FUNCTIONAL A POSTERIORI ERROR ESTIMATES FOR PROBLEMS WITH NONLINEAR BOUNDARY CONDITIONS

S. REPIN AND J. VALDMAN

ABSTRACT. In this paper, we consider variational inequalities related to problems with nonlinear boundary conditions. We are focused on deriving a posteriori estimates of the difference between exact solutions of such type variational inequalities and any function lying in the admissible functional class of the problem considered. These estimates are obtained by an advanced version of the variational approach earlier used for problems with uniformly convex functionals (see [13, 15]). It is shown that the structure of error majorants reflects properties of the exact solution. The majorants provide guaranteed upper bounds of the error for any conforming approximation and possess necessary continuity properties. In the series of numerical tests performed, it was shown that the estimates are explicitly computable, provide sharp bounds of approximation errors, and give high quality indication of the distribution of local (elementwise) errors.

1. INTRODUCTION

The problem of how to properly define boundary conditions in a certain mathematical model is of utmost importance in the mathematical modeling. In many cases, commonly used Dirichlét or Neumann boundary conditions cannot properly describe the behavior of a model and should be replaced by more sophisticated conditions that reflect real physical situations. Typical examples are presented by problems with unilateral boundary conditions and friction (see, e.g., [1, 3, 4, 7, 8, 12]). The respective boundary–value problems are formulated as variational inequalities and can be solved numerically by known (regularization or saddle-point) methods. Error estimates for finite element and other approximations form an important part of the numerical analysis of these problems. A priori rate convergence estimates for finite element approximations of such problems has been investigated in 70s-80s (see, e.g., [5]). However, the necessity of using adaptive multi-level algorithms requires a posteriori estimates able to (a) provide a reliable and directly computable estimate of the approximation error and (b) efficient error indicator able to detect the regions with excessively high errors. First

1991 *Mathematics Subject Classification.* Primary 65N30.

Key words and phrases. functional a posteriori error estimates, problems with nonlinear boundary conditions, variational inequalities, friction type conditions.

a posteriori error estimates for FEM approximations were developed at the end of 70s (see Babuška and Rheinboldt [25, 26]). Later, this subject was investigated by many authors (readers will find a consequent exposition of the results and more references in the books by Ainsworth and Oden [20], R. Verfürth, [21], Babuška and Strouboulis [27], Neittaanmaki and Repin [28].

In this paper, we present a posteriori estimates of the difference between exact solutions of a boundary–value problem with nonlinear boundary conditions and *any* function in the admissible (energy) class of the problem considered. Estimates contain no mesh–dependent constants and provide *guaranteed upper bounds* of the approximation errors (therefore we also call them Error Majorants). They are obtained by a modification of the variational approach earlier used for problems with uniformly convex functionals [13, 14, 15]. A posteriori error estimates for the approximations that not necessarily satisfy the prescribed Dirichlét, Neumann or mixed Dirichlét-Neumann boundary conditions has been considered in [17, 18]. These conditions can be viewed as special forms of nonlinear boundary conditions considered in this paper. In the present work, we analyze the structure of the Error Majorants and show that it reflects properties of the exact solution. They possess necessary continuity properties and make it possible to obtain the upper bound as close to the actual error as it is required. In the series of numerical tests performed, it was shown that the estimates are explicitly computable, provide sharp bounds of approximation errors, and give high quality indication of the distribution of local (elementwise) errors.

2. Statement of a problem with nonlinear boundary conditions

2.1. **Classical statement.** Let $\Omega \in \mathbb{R}^d$, $d = 2, 3$ be an open bounded domain with Lipschitz continuous boundary Γ. We assume that the boundary is piecewise smooth, so that one can uniquely define the unit outward normal in almost all points of Γ. It is assumed that Γ consists of two disjoint measurable parts Γ_0 and Γ_1. In Ω we find a solution of the differential equation

$$\mathrm{div} A\nabla u + f \; = \; 0 \,, \tag{2.1}$$

where $A : \mathbb{M}_s^{d\times d} \to \mathbb{M}_s^{d\times d}$ is a symmetric positive definite matrix. We assume that its components are bounded measurable functions and that the usual coercivity conditions

$$c_\ominus \, |\kappa|^2 \; \leq \; A\kappa \cdot \kappa \; \leq c_\oplus \, |\kappa|^2 \quad \forall \kappa \in \mathbb{R}^d \tag{2.2}$$

hold. Here $|\kappa| := \sqrt{\kappa \cdot \kappa}$. Note that the symbol $\cdot$ denotes an Euclidean product of two vectors $a \cdot b := \sum_{i=1\ldots d} a_i b_i$ for any vectors $a, b \in \mathbb{R}^d$. It is assumed that

$$u(x) = u_0(x), \qquad x \in \Gamma_0. \tag{2.3}$$

The boundary conditions on Γ_1 are more complicated. We present them in one common form

$$-u_{,n}(x) \in \partial j(u(x)) \quad x \in \Gamma_1 \,, \tag{2.4}$$

where $u_{,n}$ denotes the normal derivative of u, $j : \mathbb{R}^d \to \mathbb{R}$ is a convex lower semicontinuous functional, and ∂j is the subdifferential of j. Note that if $j \equiv 0$, then (2.4) is transformed to the Neumann boundary condition.

Functional j is called the "boundary dissipative potential" (see e.g. [12]). It makes possible to present a wide spectrum of boundary conditions in one common form. The latter is especially important in the problems of continuum mechanics where "classical" Dirichlét and Neumann conditions are often unable to adequately describe a wide variety of contact phenomena (e.g., unilateral contact, contact with friction, etc.). In this case, the boundary conditions can be presented in the form

$$-\sigma_n(x) \in \partial j(u(x)) \quad x \in \Gamma_1 \,, \tag{2.5}$$

where σ is the stress tensor and u is the displacement. Our model (2.1)–(2.4) can be considered as a simplified version of the elasticity model, in which u is a scalar–valued function and (2.5) is replaced by a simpler condition (2.5). However, from the mathematical point of view these two problems are similar. Our aim is to derive functional type a posteriori estimates for approximate solutions of (2.1)–(2.4), investigate their properties and verify numerically. The elasticity problem with nonlinear boundary condition (2.5) will be considered in a subsequent publication.

3. Functional formulation of the problem

3.1. **Notation.** We denote the spaces of square summable scalar- and vector-valued functions defined on the set S by $L_2(S)$ and $L_2(S, \mathbb{R}^d)$, respectively. Their norms are associated with natural scalar products

$$\int_S uv\, ds \quad \text{and} \quad \int_S p \cdot y\, ds.$$

Since no confusion my arise, we use for these norms one common symbol $\| \ \|$. We shall use special notations Y and Y^* for the spaces that contain gradients of the solutions and their fluxes, respectively. Functions in these spaces we denote by y, q, η and y^*, q^*, η^*, respectively. In the considered case, the gradients and fluxes belong to $L_2(\Omega, \mathbb{R}^d)$. However, by reasons that will become clear later, we keep different notation for this pair of spaces.

We shall also use the space

$$Q^*(\Omega) := \{ y^* \in Y^* \ | \ \mathrm{div} y^* \in L_2(\Omega) \} \ .$$

It is known that Q^* is a Hilbert space with respect to the norm

$$\|y^*\|^2_{Q^*} := \int\limits_\Omega (|y^*|^2 + |\mathrm{div} y^*|^2)\, dx$$

and that the smooth functions $C^\infty(\overline{\Omega}, \mathbb{M}^{d\times d}_s)$ are dense in Q^*.

Let $V = H^1(\Omega, \mathbb{R}^d)$ and

$$\gamma \in \mathcal{L}\left(H^1(\Omega), H^{1/2}(\Gamma)\right), \qquad H^{1/2}(\Gamma) \hookrightarrow L_2(\Gamma)$$

be the trace operator. By $H^1_0(\Omega)$ we denote the kernel of γ.

Also, for any $\phi \in H^{1/2}(\Gamma)$, one can define the continuation operator

$$\mu \in \mathcal{L}(H^{1/2}(\Gamma), H^1(\Omega))$$

such that

$$\mu\phi = w, \qquad \phi \in H^1(\Omega), \qquad \gamma w = \phi \qquad \text{on } \Gamma$$

and (see, e.g., [10])

$$\|\phi\|_{1/2,\Gamma} \leq c_\gamma \|w\|_{1,\Omega}, \qquad \|w\|_{1,\Omega} \leq c_\mu \|\phi\|_{1/2,\Gamma}, \tag{3.1}$$

where $\|\cdot\|_{1,\Omega}$ and $\|\cdot\|_{1/2,\Gamma}$ are the norms in H^1 and $H^{1/2}$, respectively.

By means of the operator γ we define the space

$$V_0 := \{v \in V \mid \gamma v = 0 \text{ a.e. on } \Gamma_0\},$$

which is a subspace of V. The set $\gamma(V_0)$ is a subspace of $H^{1/2}(\Gamma)$. Hereafter, we denote this set by Z and the respective dual space by Z^* (also called $H^{-1/2}$), which can be identified with the set of traces on Γ_1 of functions belonging to $Q^*(\Omega)$. Indeed, for any smooth y^* and any $v \in V_0$, we have the classically relation

$$\int_{\Gamma_1} (y^* \cdot n)\gamma v \, dx = \int_\Omega (y^* \cdot \nabla v + (\text{div} y^*) v) \, dx. \tag{3.2}$$

For any $y^* \in Q^*(\Omega)$, the right–hand side of this identity is a linear continuous functional $\Lambda_{y^*} : V_0 \to \mathbb{R}$ that satisfies the relations

$$\Lambda_{y^*} v = 0 \qquad \forall v \in H^1_0(\Omega), \tag{3.3}$$

$$|\Lambda_{y^*} v| \leq c_\mu \|y^*\|_{Q^*} \|\gamma v\|_{1/2,\Gamma}. \tag{3.4}$$

In essence, Λ_{y^*}, is a linear continuous mapping defined on a factor space of V_0. Really,

$$\Lambda_{y^*}(v_1) = \Lambda_{y^*}(v_2) \qquad \text{if } v_1, v_2 \in V_0 \text{ and } \gamma v_1 = \gamma v_2.$$

Thus, in this factor space two functions belong to one class if they have the same trace on Γ_1. This means that Λ_{y^*} is a mapping from Z to $\mathbb{R}$ and, consequently, can be identified with a certain element in Z^*, which we denote $\delta_n y^*$ and call the *normal trace of* y^* *on* Γ_1.

