Multi-yield elastoplastic continuum - modeling and computations.

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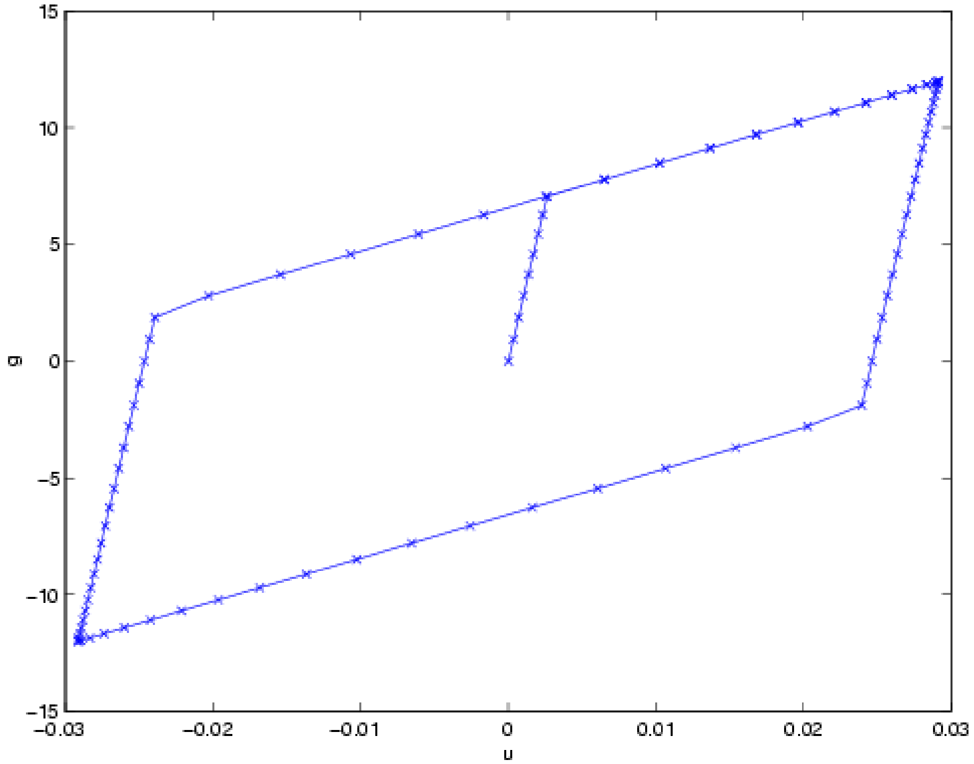
Outline

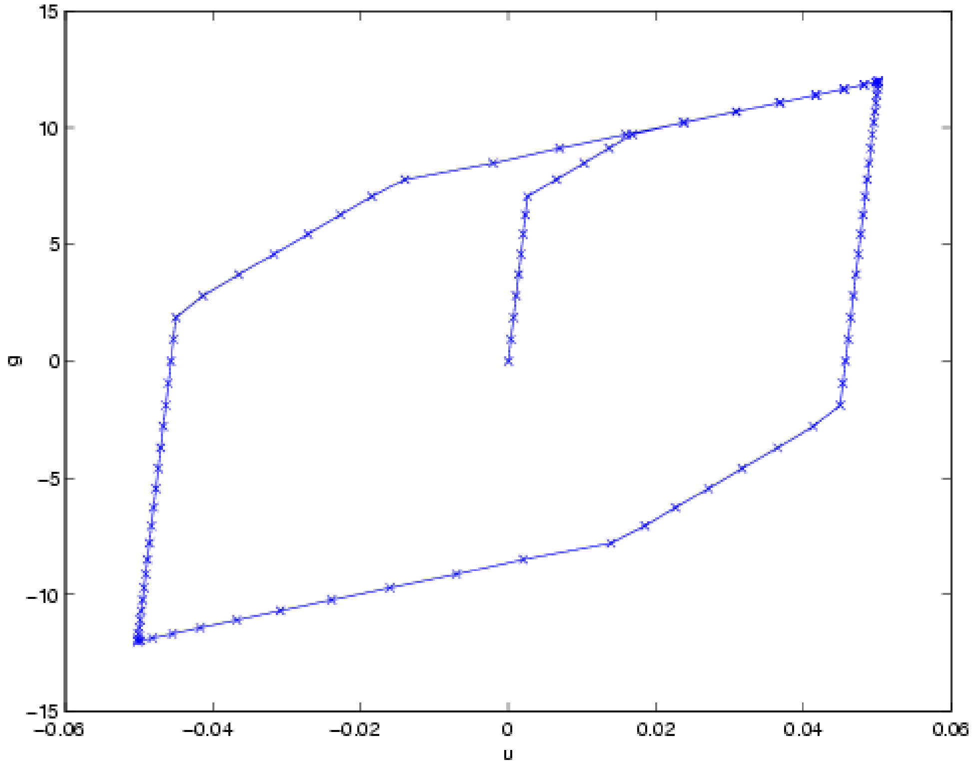
- Why multi-yield plasticity?
- Mathematical and analysis
- Discretized minimization problem
- Algorithm
- Numerical experiments
- Outlook

Why Multi-yield (Two-yield) model?

- More realistic hysteresis stress-strain relation in materials!



Kinematic hardening model.



Two-yield hardening model.

Mathematical model of M-yield elastoplasticity

Problem (Prandtl-Ishlinskii): For $l \in H^1(0,T;\mathcal{H}^*), l(0)=0$, find $w=(u,p^1,\dots,p^M):[0,T]\rightarrow \mathcal{H}, w(0)=0$ s. t.

$$\langle l(t), z-\dot{w}(t)\rangle \leq a(w(t), z-\dot{w}(t)) + j(z) - j(\dot{w}(t))$$

$\forall z=(v,\tau^1,\dots,\tau^M)\in\mathcal{H}$.

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$\forall z = (v, \tau^1, \dots, \tau^M) \in \mathcal{H}$.

Notation: $\mathcal{H} = H^1_D(\Omega) \times \underbrace{L^2(\Omega)^{d\times d}_{sym} \times \dots \times L^2(\Omega)^{d\times d}_{sym}}_{M \text{ times}},$

$$a(w,z) = \int_\Omega \Big(\mathbb{C}(\varepsilon(u) - \sum_{r=1}^{M} p^r)\Big) : \Big(\varepsilon(v) - \sum_{r=1}^{M} \tau^r\Big)\,\mathrm{d}x + \sum_{r=1}^{M} \int_\Omega \mathbb{H}^r p^r : \tau^r \,\mathrm{d}x,$$

$$\langle l(t), z\rangle = \int_\Omega f(t)\cdot v\,\mathrm{d}x + \int_{\Gamma_N} g(t)\cdot v\,\mathrm{d}x,$$

$$j(z) = \int_\Omega \sum_{r=1}^{M} D^r(\tau^r)\,\mathrm{d}x, \qquad D^r(x) = \begin{cases} \sigma^r_y ||x|| & \text{if } \operatorname{tr} x = 0, \\ +\infty & \text{otherwise.} \end{cases} \quad \text{(von Mises criterion)}$$

