Multigrid Preconditioned Solvers for Some Elastoplastic Type Problems

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Outline

- Motivation
- Modeling
- Algorithm for isotropic hardening
- Numerical results
- Conclusions
- Outlooks

Motivation

Computing solutions numerically avoids e.g. expensive crash tests:

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Computing solutions numerically avoids e.g. expensive crash tests:

Literature:

- Plasticity: Carstensen, Han/Reddy
- Variational inequalities: Ekeland/Teman, Glowinski et al.
- FEM and multigrid: Braess, Bramble, Brenner/Scott, Hackbusch

Modeling

Find $u \in W^{1,2}(0,T;\ H_0^1(\Omega)^n)$, $p \in W^{1,2}(0,T; L^2(\Omega, \mathbb{R}^{n\times n}))$,
$\sigma \in W^{1,2}(0,T; L^2(\Omega, \mathbb{R}^{n\times n}))$, $\alpha \in W^{1,2}(0,T; L^2(\Omega, \mathbb{R}^m))$ such that

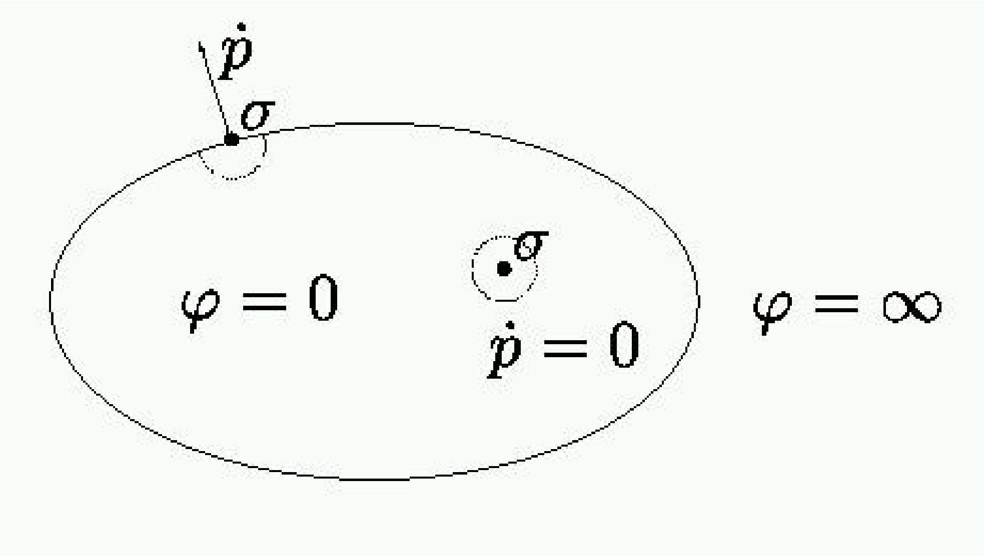
$$\begin{aligned} -\operatorname{div}\sigma &= b \\ \sigma &= \sigma^T \\ \varepsilon(u) &= \frac{1}{2}\left(\nabla u + (\nabla u)^T\right) \\ \varepsilon(u) &= \mathbb{C}^{-1}\sigma + p \\ \varphi(\sigma,\alpha) &< \infty \\ \dot{p} : (\tau - \sigma) - \dot{\alpha} : (\beta - \alpha) &\leq \varphi(\tau,\beta) - \varphi(\sigma,\alpha) \end{aligned}$$

are satisfied in the variational sense with $(u,p,\sigma,\alpha)(0) = 0$ for all (τ,β).
b and $\mathbb{C}^{-1}$ are given, $b(0) = 0$.