Hereafter, we follow the usual convention and denote the value of the functional $\xi^* \in Z^*$ on $\xi \in Z$ by means of duality pairing $\langle \xi^*, \xi \rangle_{\Gamma_1}$. Then, (3.2) comes in a more general form

$$\Lambda_{y^*}(\gamma v) = \langle \delta_n y^*, \gamma v \rangle_{\Gamma_1} = \int_\Omega (y^* \cdot \nabla v + \text{div} y^* \cdot v) \, dx. \tag{3.5}$$

The norm of such a functional is given by the standard relation

$$\|\delta_n y^*\|_{Z^*} = \sup_{v\in V_0} \frac{\int\limits_\Omega (y^*\cdot\nabla v + \mathrm{div} y^*\cdot v)\,dx}{\|\gamma v\|_Z} \tag{3.6}$$

In view of (3.4), this norm is bounded:

$$\|\delta_n y^*\|_{Z^*} \le c_\mu \|y^*\|_{Q^*} . \tag{3.7}$$

3.2. **Conjugate functionals defined on spaces of traces.** For any $\xi \in Z$ we define the functional

$$\Upsilon(\xi) := \int\limits_{\Gamma_1} j(\xi)\,d\Gamma .$$

We assume that the integrand $j : \mathbb{R}^d \to \mathbb{R}^d$ is a nonnegative, convex, and lower semicontinuous (l.s.c.) function. In addition, we assume that $j(0) = 0$ and

$$\mathrm{dom}\, j := \{p \in \mathbb{R}^d \,\|\, j(p) < +\infty\} \ne \emptyset ,$$

so that j belongs to the class of so–called *proper* convex functionals.

In this case, the functional $\Upsilon(\xi)$ is also nonnegative, convex and l.s.c. on Z. Since γ is a bounded linear operator, the functional $\Upsilon(\gamma v)$ also possesses the above properties as the functional on V_0.

Let us introduce a new functional

$$\Upsilon^*(\xi^*) := \sup_{\xi\in Z}\left\{\langle\xi^*,\xi\rangle_{\Gamma_1} - \Upsilon(\xi)\right\} , \tag{3.8}$$

which we call *conjugate* (in the sense of Young–Fenchel) to the functional Υ.

Under the above assumptions, the functional $\Upsilon : Z \to \mathbb{R}$ coincides with pointwise supremum of all its affine minorants. It is easy to see that

$$\Upsilon(\xi) \ge \langle\xi^*,\xi\rangle_{\Gamma_1} + \lambda \qquad \forall\, \lambda \le -\Upsilon^*(\xi^*).$$

This effectively means that

$$\Upsilon(\xi) = \sup_{\xi^*\in Z^*}\left\{\langle\xi^*,\xi\rangle_{\Gamma_1} - \Upsilon^*(\xi^*)\right\} \tag{3.9}$$

By recalling (3.6), we see that

$$\Upsilon(\gamma v) = \sup_{y^*\in Q^*}\left\{\int\limits_\Omega (y^*\cdot\nabla v + \mathrm{div} y^*\cdot v)\,dx - \Upsilon^*(\delta_n y^*)\right\} \tag{3.10}$$

$$\Upsilon^*(\delta_n y^*) = \sup_{v\in V_0}\left\{\int\limits_\Omega (y^*\cdot\nabla v + \mathrm{div} y^*\cdot v)\,dx - \Upsilon(\gamma v)\right\} . \tag{3.11}$$

In what follows we use the *compound functional*

$$\mathrm{D}_{\Gamma_1}(\gamma v, \delta_n y^*) := \Upsilon(\gamma v) + \Upsilon^*(\delta_n y^*) - \langle\gamma v, \delta_n y^*\rangle_{\Gamma_1}.$$

It is easy to see that

$$\mathrm{D}_{\Gamma_1}(\gamma v, \delta_n y^*) := \sup_{w\in V_0}\Big[\int\limits_\Omega (y^*\cdot\nabla(w-v) + \mathrm{div} y^*\cdot(w-v))\,dx + \tag{3.12}$$
$$+\int_{\Gamma_1}(j(\gamma v)-j(\gamma w))\,d\Gamma\Big],$$

$$\mathrm{D}_{\Gamma_1}(\gamma v, \delta_n y^*) \geq 0. \tag{3.13}$$

Moreover,

$$\mathrm{D}_{\Gamma_1}(\gamma v, \delta_n y^*) = 0 \;\Rightarrow\; \delta_n y^* \in \partial\Upsilon(\gamma v)$$

and if $\delta_n y^* \in L_2(\Gamma_1, \mathbb{R}^d)$, then

$$\Upsilon^*(\delta_n y^*) = \int\limits_{\Gamma_1} j^*(\delta_n y^*)\,dx\,,$$

where $j^* : \mathbb{R}^d \to \mathbb{R}$ is the function conjugate to j, i.e.

$$j^*(q^*) = \sup_{q\in\mathbb{R}^d}\{q^*\cdot q - j(q)\}\;.$$

3.3. **Variational inequality.** On $V \times V$ we define the bilinear form

$$a(u,v) := \int\limits_\Omega A\nabla(u)\cdot\nabla v\,dx.$$

The action of external forces is described by the linear functional

$$\ell(v) := \int\limits_\Omega f v\,dx\,.$$

Henceforth, we assume that

$$f \in L_2(\Omega), \tag{3.14}$$

$$u_0 \in V(\Omega)\,. \tag{3.15}$$

Now we may formulate the above contact problem in the form of variational inequality (see, e.g., [4, 8]).

Problem $\mathcal{P}$. Find $u \in V_0 + u_0 := \{w \mid\; w = w_0 + u_0,\; w_0 \in V_0\}$ such that

$$a(u, w-u) + \Upsilon(w) - \Upsilon(u) \geq \ell(w-u)\;\;\forall w \in V_0 + u_0\,. \tag{3.16}$$

In view of the Lions-Stampacchia Theorem, this problem is equivalent to the variational problem: find $u \in V_0 + u_0$ such that

$$J(u) = \inf_{w\in V_0+u_0} J(w), \quad J(w) = \frac{1}{2}a(w,w) + \Upsilon(w) - \ell(w). \tag{3.17}$$

Since the functional J is strictly convex, continuous, and coercive on V and the set $V_0 + u_0$ is a convex closed subset of V, we arrive at the conclusion that Problem $\mathcal{P}$ is uniquely solvable.

It is not difficult to see that on Γ_1 u and its normal derivative $u_{,n}$ satisfy the boundary condition (2.4).

4. Estimates of deviations

4.1. **General estimate.** The minimizer u to problem $\mathcal{P}$ meets the variational inequality (3.16). This leads to the inequality

$$\begin{aligned} J(v) - J(u) &= \frac{1}{2}a(v-u, v-u) + \\ &+ a(u, v-u) - \langle f, v-u\rangle + \Upsilon(v) - \Upsilon(u) \geq \\ &\geq \frac{1}{2}a(v-u, v-u) \qquad \forall\, v \in V_0 + u_0\,, \end{aligned} \tag{4.1}$$

which implies the basic "deviation" estimate

$$\frac{\mathbf{1}}{\mathbf{2}} \,|\!|\!|\, \boldsymbol{v} - \boldsymbol{u} \,|\!|\!|^{\mathbf{2}} \leq \boldsymbol{J(v)} - \mathbf{inf}\, \boldsymbol{\mathcal{P}} \quad \boldsymbol{\forall v \in V_0 + u_0}\,, \tag{4.2}$$

where $\inf \mathcal{P}$ denotes the exact lower bound of the functional J and $|\!|\!|\, v \,|\!|\!| := (a(v,v))^{1/2}$. In general, the quantity $\inf \mathcal{P}$ is unknown so that (4.2) has little to offer as a practical tool of error estimation. Our aim is to show that the right–hand side of (4.2) can be estimated from above by a quantity which is practically computable, possesses necessary continuity properties and has clear physical motivation.

For this purpose, we apply the techniques earlier used in [15, 16] based on the consideration of the so–called *perturbed* functionals. In our case, such a functional has the form

$$J_{\xi^*}(v) = \frac{1}{2}a(v,v) - \ell(v) + \langle \xi^*, \gamma v\rangle_{\Gamma_1} - \Upsilon^*(\xi^*)\,. \tag{4.3}$$

It is easy to see that

$$\sup_{\xi^* \in Z^*} J_{\xi^*}(v) = J(v)$$

and, consequently, for any $\xi^* \in Z^*$

$$\inf_{v \in V_0 + u_0} J_{\xi^*}(v) \;\leq\; \inf_{v \in V_0 + u_0} J(v) \;=\; \inf \mathcal{P}. \tag{4.4}$$

The perturbed Problem $\mathcal{P}_{\xi^*}$ is to find $u_{\xi^*} \in V_0 + u_0$ such that

$$J_{\xi^*}(u_{\xi^*}) \;=\; \inf_{v \in V_0 + u_0} J_{\xi^*}(v) \;=\; \inf\, \mathcal{P}_{\xi^*}\,.$$

This problem is a simple quadratic problem, which has a unique solution for any $\xi^* \in Z^*$. The perturbed problem has a dual counterpart.
Problem ${\mathcal{P}^*}_{\xi^*}$: Find $y^*_{\xi^*} \in Q^*_{\ell_{\xi^*}}$ such that

$$I^*_{\xi^*}(y^*_{\xi^*}) \;=\; \sup_{\eta^* \in Q^*_{\ell_{\xi^*}}} I^*_{\xi^*}(\eta^*)\,,$$

where

$$I^*_{\xi^*}(\eta^*) = \int\limits_\Omega \nabla(u_0)\cdot\eta^*\,dx - \frac{1}{2}a^*(\eta^*,\eta^*) - \ell_{\xi^*}(u_0) - \Upsilon^*(\xi^*)\,,$$

a^* is a bilinear form conjugate to a, $\ell_{\xi^*}(\cdot) = \ell(\cdot) - \langle \xi^*, \cdot\rangle_{\Gamma_1}$ is a linear functional and

$$Q^*_{\ell_{\xi^*}} := \left\{ \eta^* \in Y^* \;\mid\; \int\limits_\Omega \eta^*\cdot\nabla v\,dx = \ell_{\xi^*}(v)\,,\quad \forall v \in V_0 \right\}.$$

This problem also has a unique solution. Moreover,

$$\inf \mathcal{P}_{\xi^*} = \sup {\mathcal{P}^*}_{\xi^*}\,.$$

In view of the above connection between lower and upper bounds in Problems $\mathcal{P}_{\xi^*}$ and ${\mathcal{P}^*}_{\xi^*}$, we obtain

$$\frac{1}{2}\,\|v-u\|_a^2 \le J(v) - \sup {\mathcal{P}^*}_{\xi^*} \le J(v) - I^*_{\xi^*}(\eta^*)\;\; \forall \eta^* \in Q^*_{\ell_{\xi^*}}. \tag{4.5}$$

