An efficient solution algorithm for elastoplasticity and its first implementation towards uniform h- and p- mesh refinements

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Summary. The main subject of this paper is the detailed description of an algorithm solving elastoplastic deformations. Our concern is a one time-step problem, for which the minimization of a convex but non-smooth functional is required. We propose a minimization algorithm based on the reduction of the functional to a quadratic functional in the displacement and the plastic strain increment omitting a certain nonlinear dependency. The algorithm also allows for an easy extension to higher order finite elements. A numerical example in 2D reports on first results for uniform h- and p- mesh refinements.

1 Introduction

We consider the quasi-static initial-boundary value problem for small strain elastoplasticity with a linear hardening constitutive law, which can be abstractly formulated as a time-dependent variational inequality for unknown displacements and plastic strains fields. The question of the existence and uniqueness of the solution has been positively answered in [4] under the presence of hardening. It has been showed that a time-dependent variational inequality can be sufficiently approximated by a sequence of variational inequalities in given discrete times. Each of these variational inequalities contains a dissipative term coming from the plastic part of the model and represents an inequality of the second type according to [3]. Furthermore, the solution of each of these inequalities can be alternatively obtained as the minimizer of a certain convex energy functional, which is a functional depending on the unknown displacement smoothly and on the unknown plastic strain non-smoothly. The energy functional possesses a unique solution due to its strict convexity and coercivity.

Our main task here is the description of a new effective algorithm for finding such a solution. In addition to [6], where the basic parts of the algorithm have been explained, we concentrate on providing a more detailed description allowing straightforward implementation. The algorithm is based on the reduction of the functional to a quadratic functional in the displacement and the plastic strain omitting a certain nonlinear dependency. This can be understood as a linearization of the nonlinear elastoplastic problem. Then, the displacement field satisfies the linear Schur complement system after the elimination of plastic strains. The solution of this linear system can be efficiently computed by a multi-grid preconditioned conjugate gradient solver, whose convergence is already guaranteed [5].

The structure of the algorithm also allows for a direct generalization for higher degree polynomial finite elements. This is demonstrated in the numerical example, where the calculation for meshes of different sizes (h- uniform method) and polynomial degrees (p- uniform method) are presented.

2 The Model of Elastoplasticity

The elastoplastic body is assumed to occupy a bounded domain $\Omega \subset \mathbb{R}^d$ with a Lipschitz boundary $\Gamma = \partial\Omega$, where is the space-dimension. The local behavior is driven by the system of equalities and inequalities, see [4]:

$$\operatorname{div}\sigma + b = 0 \tag{1}$$

$$\sigma = \sigma^T \tag{2}$$

$$\varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T) \tag{3}$$

$$\sigma = \mathbb{C}(\varepsilon(u) - p) \tag{4}$$

$$\varphi(\sigma, \alpha) < \infty \tag{5}$$

$$\dot{p} : (\tau - \sigma) - \dot{\alpha}(\beta - \alpha) \leq \varphi(\tau, \beta) - \varphi(\sigma, \alpha) \tag{6}$$

Equation (1) describes the equilibrium of the stress tensor σ and outer volume body force b, equation (2) states the stress tensor's symmetry. The linearized elastic strain ε is defined in equation (3), whereas equation (4) represents the additive decomposition of the strain into its elastic part ε and its plastic part p. It also states the linear relation between the strain and the stress given by the elasticity tensor $\mathbb{C}$ which is defined for isotropic continua as

