Computational plasticity: h and p elastoplastic interface adaptivity

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Outline

- Motivation
- Modeling
- Algorithm
- h-adaptivity (interfaces)
- First p-version results
- Conclusions and outlook

Motivation

- Computing solutions numerically avoids e.g. expensive crash tests:

- A fast and robust solver for the direct field problem is required!

Literature:

Han/Reddy, Carstensen

Modeling

Find $u \in W^{1,2}(0,T;\ H^1_0(\Omega)^n)$, $p \in W^{1,2}(0,T; L^2(\Omega, \mathbb{R}^{n\times n}))$,
$\sigma \in W^{1,2}(0,T; L^2(\Omega, \mathbb{R}^{n\times n}))$, $\alpha \in W^{1,2}(0,T; L^2(\Omega, \mathbb{R}^{m}))$ such that

$$\begin{aligned}
-\operatorname{div} \sigma &= b \\
\sigma &= \sigma^T \\
\varepsilon(u) &= \frac{1}{2}\left(\nabla u + (\nabla u)^T\right) \\
\varepsilon(u) &= \mathbb{C}^{-1}\sigma + p \\
\varphi(\sigma, \alpha) &< \infty \\
\dot{p} : (\tau - \sigma) - \dot{\alpha} : (\beta - \alpha) &\leq \varphi(\tau, \beta) - \varphi(\sigma, \alpha)
\end{aligned}$$

are satisfied in the variational sense with $(u, p, \sigma, \alpha)(0) = 0$ for all (τ, β).
b and $\mathbb{C}^{-1}$ are given, $b(0) = 0$.

Numeric-analytic steps

- Time discretization: $t_1 = t_0 + \Delta t$
- Reformulation of the problem using functional-analytic arguments (switching arguments in variational inequalities using a dual functional)
- Equivalent minimization problem:

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  Find the minimizer $(u, p, \alpha) \in H \times L^{n\times n}_{sym} \times L^m$ of

$$f(u,p,\alpha) := \frac{1}{2}\int_\Omega \mathbb{C}[\varepsilon(u)-p] : (\varepsilon(u)-p)dx + \frac{1}{2}\int_\Omega |\alpha|^2 dx + \Delta t \int_\Omega \varphi^*(\frac{p-p_0}{\Delta t}, \frac{\alpha_0-\alpha}{\Delta t})dx - \int_\Omega b\, u\, dx$$

with φ describing the hardening law.

Minimization problem for isotropic hardening

The minimization problem is

$$f(u,p) := \frac{1}{2}\int_\Omega \mathbb{C}[\varepsilon(u)-p] : (\varepsilon(u)-p)dx + \frac{1}{2}\int_\Omega (\alpha_0 + \sigma_y H|p-p_0|)^2 dx + \int_\Omega \sigma_y |p-p_0| dx - \int_\Omega b(t)\, u\, dx$$

under the constraint $\mathrm{tr}(p-p_0) = 0$

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New variable: $\tilde{p} = p - p_0$

A differentiable approximation of $|\tilde{p}|$:

$$|p|_\epsilon := \begin{cases} |p| & \text{if } |p| \geq \epsilon \\ \frac{1}{2\epsilon}|p|^2 + \frac{\epsilon}{2} & \text{if } |p| < \epsilon \end{cases}$$

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Minimization strategy in each time step:

$$u = \mathrm{argmin}_v \min_q \tilde{f}(v,q) = \mathrm{argmin}_v \tilde{f}(v, q_{\mathsf{opt}}(v))$$

Then $p = p_0 + \tilde{p}$

Minimization in u

FEM-Discretization of the unconstrained objective is equivalent to

$$\frac{1}{2}(Bu - \tilde{p})^T \mathbb{C}(Bu - \tilde{p}) + \frac{1}{2}\tilde{p}^T \mathbb{H}(|\tilde{p}|_\epsilon)\tilde{p} - bu \longrightarrow \min!$$

Matrix notation:

$$\frac{1}{2}\begin{pmatrix} u \\ \tilde{p} \end{pmatrix}^T \begin{pmatrix} B^T\mathbb{C}B & -B^T\mathbb{C} \\ -\mathbb{C}B & \mathbb{C}+\mathbb{H} \end{pmatrix} \begin{pmatrix} u \\ \tilde{p} \end{pmatrix} + \begin{pmatrix} -b - B^T\mathbb{C}p_0 \\ \mathbb{C}p_0 \end{pmatrix}^T \begin{pmatrix} u \\ \tilde{p} \end{pmatrix} \longrightarrow \min!$$

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Necessary condition:

$$\begin{pmatrix} B^T\mathbb{C}B & -B^T\mathbb{C} \\ -\mathbb{C}B & \mathbb{C}+\mathbb{H} \end{pmatrix} \begin{pmatrix} u \\ \tilde{p} \end{pmatrix} + \begin{pmatrix} -b - B^T\mathbb{C}p_0 \\ \mathbb{C}p_0 \end{pmatrix} = 0$$

The Schur-Complement system in u with the matrix

$$S = B^T(\mathbb{C} - \mathbb{C}(\mathbb{C}+\mathbb{H})^{-1}\mathbb{C})B$$

is solved by a multigrid preconditioned conjugate gradient method.