The right–hand side of (4.5) can be estimated as follows

$$\begin{aligned} J(v) - I^*_{\xi^*}(\eta^*) \;=\; & \frac{1}{2}a(v,v) + \frac{1}{2}a^*(y^*,y^*) - \int\limits_\Omega \nabla v\cdot y^*\,dx + \\ & + \Upsilon(\gamma v) + \Upsilon^*(\xi^*) - \ell(v) - \int\limits_\Omega \nabla(u_0)\cdot\eta^*\,dx - \ell_{\xi^*}(\gamma u_0) + \\ & + \int\limits_\Omega \nabla v\cdot y^*\,dx + \frac{1}{2}a^*(\eta^*,\eta^*) - \frac{1}{2}a(y^*,y^*)\,, \end{aligned} \tag{4.6}$$

where y^* is an arbitrary element of Y^*. Since

$$\ell(v-u_0) = \int\limits_\Omega \eta^*\cdot\nabla(v-u_0)\,dx + \langle \xi^*, \gamma(v-u_0)\rangle_{\Gamma_1}, \tag{4.7}$$

we obtain

$$\begin{aligned} J(v) - I^*_{\xi^*}(\eta^*) \;=\; & \frac{1}{2}a(v,v) + \frac{1}{2}a^*(y^*,y^*) - \int\limits_\Omega \nabla v\cdot y^*\,dx + \\ & + \Upsilon(\gamma v) + \Upsilon^*(\xi^*) - \langle \xi^*, \gamma v\rangle_{\Gamma_1} + \\ & + \int\limits_\Omega \nabla v\cdot(y^*-\eta^*)\,dx + \frac{1}{2}a^*(\eta^*,\eta^*) - \frac{1}{2}a(y^*,y^*)\,. \end{aligned} \tag{4.8}$$

This identity has an equivalent form

$$
\begin{aligned}
(4.9)\quad J(v) - I^*_{\xi^*}(\eta^*) \;=\; &\frac{1}{2}\int\limits_{\Omega} (A\nabla v\cdot\nabla v + A^{-1}y^*\cdot y^* - 2\nabla v\cdot y^*)\,dx + \\
&+ \Upsilon(\gamma v) + \Upsilon^*(\xi^*) - \langle \xi^*, \gamma v\rangle_{\Gamma_1} + \int\limits_{\Omega} (\nabla v - A^{-1}y^*)\cdot(y^*-\eta^*)\,dx + \\
&+ \frac{1}{2}\int\limits_{\Omega} A^{-1}(\eta^*-y^*)(\eta^*-y^*)\,dx\,.
\end{aligned}
$$

Now we use the inequality

$$
\eta\cdot\eta^* \;\le\; \frac{\beta}{2}A\eta\cdot\eta + \frac{1}{2\beta}A^{-1}\eta^*\cdot\eta^*\,,
$$

which is valid for all vectors η and η^* and any $\beta > 0$. We obtain the estimate

$$
\begin{aligned}
\int\limits_{\Omega}(\nabla v - A^{-1}y^*)\cdot(y^*-\eta^*)\,dx \quad \le\; &\frac{\beta}{2}\int\limits_{\Omega} A(\nabla v - A^{-1}y^*)\cdot(\nabla v - A^{-1}y^*)\,dx + \\
&+\frac{1}{2\beta}\int\limits_{\Omega} A^{-1}(y^*-\eta^*)\cdot(y^*-\eta^*)\,dx,
\end{aligned}
$$

which gives the relation

(4.10)

$$
\begin{aligned}
J(v) - I^*_{\xi^*}(\eta^*) \;=\; &\frac{1}{2}(1+\beta)\int\limits_{\Omega} (A\nabla v\cdot\nabla v + A^{-1}y^*\cdot y^* - 2\nabla v\cdot y^*)\,dx + \\
&+ \Upsilon(\gamma v) + \Upsilon^*(\xi^*) - \langle \xi^*, \gamma v\rangle_{\Gamma_1} + \\
&+ \frac{1}{2}\left(1+\frac{1}{\beta}\right)\int\limits_{\Omega} A^{-1}(\eta^*-y^*)\cdot(\eta^*-y^*)\,dx.
\end{aligned}
$$

Let us introduce the following quantities

$$
\begin{aligned}
(4.11)\qquad & M_1(v, y^*) = D_A(\nabla v, y^*) = \\
&\quad = \frac{1}{2}\int\limits_{\Omega} (A\nabla v\cdot\nabla v + A^{-1}y^*\cdot y^* - 2\nabla v\cdot y^*)\,dx, \\
(4.12)\qquad & M_2(\gamma v, \xi^*) = D_\Upsilon(\gamma v, \xi^*) = \Upsilon(\gamma v) + \Upsilon^*(\xi^*) - \langle \xi^*, \gamma v\rangle_{\Gamma_1}, \\
(4.13)\qquad & M_3(y^*, \xi^*) = \frac{1}{2}\inf_{\eta^*\in Q^*_{\ell_{\xi^*}}}\int\limits_{\Omega} A^{-1}(\eta^*-y^*)\cdot(\eta^*-y^*)\,dx.
\end{aligned}
$$

Then (4.5), (4.10)–(4.13) result in the estimate

$$\frac{1}{2}\,\|v-u\|_a^2 \leq (1+\beta)M_1(v,y^*) + M_2(\gamma v,\xi^*) + \\ +\left(1+\frac{1}{\beta}\right)M_3(y^*,\xi^*), \tag{4.14}$$

where y^*, ξ^* and β are arbitrary elements of the sets Y^*, Z^* and $\mathbb{R}_+$, respectively.

Let us discuss the meaning of three quantities in the right–hand side of (4.14). In view of the Young–Fenchel inequality, M_1 and M_2 are evidently nonnegative. Since A^{-1} is positive definite, M_3 is also nonnegative.

The quantity $M_1(v,y^*)$ vanishes if and only if v and y^* satisfy the relation (2.4). Therefore, this term presents the error in the relation

$$p^* = A\nabla u.$$

It is easy to see that $M_2(\gamma v,\xi^*) = 0$ if and only if

$$\xi^* = \partial\Upsilon(\gamma v) \qquad \text{on } \Gamma_1,$$

so that M_2 is a measure of the error in the boundary condition (2.3) computed on Γ_1 for the function $-\xi^* \in Z^*$ (which can be thought of as an image of the normal component of the flux) and the trace of v .

The quantity $M_3(y^*)$ vanishes if and only if $y^* \in Q^*_{\ell_{\xi^*}}$, i.e., if

$$\int\limits_\Omega y^*\cdot\nabla v dx = \int\limits_\Omega f\cdot v dx - \langle\xi^*,\gamma v\rangle_{\Gamma_1} \qquad \forall v\in V_0\,.$$

However

$$\int\limits_\Omega y^*\cdot\nabla v dx = \langle\delta_n y^*,\gamma v\rangle_{\Gamma_1} - \int\limits_\Omega \mathrm{div} y^*\cdot v dx.$$

Thus, we arrive at the conclusion that this term vanishes if and only if

(i) the equilibrium equation (2.5) holds;
(ii) the relation $\delta_n y^* = -\,\xi^*$ on Γ_1 holds.

It is worth remarking that the above relations are understood in a generalized sense.

4.2. **Another form of the estimate.** To obtain the estimate in a more convenient form, we assume that y^* belongs to the set

$$Q^*_{\Gamma_1} := \{y^*\in Y^* \mid \mathrm{div} y^*\in L_2(\Omega),\ \delta_n y^*\in L_2(\Gamma_1)\}\ .$$

Note that $p^*\in Q^*_{\Gamma_1}$ provided that $f\in L_2(\Omega,\mathbb{R}^d)$ and the trace $\delta_n p^*$ on Γ_1 is a square summable function.

Now we concentrate on finding another form of the term M_3. For this purpose we consider an auxiliary problem in the domain Ω. This problem is to find $\widetilde{u}$ and $\widetilde{p}^*$ that satisfy the relations (2.1)–(2.4) where

$$f = g\in L_2(\Omega)$$

and the boundary condition on Γ_1 is given by the relation

$$p_n^* = G \in L_2(\Gamma_1).$$

Then, in view of the duality relation (see e.g. [6])

$$\sup_{\eta^* \in Q^*_{gG}} \left[-\frac{1}{2} \int\limits_\Omega A^{-1}\eta^* \cdot \eta^* \, dx \right] = \tag{4.15}$$

$$= \inf_{w \in V_0} \left[\int\limits_\Omega \left(\frac{1}{2} A\nabla(w) \cdot \nabla(w) - g \cdot w \right) dx - \int\limits_{\Gamma_1} G \cdot \gamma w \, d\Gamma \right],$$

where

$$Q^*_{gG} := \left\{ \eta^* \in Y^* \mid \int\limits_\Omega \eta^* \cdot \nabla v dx = \int\limits_\Omega g \cdot w dx + \int\limits_{\Gamma_1} G \cdot \gamma w d\Gamma \;\; \forall w \in V_0 \right\}.$$

Take some functions $y^* \in Q^*_{\Gamma_1}$ and $\eta^* \in Q^*_{gG}$. Then

$$\int\limits_\Omega (\eta^* - y^*) \cdot \nabla w \, dx \; = \tag{4.16}$$

$$= \int\limits_\Omega (\mathrm{divy} + g) \cdot w \, dx + \int\limits_{\Gamma_1} (G - \delta_n y^*) \cdot \gamma w \, d\Gamma \;\forall w \in V_0 \, .$$

Let us set

$$\widetilde{g} = \mathrm{div} y^* + g \in L_2(\Omega, \mathbb{R}^d)$$

and

$$\widetilde{G} = G - \delta_n y^* \in L_2(\Gamma_1, \mathbb{R}^d).$$

We observe that $\kappa^* = \eta^* - y^*$ belongs to the set Q^*_{gG} with $g = \widetilde{g}$ and $G = \widetilde{G}$ (hereafter it is called $Q^*_{\widetilde{g}\widetilde{G}}$). By the equality (4.15), we see that

$$\sup_{\kappa^* \in Q^*_{\widetilde{g}\widetilde{G}}} \left[-\frac{1}{2} \int\limits_\Omega A^{-1}\kappa^* \cdot \kappa^* \, dx \right] = \tag{4.17}$$

$$= \inf_{w \in V_0} \left[\int\limits_\Omega (\frac{1}{2} A\nabla(w) \cdot \nabla(w) - \widetilde{g} \cdot w) \, dx - \int\limits_{\Gamma_1} \widetilde{G} \cdot \gamma w \, d\Gamma \right].$$