$$\mathbb{C}e = 2\mu e + \lambda(\operatorname{tr} e)\mathrm{i}, \tag{7}$$

where μ and λ are the Lamé coefficients, i is the identity matrix in $\mathbb{R}^{d\times d}$ and e a strain tensor. The trace operator $\operatorname{tr} : \mathbb{R}^{d\times d} \to \mathbb{R}^d$ of a matrix e is given by $\operatorname{tr} e := \sum_{i=1}^{d} e_{ii}$. The set of admissible stresses σ is steered by the the dissipational functional φ of equation (5), where the hardening parameter $\alpha \in \mathbb{R}_+$ represent memory (hysteresis) effects throughout the plastic deformations. The time development (the time derivative is denoted by $\dot{p} = \frac{\partial p}{\partial t}$) is given by the Prandtl-Reuß normality law in equation (6). The scalar product of matrices $A, B \in \mathbb{R}^{d\times d}$ is defined such that $A : B = \sum_{i,j=1}^{d} A_{ij}B_{ij}$. Consequently, the induced matrix norm is the Frobenius norm $|A| := (\sum_{i,j=1}^{d} A_{ij}^2)^{\frac{1}{2}}$. For the local model above, the global initial value problem reads, see [4]:

Problem 1 (Variational formulation) *Let $b \in W^{1,2}(0,T; L^2(\Omega, \mathbb{R}^d))$ with $b(0) = 0$ be a given volume force. Find the displacement $u \in W^{1,2}(0,T; H_0^1(\Omega)^d)$, the plastic strain $p \in W^{1,2}(0,T; L^2(\Omega, \mathbb{R}^{d\times d}))$ such that $p(0) = 0$, the hardening parameter $\alpha \in W^{1,2}(0,T; L^2(\Omega, \mathbb{R}^m))$ such that $\alpha(0) = 0$, and the stress $\sigma \in W^{1,2}(0,T; L^2(\Omega, \mathbb{R}^{d\times d}))$ such that $\sigma(0) = 0$, and such that the system (1)-(6) is satisfied in a weak sense.*

It has been shown in [4] that Problem 1 can be reformulated as a single time-dependent variational inequality which possesses a unique solution under the presence of the positive hardening H given later. Using an implicit Euler time-discretization with an uniformly chosen Δt, we obtain a sequence of one time-step variational inequalities. The solution of each of these inequalities satisfies a minimization problem, which is obtained using the dual functional φ^* calculated by the Legendre-Fenchel transformation $\varphi^*(y) := \sup_x \{y : x - \varphi(x)\}$.

Problem 2 (One time step) *Find the minimizer $(u, p, \alpha) \in H_0^1(\Omega)^d \times L^2(\Omega, \mathbb{R}^{d\times d}_{sym}) \times L^2(\Omega, \mathbb{R}^m)$ of*

$$\begin{aligned} f(u,p,\alpha) := \frac{1}{2}\int_\Omega \mathbb{C}[\varepsilon(u) - p] : (\varepsilon(u) - p)dx + \frac{1}{2}\int_\Omega |\alpha|^2 dx \\ +\Delta t \int_\Omega \varphi^*(\frac{p - p_0}{\Delta t}, \frac{\alpha_0 - \alpha}{\Delta t})dx - \int_\Omega b\, u\, dx. \end{aligned} \tag{8}$$

In comparison with Problem 1, σ has been replaced by $\mathbb{C}(\varepsilon(u) - p)$ and therefore it is no longer an unknown. $\mathbb{R}^{d\times d}_{sym}$ denotes real, symmetric $d \times d$ matrices. The values α_0 and p_0 are given from the previous time step t_0. The dissipational functional φ, its dual functional φ^*, as well as the hardening with parameter dimension m are specific for each hardening law such as isotropic hardening, kinematic hardening, its combination and the perfect plasticity as the limit case. For deriving an algorithm, the dual functional is calculated explicitely. There is $m = 1$ and the local minimization with respect to the hardening parameter α yields $\alpha = \alpha_0 + \sigma_y H\,|p - p_0|$ in the case of isotropic hardening. Problem 2 reduces to

Problem 3 (Isotropic hardening) *Find the minimizer $(u, p) \in H_0^1(\Omega)^d \times L^2(\Omega, \mathbb{R}^{d\times d}_{sym})$ of*

$$\begin{aligned} f(u,p) := \frac{1}{2}\int_\Omega \mathbb{C}[\varepsilon(u) - p] : (\varepsilon(u) - p)dx + \frac{1}{2}\int_\Omega (\alpha_0 + \sigma_y H|p - p_0|)^2 dx \\ + \int_\Omega \sigma_y |p - p_0| dx - \int_\Omega b\, u\, dx \end{aligned} \tag{9}$$

under the local constraint $tr\ (p - p_0) = 0$.