Constraint $\operatorname{tr}\tilde{p} = 0$

in 2D: $\tilde{p}_{22} = -\tilde{p}_{11}$, in 3D: $\tilde{p}_{33} = -\tilde{p}_{11} - \tilde{p}_{22}$.

Projection matrix P: $\tilde{p} = P\bar{p}$

$$\text{2D: } P_E = \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \text{3D: } P_E = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Modified Schur-Complement Matrix:

$$S = B^T(\mathbb{C} - \mathbb{C}P(P^T(\mathbb{C} + \mathbb{H})P)^{-1}P^T\mathbb{C})B$$

Minimization in $\tilde{p}$

The objective in each integration point writes as

$$F(\tilde{p}) = \frac{1}{2}\tilde{p}^T\mathbb{C}\tilde{p} + p_0^T\mathbb{C}\tilde{p} - \tilde{p}^T\mathbb{C}\varepsilon(u) + \frac{1}{2}\sigma_y^2 H^2|\tilde{p}|^2 + \sigma_y(1+\alpha_0 H)|\tilde{p}|_\epsilon$$

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This problem (without regularization, i.e. $\epsilon = 0$) has a unique solution

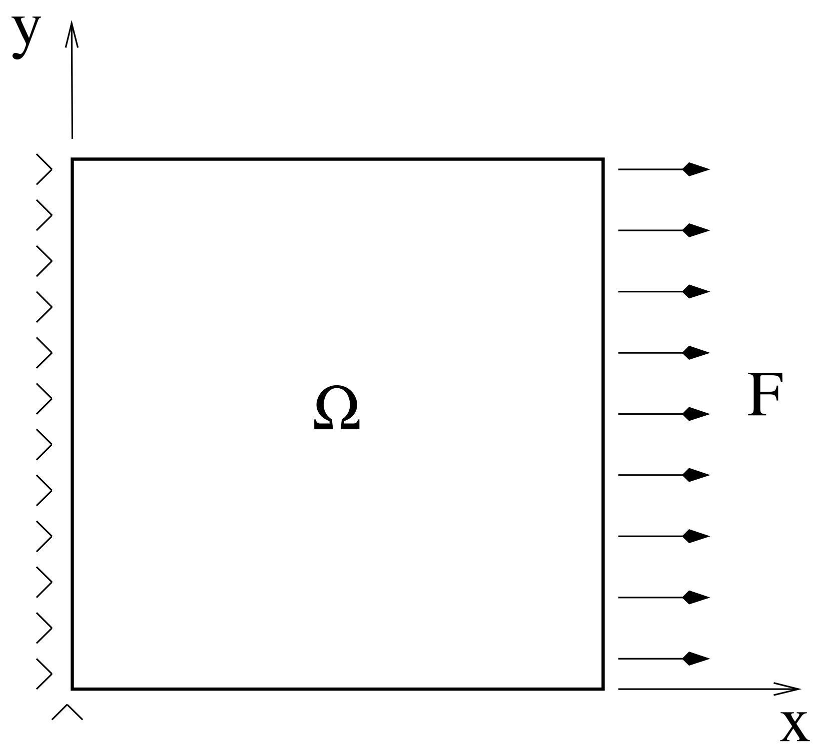
$$\tilde{p} = \frac{(||\operatorname{dev} A|| - b)_+}{2\mu + \sigma_y^2 H^2}\frac{\operatorname{dev} A}{||\operatorname{dev} A||},$$

