Multi-yield elastoplastic continuum - modeling and computations.

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Summary. The quasi-static evolution of an elastoplastic body with a multi-surface constitutive law of linear kinematic hardening type allows the modeling of curved stress-strain relations. It generalises classical small-strain elastoplasticity from one to various plastic phases. Firstly, we briefly recall a mathematical model represented by an initial-boundary value problem in the form a variational inequality. Then, the main concern of this paper is focused on an efficient numerical implementation of a one time-step problem. Based on the minimisation problem we describe an iterative non-linear algorithm whose linear subsystems are solved by a geometrical multigrid method. Finally, the numerical computations in 2D and 3D are presented.

Keywords Variational inequalities, elastoplasticity, kinematic hardening, multi-surface model, Prandtl-Ishlinskii model, multigrid preconditioned solver
AMS Subject Classification 47J40, 49J40, 65M55, 74C05

1 Introduction

In this paper we consider the quasi-static initial-boundary value problem for small strain elastoplasticity with a multi-surface constitutive law. We treat here a Prandtl-Ishlinskii model of a play type which goes back in the 1D case to Prandtl [Pra28] and Ishlinskii [Ish54] and in the multidimensional case to Besseling [Bes58] and Iwan [Iwa66]. The model extends the classical linear kinematic harding model (single-yield model), that goes back to Melan [Mel38] and Prager [Pra49] in the sense, that is operates with more plastic strains (multi-yield model). Hysteresis properties have been intensively studied by Visintin [Vis94] or Krejčí [Kre96] amongst others. Our functional formulation of the model and its analysis is based on a direct extention of the work of Han and Reddy [HR99] for the linear kinematic hardening model in terms of a time dependent variational inequality. Our numerical approximation for one time-step problem uses the formulation of Alberty, Carstensen, and Zarrabi [ACZ99] extended for a two-yield model, where the solution parameters, i.e., the displacement and two plastic strains, are sought as minimisers of a convex but non-smooth functional. For our approach we regularise this functional, thus standard methods can be applied to the quadratic optimisation problem. The main idea for the algorithm is the use of the Schur-Complement form of the discretised problem in the displacements. The arising linear system is solved by a multi-grid preconditioned conjugate gradient solver.

The paper is organised as follows: In Section 2, the local material model is presented, which is the basis for the boundary value problem in Section 3. The numerical algorithm

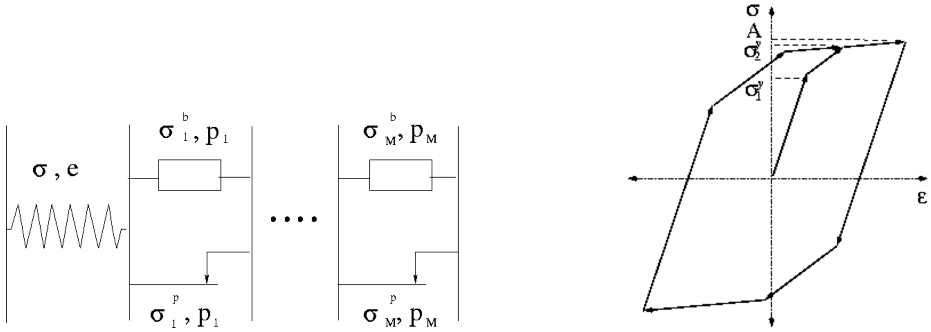

Fig. 1. Prandtl-Ishlinskii model of play type (left) and its $\sigma - \varepsilon$ hysteresis type behaviour for a periodical stress $\sigma(t) = A\sin(t), t \in (0, 2\pi)$ (right).

is designed in Section 4, the numerical experiments are presented in Section 5. Finally, an outlook on the work still to do is given.

2 The Local Material Model

The constitutive law furnishes the relationship between the stress tensor σ and the strain tensor ε. The model discussed here is the Prandtl-Ishlinskii model of play type described by Visintin [Vis94] and Krejčí [Kre96] among others. It contains finitely many surfaces and its rheological structure and typical hysteresis behaviour are depicted in Figure 1. It is local in the sense that for any given material point x it involves only the time histories $\sigma = \sigma(t)$ and $\varepsilon = \varepsilon(t)$ at that point. It is given by the following system of equations and an evolution variational inequality:

$$\begin{aligned} &\varepsilon = e + p \\ &p = \sum_{r \in I} p_r \\ &\sigma = \sigma_r^b + \sigma_r^p, \quad r \in I \end{aligned} \tag{1}$$

$$\sigma = \mathbb{C}e \tag{2}$$

$$\sigma_r^b = \mathbb{H}_r p_r, \quad r \in I \tag{3}$$

$$\sigma_r^p \in Z, \quad \dot{p}_r : (\tau_r - \sigma_r^p) \leq 0 \quad \text{for all } \tau_r \in Z_r, r \in I, \tag{4}$$

