Some experiments using quantifier elimination to prove the non-existence of dynamically balanced spherical linkages*

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Abstract

The goal of this report is twofold. First, we describe and compare different strategies that can be used to solve a decision problem arising from engineering using quantifier elimination. Second, the decision problem stated in this report corresponds to the last step in proving that it is not possible to dynamically balance a special type of spherical 4R linkage.

1 Problem statement

A spherical 4R linkage is shown in Fig. 1. It consists of four bars: the base which is fixed, the input crank and the output crank which are connected to the base, and the coupler. The joints are revolute joints and their axes of rotation intersect in a point. This point is considered as the origin of a sphere of radius 1, on which the joints are moving. Although the joints do not have to move on the sphere, this model is equivalent. The length of the bars are measured by the angles between two successive axis of rotation. The base have length α, the input crank β, the output crank γ and the coupler δ. The design parameters (which will be called variable for the quantifier elimination problem) α, β, γ and δ can be replaced by a, b, c and d respectively using the tangent half-angles substitutions, i.e. for α we obtain:

$$\cos(\alpha) = \frac{1-a^2}{1+a^2} \qquad \sin(\alpha) = \frac{2a}{1+a^2} \tag{1}$$

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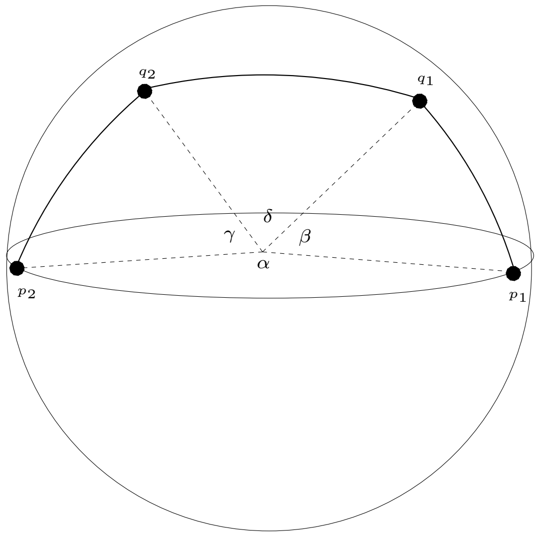

Two joints, p_1 and p_2 are fixed on the base and the two other joints, q_1 and q_2, are moving on two circles inscribed in a plane perpendicular to their axis of rotation.

The mass properties of each moving bars can be represented by its design static parameters (mass and centre of mass position) and its design dynamic variables (inertia matrix). The design parameters consists of the geometric, static and dynamic parameters. The problem consists of finding all possible design parameters such that the angular momentum of the linkage is zero for any motion of the linkage. Using an equivalent model, the equations of the angular momentum can be written in terms of the geometric and dynamic parameters only, that is, independent of the static parameters. The inertia matrices can be written in the following form:

$$I_j = \begin{bmatrix} I_{jxx} & I_{jxy} & I_{jxz} \\ I_{jxy} & I_{jyy} & I_{jyz} \\ I_{jxz} & I_{jyz} & I_{jzz} \end{bmatrix} \tag{2}$$

where I_1, I_2, I_3 are the inertia matrix of the input crank, output crank and coupler respectively.

The method for finding dynamically balanced linkages yield a set of equalities and inequalities in terms of the design parameters. This set of equations form a set of underdetermined linear system in terms of the dynamic parameters and can be solved symbolically in terms of the dynamic parameters, where some dynamic parameters are free parameters. Since there are also inequality constraints in terms of the geometric and dynamic parameters, we must check if there exists also solutions satisfying these inequalities.