where

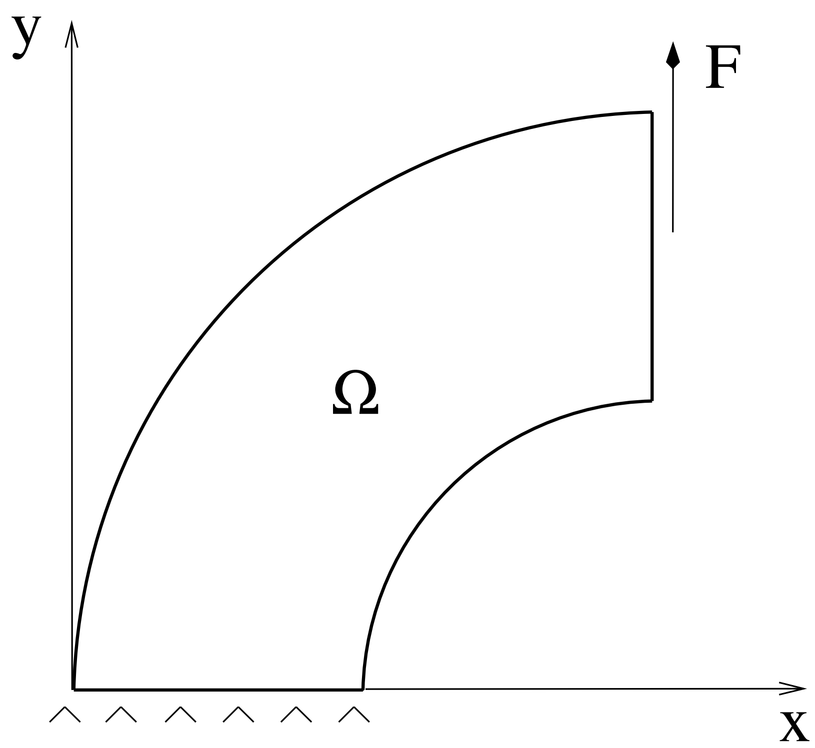
$$A = \mathbb{C}[\varepsilon(u) - p_0], \quad b = \sigma_y(1+\alpha_0 H).$$

Numerical experiments

2D
Quadratic plate



3D
Quarter of a ring

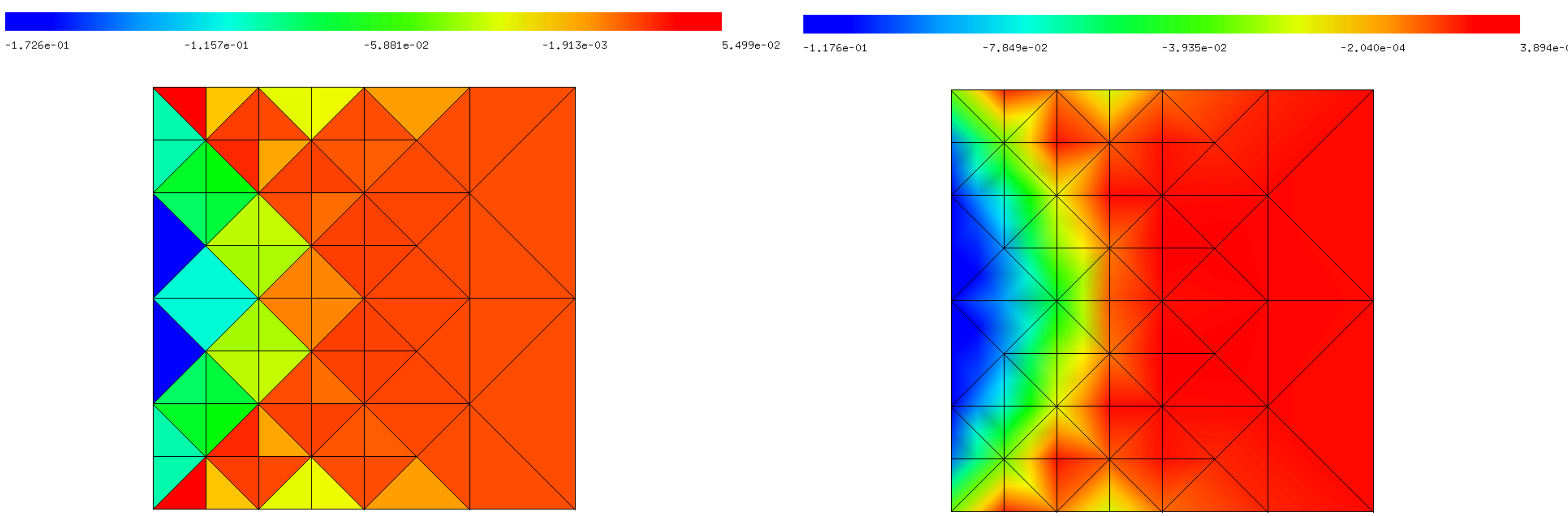


Elasto-plastic interface adaptivity for piecewise constant stresses: method of linearized yields