Existence and uniqueness

Assumption: positive definite elastic and hardening operators:

$$\mathbb{C}\xi : \xi \geq c||\xi||^2 \ \ \forall \xi \in \mathbb{R}^d$$

$$\mathbb{H}^r\xi : \xi \geq h^r||\xi||^2 \ \ \forall \xi \in \mathbb{R}^d, r = 1, \ldots, M$$

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Results: 1) $a(\cdot,\cdot)$ - *bounded, elliptic bilinear form in* $\mathcal{H}$*,*

$$|a(w,z)| \leq \Big((M+1)||\mathbb{C}|| + \max_{r=1,\dots,M} ||\mathbb{H}^r||\Big)||w||_{\mathcal{H}}||z||_{\mathcal{H}},$$
$$a(w,w) \geq \Big(k \min_{r=1,\dots,M}\{c, h^r\} \min\{1, K\}\Big)||w||^2_{\mathcal{H}},$$

where $K > 0$ *(Korn's first inequality) and* $k = k(M) = 1 + \frac{M}{2} - \frac{1}{2}\sqrt{M(M+4)}$*.*

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2) $j(\cdot)$ - nonnegative, positive homogeneous and Lipschitz continuous functional in $\mathcal{H}$

$$|j(z_1) - j(z_2)| \leq \ \ \Big(\max_{r=1,\ldots,M}\{\sigma_y^r\}\mathit{meas}(\Omega)^{\frac{1}{2}} M^{\frac{1}{2}}\Big)||z_1 - z_2||_{\mathcal{H}}.$$

Existence and uniqueness

Theorem: Let $l \in H^1(0,T;\mathcal{H}')$ with $l(0) = 0$. $\exists!\ w = (u, p^1, \dots, p^M)(t) \in H^1(0,T;\mathcal{H})$ of **Problem** (Prandtl-Ishlinskii).

Proof: *[Han, Reddy '99.]*

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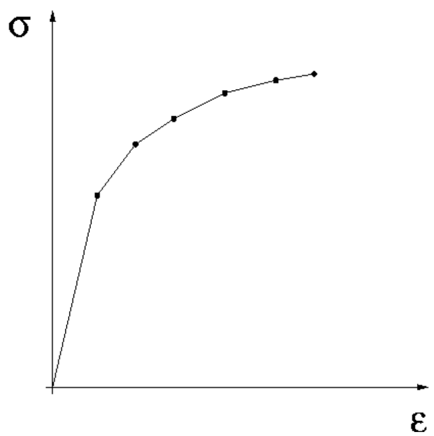
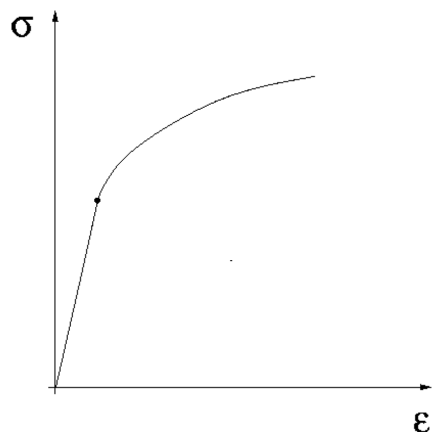
Generalization to Measure case:



$$p = \sum_{r=1}^{M} p^r \quad \Rightarrow \quad p = \int_{r \in I} p(r)\, d\mu(r)$$



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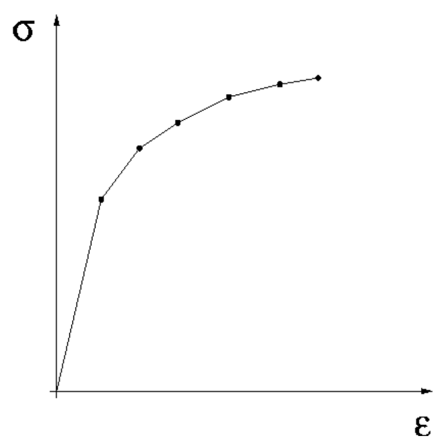

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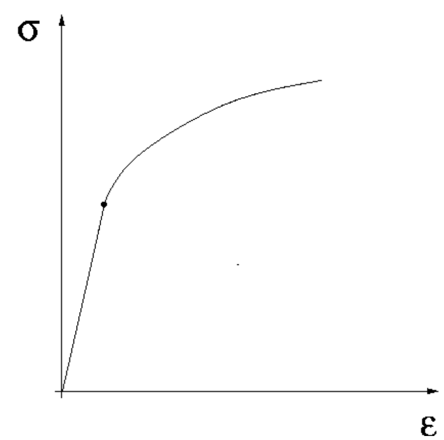

- *The same existence and uniqueness result with modified constants.*

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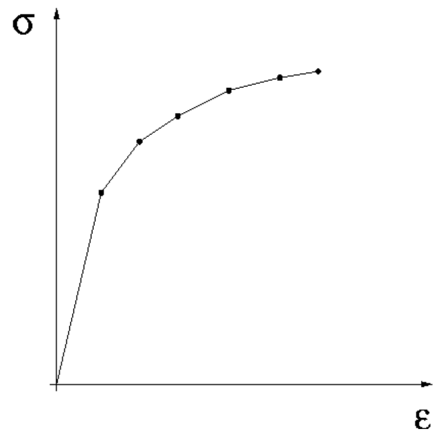

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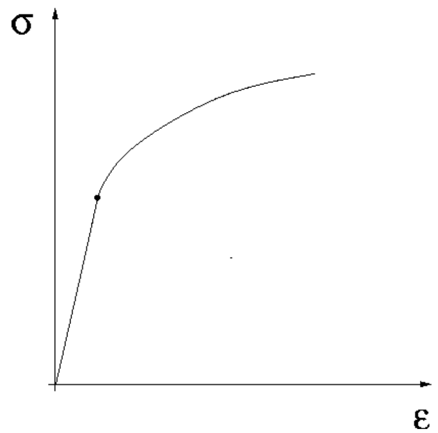

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Paper: Brokate, Carstensen, Valdman: Mathematical Modeling of the multi-yield elastoplastic material: Part 1 - Analysis

Discretization

- *in time:* *net* $0 = t_0 < t_1 < \cdots < t_N = T$, *and implicit Euler scheme*

$$\dot{X}(t_j) = \frac{X(t_j) - X(t_{j-1})}{t_j - t_{j-1}} \quad \text{for } j = 1, \ldots, N.$$

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- *in space: regular triangulation* $\mathcal{T}$ *of* Ω

  - *displacement* u : $H^1_D(\Omega)$ *approximated by* $\mathcal{S}^1_D(\mathcal{T}) := \{v \in H^1_D(\Omega) : \forall T \in \mathcal{T}, v|_T \in \mathbb{P}_1(T)^d\}$