Normality law

Formulas without α (perfect plasticity)

$$\begin{aligned} \varphi(\sigma) &< \infty \\ \dot{p} : (\tau - \sigma) &\leq \varphi(\tau) - \varphi(\sigma) \end{aligned}$$



Numeric-analytic steps

- Time discretization: $t_1 = t_0 + \Delta t$
- Reformalution of the problem using functional-analytic arguments (switching arguments in variational inequalities using a dual functional)
- Equivalent minimization problem

Minimization problem for isotropic hardening

The minimization problem is

$$f(u,p) := \frac{1}{2}\int_\Omega \mathbb{C}[\varepsilon(u)-p] : (\varepsilon(u)-p)dx + \frac{1}{2}\int_\Omega (\alpha_0 + \sigma_y H|p-p_0|)^2 dx + \int_\Omega \sigma_y |p-p_0| dx - \int_\Omega b(t)\, u\, dx$$

under the constraint $\mathrm{tr}(p-p_0) = 0$

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New variable: $\tilde{p} = p - p_0$

A differentiable approximation of $|\tilde{p}|$:

$$|p|_\epsilon := \begin{cases} |p| & \text{if } |p| \geq \epsilon \\ \frac{1}{2\epsilon}|p|^2 + \frac{\epsilon}{2} & \text{if } |p| < \epsilon \end{cases}$$

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Minimization strategy in each time step:

$$u^{k+1} = \mathrm{argmin}_v \min_q \bar{f}(v,q) = \mathrm{argmin}_v \tilde{f}(v, q_{\mathsf{opt}}(v))$$

Then $p = p_0 + \tilde{p}$

Minimization in u

FEM-Discretization of the unconstrained objective is equivalent to

$$\frac{1}{2}(Bu - \tilde{p})^T \mathbb{C}(Bu - \tilde{p}) + \frac{1}{2}\tilde{p}^T \mathbb{H}(|\tilde{p}|_\epsilon)\tilde{p} - bu \longrightarrow \min!$$

Matrix notation:

$$\frac{1}{2}\begin{pmatrix} u \\ \tilde{p} \end{pmatrix}^T \begin{pmatrix} B^T\mathbb{C}B & -B^T\mathbb{C} \\ -\mathbb{C}B & \mathbb{C} + \mathbb{H} \end{pmatrix} \begin{pmatrix} u \\ \tilde{p} \end{pmatrix} + \begin{pmatrix} -b - B^T\mathbb{C}p_0 \\ \mathbb{C}p_0 \end{pmatrix}^T \begin{pmatrix} u \\ \tilde{p} \end{pmatrix} \longrightarrow \min!$$

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Necessary condition:

$$\begin{pmatrix} B^T\mathbb{C}B & -B^T\mathbb{C} \\ -\mathbb{C}B & \mathbb{C}+\mathbb{H} \end{pmatrix}\begin{pmatrix} u \\ \tilde{p} \end{pmatrix} + \begin{pmatrix} -b - B^T\mathbb{C}p_0 \\ \mathbb{C}p_0 \end{pmatrix} = 0$$

The Schur-Complement system in u with the matrix

$$S = B^T(\mathbb{C} - \mathbb{C}(\mathbb{C}+\mathbb{H})^{-1}\mathbb{C})B$$

is solved by multigrid preconditioned conjugate gradient method.

Minimization in $\tilde{p}$

The objective in each integration point writes as

$$F(\tilde{p}) = \frac{1}{2}\tilde{p}^T\mathbb{C}\tilde{p} + p_0^T\mathbb{C}\tilde{p} - \tilde{p}^T\mathbb{C}\varepsilon(u) + \frac{1}{2}\sigma_y^2 H^2|\tilde{p}|^2 + \sigma_y(1+\alpha_0 H)|\tilde{p}|_{\epsilon} \longrightarrow \min!$$

$\tilde{p}$ is determined by a modified Newton Algorithm in each integration point.

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Are there symbolic methods (as in the unregularized case)?

$$\tilde{p} = \frac{(||\operatorname{dev} A|| - b)_+}{2\mu + \sigma_y^2 H^2} \frac{\operatorname{dev} A}{||\operatorname{dev} A||},$$

where

$$A = \mathbb{C}[\varepsilon(u) - p_0], b = \sigma_y(1+\alpha_0 H).$$

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What about the constraint?

Constraint $\operatorname{tr} \tilde{p} = 0$

in 2D: $\tilde{p}_{22} = -\tilde{p}_{11}$, in 3D: $\tilde{p}_{33} = -\tilde{p}_{11} - \tilde{p}_{22}$.