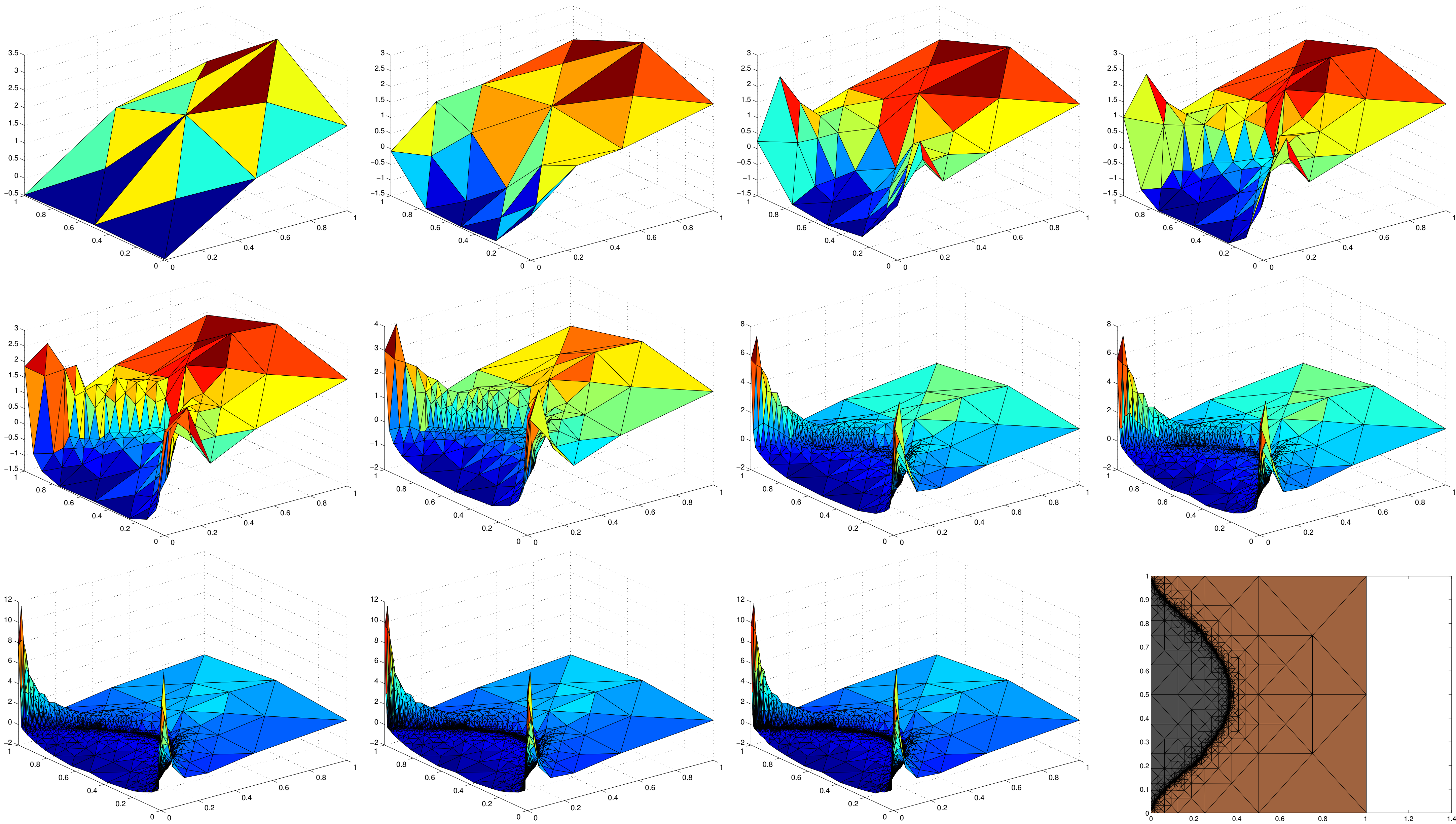
Yield function in the isotropic hardening case $\Phi(\sigma,\alpha) = |\operatorname{dev}\sigma| - \sigma_y(1+\alpha H)$, where the hardening parameter $\alpha = \alpha_0 + \sigma_y H|p - p_0|$. Nodal linear projection $\bar{\Phi} \in H^1(\Omega)^{d\times d}_{\text{sym}}$ to the piecewise constant yield function $\Phi \in L^2(\Omega)^{d\times d}_{\text{sym}}$:

$$\bar{\Phi}(N) := \frac{\sum_{T\in\mathcal{T}:N\in T} \Phi(\sigma|_T, \alpha|_T)|T|}{\sum_{T\in\mathcal{T}:N\in T} |T|}$$

Elements with both negative and positive nodal values of $\bar{\Phi}(N)$ are refined.



Yield function (left) and its nodal linear projection (right)