$\sigma_y > 0$ is the initial yield stress and $H > 0$ the modulus of hardening. dev denotes the matrix deviator defined by $\operatorname{dev} A := A - \frac{1}{d}\operatorname{tr}(A) \cdot \mathrm{i}$.

3 The Algorithm

The solution algorithm is derived for Problem 3, i.e., for the isotropic hardening only. Modification to the kinematic hardening case is straightforward.

The objective functional in (9) contains the matrix norm term $|p - p_0|$, which is non-differentiable in the origin. Thus standard methods, e.g. Newton's method, do not apply. A remedy is the following regularization:

$$|\cdot|_\epsilon := \begin{cases} |\cdot| & \text{if } |\cdot| \geq \epsilon, \\ \frac{1}{2\epsilon}|\cdot|^2 + \frac{\epsilon}{2} & \text{if } |\cdot| < \epsilon, \end{cases} \tag{10}$$

for some positive small ϵ. By this regularization we replace the original non-smooth objective $f(u,p)$ in (9) by an already smooth objective denoted as $\bar{f}(u,p)$. Thus, by introducing the plastic strain increment $\tilde{p} = p - p_0$, the modified problem writes:

Problem 4 (Isotropic hardening regularized) *Find the minimizer* $(u, \tilde{p}) \in H_0^1(\Omega)^d \times L^2(\Omega, \mathbb{R}^{d\times d}_{sym})$ *of*

$$\begin{aligned} \bar{f}(u,\tilde{p}) := \frac{1}{2}\int_\Omega \mathbb{C}[\varepsilon(u) - \tilde{p} - p_0] : (\varepsilon(u) - \tilde{p} - p_0)\,dx - \int_\Omega b\,u\,dx \\ + \frac{1}{2}\int_\Omega \alpha_0^2\,dx + \frac{1}{2}\int_\Omega \sigma_y^2\,H^2\,|\tilde{p}|^2\,dx + \int_\Omega \sigma_y\,(1 + \alpha_0\,H)|\tilde{p}|_\epsilon\,dx \end{aligned} \tag{11}$$

under the constraint $\operatorname{tr}\tilde{p} = 0$.

The spatial discretization is carried out by the standard finite element method using finite elements of a fixed polynomial degree. For computational reasons the symmetric matrices, e.g. $\tilde{p}$, are transformed into vectors $\tilde{p} = (\tilde{p}_{11}, \tilde{p}_{22}, \tilde{p}_{12})^T$ (in 2D) or $\tilde{p} = (\tilde{p}_{11}, \tilde{p}_{22}, \tilde{p}_{33}, \tilde{p}_{12}, \tilde{p}_{13}, \tilde{p}_{23})^T$ (in 3D). The objective $\bar{f}(u,\tilde{p})$ can now be discretized using the matrix and vector notation:

$$\frac{1}{2}(Bu - \tilde{p})^T\mathbb{C}(Bu - \tilde{p}) + \frac{1}{2}\tilde{p}^T\mathbb{H}(|\tilde{p}|_\epsilon)\tilde{p} + (-B^T\mathbb{C}p_0 - b)^T u \longrightarrow \min! \tag{12}$$

under the constraint tr $\tilde{p} = 0$. Here, Bu denotes the discretized strain $\varepsilon(u)$. $\mathbb{H}$ is the Hessian of the discretized objective with respect to $\tilde{p}$, it depends on $|\tilde{p}|_\epsilon$ only and is computed as