For some particular choice of the geometric parameters, we obtain different sets of constraints. In this report, we are interested in the "parallelogram" linkage, i.e. $d = a \neq c = b$ [1]. Let

$$
\begin{aligned}
I_{1xx} &= b^2 \frac{T_1 I_{3xx} + T_2 I_{3yy} + T_3 I_{3zz}}{a\left(1+b^2\right)^2 D} \\
I_{2xx} &= b^2(1+a^2) \frac{T_4 I_{3xx} + T_5 I_{3yy} + T_6 I_{3zz}}{(a+b)\, a\left(1+b^2\right)^2 D} \\
I_{3xz} &= \frac{T_7 I_{3xx} + T_8 I_{3yy} + T_9 I_{3zz}}{2(a+b)D} \\
I_{3xy} &= 0 \\
I_{3yz} &= 0
\end{aligned}
\tag{3}
$$

with

$$
\begin{aligned}
T_1 &= 2\left(1+a^2\right)(ba-1)\left(2\,ba^2 - ab^4 + 2\,b^2a - a + 2\,b^3\right) \\
T_2 &= (a+b)\left(a^4b^4 + a^4 - 6\,a^4b^2 - 4\,a^3b^3 + 12\,a^3b - 4\,a^2 + 4\,a^2b^4 - 12\,b^3a + 4\,ba - 1 + 6\,b^2 - b^4\right) \\
T_3 &= -(b-1)(b+1)\left(1+a^2\right)(ba + a - 1 + b)(a+b)(ba - a - 1 - b) \\
T_4 &= -2\,b\left(2\,ba^2 + ab^4 - a - 2\,b^3\right) \\
T_5 &= -a^2 - a^2b^4 + 6\,b^2a^2 - 2\,ba + 2\,b^5a + b^2 + b^6 - 6\,b^4 \\
T_6 &= (b-1)(b+1)\left(-b^4 - 2\,b^3a + b^2 + b^2a^2 - 2\,ba - a^2\right) \\
T_7 &= a^2 + 4\,b^4 + 4\,b^3a - 4\,a^3b + a^2b^6 - a^2b^4 + 4\,a^4b^2 - b^2a^2 + 4\,a^3b^3 - 4\,b^5a \\
T_8 &= -\left(1+b^2\right)^2\left(b^2 + a^4\right) \\
T_9 &= b^2 - 2\,b^4 + a^4b^4 + b^6 + a^4 - 4\,a^3b^3 + 4\,b^2a^2 + 4\,a^2b^4 - 4\,b^3a + 4\,a^3b - 2\,a^4b^2 + 4\,b^5a \\
D &= \left(2\,ba + b^2 - 1\right)\left(b^2a - a - 2\,b\right)(a-b)
\end{aligned}
\tag{4}
$$

Let E be the identity matrix and the matrix A defined as

$$A := E \ \mathrm{trace}(I_3) - 2I_3 \tag{5}$$

For a valid inertia matrix, the determinant of A should be positive[2]. Such dynamically balanced linkage exists if and only if

$$\exists_{a,b,I_{3xx},I_{3yy},I_{3zz}\in\mathbb{R}^+}\, I_{1xx} > 0 \,\wedge\, I_{2xx} > 0 \wedge \det(A) > 0 \tag{6}$$

Clearly, we can simplify the values of I_{1xx} and I_{2xx} by removing terms in the numerator and denominator that are strictly positive, i.e. that do not change the sign of the expression. For example, we can omit the factors a, $a+b$, b^2 and $1+a^2$. and obtain simplified expressions for I_{1xx} and I_{2xx}, i.e.

$$
\begin{aligned}
I_{1xxs} &= \frac{T_1 I_{3xx} + T_2 I_{3yy} + T_3 I_{3zz}}{D} \\
I_{2xxs} &= \frac{T_4 I_{3xx} + T_5 I_{3yy} + T_6 I_{3zz}}{D}
\end{aligned}
\tag{7}
$$

[1] There are actually two possible kinematic modes[6] and we will investigate one of them. The other case is very similar.

[2] This is equivalent to the fact that the inertia matrix is positive definite and that every eigenvalue is smaller than the sum of the two other eigenvalues.