  - *plastic strains* $p_1, \ldots, p_M$: $L^2(\Omega)$ *approximated by* $\mathcal{S}^0(\mathcal{T}) := \{a \in L^2(\Omega) : \forall T \in \mathcal{T}, a|_T \in \mathbb{R}\}$

Minimization problem in $u, p^1, \dots, p^M$

$$f(u,p^1,\dots,p^M) := \frac{1}{2}\int_\Omega \mathbb{C}(\varepsilon(u) - \sum_{r=1}^M p^r) : (\varepsilon(u) - \sum_{r=1}^M p^r)\, dx + \frac{1}{2}\int_\Omega \sum_{r=1}^M |\alpha_0^r|^2\, dx + \frac{1}{2}\int_\Omega \sum_{r=1}^M |p^r - p_0^r|^2\, dx$$

$$+ \int_\Omega \sum_{r=1}^M \alpha_0^r : (p^r - p_0^r)\, dx + \int_\Omega \sum_{r=1}^M \sigma_y^r |p^r - p_0^r|\, dx - \int_\Omega bu\, dx \to \mathit{min}.$$

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New variables: $\tilde{p}^1 = p^1 - p_0^1, \ldots.$

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A differentiable approximation of $|\tilde{p}|$*:*

$$|\tilde{p}|_\epsilon := \begin{cases} |\tilde{p}| & \textit{if } |\tilde{p}| \geq \epsilon \\ \frac{1}{2\epsilon}|\tilde{p}|^2 + \frac{\epsilon}{2} & \textit{if } |\tilde{p}| < \epsilon \end{cases}$$

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Minimization strategy in each time step:

$$u^{k+1} = \operatorname{argmin}_v \min_q \bar{f}(v,q) = \operatorname{argmin}_v \tilde{f}(v, q_{opt}(v))$$

Then $p^1 = p_0^1 + \tilde{p}^1, \dots$

Direct minimization problem in u: Two-yield model

Matrix form:

$$f(u,\tilde{p}^1,\tilde{p}^2) = \frac{1}{2}\begin{pmatrix} u \\ \tilde{p}^1 \\ \tilde{p}^2 \end{pmatrix}^T \begin{pmatrix} B^T\mathbb{C}B & -B^T\mathbb{C} & -B^T\mathbb{C} \\ -\mathbb{C}B & \mathbb{C}+\mathcal{D}^1 & \mathbb{C} \\ -\mathbb{C}B & \mathbb{C} & \mathbb{C}+\mathcal{D}^2 \end{pmatrix} \begin{pmatrix} u \\ \tilde{p}^1 \\ \tilde{p}^2 \end{pmatrix} + \begin{pmatrix} -b - B^T\mathbb{C}(p_0^1+p_0^2) \\ \mathbb{C}(p_0^1+p_0^2)+\mathbb{Q}\alpha_0^1 \\ \mathbb{C}(p_0^1+p_0^2)+\mathbb{Q}\alpha_0^2 \end{pmatrix}^T \begin{pmatrix} u \\ \tilde{p}^1 \\ \tilde{p}^2 \end{pmatrix}$$
$$+\frac{1}{2}\mathbb{C}p_0^1 : p_0^1 + \frac{1}{2}\mathbb{C}p_0^2 : p_0^2 + \frac{1}{2}|\alpha_0^1|^2 + \frac{1}{2}|\alpha_0^2|^2 \rightarrow \textit{min},$$

where $\mathcal{D}^1 = \mathbb{Q}(1+\frac{2\sigma_y^1}{|\tilde{p}^1|_\epsilon}), \mathcal{D}^2 = \mathbb{Q}(1+\frac{2\sigma_y^2}{|\tilde{p}^2|_\epsilon})$.

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The Schur-Complement system in u

$$B^T(\mathbb{C} - \begin{pmatrix} \mathbb{C} \\ \mathbb{C} \end{pmatrix}^T \begin{pmatrix} \mathbb{C}+\mathcal{D}^1 & \mathbb{C} \\ \mathbb{C} & \mathbb{C}+\mathcal{D}^2 \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{C} \\ \mathbb{C} \end{pmatrix})Bu = b + B^T\mathbb{C}(p_0 - \begin{pmatrix} I \\ I \end{pmatrix}^T \begin{pmatrix} \mathbb{C}+\mathcal{D}^1 & \mathbb{C} \\ \mathbb{C} & \mathbb{C}+\mathcal{D}^2 \end{pmatrix}^{-1} \begin{pmatrix} \mathbb{C}p_0 + \mathbb{Q}\alpha_0^1 \\ \mathbb{C}p_0 + \mathbb{Q}\alpha_0^2 \end{pmatrix}$$

where $p_0 = p_0^1 + p_0^2$. $\Rightarrow$ *Multigrid-PCG*

Direct minimization problem in $\tilde{p}$: analytical approach

Kinematic hardening model $(M = 1)$*:*

$$f(q) = \frac{1}{2}(\mathbb{C} + \mathbb{H})q : q - q : A + \sigma_y ||q|| \to \textit{min}$$

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Two-yield hardening model $(M=2)$*:*

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minimizer $\tilde{p} = \frac{(||\operatorname{dev} A|| - \sigma_y)_+}{2\mu + h} \frac{\operatorname{dev} A}{||\operatorname{dev} A||}$

Two-yield hardening model $(M = 2)$*:*

$$f\begin{pmatrix} q^1 \\ q^2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \mathbb{C} + \mathbb{H}^1 & \mathbb{C} \\ \mathbb{C} & \mathbb{C} + \mathbb{H}^2 \end{pmatrix}\begin{pmatrix} q^1 \\ q^2 \end{pmatrix} : \begin{pmatrix} q^1 \\ q^2 \end{pmatrix} - \begin{pmatrix} q^1 \\ q^2 \end{pmatrix} : \begin{pmatrix} A^1 \\ A^2 \end{pmatrix} + \sigma_y^1 ||q^1|| + \sigma_y^2 ||q^2|| \rightarrow \textit{min}$$

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$$\textit{minimizer } (\tilde{p}^1, \tilde{p}^2) = ?$$

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$\tilde{p}^2 \neq 0 \Rightarrow ||\tilde{p}^2||$ *is a root of a 6-th degree polynomial.*

Symbolic methods (Gröbner basis?)

Direct minimization problem in $\tilde{p}$: Two-yield model - iterative approach

Algorithm (*): *Given* $tolerance \geq 0$.

(a) Choose $(p_1^0, p_2^0) \in \operatorname{dev} \mathbb{R}^{d\times d}_{sym} \times \operatorname{dev} \mathbb{R}^{d\times \longrightarrow \min! d}_{sym}$, *set* $i := 0$.