Equation (1) represents the additive decomposition of the strain ε into its elastic part e and its plastic part p as well as of the stress σ into the backstresses σ_r^b and the plastic stress σ_r^p, where $r \in I = \{1, \dots, M\}$. The plastic strain p is additively decomposed to internal plastic strains p_r. The equation (2) denotes a linear elastic law, in the isotropic case one has

$$\mathbb{C}\varepsilon = 2\mu\varepsilon + \lambda(\operatorname{tr}\varepsilon)\mathbb{I}, \tag{5}$$

where the (positive) coefficients μ and λ are called Lamé coefficients. Here $\mathbb{I}$ denotes the second order identity tensor (an identity matrix) and $\operatorname{tr} : \mathbb{R}^{d\times d} \to \mathbb{R}$ defines the trace of a matrix, $\operatorname{tr}\varepsilon := \sum_{j=1}^{d} \varepsilon_{jj}$, for $\varepsilon \in \mathbb{R}^{d\times d}$, where d is the problem dimension. Equation (3) couples the backstresses σ_r^b and the plastic strains p_r through linear mappings with positive definite hardening matrices $\mathbb{H}_r, r \in I$. A typical choice will be $\mathbb{H}_r = h_r\mathbb{I}$, where

$h_r > 0, r \in I$ are hardening coefficients. Variational inequality (4) formalises the Prandtl-Reuß normality law, also called the principle of maximal dissipation. The sets $Z_r \subset \mathbb{R}^{d\times d}_{sym}, r \in I$ describe the admissible (plastic) stresses, their boundaries ∂Z_r are called the yield surfaces. We will exclusively use the standard von Mises cylinder with yield stress σ^y

$$Z_r = \{\sigma \in \mathbb{R}^{d\times d}_{sym} : ||\operatorname{dev}\sigma|| \le \sigma^y_r\}. \tag{6}$$

Here, $||a||^2 = a : a, \quad a : b = \sum_{i,j=1}^d a_{ij} b_{ij}$ defines the (Frobenius) norm and the corresponding scalar product, and the deviator of σ is defined as $\operatorname{dev}\sigma := \sigma - \frac{1}{d}(\operatorname{tr}\sigma)\mathbb{I}$. Since this model is described by more (namely M) yield stresses σ^y_r, we classify the model as a multi-yield model or as M-yield model in order to express the number of yield stresses. If $M = 1$ then we speak about a single-yield model, which represents a classical linear kinematic hardening model.

3 The Boundary Value Problem

The elastoplastic continuum is assumed to occupy a bounded domain $\Omega \subset \mathbb{R}^d$, with a Lipschitz boundary $\Gamma = \partial\Omega$. The boundary Γ is split into a Dirichlet boundary Γ_D, a closed subset of Γ with a positive surface measure, and the remaining (relatively open and possibly empty) Neumann part $\Gamma_N := \Gamma \setminus \Gamma_D$. We pose essential and static boundary conditions, namely

$$u = 0 \quad \text{on } \Gamma_D \qquad \text{and} \qquad \sigma \cdot n = g \quad \text{on } \Gamma_N,$$

where g is a given applied surface force and n denotes the outer normal to the boundary Γ_N. Our analysis will be restricted to the study of a boundary value problem defined in these functional spaces:

$$H^1_D(\Omega) = \{v \in H^1(\Omega)^d | v = 0 \text{ on } \Gamma_D\},$$
$$Q = \{q : q \in \operatorname{dev}\mathbb{R}^{d\times d}_{sym}, q_{ij} \in L^2(\Omega)\},$$

where $H^1(\Omega)$ and $L^2(\Omega)$ are the usual Sobolev and Lebesgue spaces. The condition $q \in \operatorname{dev}\mathbb{R}^{d\times d}_{sym}$ in the definition of Q implies that $\operatorname{tr} q = 0$, i.e., q is a trace free matrix. It is shown by Brokate, Carstensen, Valdman [BCV03] that the combination of the system (1)-(4) describing the Prandtl-Ishlinskii model of the play type together with (quasi-static) equilibrium between external (denoted as f) and internal forces, i.e.,