Projection matrix P: $\tilde{p} = P\bar{p}$

$$\text{2D: } P = \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \text{3D: } P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Modified Newton system:

$$P^T F''(\tilde{p}) P \bar{p} = P^T F'(\tilde{p})$$

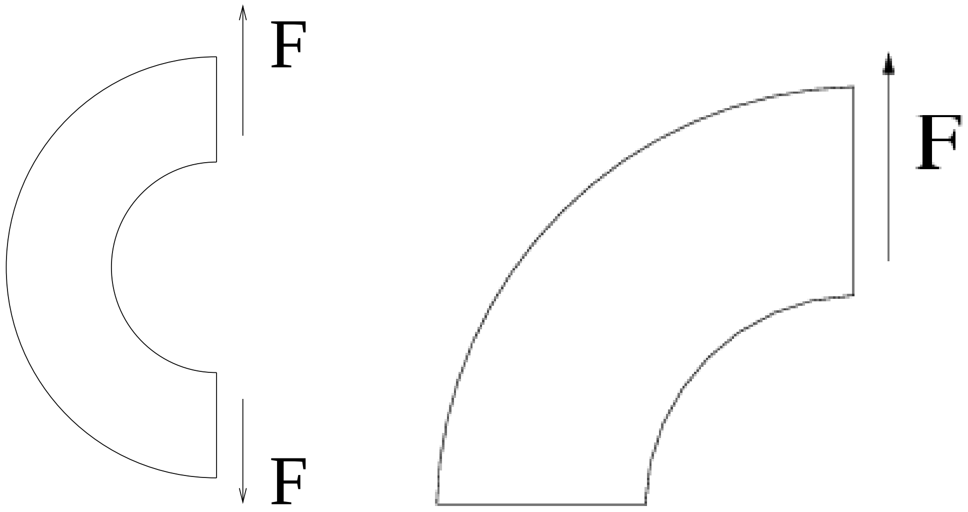
Modified Schur-Complement Matrix:

$$S = B^T(\mathbb{C} - \mathbb{C}P(P^T(\mathbb{C} + \mathbb{H})P)^{-1}P^T\mathbb{C})B$$