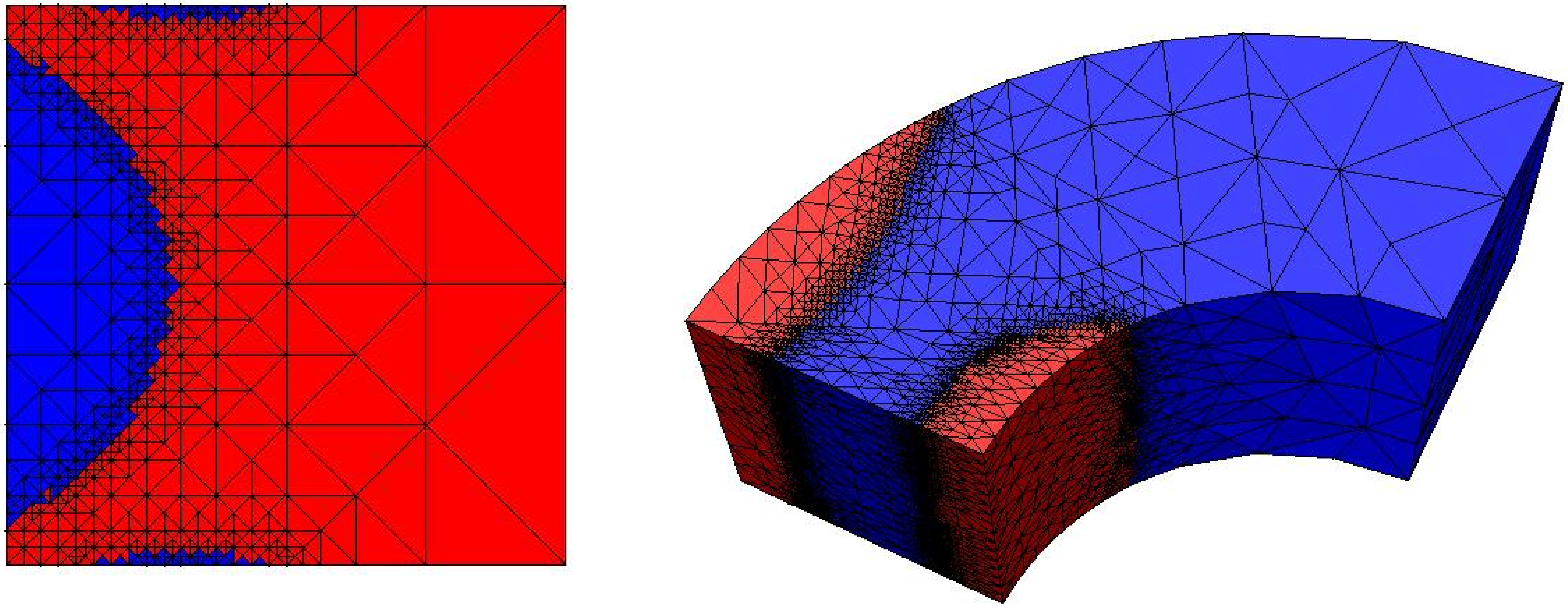


Nodal linear approximations of the yield function for graduated refinements and the elastoplastic zones for the finest refinement. Implemented in Matlab.

Elasto-plastic interface adaptivity for piecewise constant stresses: method of neighboring elements with different phases

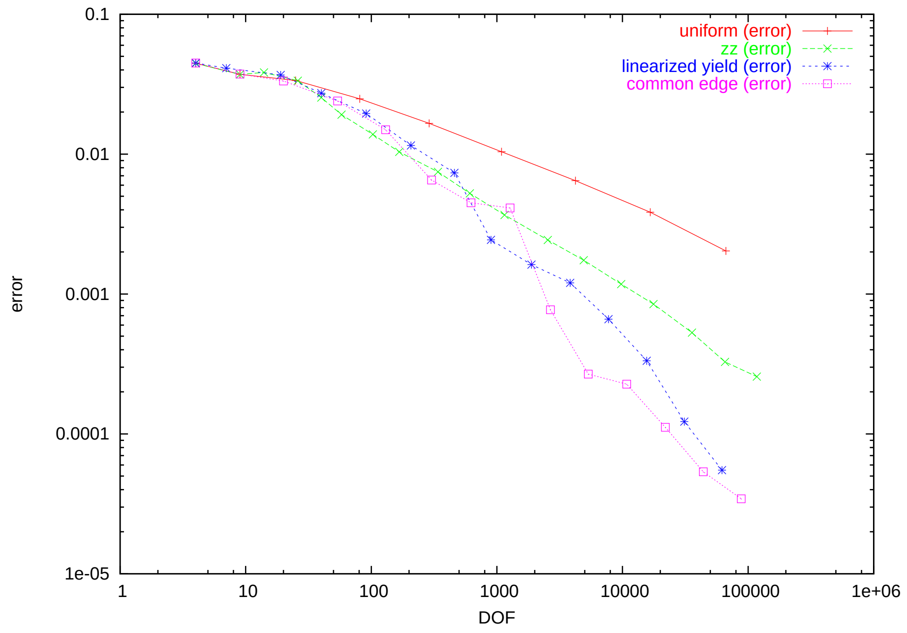
Error estimator marks both neighboring elements T_1 and T_2 with different phases, i.e where