$$\operatorname{div}\sigma(x,t) + f(x,t) = 0, \quad x \in \Omega, t \in (0,T) \tag{7}$$

results in the time-dependent variational inequality for the state variable $w = (u, (p_r)_{r\in I})$:

$$a(w(t), z - \dot{w}(t)) + \psi(z) - \psi(\dot{w}(t)) \ge \langle \ell(t), z - \dot{w}(t) \rangle, \quad \text{for all } z \in \mathcal{H}. \tag{8}$$

w is considered to be an element of the Hilbert space $\mathcal{H} = H^1_D(\Omega) \times \prod_{r\in I} Q$ and to satisfy the zero initial condition $w(0) = 0$. Writing $z = (v, (q_r)_{r\in I})$, a bilinear form $a(\cdot,\cdot)$, a linear functional $\ell(\cdot)$ and a nonlinear functional $\psi(\cdot)$ are defined as:

$$a : \mathcal{H}\times\mathcal{H} \to \mathbb{R}, \quad a(w,z) = \int_\Omega \mathbb{C}(\varepsilon(u) - \sum_{r\in I} p_r) : (\varepsilon(v) - \sum_{r\in I} q_r)\,\mathrm{d}x + \\ + \sum_{r\in I}\int_\Omega \mathbb{H} p_r : q_r \,\mathrm{d}x, \tag{9}$$

$$\ell(t) : \mathcal{H} \to \mathbb{R}, \quad \langle \ell(t), z \rangle = \int_{\Omega} f(t) \cdot v \, \mathrm{d}x + \int_{\Gamma_N} g(t) \cdot v \, \mathrm{d}S(x), \tag{10}$$

$$\psi : \mathcal{H} \to \mathbb{R}, \quad \psi(z) = \sum_{r \in I} \int_{\Omega} \sigma_r^y ||q_r|| \, \mathrm{d}x. \tag{11}$$

Thus we can formulate the following formulation of the boundary value problem of quasi-static elastoplasticity.

Problem 1 (BVP of quasi-static multi-surface elastoplasticity).
For given $l \in H^1(0, T; \mathcal{H}^*)$ with $\ell(0) = 0$, find $w \in H^1(0, T; \mathcal{H})$ with $w(0) = 0$, such that (8) holds for almost all $t \in (0, T)$.

The unique solvability of Problem 1 under the assumption that the elastic and hardening tensors are symmetric and positive definite bases on the extension on the proof of Han and Reddy [HR95, HR99] and can be found in works of Valdman [Val02] or Brokate, Carstensen, Valdman [BCV03]:

Theorem 1. *Let $l \in H^1(0, T; \mathcal{H}^*)$ with $\ell(0) = 0$. Then there exists a unique solution $w \in H^1(0, T; \mathcal{H})$ of Problem 1.*

4 Numerical Algorithm

The starting point for the finite element method is the time-discretised form of the variational problem. Problem 1 is solved by an implicit time discretisation, we use the implicit Euler scheme with equidistant time intervals.

It was shown by Alberty, Carstensen, and Zarrabi [ACZ99] that in the single-yield case, i.e., $M = 1$, the time-discretised dual formulation in each time step is equivalent to an optimisation problem depending only on the displacement u and the plastic strain p. This result was obtained by using functional analytic arguments, as the variational inequality is regarded as a sub-differential, for which the dual sub-differential exists and can be reformulated. The resulting objective depends on the chosen hardening law (linear kinematic hardening or isotropic hardening), though the structure remains the same. In Kienesberger [Kie03] an algorithm solving the single-yield problems was developed using the results and notation of Alberty, Carstensen, and Zarrabi [ACZ99]. Since the multi-yield hardening model structurally generalises the linear kinematic hardening model, authors managed to extend the original code using templates in C++ effectively in the way, that the multi-yield hardening model becomes a new hardening model. For computational reasons new parameters α_r are introduced, which are internal hardening parameters of the the same dimension as the plastic strains p_r and are defined by $\alpha_r = \mathbb{H}_r p_r$.