(b) Find $p_2^{i+1} \in \operatorname{dev} \mathbb{R}^{d\times d}_{sym}$ *s. t.*

$$f(p_1^i, p_2^{i+1}) = \min_{Q_2 \in \text{ dev} \mathbb{R}^{d\times d}_{sym}} f(p_1^i, q_2).$$

(c) Find $p_1^{i+1} \in \operatorname{dev} \mathbb{R}^{d\times d}_{sym}$ *s. t.*

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(d) If $\frac{||p_1^{i+1} - p_1^i|| + ||p_2^{i+1} - p_2^i||}{||p_1^{i+1}|| + ||p_1^i|| + ||p_2^{i+1}|| + ||p_2^i||} > tolerance$ *set* $i := i + 1$ *and goto (b), otherwise output* (p_1^{i+1}, p_2^{i+1}).

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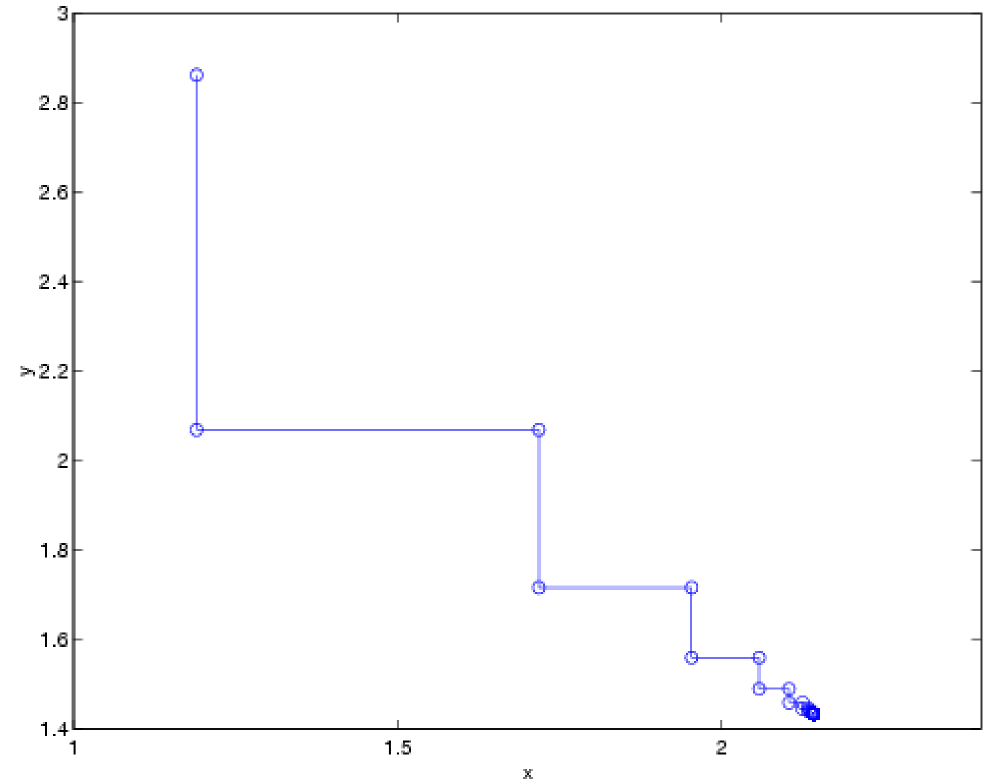
$$f(p_1^{i+1}, p_2^{i+1}) = \min_{q_1 \in \, dev \, \mathbb{R}^{d\times d}_{sym}} f(q_1, p_2^{i+1}).$$

(d) If $\frac{||p_1^{i+1} - p_1^i|| + ||p_2^{i+1} - p_2^i||}{||p_1^{i+1}|| + ||p_1^i|| + ||p_2^{i+1}|| + ||p_2^i||} > tolerance$ *set* $i := i + 1$ *and goto (b), otherwise output* (p_1^{i+1}, p_2^{i+1}).

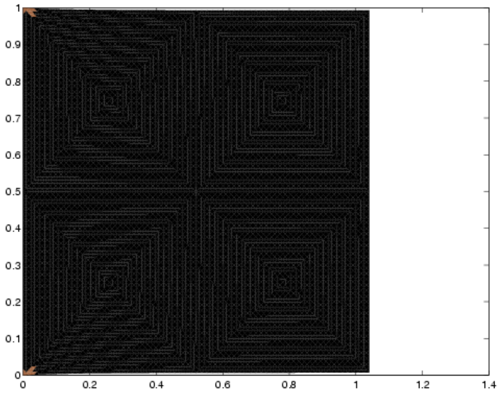


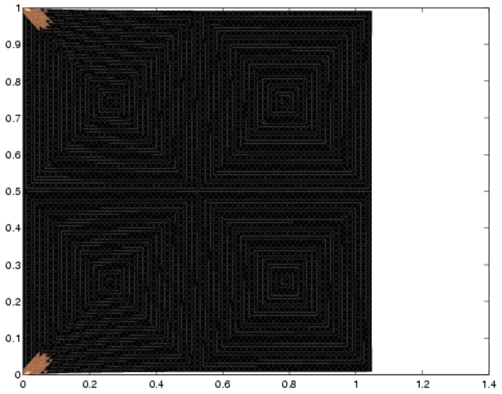
The approximations $p_1^i = (x^i, 0; 0, -x^i), p_2^i = (y^i, 0; 0, -y^i), i = 0, 1, \ldots$ *of Algorithm (*) in the* $x - y$ *coordinate system.*

- *global convergence with the rate* $1/2$: $||p_1^i - p_1||^2 + ||p_2^i - p_2||^2 \leq C_0 \cdot q^i$

Matlab calculations in 2D

Elastoplastic domains (black - elastic, brown - first plastic, yellow - second plastic)





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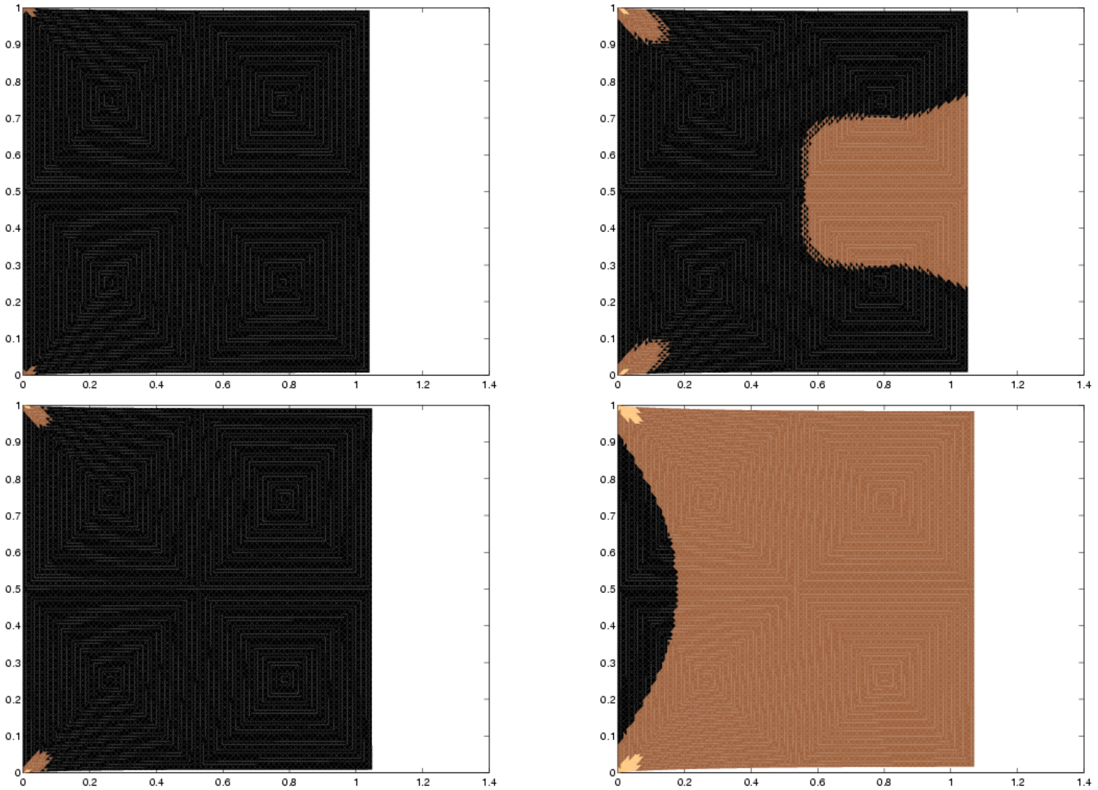