$$\mathbb{H}(|\tilde{p}|_\epsilon) = \left(\sigma_y^2 H^2 + \frac{2\sigma_y(1 + \alpha_0 H)}{|\tilde{p}|_\epsilon}\right)\mathbf{N}, \tag{13}$$

where the matrix $\mathbf{N}$ is defined, so that it holds $|p| = \left(p^T\mathbf{N}p\right)^{\frac{1}{2}}$. Thus

$$\text{2D: } \mathbf{N} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad \text{3D: } \mathbf{N} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix},$$

where the symbol $\oplus$ denotes the direct sum of two matrices, so $\mathbf{N}$ is a 6×6 matrix in 3D. The trace-free plastic strain constraint is explicitely satisfied as follows: In 3D, it must hold that $\tilde{p}_{33} = -\tilde{p}_{11} - \tilde{p}_{22}$. This linear condition is easily realized by projecting the arbitrary $\tilde{p}$ onto a hyperplane, where the constraint $\operatorname{tr} \tilde{p} = 0$ is satisfied. The projection matrix is denoted by P, the result of projection is then

$$\tilde{p} = P\bar{p}. \tag{14}$$

In 2D plane strain model, which is of our interest, the "third" dimension components of the elastic strain are zeros, i.e., $\varepsilon_{13} = \varepsilon_{23} = \varepsilon_{33} = 0$. However, the plastic strain increment $\tilde{p}$ as well as the stress σ have non-zero components

$$\tilde{p}_{33} = -\tilde{p}_{11} - \tilde{p}_{22} \quad \text{and} \quad \sigma_{33} = \frac{\lambda}{2(\lambda + \mu)}(\sigma^e_{11} + \sigma^e_{22}) + 2\mu(\tilde{p}_{11} + \tilde{p}_{22}),$$

where $\sigma^e := \mathbb{C}Bu$ is the elastic part of the stress tensor. Therefore, $\tilde{p}_{11}$ and $\tilde{p}_{22}$ are arbitrary and no special projection as in 3D is required, i.e., $P = I$. The dependence of $\mathbb{H}$ on $|p|_\epsilon$ in (13) is "frozen" and the nonlinear functional (12) becomes a quadratic one. Its minimizer must fulfill the necessary condition

$$\begin{pmatrix} B^T\mathbb{C}B & -B^T\mathbb{C}P \\ -P^T\mathbb{C}B & P^T(\mathbb{C} + \mathbb{H})P \end{pmatrix} \begin{pmatrix} u \\ \bar{p} \end{pmatrix} + \begin{pmatrix} -b - B^T\mathbb{C}p_0 \\ P^T\mathbb{C}p_0 \end{pmatrix} = 0. \tag{15}$$

By eliminating $\bar{p}$ in (15) we obtain the Schur-Complement system in u:

$$\begin{aligned} &B^T(\mathbb{C} - \mathbb{C}P(P^T(\mathbb{C} + \mathbb{H})P)^{-1}P^T\mathbb{C})Bu = \\ &\qquad\qquad - b - B^T(\mathbb{C} + \mathbb{C}P(P^T(\mathbb{C} + \mathbb{H})P)^{-1}P^T\mathbb{C})\, p_0. \end{aligned} \tag{16}$$