The notation is as follows: For given variables with index 0 of an initial time step t^0, the upgrades of the variables at the time step $t^1 = t^0 + \Delta t$ have to be determined. The already time-discretised generalised optimisation problem for the multi-yield case in each time step, subject to the modifications for fitting to the single-yield algorithm, reads as:

$$
\begin{aligned}
f(u,p_1,\dots,p_M) :=& \frac{1}{2}\int_\Omega \mathbb{C}(\varepsilon(u)-\sum_{r\in I}p_r):(\varepsilon(u)-\sum_{r\in I}p_r)\,\mathrm{d}x \\
&+\frac{1}{2}\int_\Omega\sum_{r\in I}|\alpha_r^0|^2\,\mathrm{d}x+\frac{1}{2}\int_\Omega\sum_{r\in I}|p_r-p_r^0|^2\,\mathrm{d}x+\int_\Omega\sum_{r\in I}\alpha_r^0:(p_r-p_r^0)\,\mathrm{d}x \\
&+\int_\Omega\sum_{r\in I}\sigma_r^y|p_r-p_r^0|\,\mathrm{d}x-\int_\Omega fu\,\mathrm{d}x\to\min,
\end{aligned}
\tag{12}
$$

where α^0 is the internal hardening variable from the initial time step.

The basic idea idea for solving the quasi-static problem is using a uniform time discretisation and iterate in each time step until the minimisers, i.e., the displacement u and the plastic strains p_r are determined. Then these values and the separately calculated α_r are used as the reference values with index 0 for the next time step t^2.

The fifth term in (12) contains a norm the sharp bend of which may cause trouble, as the function f is not differentiable. To apply standard methods, the objective is desired to be differentiable and quadratic, thus the function is regularised as follows: The term $|.|$ is regularised by smoothing the norm function, i.e.,

$$
|.|_\epsilon := \begin{cases} |.| & \text{if } |.|\geq\epsilon, \\ \frac{1}{2\epsilon}|.|^2+\frac{\epsilon}{2} & \text{if } |.|<\epsilon. \end{cases}
\tag{13}
$$

For small ϵ, the quadratised function $f(u,p_1,\dots,p_M)$ is very similar to the original one, but its properties change enormously. Therefore, it will be referred to by the new symbol $\bar{f}$.

Another simplification is defining the change of p_r by $\tilde{p}_r = p_r - p_r^0$, and using it as an argument of the objective instead of p_r:

The spatial discretisation is carried out by the standard finite element method using linear triangular, resp. tetrahedral finite elements. For reasons of better readability and coherence, the name of the vector denoting the discretised displacement u is again u. The same is valid for $\tilde{p}_r$, p_r^0, furthermore the symmetric matrices are transformed to vectors, e.g. in 2D

$$
\begin{pmatrix} \tilde{p}_r^{11} & \tilde{p}_r^{12} \\ \tilde{p}_r^{12} & \tilde{p}_r^{22} \end{pmatrix} \Longrightarrow \begin{pmatrix} \tilde{p}_r^{11} \\ \tilde{p}_r^{22} \\ \tilde{p}_r^{12} \end{pmatrix},
$$

such that the objective and other equations can be written in a matrix and vector notation.

For the derivation of the algorithm and numerical experiments we will consider only the two-yield case, i.e., $M=2$, as it shows the characteristics of the multi-yield problem and can be extended easily. Now, the objective reads as

$$
\begin{aligned}
\bar{f}(u,\tilde{p}_1,\tilde{p}_2) =& \frac{1}{2}\begin{pmatrix} u \\ \tilde{p}_1 \\ \tilde{p}_2 \end{pmatrix}^T \begin{pmatrix} B^T\mathbb{C}B & -B^T\mathbb{C} & -B^T\mathbb{C} \\ -\mathbb{C}B & \mathbb{C}+\mathcal{D}^1 & \mathbb{C} \\ -\mathbb{C}B & \mathbb{C} & \mathbb{C}+\mathcal{D}^2 \end{pmatrix} \begin{pmatrix} u \\ \tilde{p}_1 \\ \tilde{p}_2 \end{pmatrix} + \begin{pmatrix} -f-B^T\mathbb{C}(p_1^0+p_2^0) \\ \mathbb{C}(p_1^0+p_2^0)+\mathbb{Q}\alpha_1^0 \\ \mathbb{C}(p_1^0+p_2^0)+\mathbb{Q}\alpha_2^0 \end{pmatrix}^T \begin{pmatrix} u \\ \tilde{p}_1 \\ \tilde{p}_2 \end{pmatrix} \\
&+\frac{1}{2}\mathbb{C}p_1^0:p_1^0+\frac{1}{2}\mathbb{C}p_2^0:p_2^0+\frac{1}{2}|\alpha_1^0|^2+\frac{1}{2}|\alpha_2^0|^2\to\min,
\end{aligned}
\tag{14}
$$

where Bu denotes the discretised strain $\varepsilon(u)$, and $\mathbb{Q}$ is the result of regarding $\tilde{p}_r$ as vectors, i.e., the matrix norm is defined by $|p| = (p^T\mathbb{Q}p)^{\frac{1}{2}}$.

