mn+t)\equiv_p 0, n\geq 0$. The column labeled by N specifies the integer N as in Lemma 4.4. The last column specifies the (a_δ) in $R(N)$ such that Lemma 4.4 applies; this is also listed in the form $\prod_{\delta|N} \delta^{a_\delta}$. Finally the column ν shows the bound for the "verification proof"; i.e., that, number such that if $a(mn+t) \equiv_p 0$ is true for all $0 \le n \le \nu$ and all t in column 4, then it is true for all $n \ge 0$.

Ex.	gen. funct.	m	t	p	ν	N	$(a_\delta) \in R(N)$
1	$1^{-3}2^1 5^1 10^{-1}$	25	14,24	5	30	10	$1^{73}2^{-21}5^{-15}10^5$
2	$1^{-3}2^1 5^1 10^{-1}$	10	2,6	2	39	40	$1^{33}2^{-15}5^{-6}10^3$
3	$3^{-1}1^{-3}$	63	22,40,49	7	182	21	$\frac{1^{240}21^{11}}{3^{77}7^{33}}$
4	$1^4 3^{-1}7^{-1}$	63	22,40,49	7	8	21	$1^5 7^{-1}$
5	1^{-1}	5	4	5	1	5	1^5
6	1^{-1}	7	5	7	2	7	$1^8 7^{-1}$
7	1^{-1}	11	6	11	5	11	1^{11}
8	1^{-1}	25	24	25	5	5	$1^{26}5^{-5}$
9	1^{-1}	49	47	49	14	7	$1^{50}7^{-7}$
10	1^{-1}	$11^3 \cdot 13$	$t \in P(237)$	13	103145	143	$\frac{1^{17551}143^{122}}{11^{1595}13^{1342}}$
11	$1^{12}13^{-1}$	$11^3 \cdot 13$	$t \in P(237)$	13	742	143	$1^{104}11^{-9}$
12	1^{-1}	125	74,124	125	26	5	$1^{130}5^{-25}$
13	1^{-2}	5	3	5	2	5	$1^{11}5^{-2}$
14	1^{-2}	25	23	25	10	5	$1^{52}5^{-10}$
15	1^{-8}	11	4	11	37	11	$1^{89}11^{-8}$
16	$1^3 11^{-1}$	11	4	11	2	11	1^1
17	$2^5 1^{-4}4^{-2}$	625	573	625	1301	20	$\frac{1^{1736}4^{434}10^{217}}{2^{1085}5^{347}20^{86}}$
18	$\frac{1^{-3+5}9^3}{3^1 5^1}$	45	22,40	5	7	15	$1^6 3^{-2}15^1$
19	$\frac{1^{-3+7}9^3}{3^1 7^1}$	63	49	7	12	21	$1^5 3^{-1}$
20	$\frac{1^{-3+11}9^3}{3^1 11^1}$	99	94	11	22	33	$1^3 3^{-1}$
21	$\frac{1^{-3+19}9^3}{3^1 19^1}$	171	49	19	63	22	1^2

Remark 5.4. Note that the examples $5, 6$ and 7 are the famous Ramanujan congruences. Let $p(n)$ denote the number of partitions of $n \in \mathbb{N}$; then the entries in example 5 show that in order to prove $p(5n+4) \equiv_5 0$ for all $n \ge 0$, it is sufficient to verify that $5|p(4)$. Similarly if $7|p(5)$ and $7|p(12)$ then $p(7n+5) \equiv_7 0$ for all $n \ge 0$. Finally in order to prove $p(11n+6) \equiv_{11} 0$ for all $n \ge 0$ we need to verify that $p(11n+6) \equiv_{11} 0$ for $0 \le n \le 5$. Ono [3] obtains twice as big bounds for the same congruences.

Remark 5.5. Generally, for some congruences one obtains a much better bound if one multiplies the generating function by $\prod_{n=1}^{\infty} \frac{(1-q^n)^p}{1-q^{pn})} \equiv_p 1$ for some prime p when one wants to prove a congruence modulo p. This trick has been found by Ono [3]. In the table above examples 3 and 4 prove the same congruence because their generating functions are equal modulo p; the same holds for examples $10, 11$ and examples $15, 16$; however the bounds ν differ.

Remark 5.6. Example 17 in the table has been studied by Eichhorn and Sellers [4]. The generating function is denoted in their paper by $\sum_{n=0}^{\infty} c\phi_2(n)$ and corresponds to 2-colored Frobenius partitions. They conclude that $c\phi_2(625n+573) \equiv_{625} 0, n \ge 0$ iff $c\phi_2(625n+573) \equiv_{625} 0, 0 \le n \le 198745$. As seen in the table we only require that $c\phi_2(625n+573) \equiv_{625} 0, 0 \le n \le 1301$. This improves the number of coefficients needed to be checked by a factor of approximately 152. In the end of the paper they are stating that the computation took 147 hours while with our bound we are decreasing the computation time to less then one hour!

Remark 5.7. The congruences in examples $18, 19, 20$ and 21 are studied by Lovejoy [9]. If we multiply the generating function in examples $18, 19, 20$ and 21 by $\prod_{n=1}^{\infty} \frac{1-q^{pn}}{(1-q^n)^p}$ for $p = 5, 7, 11, 19$ we then obtain the same generating function $f(q)$ (and $9qf(q)$ is the generating function for 3-colored Frobenius partitions, e.g., [7]).

For examples $18, 20$ and 21, Lovejoy proves the congruences by checking the first $181, 505$ and 841 initial values while with the methods developed here we only need to check the first $7, 22$ and 63 initial values. This gives an improvement by a factor of $25, 22$ and 13, respectively.

Remark 5.8. It should be noted that there is a difference between what Ono and Eichhorn [3] do and the approach here. Let $f(q) = \sum_{n=0}^{\infty} a(n)q^n$ and assume that we want to prove that $\sum_{n=0}^{\infty} a(mn+t)q^n \equiv_p 0$. Ono multiplies $f(q)$ be a suitable η product and gets a new generating function $\sum_{n=0}^{\infty} b(n)q^n$ which is a modular form. Then he shows that

$$\sum_{n=0}^{\infty} a(mn+t)q^n \equiv_p 0 \Leftrightarrow \sum_{n=0}^{\infty} b(m'n+t')q^n \equiv_p 0$$

for suitable m' and t'. Finally he uses a lemma which says that if $\sum_{n=0}^{\infty} b(n)q^n$ is a modular form for a group Γ' then also $\sum_{n=0}^{\infty} b(m'n+t')q^{m'n+t'}$ is a modular form for another group for which he applies the theorem of Sturm. We on the other hand are transforming $\sum_{n=0}^{\infty} a(mn+t)q^n$ into a modular form by multiplying with a suitable function $h_1(q)$. As we have seen, our method which is a generalization of the method in Rademacher [12] in practice gives much better bounds ν.

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