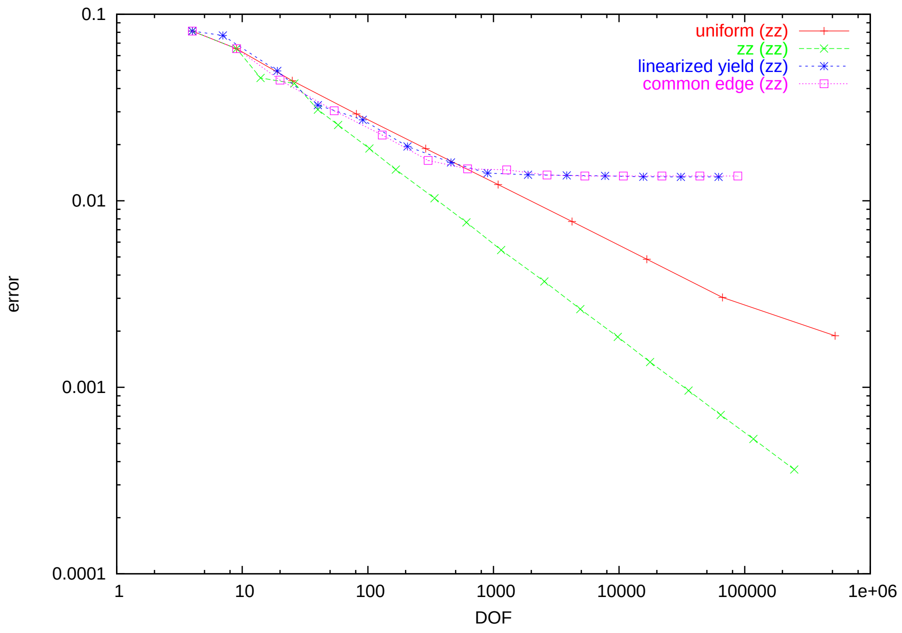
$$\Phi(\sigma|_{T_1}, \alpha|_{T_1})\Phi(\sigma|_{T_2}, \alpha|_{T_2}) \leq 0$$



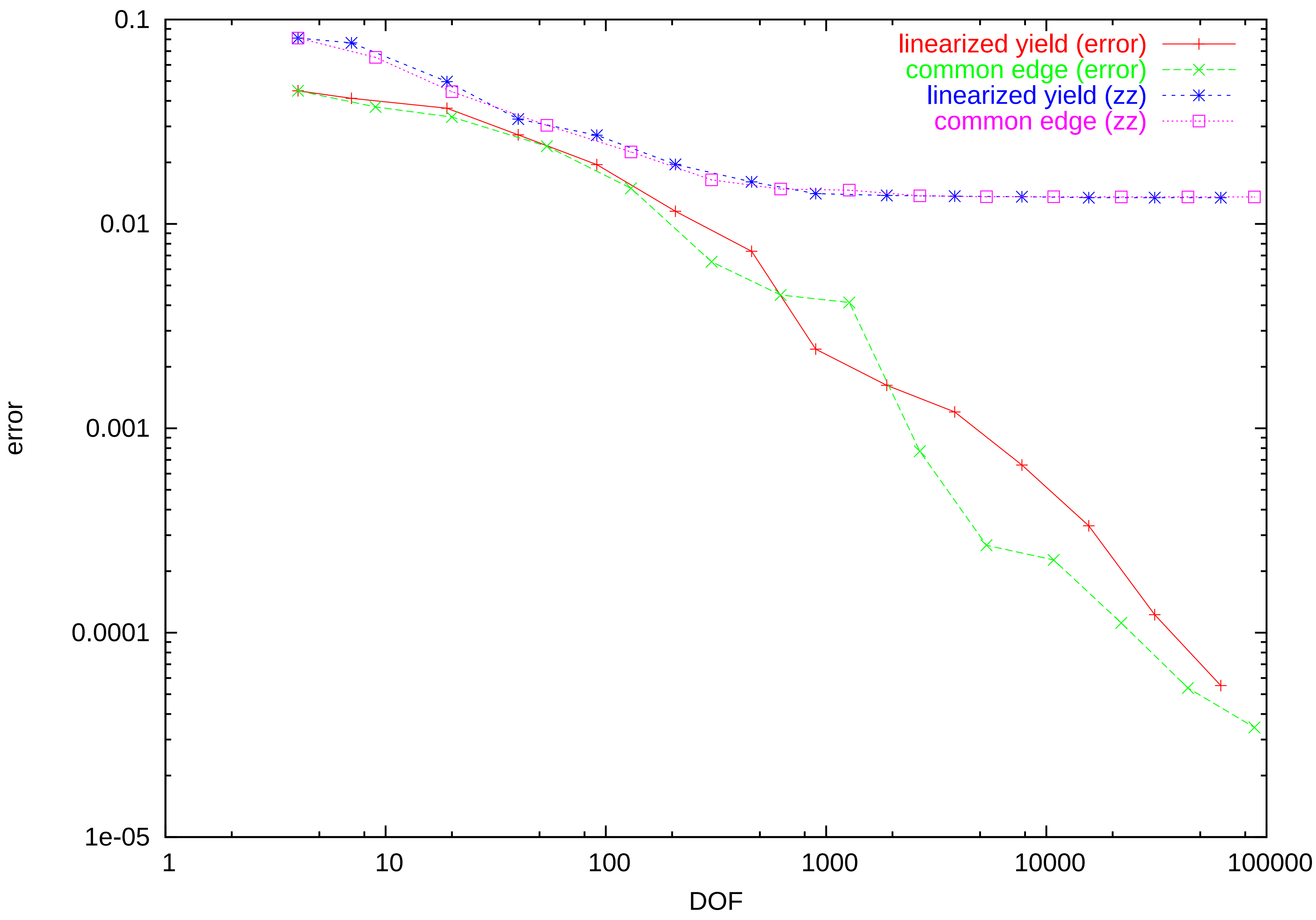
Elastoplastic zones (blue - elastic, red - plastic)