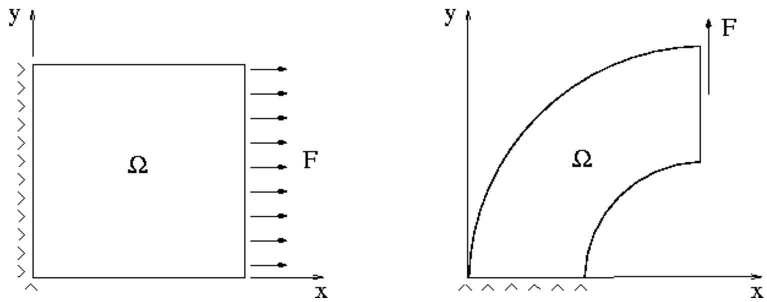

Fig. 2. Geometry of a beam (left) and the quarter of a ring (right) problems.

$\mathcal{D}^1 = \mathbb{Q}(1 + \frac{2\sigma_1^y}{|\tilde{p}_1|_\epsilon})$ is the non-linear iteration matrix of $\bar{f}$ with respect to $\tilde{p}_1$, and analogous for $\mathcal{D}^2$ and $\tilde{p}_2$. These matrices are computed in every iteration step using the current p_r, but apart from that the dependencies on $|\tilde{p}_r|_\epsilon$ will be neglected. This is not an exact method for determining the change of the plastic strain, but its error will be corrected later on as the $\tilde{p}_r$ will be calculated separately and iteratively with the alternating direction method. The matrix in (14) is positive definite, thus the minimiser $(u, \tilde{p}_1, \tilde{p}_2)$ has to fulfill the necessary condition of the derivative being equal to zero:

$$\begin{pmatrix} B^T\mathbb{C}B & -B^T\mathbb{C} & -B^T\mathbb{C} \\ -\mathbb{C}B & \mathbb{C}+\mathcal{D}^1 & \mathbb{C} \\ -\mathbb{C}B & \mathbb{C} & \mathbb{C}+\mathcal{D}^2 \end{pmatrix} \begin{pmatrix} u \\ \tilde{p}_1 \\ \tilde{p}_2 \end{pmatrix} + \begin{pmatrix} -f - B^T\mathbb{C}(p_1^0 + p_2^0) \\ \mathbb{C}(p_1^0 + p_2^0) + \mathbb{Q}\alpha_1^0 \\ \mathbb{C}(p_1^0 + p_2^0) + \mathbb{Q}\alpha_2^0 \end{pmatrix} = 0. \tag{15}$$

Extracting the vector $(\tilde{p}_1, \tilde{p}_2)^T$ from the two lower lines in (15) and inserting it into the first one yields the Schur-Complement system in u:

$$\begin{aligned} B^T(\mathbb{C} - &\begin{pmatrix} \mathbb{C} \\ \mathbb{C} \end{pmatrix}^T \begin{pmatrix} \mathbb{C}+\mathcal{D}^1 & \mathbb{C} \\ \mathbb{C} & \mathbb{C}+\mathcal{D}^2 \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{C} \\ \mathbb{C} \end{pmatrix})Bu \\ &= f + B^T\mathbb{C}((p_1^0 + p_2^0) - \begin{pmatrix} I \\ I \end{pmatrix}^T \begin{pmatrix} \mathbb{C}+\mathcal{D}^1 & \mathbb{C} \\ \mathbb{C} & \mathbb{C}+\mathcal{D}^2 \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{C}(p_1^0 + p_2^0) + \mathbb{Q}\alpha_1^0 \\ \mathbb{C}(p_1^0 + p_2^0) + \mathbb{Q}\alpha_2^0 \end{pmatrix}). \end{aligned} \tag{16}$$

This linear system is solved by a multigrid preconditioned conjugate gradient method, see e.g. Bramble [Bra95]. From the numerical tests we have seen that it is not necessary to use the multigrid preconditioner arising from the plasticity problem, the preconditioner for the related problem of elasticity is sufficient and much faster.