Note that

$$\sup_{\kappa^* \in Q^*_{\widetilde{g}\widetilde{G}}} \left[-\frac{1}{2} \int_\Omega A^{-1}\kappa^* \cdot \kappa^* \, dx \right] = \tag{4.18}$$

$$= \sup_{\eta^* \in Q^*_{gG}} \left[-\frac{1}{2} \int_\Omega A^{-1}(\eta^* - y^*) \cdot (\eta^* - y^*) \, dx \right].$$

Thus, (4.17) and (4.18) means that

$$\sup_{\eta^* \in Q^*_{gG}} \left[-\frac{1}{2} \int_\Omega A^{-1}(\eta^* - y^*) \cdot (\eta^* - y^*) \, dx \right] =$$

$$= \inf_{w \in V_0} \left[\int_\Omega (\frac{1}{2} A\nabla w \cdot \nabla w - \widetilde{g} \cdot w) \, dx - \int_{\Gamma_1} \widetilde{G} \cdot \gamma w \, d\Gamma \right] =$$

$$= \inf_{w \in V_0} \left[\int_\Omega (\frac{1}{2} A\nabla(w) \cdot \nabla w - (\mathrm{div} y^* + g) \cdot w) \, dx - \int_{\Gamma_1} (G - \delta_n y^*) \cdot \gamma w d\Gamma \right]$$

what gives the relation

$$\inf_{\eta^* \in Q^*_{gG}} \left[\frac{1}{2} \int_\Omega A^{-1}(\eta^* - y^*) \cdot (\eta^* - y^*) dx \right] = \tag{4.19}$$

$$= -\inf_{w \in V_0} \left[\int_\Omega \left(\frac{1}{2} A\nabla w \cdot \nabla w - (\mathrm{div} y^* + g) \cdot w \right) dx \right.$$

$$\left. - \int_{\Gamma_1} (G - \delta_n y^*) \cdot \gamma w d\Gamma \right].$$

The set $Q^*_{\ell_{\xi^*}}$ coincides with Q^*_{gG} if $g = f$ and $G = -\xi^* \in L_2(\Gamma_1)$. By applying (4.19), we obtain

$$\inf_{\eta^* \in Q^*_{\ell_{\xi^*}}} \left[\frac{1}{2} \int_\Omega A^{-1}(\eta^* - y^*) \cdot (\eta^* - y^*) dx \right] = \tag{4.20}$$

$$= -\inf_{w \in V_0} \left[\int_\Omega \left(\frac{1}{2} A\nabla w \cdot \nabla w - (\mathrm{div} y^* + f) \cdot w \right) dx \right.$$

$$\left. + \int_{\Gamma_1} (\xi^* + \delta_n y^*)(\gamma w) d\Gamma \right].$$

It is easy to see that

(4.21)
$$\inf_{w\in V_0}\left[\int\limits_\Omega\left(\frac{1}{2}A\nabla w\cdot\nabla w-(\mathrm{div}y^*+f)\cdot w\right)dx+\int\limits_{\Gamma_1}(\xi^*+\delta_n y^*)(\gamma w)d\Gamma\right]\geq$$
$$\geq\inf_{w\in V_0}\left[\frac{1}{2}a(w,w)-\boldsymbol{R}_\Omega(y^*)\,\|w\|_\Omega-\boldsymbol{R}_{\Gamma_1}(y^*,\xi^*)\,\|\gamma w\|_{\Gamma_1}\right],$$

where

$$\boldsymbol{R}_\Omega(y^*)^2:=\int\limits_\Omega(\mathrm{div}y^*+f)^2dx,$$
$$\boldsymbol{R}_{\Gamma_1}(\delta_n y^*,\xi^*)^2:=\int\limits_{\Gamma_1}(\xi^*+\delta_n y^*)^2d\Gamma.$$

In view of the embedding theorems for functions and their traces, there exist constants C_Ω, and C_{Γ_1} such that

$$\|w\|^2_\Omega\leq C^2_\Omega\,a(w,w)\,, \tag{4.22}$$
$$\|\gamma w\|^2_{\Gamma_1}\leq C^2_{\Gamma_1}a(w,w) \tag{4.23}$$

for all $w\in V_0$. Estimate (4.22) follows from (2.2) and the Friedrichs' type inequality for the functions vanishing at Γ_1. Estimate (4.23) follows from the trace theorem. More detailed information concerning such type inequalities and the constants can be found in the works of Sauter and Carstensen [2], S. G. Mikhlin [11] among others.

Then the right–hand side of (4.21) is bounded from below by the quantity

$$\inf_{z\in\mathbb{R}_+}\left\{\frac{z^2}{2}-\left(C_\Omega\boldsymbol{R}_\Omega(y^*)+C_{\Gamma_1}\boldsymbol{R}_{\Gamma_1}(\delta_n y^*,\xi^*)\right)z\right\}=$$
$$=-\tfrac{1}{2}\left(C_\Omega\boldsymbol{R}_\Omega(y^*)+C_{\Gamma_1}\boldsymbol{R}_{\Gamma_1}(\delta_n y^*,\xi^*)\right)^2\,.$$

Thus, we have

$$\boldsymbol{\tfrac{1}{2}\,\|v-u\|^2_a\leq M_\oplus(v,y^*,\xi^*,\beta):=(1+\beta)M_1(v,y^*)+}$$
$$\boldsymbol{+M_2(\gamma v,\xi^*)+\tfrac{1}{2}\Big(1+\tfrac{1}{\beta}\Big)\Big(C_\Omega R_\Omega(y^*)+C_{\Gamma_1}R_{\Gamma_1}(\delta_n y^*,\xi^*)\Big)^2} \tag{4.24}$$

Here, $y^*\in Q^*_{\Gamma_1}$, $\xi^*\in L_2(\Gamma_1)$, and $\beta>0$. Let us discuss the meaning of this estimate. We see that the Majorant $M_\oplus$ depends on the approximate solution v and also on two other functions: y^* and ξ^*. The first one can be regarded as an image of the true flux p^* and the second one is the image of the normal trace $p^*\cdot n$ on the boundary γ_1. Assume that

$$M_\oplus(v,y^*,\xi^*,\beta)=0.$$

Since all the terms are nonnegative, we arrive at the conclusion that

$$y^* = A\nabla v, \tag{4.25}$$

$$\xi^* \subset \partial\Upsilon(\gamma v), \tag{4.26}$$

$$\mathrm{div} y^* + f = 0, \qquad \xi^* = -\delta_n y^*. \tag{4.27}$$

The relations (4.25), (4.26), and (4.27) means that v is the exact solution, p^* is its flux and $\xi^* = \delta_n p^*$ on γ_1.

Note that (4.21) also leads to a somewhat different estimate. Indeed,

$$\inf_{w\in V_0}\left[\frac{1}{2}a(w,w) - \boldsymbol{R}_\Omega(y^*)\,\|w\|_\Omega - \boldsymbol{R}_{\Gamma_1}(y^*,\xi^*)\,\|\gamma w\|_{\Gamma_1}\right] \geq$$
$$\geq \inf_{w\in V_0}\left[\frac{1}{2}a(w,w) - \sqrt{\boldsymbol{R}^2_{\Gamma_1}(y^*,\xi^*) + \boldsymbol{R}^2_\Omega(y^*)}\sqrt{\|w\|^2_\Omega + \|\gamma w\|^2_{\Gamma_1}}\right].$$

It is easy to see that

$$\|w\|^2_\Omega + \|\gamma w\|^2_{\Gamma_1} \leq C^2_{(\Omega,\Gamma_1)} a(w,w)$$

with a certain constant $C_{(\Omega,\Gamma_1)}$. Therefore the value of inf is bounded from below by the quantity

$$-\frac{1}{2}C^2_{(\Omega,\Gamma_1)}\left(\boldsymbol{R}^2_{\Gamma_1}(y^*,\xi^*) + \boldsymbol{R}^2_\Omega(y^*)\right).$$

Thus, instead of (4.24), we have

$$\begin{aligned}\boldsymbol{\tfrac{1}{2}\,\|v-u\|^2_a} &\boldsymbol{\leq} \\ \boldsymbol{\leq \widetilde{M}_\oplus(v,y^*,\xi^*,\beta)} &\boldsymbol{:= (1{+}\beta)M_1(v,y^*) + M_2(\gamma v,\xi^*) +} \\ &\boldsymbol{+\tfrac{1}{2}\left(1{+}\tfrac{1}{\beta}\right)C^2_{(\Omega,\Gamma_1)}\left(R^2_{\Gamma_1}(y^*,\xi^*) + R^2_\Omega(y^*)\right)}\end{aligned} \tag{4.28}$$

Let us now consider particular forms of the estimates (4.24) and (4.28).

First, we set

$$\xi^* = -\delta_n y^*.$$

In this case,

$$\boldsymbol{R}_{\Gamma_1}(\delta_n y^*\xi^*) = 0$$

and by (4.24) we obtain the estimate

$$\tfrac{1}{2}\,\|v-u\|^2_a \leq (1{+}\beta)M_1(v,y^*) + \\ + M_2(\gamma v,\delta_n y^*) + \tfrac{1}{2}\left(1{+}\frac{1}{\beta}\right)C^2_\Omega \boldsymbol{R}^2_\Omega(y^*). \tag{4.29}$$

Note that this estimate is sharper than the one that follows from (4.28) because $C_\Omega \leq C_{(\Omega,\Gamma_1)}$.

Another estimate, if the last term of (4.24) is estimated from above by means of the Young's inequality. Then, we obtain the following inequality which involves a new positive constant α:

$$\frac{1}{2}\|v-u\|_a^2 \le (1+\beta)M_1(v,y^*)+M_2(\gamma v,\xi^*)+ \\ +\frac{1}{2}\left(1+\frac{1}{\beta}\right)(1+\alpha)C_\Omega^2\boldsymbol{R}_\Omega^2(y^*) \\ +\frac{1}{2}\left(1+\frac{1}{\beta}\right)\left(1+\frac{1}{\alpha}\right)C_{\Gamma_1}^2\boldsymbol{R}_{\Gamma_1}^2(\delta_n y^*,\xi^*)\,. \tag{4.30}$$

For $\alpha=1$, we can view (4.30) as a form of (4.28) with

$$C_{(\Omega,\Gamma_1)}=\sqrt{2(C_\Omega^2+C_{\Gamma_1}^2)}.$$

Let us gather in (4.30) all the terms related to the boundary condition on Γ_1 and denote them

$$I_{\Gamma_1}(\gamma v,\delta_n y^*,\xi^*) = = \int\limits_{\Gamma_1}(j(\gamma v)+j^*(\xi^*)-(\gamma v)\,\xi^*+\frac{\theta}{2}\,|\delta_n y^*+\xi^*|^2)d\Gamma, \tag{4.31}$$

where $\theta=\left(1+\frac{1}{\beta}\right)\left(1+\frac{1}{\alpha}\right)C_{\Gamma_1}^2$.