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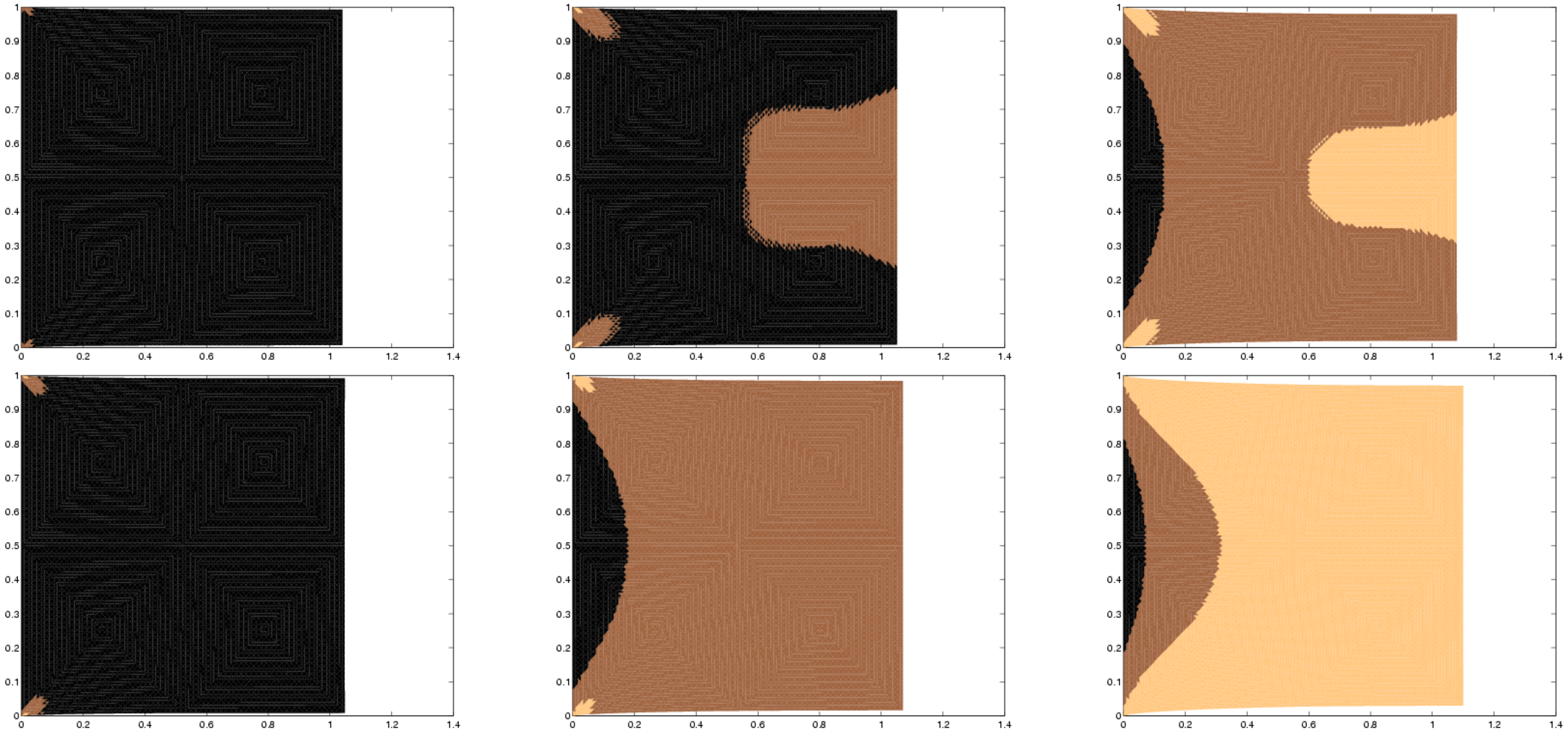