Numerical results - Quarter of a ring

FEM shape functions: u piecewise quadratic, p piecewise constant

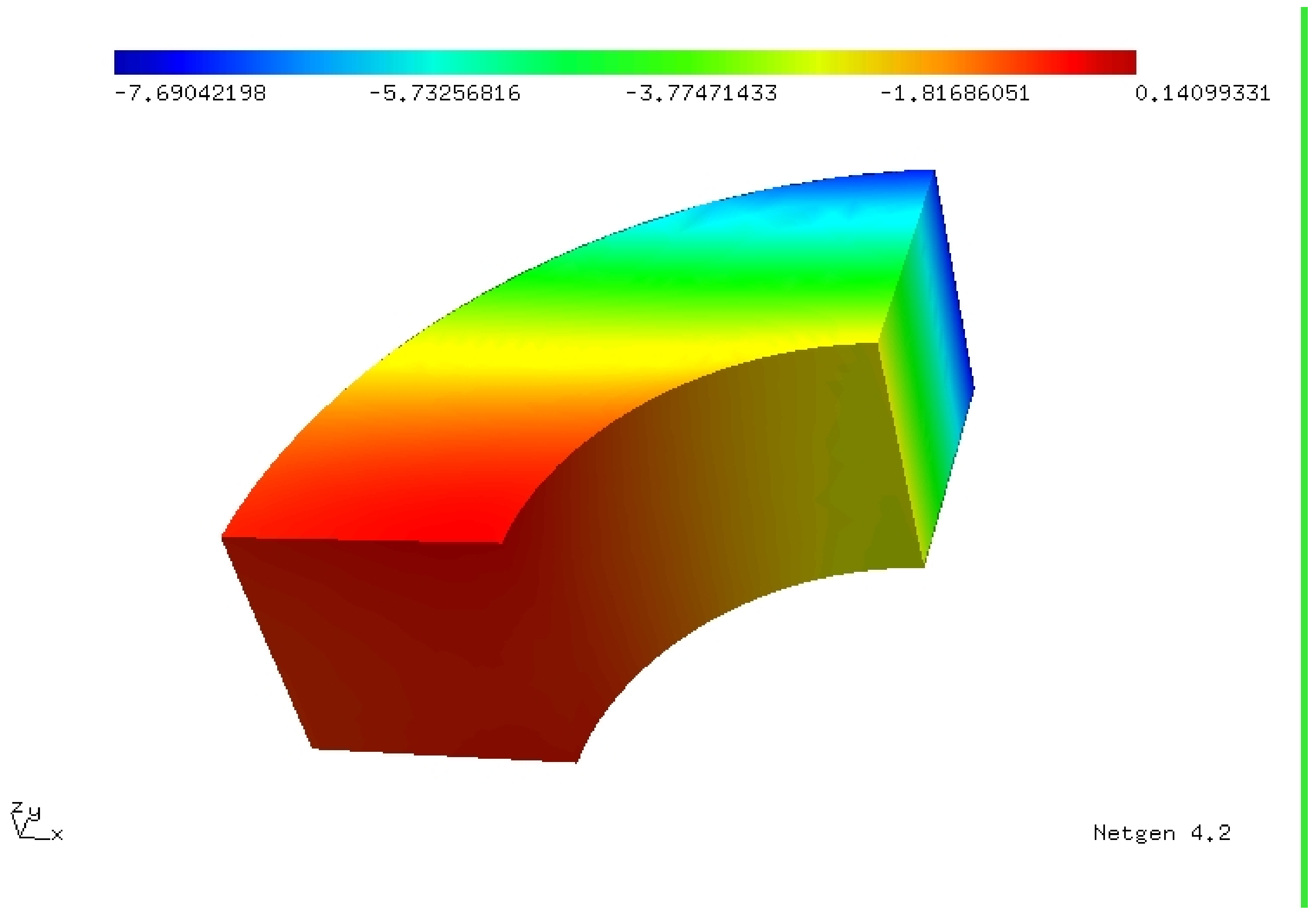
Symmetric problem:



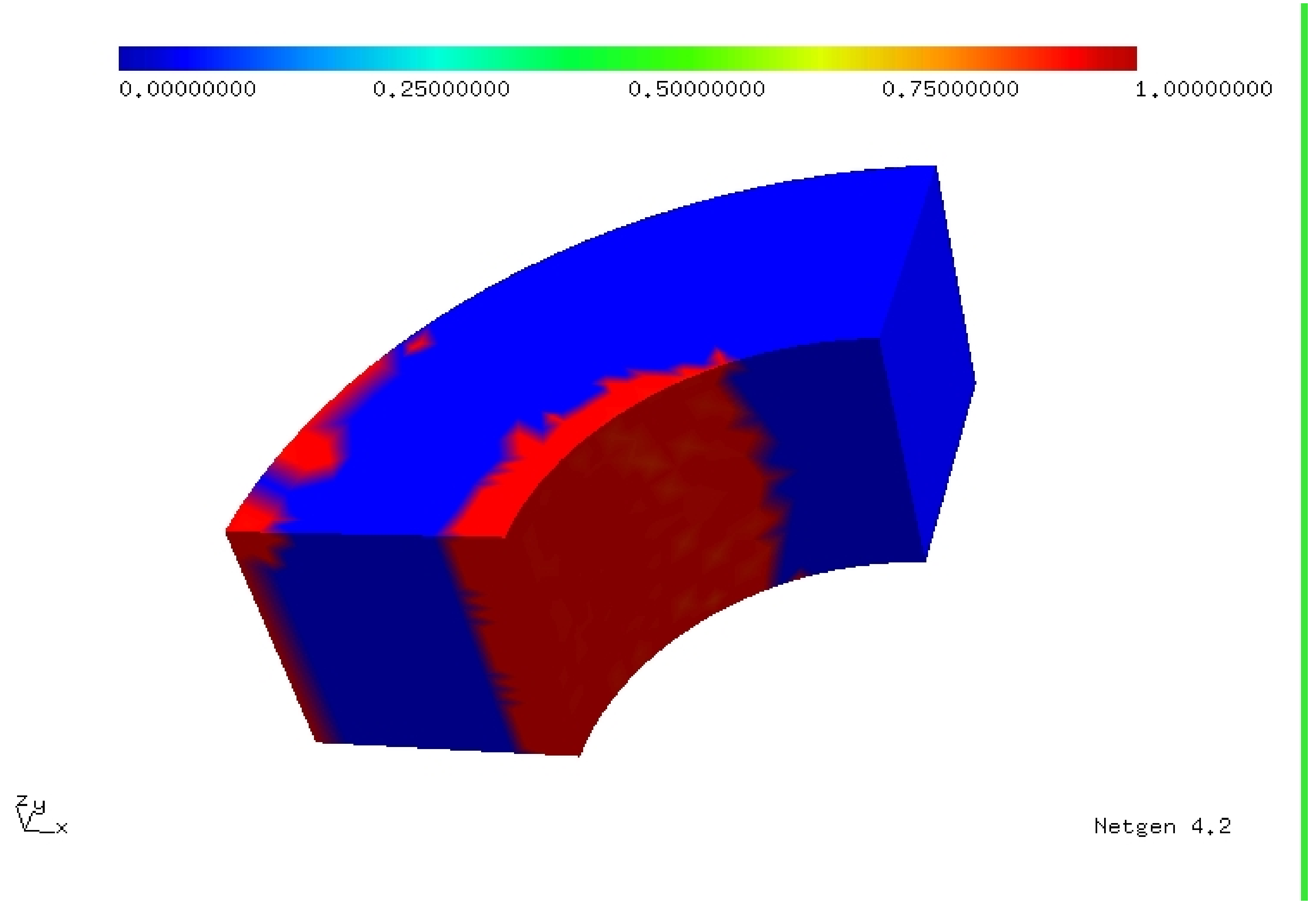
Constants:
E=1, $\nu = 0.2$, H = 0.01, $\sigma_y = 1$, F = 0.25