To minimize the right–hand side of (4.30) we should minimize I_{Γ_1} with respect to ξ^*. Now the estimate (4.30) comes in the form

$$\frac{1}{2}\|v-u\|_a^2 \le (1+\beta)M_1(v,y^*)+\inf_{\xi^*} I_{\Gamma_1}(\gamma v,\delta_n y^*,\xi^*)+ \\ +\frac{1}{2}\left(1+\frac{1}{\beta}\right)(1+\alpha)C_\Omega^2\boldsymbol{R}_\Omega^2(y^*)\,, \tag{4.32}$$

5. Particular cases

5.1. **Neumann type of boundary condition.** This type boundary conditions correspond to the case, in which Υ is a linear functional, i.e.

$$\Upsilon(\xi) := \langle \eta^*,\xi\rangle_{\Gamma_1} \tag{5.1}$$

where $\eta^*\in Z^*$. In particular, if η^* is associated with a square summable (on Γ_1) function F, then one can set

$$j(v)=F\,v, \qquad -\delta_n p^*=F \qquad \text{a.e. on}\,\Gamma_1\,. \tag{5.2}$$

Then

$$\Upsilon(\xi)=\int\limits_{\Gamma_1}F\,\xi\,d\Gamma,$$

$$\Upsilon^*(\xi^*)=\begin{cases}0, & \text{if}\,\xi^*=F\ \text{a.e. on}\,\Gamma_1,\\ +\infty & \text{otherwise.}\end{cases}$$

In this case,

$$I_{\Gamma_1}=\int\limits_{\Gamma_1}\left(F\,\gamma v+0-F\,\gamma v+\frac{\theta}{2}\,|\delta_n y^*+F|^2\right)d\Gamma=\frac{\theta}{2}\int\limits_{\Gamma_1}|\delta_n y^*+F|^2\,d\Gamma\,.$$

Now the estimate comes in the form

(5.3)
$$\tfrac{1}{2}\,\|v-u\|_a^2 \le (1+\beta)M_1(v,y^*) + \tfrac{1}{2}\left(1+\tfrac{1}{\beta}\right)\left(1+\tfrac{1}{\alpha}\right) C_{\Gamma_1}^2 \int\limits_{\Gamma_1} |\delta_n y^* + F|^2\, d\Gamma$$
$$+ \tfrac{1}{2}\left(1+\tfrac{1}{\beta}\right)(1+\alpha) C_\Omega^2 \boldsymbol{R}_\Omega^2(y^*).$$

5.2. **Robin type of boundary condition.** In this case we have

$$j(v) = F\,v + \frac{c}{2}\,v^2 \qquad \text{a.e. on}\,\Gamma_1\,.$$

where F is a square summable (on Γ_1) function and c is a positive constant. It is easy to calculate

$$j^*(\xi^*) = \frac{1}{2c}(\xi^* - F)^2$$

and therefore

$$I_{\Gamma_1} = \int\limits_{\Gamma_1} \left(F\,\gamma v + \frac{c}{2}\,(\gamma v)^2 + \frac{1}{2c}\,(\xi^* - F)^2 - \gamma v\,\xi^* + \tfrac{\theta}{2}\,|\delta_n y^* + \xi^*|^2 \right) d\Gamma.$$

If we choose $\xi^* = -\delta_n y^*$, then the θ dependent term drops out and we obtain

$$\begin{aligned} I_{\Gamma_1} &= \int\limits_{\Gamma_1} \left(F\,\gamma v + \frac{c}{2}\,(\gamma v)^2 + \frac{1}{2c}\,(\delta_n y^* + F)^2 + \gamma v\,\delta_n y^* \right) d\Gamma \\ &= \frac{1}{2c} \int\limits_{\Gamma_1} (F + c\,\gamma v + \delta_n y^*)^2\, d\Gamma. \end{aligned}$$

Then, the majorant estimate reads (by taking the limit case $\alpha \to 0$)

$$\tfrac{1}{2}\,\|v-u\|_a^2 \le (1+\beta)M_1(v,y^*) + \frac{1}{2c} \int\limits_{\Gamma_1} (F + c\,\gamma v + \delta_n y^*)^2\, d\Gamma + \tag{5.4}$$
$$+ \tfrac{1}{2}\left(1+\tfrac{1}{\beta}\right) C_\Omega^2 \boldsymbol{R}_\Omega^2(y^*).$$

5.3. **Friction type of boundary condition.** Here we have

$$j(v) = \mu\,|v|, \qquad \mu > 0. \tag{5.5}$$

In this case,

$$\Upsilon(\xi) = \int\limits_{\Gamma_1} \mu\,|\xi|\, d\Gamma,$$

$$\Upsilon^*(\xi^*) = \begin{cases} 0, & \text{if } |\xi^*| \le \mu \text{ a.e. on}\,\Gamma_1, \\ +\infty & \text{otherwise} \end{cases}$$

and, therefore,

$$I_{\Gamma_1} = \int\limits_{\Gamma_1} \left(\mu\,|\gamma v| + 0 - \xi^*\,(\gamma v) + \tfrac{\theta}{2}\,|\delta_n y^* + \xi^*|^2\right)\,d\Gamma \tag{5.6}$$

under the assumption $|\xi^*| \le \mu$. If $|\delta_n y^*| \le \mu$, then we set $\xi^* = -\delta_n y^*$ and

$$I_{\Gamma_1} = \int\limits_{\Gamma_1} (\mu\,|\gamma v| + (\delta_n y^*)(\gamma v))\,d\Gamma.$$

If $\delta_n y^* \ge \mu$, then to minimize the quadratic term we take $\xi^* = -\mu$ and

$$I_{\Gamma_1} = \int\limits_{\Gamma_1} \left(\mu\,|\gamma v| + \mu\,(\gamma v) + \tfrac{\theta}{2}\,|\delta_n y^* - \mu|^2\right)\,d\Gamma.$$

Analogously, if $\delta_n y^* \le \mu$, then we set $\xi^* = +\mu$ and

$$I_{\Gamma_1} = \int\limits_{\Gamma_1} \left(\mu\,|\gamma v| - \mu\,(\gamma v) + \tfrac{\theta}{2}\,|\delta_n y^* + \mu|^2\right)\,d\Gamma.$$

All three cases can be written in one form if one introduces the function

$$\phi(\gamma v, \delta_n y^*, \mu) := \begin{cases} \frac{\theta}{2}(\delta_n y^* + \mu)^2 - \mu\,(\gamma v) & \text{if} \quad \delta_n y^* < -\mu, \\ (\delta_n y^*)\,(\gamma v) & \text{if} \quad |\delta_n y^*| < \mu, \\ \frac{\theta}{2}(\delta_n y^* - \mu)^2 + \mu\,(\gamma v) & \text{if} \quad \delta_n y^* > \mu. \end{cases} \tag{5.7}$$

Then, the majorant estimate reads

$$\tfrac{1}{2}\,\|v-u\|_a^2 \le (1+\beta)M_1(v,y^*) + \int\limits_{\Gamma_1} (\mu|\gamma v| + \phi(\gamma v, \delta_n y^*, \mu))\,d\Gamma + \\ + \tfrac{1}{2}\left(1+\tfrac{1}{\beta}\right)(1+\alpha)C_\Omega^2 \boldsymbol{R}_\Omega^2(y^*). \tag{5.8}$$

Remark 1. Since the quadratic and linear terms in ξ^* in (5.6) were not minimized simultaneously, the estimate can be further improved. A careful minimization provides a sharper estimate

$$\phi(\gamma v, \delta_n y^*, \mu) := \begin{cases} \frac{\theta}{2}(\delta_n y^* + \mu)^2 - \mu\,(\gamma v) & \text{if} \quad \frac{(\gamma v)}{\theta} - \delta_n y^* \ge \mu, \\ (\delta_n y^*)\,(\gamma v) - \frac{(\gamma v)^2}{2\theta} & \text{if} \quad |\frac{(\gamma v)}{\theta} - \delta_n y^*| \le \mu, \\ \frac{\theta}{2}(\delta_n y^* - \mu)^2 + \mu\,(\gamma v) & \text{if} \quad \frac{(\gamma v)}{\theta} - \delta_n y^* \le -\mu. \end{cases} \tag{5.9}$$

5.4. **Winkler type boundary condition.** Another problem arises if we define j as follows

$$j(v) = \frac{1}{2}\kappa\,|v|^2\,, \tag{5.10}$$

where κ and is a positive constant. This case can be viewed as a simplified variant of the Winkler's boundary condition widely used in solid mechanics. In this condition, on Γ_1 a body is connected with an elastic foundation which

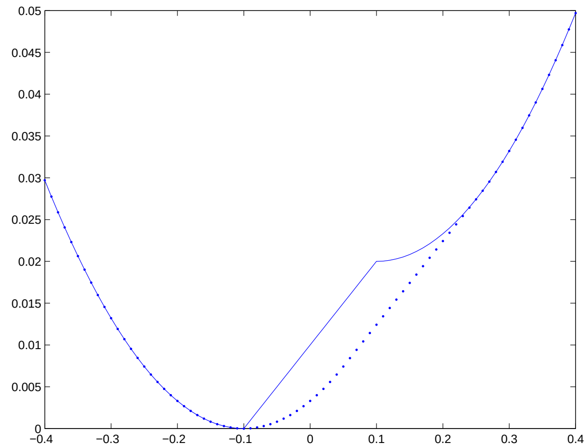

FIGURE 1. Comparison of majorant parts (5.7) and (5.9). The values of $\phi(\gamma v, \delta_n y^*, \mu)$ are displayed versus variable $\delta_n y^*$ for given $\gamma v, \mu$ and θ. Here, $\gamma v = 0.05, \mu = 0.01$ and $\theta = 0.33$. The majorant part (5.9) displayed as a dotted line provides sharper estimate then (5.7) displayed as a full line.

provides a certain response to boundary deflections (such a condition can be modeled by a large amount of springs connected with Γ_1). Now, we have

$$-p_n^* = \kappa u, \qquad \text{a.e. on}\, \Gamma_1\,. \tag{5.11}$$

and

$$j^*(\xi^*) = \sup_{\xi\in\mathbb{R}^d}\left\{\xi^*\cdot\xi - \frac{1}{2}\kappa\,|\xi|^2\right\} = \frac{1}{2\kappa}\,|\xi^*|^2\,. \tag{5.12}$$

Consider the quantity

$$I_{\Gamma_1} = \tfrac{1}{2}\int\limits_{\Gamma_1}\left(\kappa\,|\gamma v|^2 + \tfrac{1}{\kappa}\,|\xi^*|^2 - 2\xi^*\,\gamma v + \theta\,|\delta_n y^* + \xi^*|^2\right)d\Gamma$$