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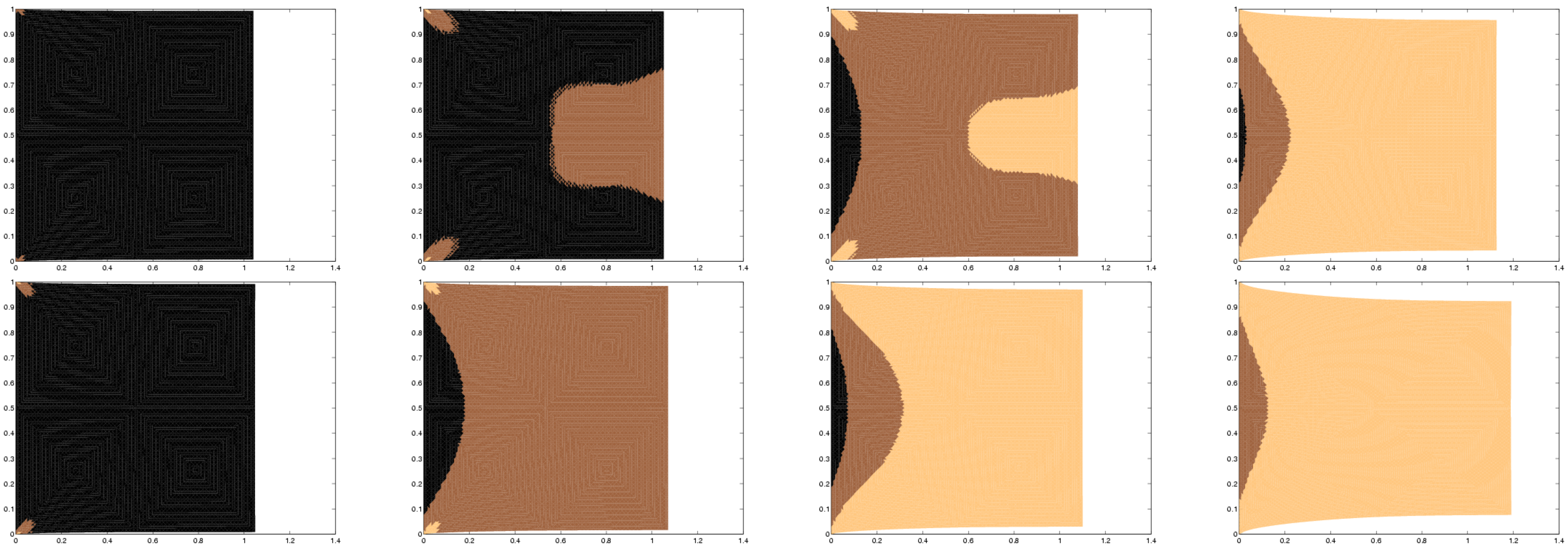
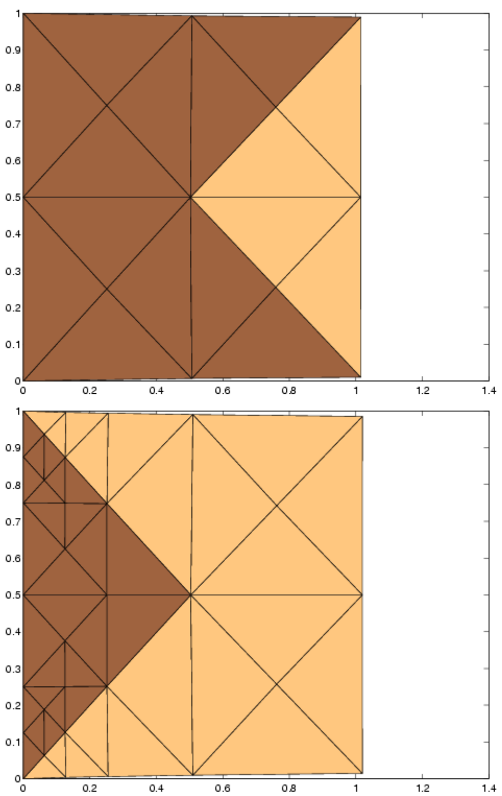


Evolution of elastoplastic zones at 8 graduate discrete times of the two-yield beam. Calculated for 16334 *elements, CPU time* $= 25.17$ *hours.*

Matlab calculations in 2D

Elastoplastic domains (black - elastic, brown - first plastic, yellow - second plastic)



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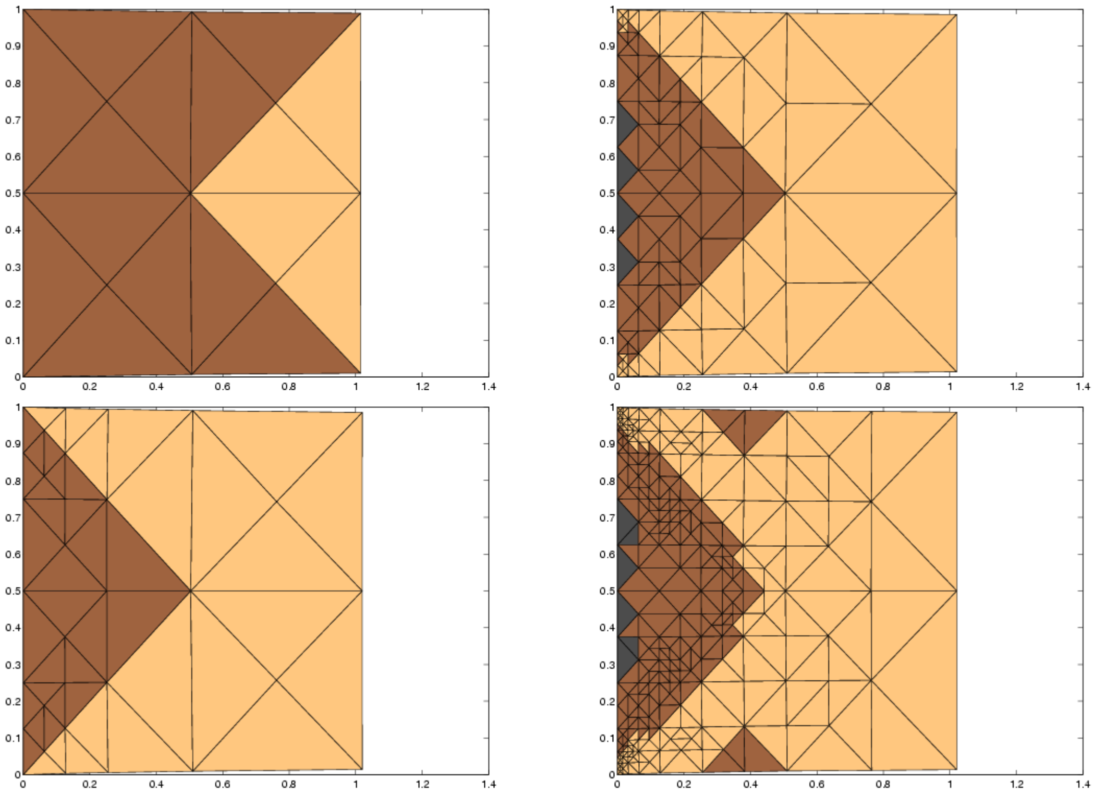