Number of time steps: 10

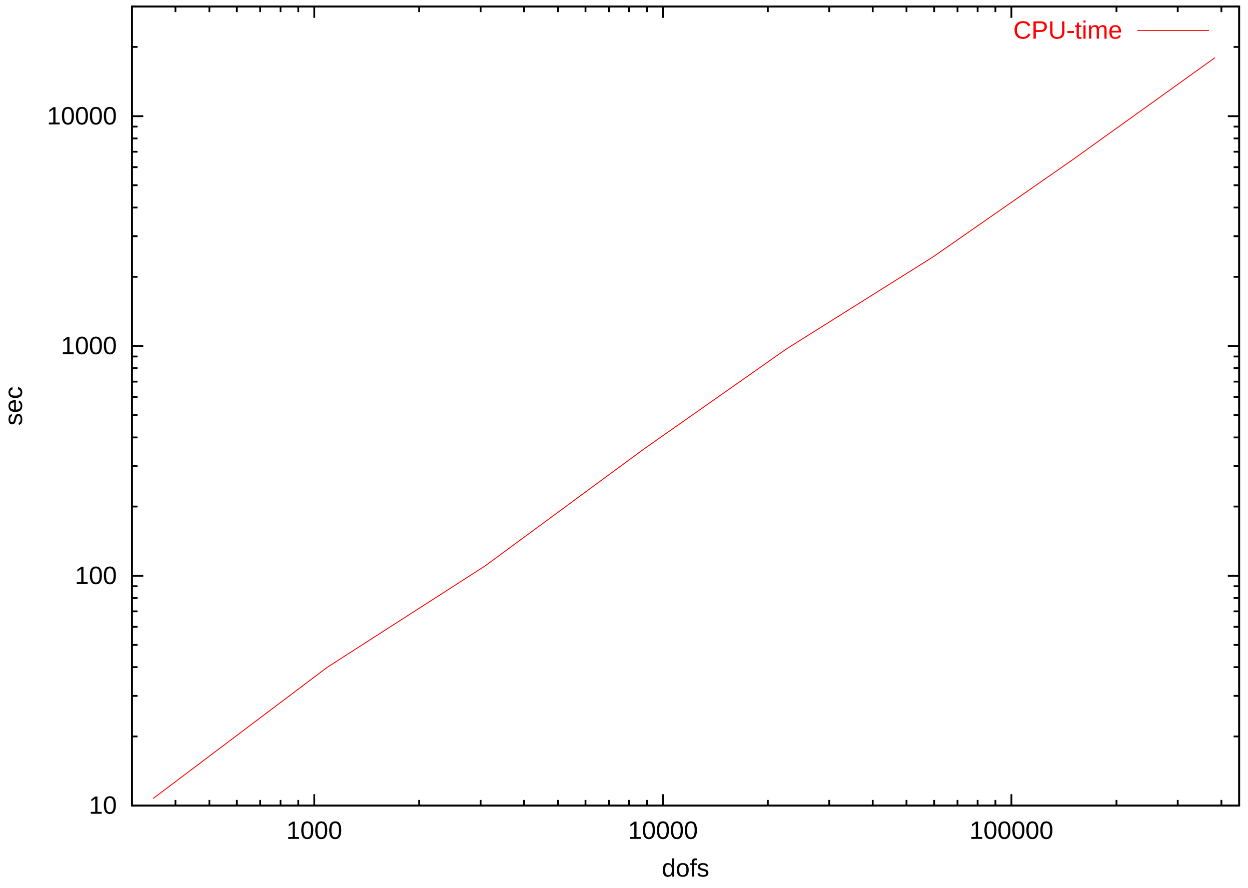
Displacement in x-direction



Plasticity domain



Linear Complexity



Conclusions

We have considered:

- Problem formulation and discretization
- Regularized minimization problem of isotropic hardening
- Minimization: 3D time-dependent algorithm

Outlooks

Future Work:

- Convergence proof

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- Convergence proof
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- Minimize $\tilde{p}$ using symbolic methods (with RISC-Linz)
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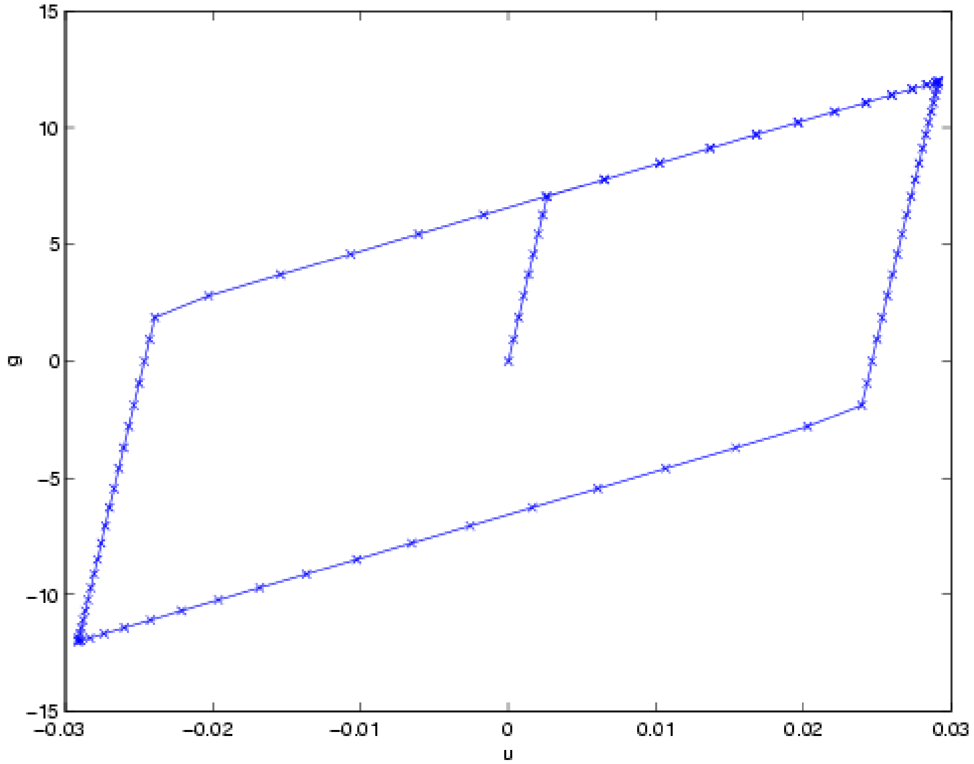
Download Netgen and NGSolve:

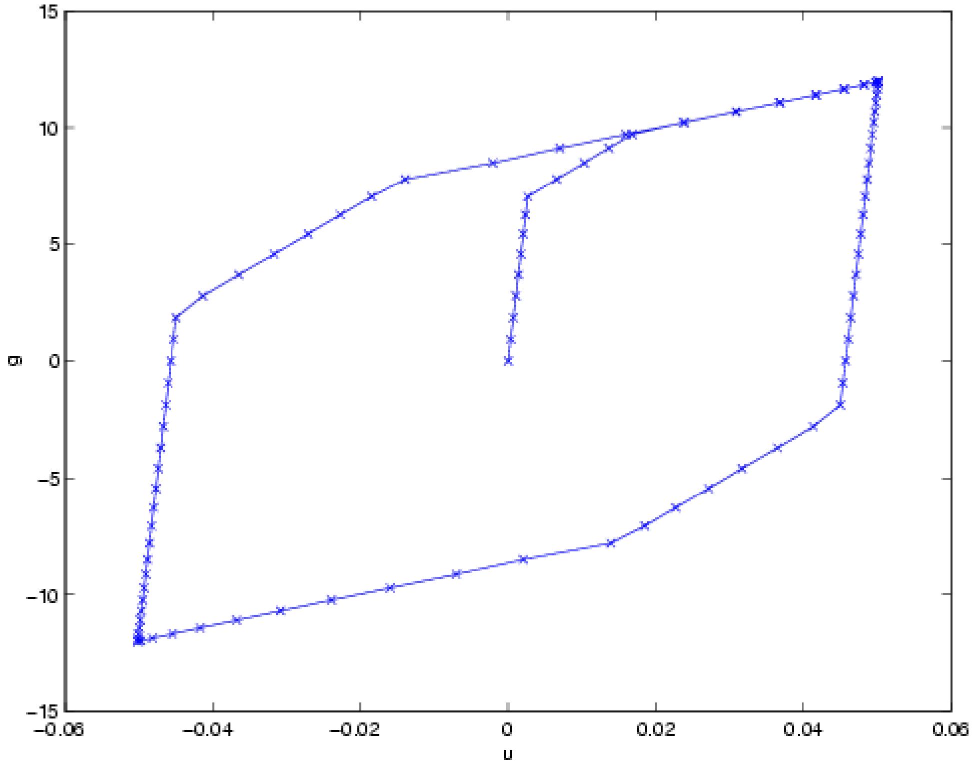
http://www.hpfem.jku.at/netgen/

Why Multi-yield (Two-yield) model?

- More realistic hysteresis stress-strain relation in materials!



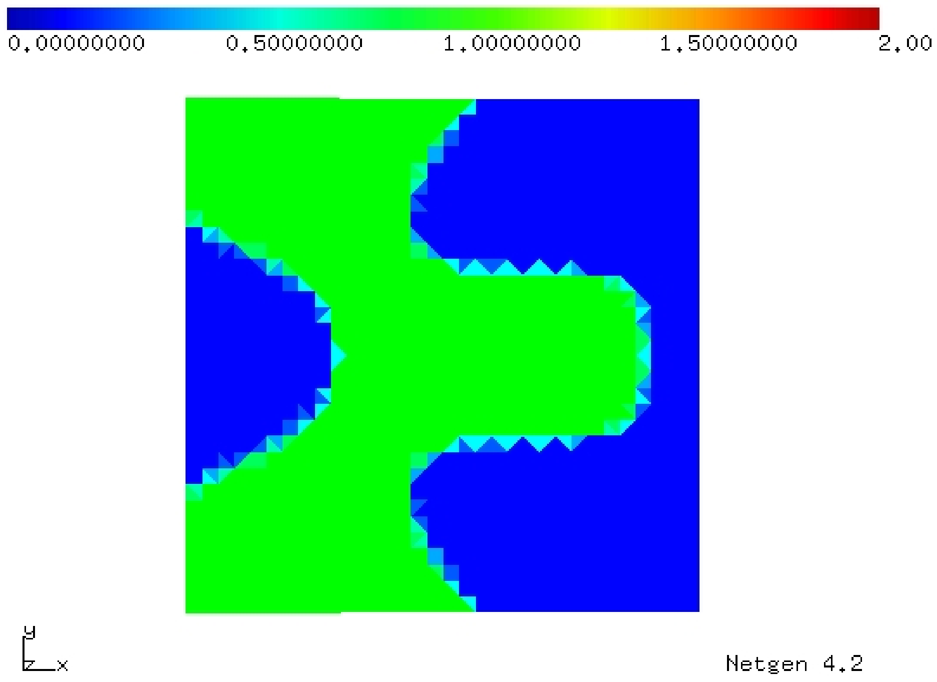
Kinematic hardening model.