This linear system of equations for the vector of displacements u is solved by the conjugate gradient method with a geometrical multigrid preconditioner [2]. The spectral equivalence of the Schur-Complement matrix and the corresponding Schur-Complement matrix for the pure elasticity has been proved in [5]. Thus the convergence of the conjugate gradient method with a geometrical multigrid preconditioner is guaranteed. From the displacements u, the plastic strain increment $\tilde{p}$ is calculated by the local minimization of (12). In the unregularized case ($\epsilon = 0$), the analytical solution from [1] states

$$\tilde{p} = \frac{(||\operatorname{dev} A|| - a)_+}{2\mu + \sigma_y^2 H^2} \frac{\operatorname{dev} A}{||\operatorname{dev} A||} \tag{17}$$

with the quantities $A = \mathbb{C}(Bu - p_0)$ and $a = \sigma_y(1 + \alpha_0 H)$. The operator $(\cdot)_+$ is defined by $(\cdot)_+ = \max(0, \cdot)$. In the regularized case ($\epsilon > 0$), $\tilde{p}$ is solved by a local Newton's method, where the analytical solution (17) is used as an initial approximation.

An important practical aspect is the use of higher order finite elements, whose implementation may lead to an exponential convergence, see [7]. Since the trace-free condition of the plastic strain increment $\tilde{p}$ is satisfied explicitely (cf. (14)), the polynomial ansatz functions for $\tilde{p}$ can be taken one degree lower than the polynomial ansatz for the displacement without the loss of stability ($\inf - \sup$ like condition). The convergence of the discrete solutions was proved for linear ansatz functions for u and piecewise constant ansatz functions for $\tilde{p}$ in [1]. For our implementation, the displacement values u on an element are computed and stored in all integration points belonging to a given Gauss rule providing exact numerical integration of (16). The same integration points are taken to determine the plastic strain increment values by (17) and to reconstruct the finite element "shape" from them. This approach works nicely for linear ansatz functions for u, where only one Gauss point (center of a triangle) is required for the exact integration of the local matrices in (16). Then, the same point is used for computation of the locally constant $\tilde{p}$. For the quadratic ansatz function for u, there are three Gauss integration points (centers of triangle edges) required per triangle and three basis functions corresponding to a locally linear $\tilde{p}$. Nevertheless, in higher polynomial degree cases, the number of Gauss integration points for the exact integration of matrices in (16) is higher than the number of the finite element basis for one degree lower polynomial and thus a certain projection is applied. Summarizing our algorithm, the solution of Problem 4 is determined by

Algorithm 1 *Given initial displacement approximation u.*

1. *Calculate $\tilde{p}$ locally using Newton's method with the initial approximation $\tilde{p} = \frac{(||\operatorname{dev} A|| - a)_+}{2\mu + \sigma_y^2 H^2} \frac{\operatorname{dev} A}{||\operatorname{dev} A||}$, where $A = \mathbb{C}(B(u) - p_0)$ and $a = \sigma_y(1 + \alpha_0 H)$.*
2. *Substitute $\tilde{p}$ into $\mathbb{H}$ in (13) and assemble the global Schur-complement system (16).*
3. *Solve u from the global linear system (16) using multigrid PCG method.*
4. *Repeat steps (1)-(3) until convergence is reached.*
5. *Output displacement u and plastic strain increment $\tilde{p}$.*

4 Numerical Experiments

Algorithm 1 was implemented in the finite element solver NGSolve which is an extension package of the mesh generator Netgen [8] developed in our group. The testing geometry considered is the unit square depicted in Figure 1. The left edge is fixed in both x and y directions and the right edge is subjected to an outward acting force. The material parameters are $\lambda = 1000$ and $\mu = 1000$ as the Lamé parameters, the modulus of hardening is $H = 100$, and the initial yield stress is $\sigma_y = 6$. Several tests have been performed with different mesh sizes and different orders of the polynomial ansatz functions for the strains and the stresses. Figure 2 shows the von-Mises stresses for uniformly refined h- and p- meshes. The columns represent results with different polynomial ansatz

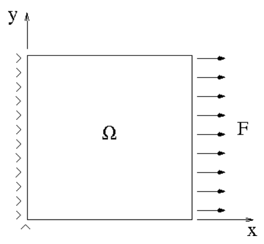

Fig. 1. Testing geometry

functions (p- method) and a fixed mesh. The rows show result for the same polynomial ansatz functions and different meshes (h- method). The stresses are scaled so that small values are darker shaded, the largest values are white colored and for most of the figures they also correspond to plasticity zones.