The minimization of this quantity over ξ^* leads to the condition

$$\frac{1}{\kappa}\xi^* - \gamma v + \theta(\delta_n y^* + \xi^*) = 0 \tag{5.13}$$

or

$$\left(\frac{1}{\kappa} + \theta\right)\xi^* = \gamma v - \theta\delta_n y^* \;\Rightarrow\; \xi^* = \frac{\kappa(\gamma v - \theta\delta_n y^*)}{1+\kappa\theta}$$

This gives a simple expression for I_{Γ_1}:

$$I_{\Gamma_1} = \tfrac{1}{2}\int\limits_{\Gamma_1}\frac{\theta}{1+\kappa\theta}(\kappa(\gamma v) + \delta_n y^*)^2 d\Gamma. \tag{5.14}$$

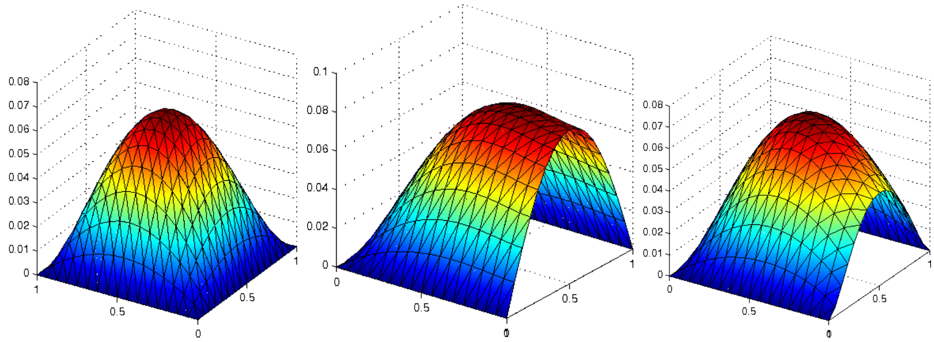

FIGURE 2. Discrete solutions of the minimization problem for $\mu \to \infty$ (left), $\mu = 0$ (middle) and $\mu = 0.1$ (right).

By (4.29), we obtain the a posteriori estimate

$$\tfrac{1}{2}\,\|v-u\|_a^2 \le (1+\beta)M_1(v,y^*) + \tfrac{1}{2}\int\limits_{\Gamma_1} \frac{\theta}{1+\kappa\theta}(\kappa(\gamma v) + \delta_n y^*)^2 d\Gamma + \tag{5.15}$$
$$+\tfrac{1}{2}\left(1+\tfrac{1}{\beta}\right)(1+\alpha)C_\Omega^2 \boldsymbol{R}_\Omega^2(y^*)\,.$$

Note the the boundary term vanishes if and only if γv and $\delta_n y^*$ satisfy the Winkler boundary condition (5.11).

6. Numerical implementation

6.1. **The nonlinear problem and its discrete solution.** We will only discuss the case of the friction boundary condition described in Section 5.3, i.e., the energy minimization problem

$$e(w) := \int\limits_{\Omega} \left(\frac{1}{2}|\nabla w|^2 - fw\right) dx + \mu \int\limits_{\Gamma_1} |w|\, d\Gamma \to \min \tag{6.1}$$

over all function $w \in V_0$. By the variation of this functional, one obtains in accordance with (2.4) the friction boundary condition for the solution $u \in V_0$ of (6.1)

$$|u|\frac{\partial u}{\partial n} + \mu u \quad = \quad 0 \quad \text{on } \Gamma_1. \tag{6.2}$$

We consider a unit square domain $\Omega := (0,1)\times(0,1) \in \mathbb{R}^2$, whose boundary Γ is split into its nonlinear boundary condition part $\Gamma_1 = \{1\}\times[0,1]$ and the purely Dirichlét part $\Gamma_0 := \Gamma \setminus \Gamma_1$. Let assume the external force

$$f(x,y) := 2x(1-x) + 2y(1-y), \quad (x,y) \in \Omega.$$

The finite element method is used to provide discrete approximations of the minimization problem above. Let divide the domain Ω by a regular triangulation $\mathcal{T}$ in triangles in the sense of Ciarlet [19], i.e. $\mathcal{T}$ is a finite partition of Ω into closed triangles; two distinct elements T_1 and T_2 are either disjoint, or $T_1 \cap T_2$ is a complete edge or a common node of both T_1 and T_2. The Hilbert space V is approximated by the set of $\mathcal{T}$- piecewise affine functions that are zero on Γ_D by

$$\mathcal{S}^1_{\Gamma_0}(\mathcal{T}) := \{w \in H^1_{\Gamma_0}(\Omega) : \forall T \in \mathcal{T}, w|_T \in \mathcal{P}_1(T)\}.$$

($\mathcal{P}_1(T)$ denotes the affine functions on T.)

Algorithm 1 (solution algorithm)**.** *Let $v \in \mathcal{S}^1_{\Gamma_0}(\mathcal{T})$ be an approximation of the discrete solution. Define $g_1 := v|_{\Gamma_1}$ as its boundary value and repeat until convergence:*
(a) Upgrade v as a solution of the minimization problem

$$\int\limits_\Omega \left(\frac{1}{2}|\nabla w|^2 - fw\right) dx \to min$$

over all functions $w \in \mathcal{S}^1_{\Gamma_0}(\mathcal{T})$ satisfying the boundary condition

$$w|_{\Gamma_1} = g_1.$$

(b) Upgrade v as a solution of the minimization problem (6.1) *over all functions $v + w \in \mathcal{S}^1_{\Gamma_0}$, $w|_{\Omega \setminus \Gamma_1} = 0$. Set $g_1 := v|_{\Gamma_1}$.*

It should be remarked that the step (a) is equivalent to a Laplace problem $-\triangle u = f$ in Ω with the Dirichlét boundary conditions $u|_{\Gamma_0} = 0, u|_{\Gamma_1} = g_1$. The step (b) is in realized through the point-wise relaxation, see [8] for more details. Note that the sequential calling of steps (a) and (b) provides monotone decreasing of the energy functional $e(w)$ on the finite element space $\mathcal{S}^1_{\Gamma_0}(\mathcal{T})$. By this fact it is possible to prove [8] that the sequence tends to the Galerkin approximation (exact minimizer on $\mathcal{S}^1_{\Gamma_0}(\mathcal{T})$).

In the dependence on the friction parameter μ, we are interested in three model cases:

- $\mu \to \infty$ - (6.2) implies the homogeneous Dirichlét boundary condition $u|_{\Gamma_1} = 0$.
- $\mu = 0$ - (6.2) implies the homogeneous Neumann boundary condition $\frac{\partial u}{\partial n}|_{\Gamma_1} = 0$.
- $\mu = 0.1$ - this is a typical friction boundary condition, i.e., the mixture of Dirichlét and Neumann boundary conditions.

Using a Matlab based code we calculated discrete solutions on uniform triangulations with 25, 81, 289, 1089, 4225, 16641, 66049, 263169 nodes. For convenience, we refere to these triangulations by their level number: level 1 stand for the triangulation with 25 nodes, level 2 for the triangulation with 81 nodes and so on. For higher levels calculations it was useful to exploit a nested iteration method which projects a solution from a coarser triangulation on the finer triangulation and takes this as a solution approximation on a finer mesh. Then, Algorithm 1 needen only about 3 iteration to achieve

level	total energy	energy in Ω	energy on Γ_1
1	-1.066241e-02	-1.325955e-02	2.597139e-03
2	-1.200491e-02	-1.456770e-02	2.562794e-03
3	-1.237813e-02	-1.492187e-02	2.543735e-03
4	-1.247626e-02	-1.500665e-02	2.530394e-03
5	-1.250161e-02	-1.502860e-02	2.526984e-03
6	-1.250798e-02	-1.503393e-02	2.525948e-03
7	-1.250958e-02	-1.503490e-02	2.525316e-03
8	-1.250999e-02	-1.503516e-02	2.525172e-03

TABLE 1. Energy functional (6.1) values for the case $\mu = 0.1$.

sufficient convergence. Table 1 reports on energy values for all triangulation levels in the case $\mu = 0.1$.

Numerically, it turns out feasible to replace the limit $\mu \to \infty$ by the choice $\mu = 1000$. Then it holds $|\frac{\partial u}{\partial n}| \leq \mu$ and the condition (6.2) implies again the homogeneous Dirichlét boundary condition $u|_{\Gamma_1} = 0$. Figure 2 displays discrete solutions for the level 4 triangulation. It is easy to check an exact solution

$$u = x(1-x)y(1-y), \quad (x, y) \in \bar{\Omega},$$

for the case $\mu \to \infty$. Exact solutions for the remaining two cases are not known to authors. Thus a reference solution is generated as a discrete solution calculated on the level 8 triangulation with 263169 nodes.

6.2. **Majorant minimization in detail.** The error of the discrete solution v, i.e, the distance to the exact solution u is measured in the energy norm by the majorant estimate (5.8). Its application requires knowledge of the constants C_Ω and C_{Γ_1} from the Friedrichs' and trace inequalities (4.22) and (4.23). The constant C_Ω can be estimated throughout the minimal eigenvalues of the operator Δ, see [11]. For the unit square domain with the right (free) edge Γ_1, we can take the value

$$C_\Omega = \frac{2}{\sqrt{5\pi^2}} \approx 0.2847.$$

For the case $\mu \to \infty$, the right edge Γ_1 represents in fact the Dirichlét boundary, for which the constant can be reduced to

$$C_\Omega = \frac{1}{\sqrt{2\pi^2}} \approx 0.2251.$$

With the help of the Cauchy-Schwarz inequality it is possible to bound the constant

$$C_{\Gamma_1} \leq 1.$$

The majorant in (5.8) is evaluated in our numerical experiments by three methods:

- Method (a): averaging on the same mesh = from a known discrete approximation v of the solution u we choose the testing function $y = \mathcal{G}v$, where $\mathcal{G}$ represents an averaged gradient operator, see e.g. [24] for more details. This is a very cheap way to get some PRELIMINARY knowledge on the upper bound of the error.
- Method (b): averaging on the refined mesh. This method is similar to method (a), only with the difference that the averaging is done for the the solution calculated on once more refined mesh. This method can be regarded as a quantitative form of the Runge's rule.
- Method (c): minimization of the majorant one the same mesh. Due to the freedom in the scalar parameters $\alpha, \beta > 0$ and the testing function $y*$ it is possible to minimize the majorant with respect to these unknowns. This is the most expensive method for the a DETAILED knowledge of the error.