For the multigrid method, we use one Gauss-Seidel pre- and post-smoothing step in a V-Cycle, the system on the coarse grid is solved exactly. Furthermore, the nested iteration approach was used, which means that the starting values for the coarse grid correction are the restrictions of the fine grid functions.

5 Numerical Experiments

The algorithm was implemented in NGSolve - the finite element solver extension package of the mesh generator tool NETGEN developed in our group. Finite element basis functions were chosen as piecewise linear for the displacement u and piecewise constant for the plastic strains p_1 and p_2. Furthermore, the full multigrid method was used, i.e.,

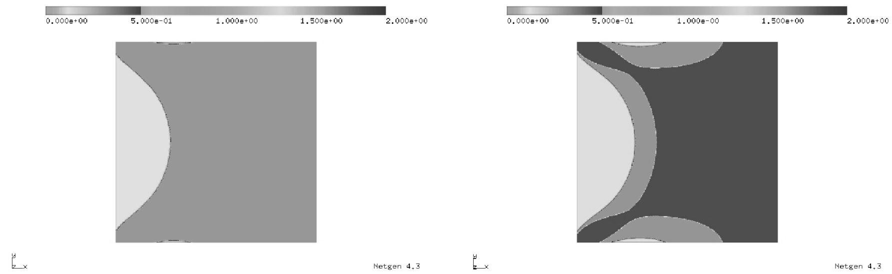

Fig. 3. Plasticity domains in the single-yield (left) and two-yield case (right) of the beam.

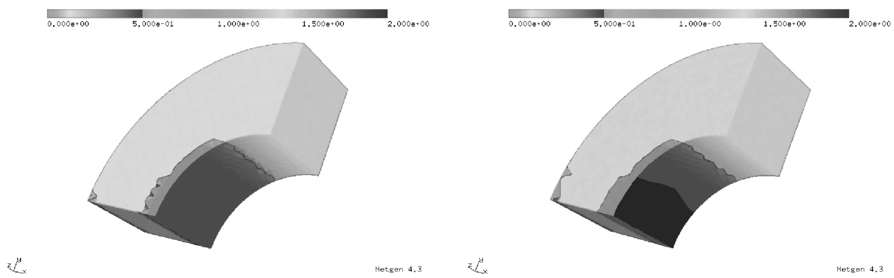

Fig. 4. Plasticity domains in the single-yield (left) and two-yield case (right) of the quarter of the ring.

we started with a coarse grid, solved the problem, refined the grid, solved the problem on the finer grid et cetera.

The algorithm was tested on two- as well as on three-dimensional domains, for both the single-yield and multi-yield case, see Figure 2 for the geometries.

The first testing geometry is the 2D beam of Figure 2 with the left edge fixed and the right edge charged with a force acting in the direction of the external normal vector. The second geometry tested is the 3D quarter of a ring from Figure 2 with constant thickness in the z-axis which is the same as the thickness of the ring in the 2D sketch. The quarter ring is fixed on the lower face and a force is acting upwards on the right face. The finest uniform mesh consists of 131 072 triangles (which corresponds to 658 428 degrees of freedom DOF in the calculation of u) for the 2D examples and 25 088 tetrahedra (122 334 DOF) for the 3D example. Figures 3 and 4 show the plasticity domains in the single-yield and in the multi-yield case. The elastic zones are colored light grey, the first plastic zones are middle-grey, and the second plastic zone is dark-grey.

6 Conclusions and Future Work

In this paper a multi-yield plasticity model and its numerical computations were shown. The nonlinear iterative algorithm that uses a multigrid preconditioned solver was presented, its performance in 2D and 3D was demonstrated.

In the future we will extend the solution idea to a quasi-Newton algorithm, i.e., the Schur-Complement matrix will have some Hessian-type entries in order to improve the computational performance. This idea is already implemented for the single-yield case, where the numerical results demonstrate the faster algorithm performance with linear complexity. We expect the same result for the multi-yield case.

Another long-term aim is to identify the interfaces between the elastic and plastic zones and to refine the mesh adaptively in such a way, that the interface is approximated by the mesh. Then we expect an even faster performance of the algorithm.

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