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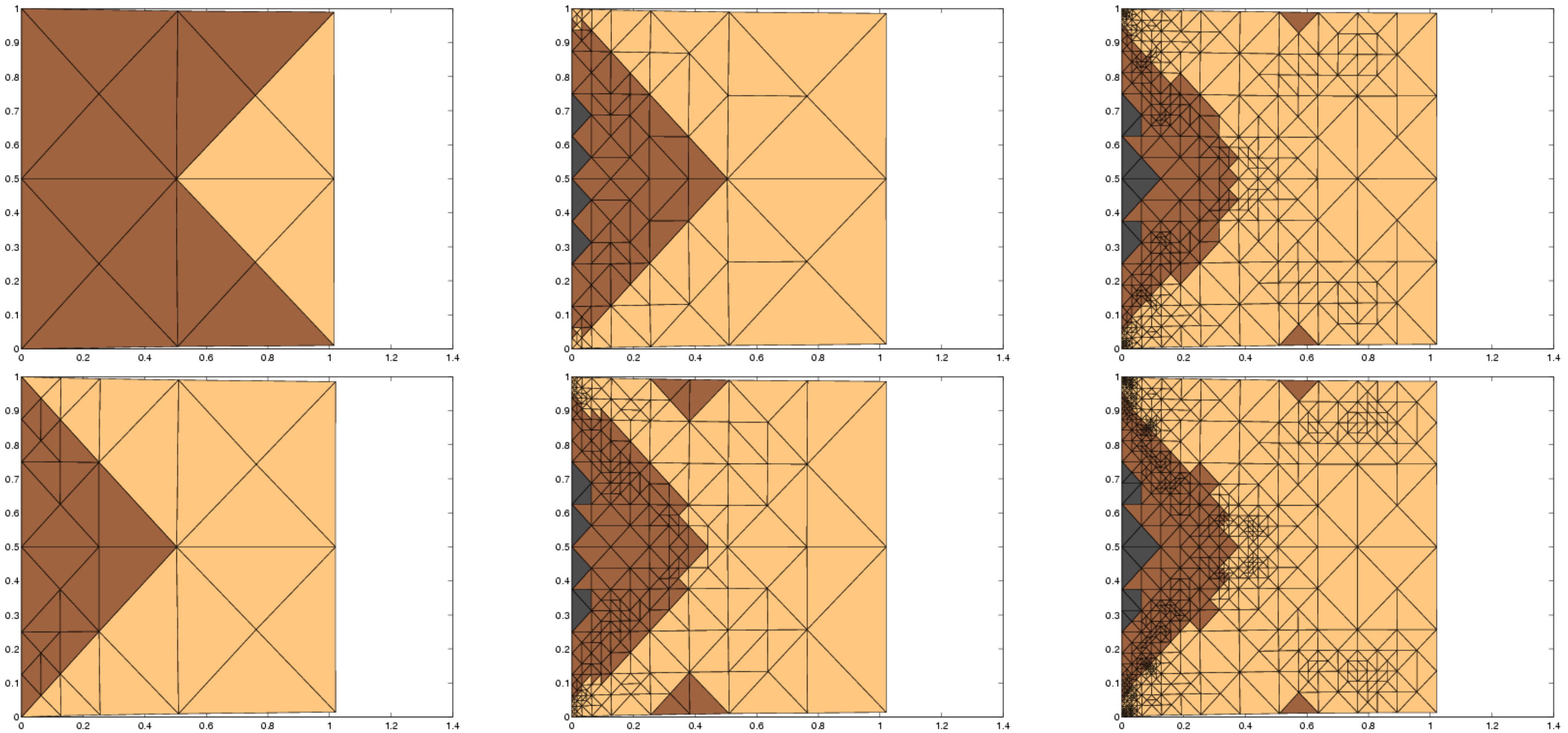

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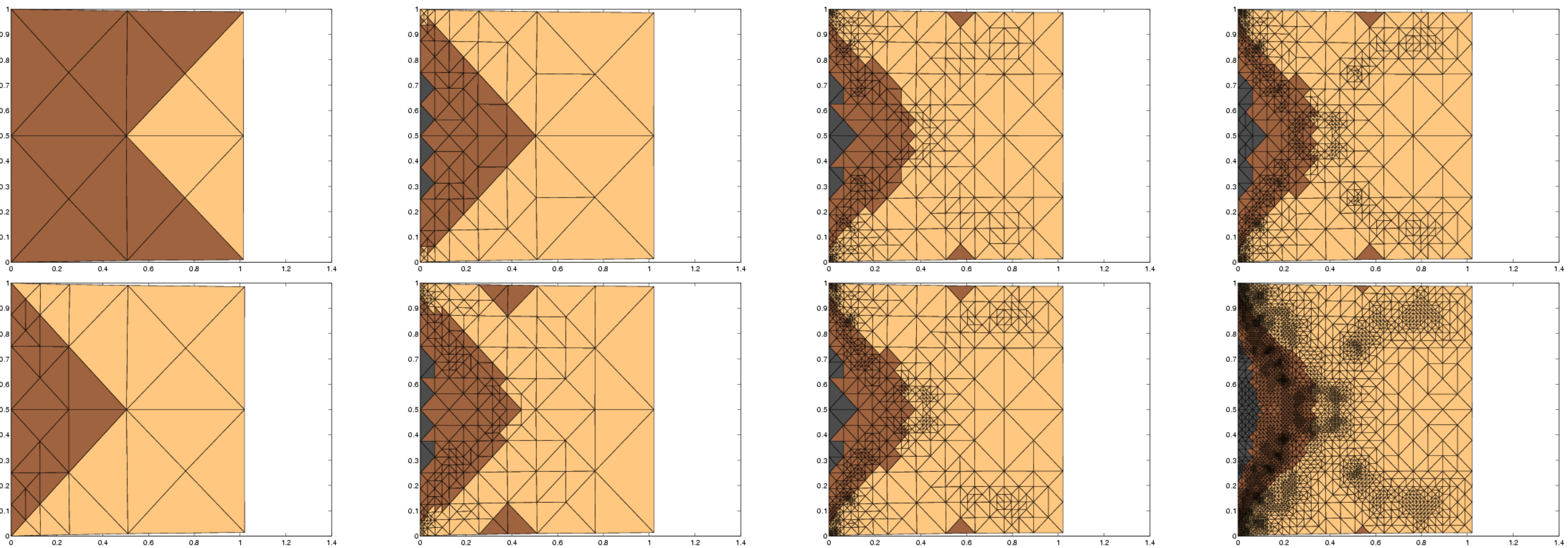
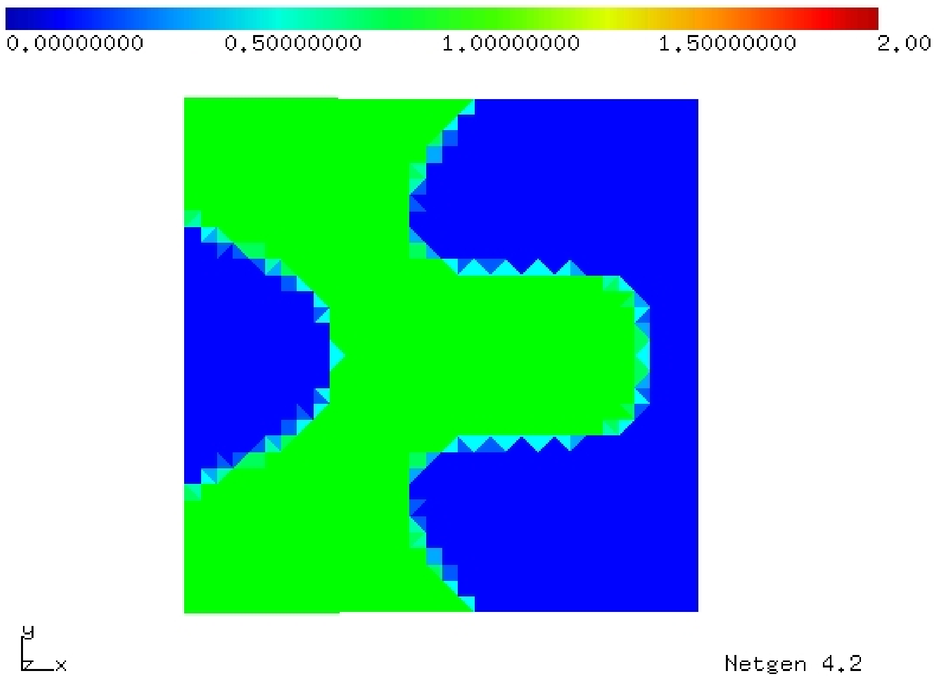