Note that the denominator of I_{1xxs} and I_{2xxs} are the same. We investigate the case where the numerator of I_{1xxs} and I_{2xxs} as well as the denominator D, are negative, i.e.

$$\exists_{a,b,I_{3xx},I_{3yy},I_{3zz}\in\mathbb{R}^+} \operatorname{num}(I_{1xxs}) < 0 \wedge \operatorname{num}(I_{2xxs}) < 0 \wedge D < 0 \wedge \det(A) > 0 \quad (8)$$

where $\operatorname{num}(I)$ is the numerator of I. The investigation of the case where $\operatorname{num}(I_{1xxs})$, $\operatorname{num}(I_{2xxs})$ and D are positive gives similar results.

2 Quantifier elimination

2.1 Methods

Since (8) is a formula from the first order theory of the real closed fields, its validity can be effectively decided [10]. For solving the decision problem, i.e., for real quantifier elimination, in theory one can use Cylindrical Algebraic Decomposition (CAD) [3, 2], Virtual Substitution (VS)[11, 5] or Hermitian Quantifier Elimination [12]. The latter is not suitable for our purpose, since we do not have equational constraints. VS can be used to eliminate variables occuring either linearly or quadratically in the input constraint system. Therefore we cannot eliminate all the variables appearing in (7) just by using VS since a and b appears with higher degree than 2. In theory, CAD works without any restriction on the degree of the variables, but it is not necessarily the most efficient in practice. Therefore we will use a combination of the VS and CAD method and study the results for different combinations (see section 2.3).

In theory one could also refine the strategies by trying out different variable orderings, but in our case since the variables $I_{3xx}, I_{3yy}, I_{3zz}$ occur quadratically and play a symmetric role in the contraint system, the best variable ordering seems to be $I_{3zz}, I_{3yy}, (I_{3xx}), a, b$ where the leftmost variable will be eliminated first.

2.2 Tools

The following systems provide implementations of at least one of the method mentioned above:

- Mathematica [13, 7, 8, 9]
- QEPCAD [1]
- Redlog [4]
- SyNRAC [14]

We have chosen the computer algebra system Mathematica[13] for solving the problem for several reasons. First, Mathematica seems to be the most efficient tool for such problem. It provides VS and CAD based real quantifier elimination (this is not the case e.g. for QEPCAD). Another reason is that the user interface is easier to use than other software. Since QEPCAD can be called from Reduce (Redlog), a viable alternative would be to use Redlog combined with QEPCAD.

In Mathematica, there is a default setting for handling real constraint systems. Inspecting the form of the input formula, Mathematica tries to guess which method to use. In order to have an explicit control over the chosen strategies, we set the inequality solving options of Mathematica directly. E.g, if we do not want to have a CAD based elimination, we set the following option:

```
Developer`SetSystemOptions["InequalitySolvingOptions" -> "CAD" -> False]
```

To eliminate a variable, let us say I_{3zz}, we use the command Resolve

```
Resolve[Exists[I3zz, cond], Reals]
```

where cond is the set of inequalities.

2.3 Solutions

To name (and refer to) the different strategies used below, we will use the following notation. The name of the strategy will be a string where the letters will be from the set $\{V, C\}$. The letter V will stand for the VS method and the letter C for the CAD method. Each letter, starting from the left describes the approach used to eliminate one variable. The last letter of the string corresponds to the strategy used to eliminate the remaining variables. The elimination order of the variables is also mentioned, where the leftmost variable will be eliminated first. Let cond be the condition as defined in (8), i.e.

$$\begin{aligned} \text{cond} = \quad & \{ \text{ I3zz} > 0 \wedge \text{I3yy} > 0 \wedge \text{I3xx} > 0 \wedge \text{a} > 0 \wedge \text{b} > 0 \wedge \\ & \text{I1xxsNumer} < 0 \wedge \text{I2xxsNumer} < 0 \wedge \text{Ds} < 0 \wedge \text{Det[A]} > 0 \ \} \end{aligned}$$

For example, if the elimination order is $\{I_{3zz}, I_{3yy}, I_{3xx}, a, b\}$, and the strategy is VC:

1. Eliminate I_{3zz} using the VS method:

```
Developer`SetSystemOptions["InequalitySolvingOptions" -> "CAD" -> False]
Developer`SetSystemOptions["InequalitySolvingOptions" -> "QuadraticQE" -> True]
Developer`SetSystemOptions["InequalitySolvingOptions" -> "LinearQE" -> True]
cond2 = Resolve[Exists[I3zz, cond], Reals]
```

2. Eliminate the remaining variables using the CAD method:

```
Developer`SetSystemOptions["InequalitySolvingOptions" -> "CAD" -> True]
Developer`SetSystemOptions["InequalitySolvingOptions" -> "QuadraticQE" -> False]
Developer`SetSystemOptions["InequalitySolvingOptions" -> "LinearQE" -> False]
Resolve[Exists[{I3yy, I3xx, a, b} , cond2], Reals]
```

To simplify the problem, it is also possible to set one of the inertia variable, I_{3zz}, I_{3yy} or I_{3xx} to 1 since the equations are homogeneous in terms of these variables. In this case, the variable which is set to 1 will be omitted from the elimination ordering list. For example, if $I_{3xx} = 1$, the elimination order could be $\{I_{3zz}, I_{3yy}, a, b\}$.

Another way to simplify the problem, is to bring the problem in the suitable form. For example, if the quantifier-free matrix of the formula is a disjunction

Method	Elimination order	Timing (s)
C	$\{I_{3zz}, I_{3yy}, a, b\}$	∞
VC	$\{I_{3zz}, I_{3yy}, a, b\}$	$3 + 146 = 149$
VVC	$\{I_{3zz}, I_{3yy}, a, b\}$	$398 + 3484 + 96775 = 100657$

Table 1: Comparison of QE strategies in Mathematica with $I_{3xx} = 1$ (Special case)

and we have to eliminate a variable which is existentially quantified, then we could use the logic equivalence

$$\exists x\,(F[x] \vee G[x]) \equiv \exists x\,F[x] \vee \exists x\,G[x] \tag{9}$$

to reduce the problem to the disjunction of a set of simpler problems. However, in our case, the elimination of one variable using the VS method leads to a problem of the form:

$$\exists x\,(D_1(x) \wedge D_2(x) \wedge ... \wedge D_n(x)) \tag{10}$$

That is, we obtain a formula where the outermost logical operator in the formula matrix is conjunction but the next variable, which has to be eliminated, is existentially quantified. However, it is possible to tranform the formula matrix into a disjunctive form, e.g. by transforming it into disjunctive normal form. In Mathematica, this can be achieved with LogicalExpand. To indicate after which step we transform the formula to its disjunctive normal form, we will put and overline over the method's name. For example, the method $\overline{V}VC$ implies that the LogicalExpand is applied after eliminating the first variable using the VS method.

In Table 1 and in Table 2, an overview of the different strategies used and the time required to prove that there exists no variables such that the linkage can be dynamically balanced, i.e. satisfying cond, is given. Note that we were able to solve the decision problem at least with one of the strategies for both the special and the general case. The timing as been done using Mathematica version 6.0.0, on a Intel(R) Xeon(TM) CPU 3.40GHz. The result ∞ means that we run it for at least 12 hours without the process terminating.

Setting $I_{3xx} = 1$, the problem can be solved efficiently using the VC approach. If none of the variables are set to 1, the problem is more difficult. It can be solved using $\overline{V}C$ in about 8.5 hours. In this case, using LogicalExpand we get 297 disjuncts, the i-th denoted by form2$[i]$, where $1 \leq i \leq 297$. We consider each subproblem/disjunct separately and solve them by method C. For example, to solve element $i = 19$, we use

Resolve[Exists[$\{I3yy, I3xx, a, b\}$, form2[[19]]], Reals]