Two-yield hardening model.

NGSOLVE calculations

Elastoplastic domains (blue -elastic, green - first plastic, red - second plastic)



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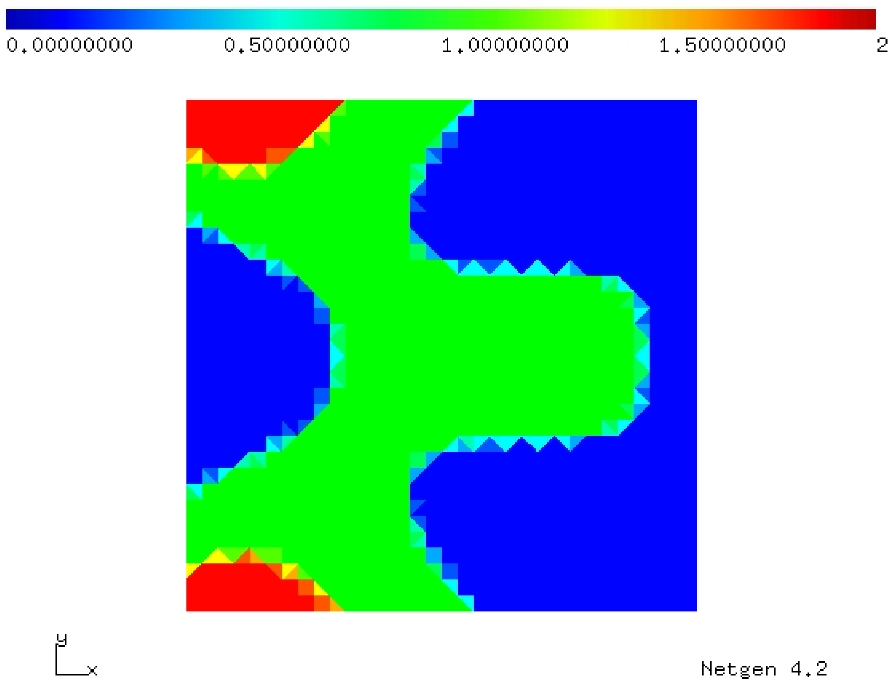

Two-yield hardening model.

Direct minimization problem in $\tilde{p}$

Kinematic hardening model:

$$f(Q) = \frac{1}{2}(\mathbb{C} + \mathbb{H})Q : Q - Q : A + \sigma^y ||Q|| \to \text{min}$$

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$$f\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \mathbb{C} + \mathbb{H}_1 & \mathbb{C} \\ \mathbb{C} & \mathbb{C} + \mathbb{H}_2 \end{pmatrix} \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} : \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} - \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} : \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} + \sigma_1^y ||Q_1|| + \sigma_2^y ||Q_2|| \rightarrow \text{min}$$

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minimizer $(\tilde{p}_1, \tilde{p}_2) = ?$

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$\tilde{p}_2 \neq 0 \Rightarrow ||\tilde{p}_2||$ is a root of a 6-th degree polynomial.

Gröbner basis?