Convergence result (2D)



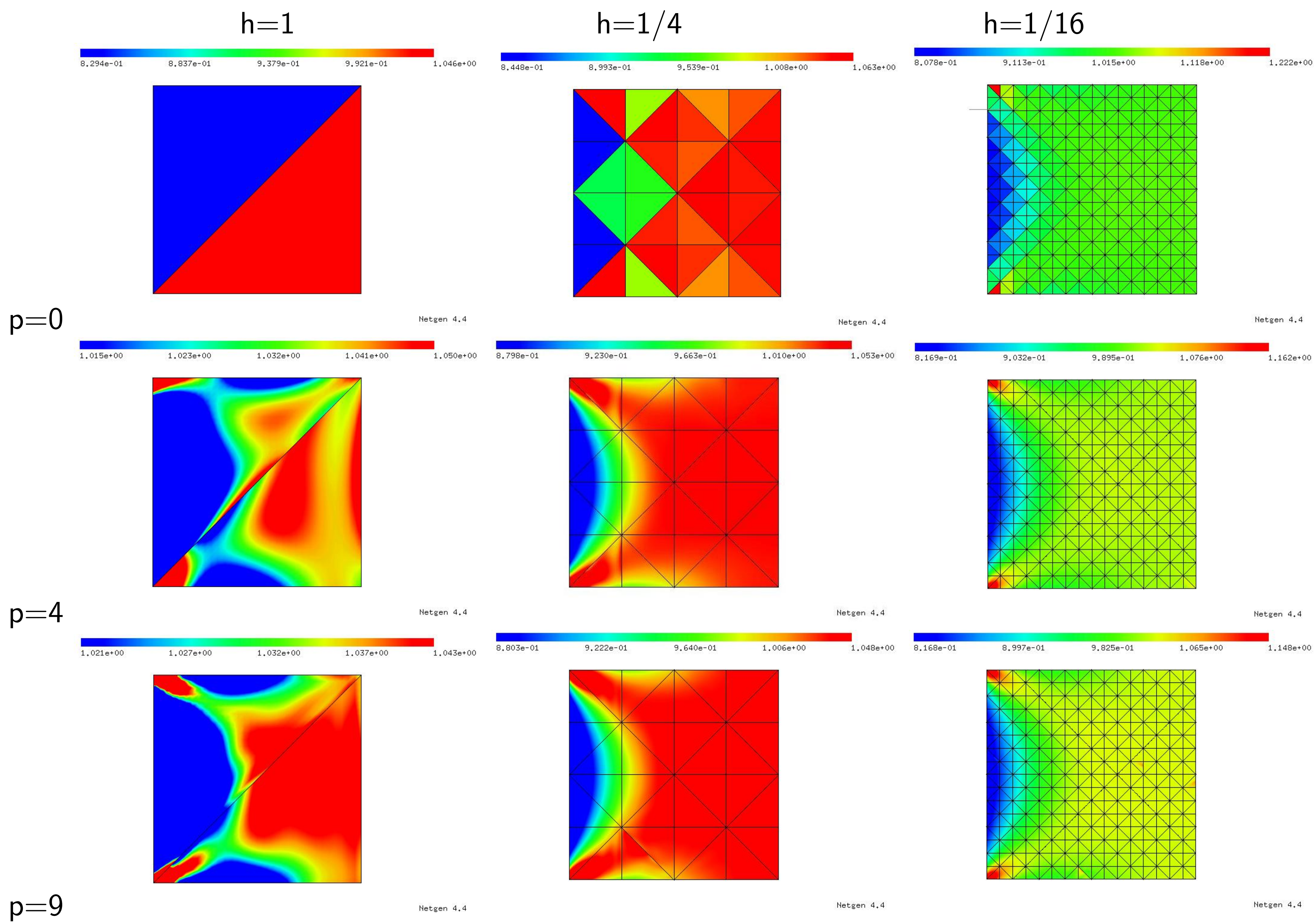


Approximated errors (left) and zz error estimates (right) for different refinement techniques



Failure of ZZ estimation in elastoplasticity???