6.3. **Majorant minimization in detail.** The majorant in the right-hand side of (5.8) represents a strictly convex functional for given scalar parameters $\alpha, \beta > 0$ and unknown function $y^* \in Q^*_{\Gamma_1}$. The majorant (5.8) is decomposed as a sum of a quadratic functional

$$(1+\beta)M_1(v, y^*) + \tfrac{1}{2}\left(1+\tfrac{1}{\beta}\right)(1+\alpha)C_\Omega^2 \boldsymbol{R}_\Omega^2(y^*) \tag{6.3}$$

defined in the domain Ω and a nonlinear functional

$$I_{\Gamma_1} := \int_{\Gamma_1} \left(\mu|\gamma v| + \phi(\gamma v, \delta_n y^*, \mu)\right) d\Gamma \tag{6.4}$$

defined on the boundary Γ_1. Due to the term $\frac{1}{\beta}$ in (6.3) and $\frac{1}{\alpha}$ occurring in the definitions of $\phi(\cdot)$ in (5.7) or (5.9), the majorant is not convex in scalar parameters α, β. Let us show that the optimal values of α and β for a given (or approximated) function y^* are calculable analytically for the case of (5.7). We decompose the boundary integral (6.4) in dependence of $\theta = \left(1+\frac{1}{\beta}\right)\left(1+\frac{1}{\alpha}\right)C^2_{\Gamma_1}$ as

$$\begin{aligned} I_{\Gamma_1} &= \frac{\theta}{2}\ I_{\Gamma_{1,1}} + I_{\Gamma_{1,2}} + \frac{1}{2\theta}\ I_{\Gamma_{1,3}} = \\ &= \frac{\theta}{2}\int_{\Gamma_1} \phi^1(\delta_n y^*, \mu) d\Gamma + \int_{\Gamma_1} \phi^2(\gamma v, \delta_n y^*, \mu) d\Gamma + \frac{1}{2\theta}\int_{\Gamma_1} \phi^3(\delta_n y^*, \mu) d\Gamma, \end{aligned} \tag{6.5}$$

where $\phi^1(\cdot), \phi^2(\cdot), \phi^3(\cdot)$ are defined using the formula (5.7) or the formula (5.9). Now the α and β dependent part of the majorant estimate has a structure

$$(1+\beta)a + (1+\frac{1}{\beta})(1+\frac{1}{\alpha})b_1 + \frac{b_3}{(1+\frac{1}{\alpha})(1+\frac{1}{\beta})} + (1+\frac{1}{\beta})(1+\alpha)c, \tag{6.6}$$

where the parameters a, b_1, b_3, c read

$$
\begin{aligned}
a &= M_1(v, y^*),\\
b_1 &= \frac{1}{2} C_{\Gamma_1}^2 I_{\mathbf{\Gamma}_{1,1}}, \quad b_3 = \frac{1}{2}\frac{I_{\mathbf{\Gamma}_{1,3}}}{C_{\Gamma_1}^2},\\
c &= \frac{1}{2} C_\Omega^2 \boldsymbol{R}_\Omega^2(y^*).
\end{aligned} \tag{6.7}
$$

We have come to the principial difference of formulae (5.7) and (5.9). For (5.7), $\phi^1(\cdot), \phi^2(\cdot), \phi^3(\cdot)$ depend additionally on θ. Thus coefficients b_1 and b_3 are functions of θ and the minimization of (6.6) with respect to α and β represents a nonlinear problem with no analytical solution.

Using the formula (5.7) instead, $\phi^1(\cdot), \phi^2(\cdot)$ are independent of θ. Since is also holds $b_3 = 0$ (or $I_{\mathbf{\Gamma}_{1,3}} = 0$) and it is possible to obtain an analytic solution $\alpha = \sqrt{\frac{b_1}{c}}, \beta = \frac{\sqrt{b_1}+\sqrt{c}}{\sqrt{a}}$ which minimizes the expression (6.6). Using back substitutions (6.7), we obtain the optimal values

$$
\alpha = \frac{C_{\Gamma_1}}{C_\Omega}\frac{\sqrt{I_{\mathbf{\Gamma}_{1,1}}}}{\boldsymbol{R}_\Omega(y^*)}, \quad \beta = \frac{C_{\Gamma_1}\sqrt{I_{\mathbf{\Gamma}_{1,1}}} + C_\Omega \boldsymbol{R}_\Omega(y^*)}{2M_1(v, y^*)}. \tag{6.8}
$$

If the boundary nonlinear terms $I_{\mathbf{\Gamma}_{1,1}}$ additionally drops out, we get the optimal values

$$
\alpha = 0, \quad \beta = \frac{C_\Omega \boldsymbol{R}_\Omega(y^*)}{\sqrt{2M_1(v, y^*)}} \tag{6.9}
$$

already known from a minimization problem with a Dirichlet boundary condition on Γ_1, i.e., the case $\mu \to \infty$.

The problem of finding a minimizer y^* of the quadratic functional (6.3) (for given α, β) is equivalent to the solution of a linear system of equations. For a given approximation v, only the boundary values $\delta_n y^*|_{\Gamma_1}$ contribute to the nonlinear functional (6.4). Therefore, point-wise relaxation [8] is applied for the minimization of the majorant (5.8) on the Γ_1 boundary. A combination of a quadratic functional minimization, a boundary point-wise relaxation and the calculation of optimal values of α and β gives rise to the following majorant optimization algorithm.

Algorithm 2 (Majorant minimization)**.** *Let $v \in V_0$ be a given discrete solution. Let $\alpha, \beta > 0$ are approximated parameters.*
(a) Set $y^* \in Q^*_{\Gamma_1}$ by minimizing the quadratic functional (6.3).
(b) Correct $\delta_n y^*|_{\Gamma_1}$ by minimizing the majorant (5.8) *with $\phi(\cdot)$ defined by (5.7) on the Γ_1 boundary only.*
(c) Go to step (a) until the convergence in y^* is achieved.
(d) Upgrade α and β from y^* using the formulae (6.9).
(e) Go to step (a) until the convergence in α and β is achieved.
(f) Recalculate the majorant value (5.8) *by using $\phi(\cdot)$ defined by (5.9).*

Due to the strict convexity of the majorant, the algorithm reduces the majorant values through steps (a)-(b) monotonically. When the convergence is achieved and y^* is known, optimal α and β are calculated, which leads to the further reduction of the majorant value. Finally, the recalculation using $\phi(\cdot)$ defined by (5.9) improves the majorant estimate (5.8) once again.

6.4. **Numerical tests with majorant evaluation.** Majorant evaluations were performed for all triangulation levels up to level 7 with 66049 triangulation nodes. Corresponding to three μ parameter cases ($\mu = 1000, \mu = 0, \mu = 0.1$) and three methods for the majorant computation (method (a), (b), (c) explained before), there are nine tables, see Tables 2 - 10. Tables columns describe majorant values (5.8) in detail:

- "l" triangulation level.
- "left" denoting the value $(1+\beta)M_1(v, y^*)$
- "right" denoting the value $\frac{1}{2}\left(1+\frac{1}{\beta}\right)(1+\alpha)C_\Omega^2 \boldsymbol{R}_\Omega^2(y^*)$
- " middle" denoting the value I_{Γ_1} defined in (6.4).
- "α, β" denoting the optimal values.
- "major." denoting the majorant defined in (5.8), i.e., the sum of "left", "right" and "middle" terms above.
- "error^2/2" denoting $\frac{1}{2}\|v-u\|_a^2 = \frac{1}{2}\int\limits_\Omega (\nabla(u-v))^2\,dx$.
- "I_{eff}" denoting the index of efficiency $I_{\text{eff}} := \frac{\text{majorant}}{\text{error}}$.

It holds $\alpha = 0$ according to (6.9) for the case $\mu \to \infty$. This value and the zero "middle" term are not displayed in Tables 2, 3, 4 for simplicity.

Tables 2, 5 and 8 confirm that the method (a), i.e., an averaging on the same mesh leads to the majorant values that cause a pessimistic majorant estimate particularly for larger triangulations. The dominating "right" and "middle" terms lead to a high index of efficiency, e.g. $I_{\text{eff}} = 22.66$ for the level 7 triangulation in the case $\mu = 0.1$.

The method (b), i.e, the averaging on once more refined mesh provides slightly better majorant values, see Tables 3, 6 and 9, however it still reaches too high values for finer triangulations.

The majorant optimization method (c) described in Algorithm 2 provides the best applicable results (see Tables 4, 7 and 10), allowing index of efficiency I_{eff} to go up to 3.32 for the level 7 triangulation in the case $\mu = 0.1$. The comparison of error distribution and the majorant distribution for the case $\mu \to \infty$ is displayed on Figure 3.

Acknowledgments The first author was supported from Johann Radon Institute for Computational and Applied Mathematics (RICAM) during his

l	left	right	β	major.	$\text{error}^2/2$	I_{eff}
1	3.4e-03	4.5e-03	1.32	7.9e-03	1.7e-03	2.14
2	1.1e-03	1.7e-03	1.54	2.9e-03	4.5e-04	2.52
3	3.5e-04	7.2e-04	2.04	1.0e-03	1.1e-04	3.06
4	1.1e-04	3.1e-04	2.82	4.2e-04	2.8e-05	3.84
5	3.6e-05	1.4e-04	3.96	1.7e-04	7.2e-06	4.97
6	1.1e-05	6.6e-05	5.57	7.8e-05	1.8e-06	6.58
7	4.0e-06	3.1e-05	7.86	3.5e-05	4.5e-07	8.87

TABLE 2. Method (a) in case $\mu \to \infty$.

l	left	right	β	major.	$\text{error}^2/2$	I_{eff}
1	2.4-03	2.2e-03	0.91	4.7e-03	1.7e-03	1.65
2	8.4-04	9.2e-04	1.10	1.7e-03	4.5e-04	1.98
3	2.6e-04	3.9e-04	1.46	6.6e-04	1.1e-04	2.40
4	8.4e-05	1.7e-04	2.01	2.5e-04	2.8e-05	2.97
5	2.7e-05	7.6e-05	2.81	1.0e-04	7.2e-06	3.78
6	8.8e-06	3.5e-05	3.95	4.4e-05	1.8e-06	4.93

TABLE 3. Method (b) in case $\mu \to \infty$.

l	left	right	β	major.	$\text{error}^2/2$	I_{eff}
1	2.3e-03	6.0e-04	0.26	2.9e-03	1.7e-03	1.30
2	6.6e-04	2.3e-04	0.35	9.0e-04	4.5e-04	1.41
3	1.7e-04	7.3e-05	0.42	2.4e-04	1.1e-04	1.47
4	4.4e-05	2.0e-05	0.47	6.4e-05	2.8e-05	1.50
5	1.1e-05	5.4e-06	0.50	1.6e-05	7.2e-06	1.51
6	2.7e-06	1.4e-06	0.51	4.1e-06	1.8e-06	1.52
7	6.9e-07	3.6e-07	0.52	1.0e-06	4.5e-07	1.53

TABLE 4. Method (c) in case $\mu \to \infty$.

stay on occasion of the Special Semester on Computational Mechanics in Linz in winter 2005. The second author acknowledges the support of Austrian Science Fund 'Fonds zur Förderung der wissenschaftlichen Forschung (FWF)' for his support under grant SFB F013/F1306 in Linz, Austria.