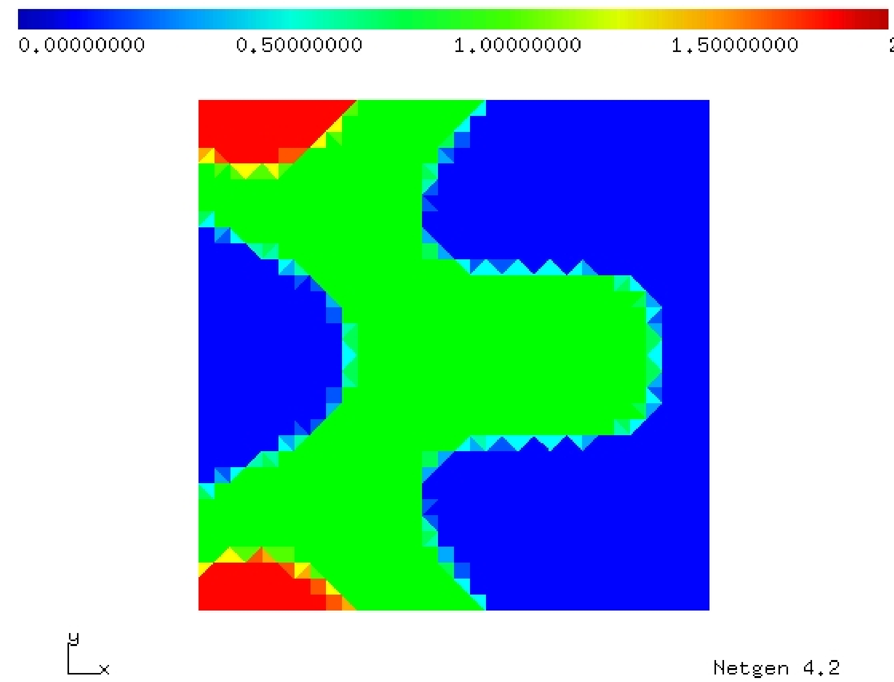
ZZ- adaptively refined meshes and elastoplastic zones, one time-step problem, two-yield beam.

NGSOLVE calculations in 2D

Elastoplastic domains (blue - elastic, green - first plastic, red - second plastic)



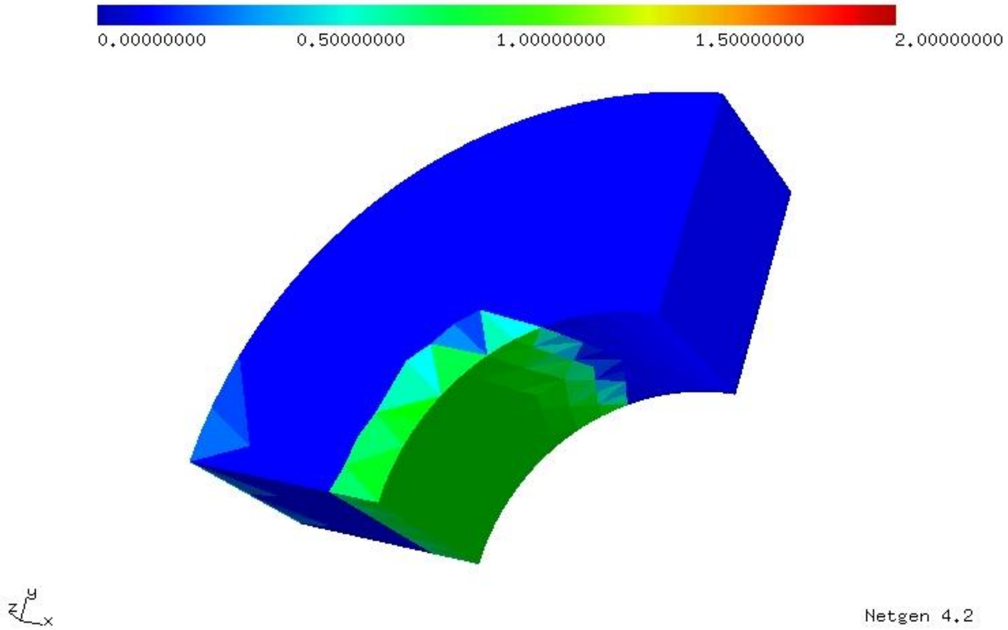
Kinematic hardening model.



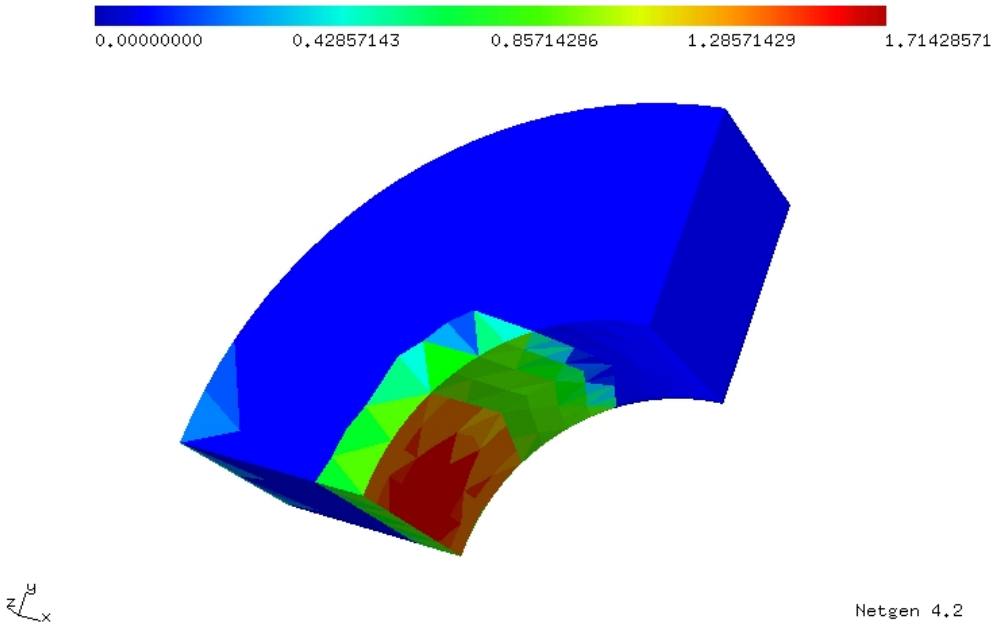
Two-yield hardening model.

NGSOLVE calculations in 3D

Elastoplastic domains (blue - elastic, green - first plastic, red - second plastic)



Kinematic hardening model.



Two-yield hardening model.

Outlook

Future Work:

- *Convergence proof*

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- *Minimize $\tilde{p}$ using symbolic methods (with RISC-Linz)*

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Download Netgen and NGSolve:

http://www.hpfem.jku.at/netgen/