For speeding up, one can also try to call the different subproblems with different variable ordering. In Mathematica default setting, the CAD-based computation of element $i = 19$ takes 9325 seconds. By explicit control of the variable order (i.e., by setting CADSortVariables to False), the following computation gives the same result and takes only 3 seconds:

Method	Elimination order	Timing (s)
C	$\{I_{3zz}, I_{3yy}, I_{3xx}, a, b\}$	∞
VC	$\{I_{3zz}, I_{3yy}, I_{3xx}, a, b\}$	$4 + \infty = \infty$
VVC	$\{I_{3zz}, I_{3yy}, I_{3xx}, a, b\}$	$4 + 3851 + \infty = \infty$
$\overline{V}C$	$\{I_{3zz}, I_{3yy}, I_{3xx}, a, b\}$	$4 + 10 + 31140 = 31154$
$V\overline{V}C$	$\{I_{3zz}, I_{3yy}, I_{3xx}, a, b\}$	$4 + 4066 + \infty = \infty$
$\overline{V}VC$	$\{I_{3zz}, I_{3yy}, I_{3xx}, a, b\}$	$4 + 10 + 8050 + 12000 \approxeq 20000$

Table 2: Comparison of QE strategies in Mathematica (General case)

$$\text{Resolve[Exists[}\{I3yy, b, I3xx, a\}\text{, form2[[19]]], Reals]}$$

Trying different choice of the variable ordering, the computation can be reduced from 8.5 hours to 10 minutes. This will be the subject of future work.

References

[1] C. W. Brown. QEPCAD B: A system for computing with semi-algebraic sets via cylindrical algebraic decomposition. *SIGSAM Bull.*, 38(1):23–24, 2004.

[2] G. E. Collins and H. Hong. Partial cylindrical algebraic decomposition for quantifier elimination. *Journal of Symbolic Computation*, 12(3):299–328, 1991.

[3] George E. Collins. Quantifier elimination for real closed fields by cylindrical algebraic decomposition. In *Automata theory and formal languages (Second GI Conf., Kaiserslautern, 1975)*, pages 134–183. Lecture Notes in Comput. Sci., Vol. 33. Springer, Berlin, 1975.

[4] A. Dolzmann and T. Sturm. Redlog: Computer algebra meets computer logic. *ACM SIGSAM Bulletin*, 31(2):2–9, 1997.

[5] R. Loos and V. Weispfenning. Applying linear quantifier elimination. *The Computer Journal*, 36(5):450–462, 1993.

[6] B. Moore, J. Schicho, and C. Gosselin. Dynamic balancing of planar mechanisms using toric geometry. Technical Report 2007-28, SFB F013, November 2007.

[7] A. W. Strzebonski. A real polynomial decision algorithm using arbitrary-precision floating point arithmetic. *Reliable Computing*, 5(3):337–346, 1999.

[8] A. W. Strzebonski. Symbolic solving of systems of real algebraic equations and inequalities with mathematica. In *Proceedings of SCI/ISAS 2000 Conference*, volume 8, pages 493–497, 2000.

[9] A. W. Strzebonski. Cylindrical algebraic decomposition using validated numerics. *Journal of Symbolic Computation*, 41(9):1021–1038, 2006.

[10] Alfred Tarski. *A decision method for elementary algebra and geometry.* University of California Press, Berkeley and Los Angeles, Calif., 1951. 2nd ed.

[11] V. Weispfenning. Quantifier elimination for real algebra–the quadratic case and beyond. *Applicable Algebra in Engineering Communication and Computing*, 8(2), 1997.

[12] V. Weispfenning. A new approach to quantifier elimination for real algebra. In *(Caviness-Johnson eds): Quantifier elimination and cylindrical algebraic decomposition*, pages 376–392, 1998.

[13] Inc. Wolfram Research. *Mathematica Edition: Version 5.2.* Wolfram Research, Inc., 2005.

[14] H. Yanami and H. Anai. SyNRAC: a maple toolbox for solving real algebraic constraints. *ACM Commun. Comput. Algebra*, 41(3):112–113, 2007.