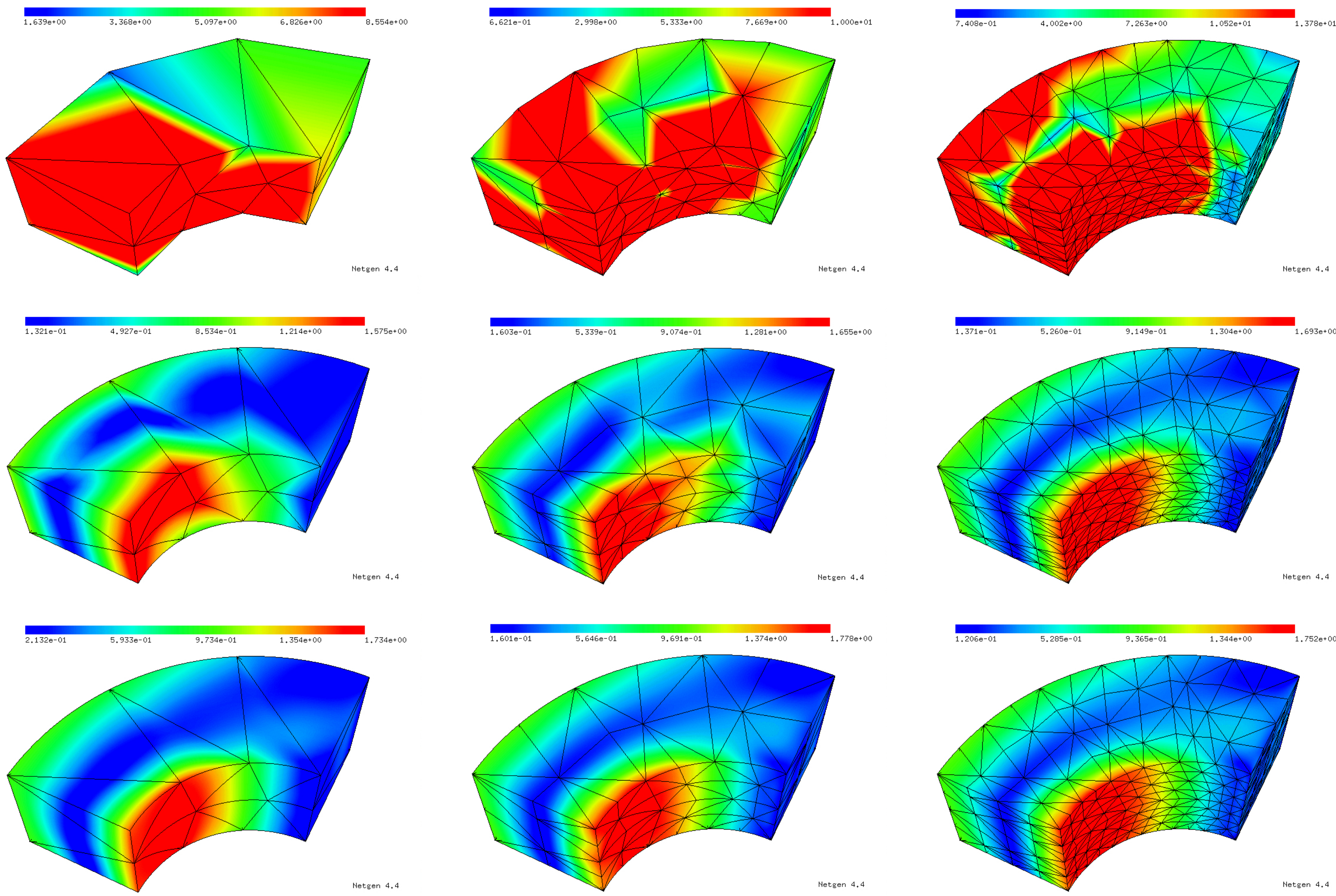
p and h method in 2D: von Mises stress



p and h method in 3D: von Mises stress

For visualization reasons the stresses are projected onto a H^1 function

h=1 h=1/2 h=1/4



p=0

p=1

p=3

Conclusions

We have considered:

- Problem formulation and discretization
- Minimization: 3D time-dependent algorithm
- Numerical experiments
  - Interface adaptivity
  - p - version

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Outlook

- Combined hpr methods
  - h, r: Singularities
  - p: Smooth solutions
- Level sets use for elastoplastic interface identification
- Application to shells