In Picture (a) both the mesh size and the order are chosen too coarse, the resulting stress is not reasonable from a physical point of view. From the Pictures (c), (f), and (i) it is obvious that for resolving singularities as they occur in the left corners, a finer local mesh-refinement (h- adaptive method) would be helpful. The great potential of higher order functions is approximating smooth functions on larger sized elements effectively, as demonstrated in Picture (g). Although there are only two elements in this calculation, a ninth degree polynomial functions for the stresses already provides the continuity on the common edge of these elements.

5 Conclusions and Future Work

In this paper, a new algorithm for the fast solution of the elastoplastic problems with hardening and its implementation together with the generalization to the higher-degree polynomial ansatz functions is presented. The future work will concentrate on theoretical analysis explaining convergence of the solution algorithm and on the implementation of the combined *hp*-adaptive method.

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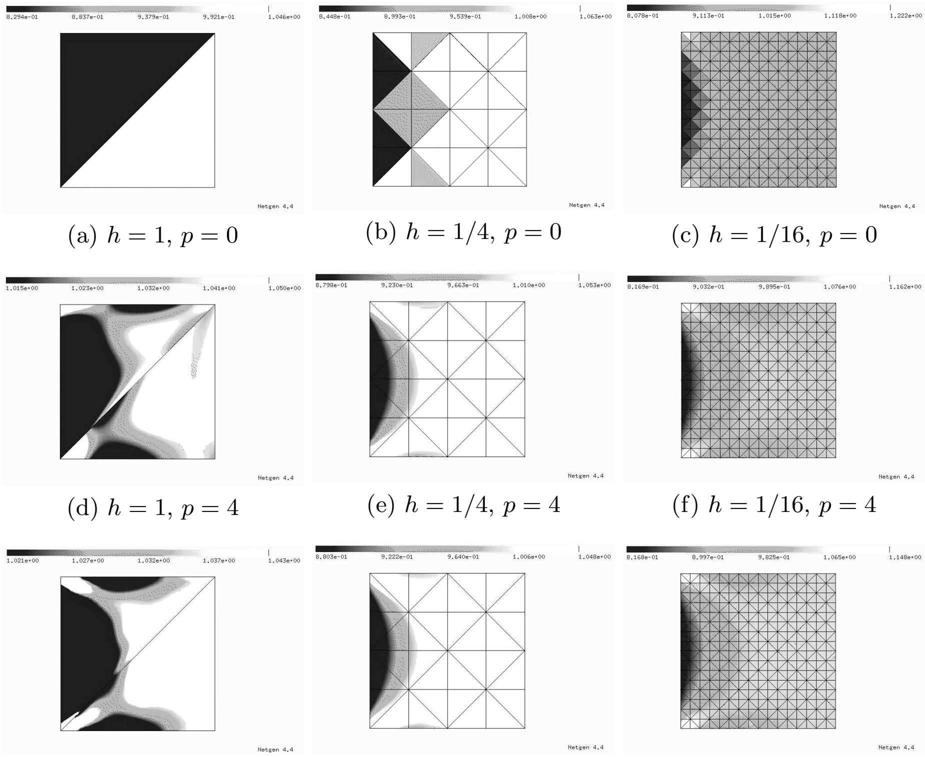

(a) $h = 1$, $p = 0$ (b) $h = 1/4$, $p = 0$ (c) $h = 1/16$, $p = 0$

(d) $h = 1$, $p = 4$ (e) $h = 1/4$, $p = 4$ (f) $h = 1/16$, $p = 4$

(g) $h = 1$, $p = 9$ (h) $h = 1/4$, $p = 9$ (i) $h = 1/16$, $p = 9$

Fig. 2. Von-Mises stress for various combinations of mesh sizes and polynomial degrees of u.

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