Multiyield models in Plasticity

Why multiyield models in plasticity?

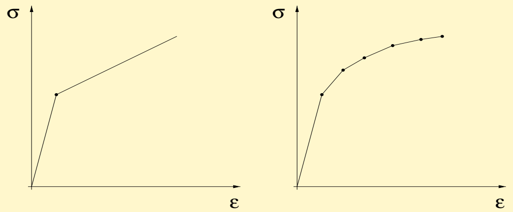

Figure 1: single yield and multiyield model

Models of plasticity involving single yield surface, such as those of linear kinematic hardening, do not provide satisfactory description of the transition between the elastic and plastic phase. In concrete experiments, the transition between the elastic and plastic regime is smooth. For this reason, so called multiyield models are introduced. The figure shows $\epsilon - \sigma$ (strain-stress) relation for the case of single and multiyield models. The multiyield model which in comparison to the single yield model contains more so called rigid-plastic elements (it is only one element for the single yield model) gives a more realistic description of change of the elastic to the plastic phase. With increasing number of rigid-plastic elements $\epsilon - \sigma$ dependence becomes smooth.

Prandtl–Ishlinskii model of the play type

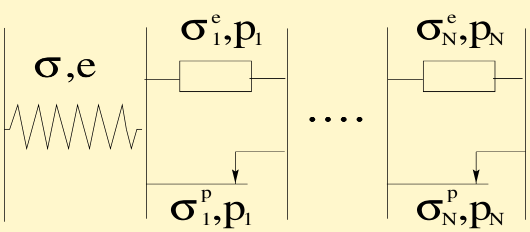

Figure 2: Prandtl–Ishlinskii model of the play type

A good example for the multiyield model is the so-called Prandtl-Ishlinskii model of the play type, Figure 2.The model can be described by the following system of variational equalities and inequalities, [5]

$$
\begin{aligned}
\epsilon &= e + p, \\
p &= \sum_{r=1}^{N} p_r, \\
\sigma &= \sigma_r^e + \sigma_r^p, && && \forall r = 1 \ldots N \\
\sigma_r^p &\in Z_r, \\
\langle \dot{p}_r, \tau_r - \sigma_r^p \rangle &\le 0, && \forall \tau_r \in Z_r, && \forall r = 1 \ldots N \\
\sigma &= Ce, \\
\sigma_r^e &= H_r p_r, && && \forall r = 1 \ldots N,
\end{aligned}
$$

ϵ denotes the strain defined as $\epsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$, e its elastic and p its plastic part. σ is then stress in the material, C the elasticity matrix and H_r, $r = 1..N$, are hardening matrices. The given system of equalities and inequalities is satisfied at every point of a bounded Lipschitz domain Ω in $\mathbb{R}^d$ which is occupied by the elastoplastic body. Variables in equalities and inequalities are time and space dependent, where $\dot{p} = \frac{dp}{dt}$.

Variational inequality

Qualitative analysis is applied to another equivalent form of our system of equalities and variational inequalities, namely only one variational inequality in the form

$$l(t)(y - \dot{x}(t)) \le a(x(t), y - \dot{x}(t)) + \psi(y) - \psi(\dot{x}(t)), \qquad \forall y \in H,$$

where $x = (u, p_1, ..., p_N)$ is the solution vector, $y = (v, q_1, ..., q_N)$ is a vector of test functions in the variational inequality and

$$
\begin{aligned}
a(x,y) &= \int_\Omega C(\epsilon(u) - p_1 - ... - p_N) : (\epsilon(v) - q_1 - ... - q_N)\ dx \\
&\quad + \int_\Omega H_1 p_1 : q_1\ dx + ... + \int_\Omega H_N p_N : q_N\ dx, \\
l(t) &: \quad H \to R,\ \langle l(t), y\rangle = \int_\Omega f(t) \cdot v\ dx, \\
\psi(y) &= \int_\Omega (D_1(q_1) + ... + D_N(q_N)) dx, \\
\text{and} &\quad D_i(q_i) = c_i |q_i|, \forall i.
\end{aligned}
$$

Let $H = V \times \underbrace{Q_0 \times Q_0 \times ... \times Q_0}_{N\ times}$, where

$$
\begin{aligned}
V &= [H_0^1(\Omega)], \\
Q_0 &= \{q = (q_{ij})_{3\times 3} : q_{ij} = q_{ji}, q_{ij} \in L^2, \mathrm{tr}(q) = 0\}.
\end{aligned}
$$

Here $a(.,.)$ is symmetric, continuous and H-elliptic. $\psi(y) : H \to \mathbb{R}$ is convex, nonnegative, positively homogeneous and Lipschitz continuous. For $l \in H^1(0, T; H'), l(0) = 0$ there exists a unique solution of the variational inequality in H, [4].

Continuous problem

More generally, corresponding to the continuous case of the Prantl-Ishlinskii model of the play type, we consider the same variational inequality with more generalized terms, namely let $I \in \mathbb{R}$ be an index set with measure μ, then:

$$x = (u, p_r), \ \ y = (v, q_r), \quad r \in I$$

$$
\begin{aligned}
a(x,y) &= \int_\Omega \Big(C(\epsilon(u) - \int_I p_r\ \ d\mu(r)) \Big) : \Big(\epsilon(v) - \int_I q_r\ \ \mu(r) \Big)\ dx \\
&\quad + \int_\Omega \int_I H_r p_r : q_r\ \ d\mu(r) dx \\
\psi(y) &= \int_\Omega \int_I D_r(q_r)\ \ d\mu(r) dx
\end{aligned}
$$

Also in this case the same conditions on $a(.,.), \psi(.)$,as before are satisfied and the variational inequality has the unique solution.

Conjecture

Our problems can be described as the variational inequality similar to the kinematic hardening problem [1, 2, 4]. Hence, we expect corresponding results, such as existence of the discrete problem, error of approximation, a-priori and a-posteriori for the error. Besides, a MATLAB program [1] for numerical solution of plasticity problem will be modified in order to compare numerical results with theoretical estimates.

References

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Dipl.–Ing. Jan Valdman, Lehrstuhl für Wissenschaftliches Rechnen, Universität Kiel, email: jva@numerik.uni–kiel.de