References

[1] H. Brezis. Problémes unilatéraux. *J. Math. Pures Appl.*, 1971, 9(72), N1, 1–168.
[2] C. Carstensen and S. Sauter. A posteriori error analysis for Elliptic PDEs on domains with complicated structures. Numer. Math. 96 (2004), No. 4, 691-721.
[3] M. Chipot. Free boundary problem. Vol. 2, Roma, 1980, 135–140.
[4] G. Duvaut and J.-L. Lions. Les inequations en mechanique et en physique. Dunod, Paris, 1972.
[5] R. S. Falk. Error estimates for the approximation of a class of variational inequalities, *Math. Comput.*, 28(1974), 963–971.

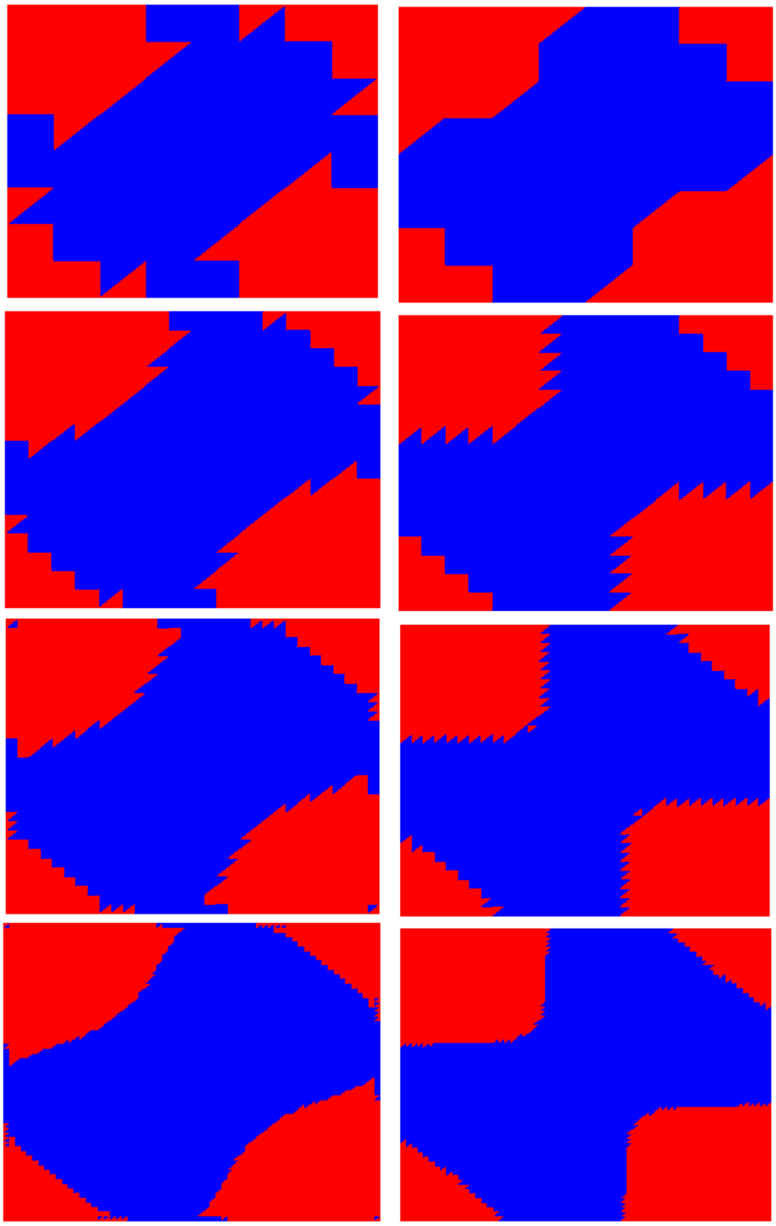

FIGURE 3. Marking elements strategy for the (optimized) majorant distribution (left) and true error indicator (right) and different uniform meshes. Marked elements are the red ones with the error value greater then the median of all error values.

l	left	right	β	middle	α	major.	error^2/2	I_{eff}
1	3.8e-03	5.3e-03	1.51	8.2e-04	7.8e-02	9.9e-03	2.0e-03	2.20
2	1.2e-03	2.1e-03	2.06	8.9e-04	2.0e-01	4.3e-03	5.1e-04	2.89
3	4.0e-04	9.3e-04	2.87	4.5e-04	2.4e-01	1.8e-03	1.3e-04	3.73
4	1.3e-04	4.3e-04	4.10	2.3e-04	2.6e-01	8.0e-04	3.2e-05	4.96
5	4.6e-05	2.1e-04	6.03	1.2e-04	2.9e-01	3.8e-04	8.0e-06	6.95
6	1.7e-05	1.2e-04	9.48	8.3e-05	3.4e-01	2.2e-04	1.9e-06	10.79
7	7.3e-06	8.7e-05	16.70	6.8e-05	3.9e-01	1.6e-04	3.8e-07	20.66

TABLE 5. Method (a) in case $\mu = 0$.

l	left	right	β	middle	α	major.	error^2/2	I_{eff}
1	2.7e-03	2.7e-03	1.17	1.1e-03	2.0e-01	6.6e-03	2.0e-03	1.79
2	9.4e-04	1.1e-03	1.52	5.6e-04	2.4e-01	2.6e-03	5.1e-04	2.27
3	3.1e-04	5.1e-04	2.10	2.7e-04	2.6e-01	1.1e-03	1.3e-04	2.91
4	1.0e-04	2.4e-04	3.05	1.4e-04	2.9e-01	4.9e-04	3.2e-05	3.90
5	3.7e-05	1.3e-04	4.77	9.1e-05	3.4e-01	2.6e-04	8.0e-06	5.71
6	1.5e-05	9.2e-05	8.39	7.2e-05	3.9e-01	1.8e-04	1.9e-06	9.71

TABLE 6. Method (b) in case $\mu = 0$.

l	left	right	β	middle	α	major.	error^2/2	I_{eff}
1	2.1e-03	8.7e-04	0.40	1.6e-06	9.3e-04	3.0e-03	2.0e-03	1.21
2	6.1e-04	2.7e-04	0.44	6.8e-07	1.2e-03	8.8e-04	5.1e-04	1.30
3	1.6e-04	7.9e-05	0.49	3.0e-07	1.9e-03	2.4e-04	1.3e-04	1.36
4	4.2e-05	2.2e-05	0.53	1.0e-07	2.3e-03	6.5e-05	3.2e-05	1.42
5	1.1e-05	6.6e-06	0.55	3.7e-08	2.8e-03	1.8e-05	8.0e-06	1.52
6	4.0e-06	2.2e-06	0.57	2.3e-08	5.1e-03	6.3e-06	1.9e-06	1.82
7	2.0e-06	1.1e-06	0.57	1.2e-08	5.5e-03	3.2e-06	3.8e-07	2.90

TABLE 7. Method (c) in case $\mu = 0$.

[6] I. Ekeland and R. Temam. *Convex analysis and variational problems.* North–Holland, Oxford, 1976.

[7] A. Friedman. Variational principles and free–boundary problems. New York, Wiley 1982.

[8] R. Glowinski, J. L. Lions and R. Tremolieres. Analyse numerique des inequations variationnelles. Dunod, Paris 1976.

[9] D. Kinderlehrer and G. Stampacchia. *An introduction to variational inequalities and their applications*, Academic Press, New York, 1980.

[10] J.-L. Lions and E. Magenes. Non-homogeneous boundary value problems and applications. Vol. I,II. Springer, New York-Heidelberg, 1972.

[11] S. G. Mikhlin. *Constants in some inequalities of analysis.* Wiley and Sons, Chiester–New York, 1986.

[12] P. Panagiotopoulos. *Inequality problems in mechanics and applications*, Birkhäuser, Boston, 1985.

[13] S. Repin. A posteriori error estimation for variational problems with uniformly convex functionals. *Mathematics of Computations*, 69 (2000), 230, 481–500.

l	left	right	β	middle	α	major.	error^2/2	I_{eff}
1	2.9e-03	3.6e-03	1.24	1.7e-03	0	8.2e-03	1.9e-03	2.06
2	9.7e-04	1.4e-03	1.50	5.2e-04	0	2.9e-03	5.1e-04	2.41
3	3.9e-04	1.0e-03	2.78	1.8e-04	1.0e-02	1.6e-03	1.3e-04	3.56
4	1.5e-04	6.6e-04	4.56	1.1e-04	3.1e-02	9.3e-04	3.3e-05	5.29
5	5.8e-05	4.1e-04	7.43	7.6e-05	4.4e-02	5.4e-04	8.2e-06	8.14
6	2.0e-05	2.1e-04	10.96	5.4e-05	6.5e-02	2.8e-04	1.9e-06	12.05
7	7.8e-06	1.2e-04	16.98	4.1e-05	9.1e-02	1.7e-04	3.9e-07	20.77

TABLE 8. Method (a) in case $\mu = 0.1$.

l	left	right	β	middle	α	major.	error^2/2	I_{eff}
1	2.1e-03	1.8e-03	0.88	5.6e-04	0	4.5e-03	1.9e-03	1.53
2	8.8e-04	1.3e-03	1.51	2.1e-04	1.0e-02	2.4e-03	5.1e-04	2.18
3	3.3e-04	7.7e-04	2.40	1.3e-04	3.1e-02	1.2e-03	1.3e-04	3.08
4	1.2e-04	4.5e-04	3.82	8.3e-05	4.4e-02	6.7e-04	3.3e-05	4.50
5	4.4e-05	2.3e-04	5.56	5.7e-05	6.5e-02	3.3e-04	8.2e-06	6.38
6	1.6e-05	1.2e-04	8.57	4.3e-05	9.1e-02	1.9e-04	1.9e-06	9.96

TABLE 9. Method (b) in case $\mu = 0.1$.

l	left	right	β	middle	α	major.	error^2/2	I_{eff}
1	1.9e-03	9.4e-04	0.48	2.5e-07	1.3e-04	2.9e-03	1.9e-03	1.22
2	6.0e-04	3.0e-04	0.51	1.6e-07	1.3e-04	9.0e-04	5.1e-04	1.33
3	1.7e-04	9.6e-05	0.54	6.8e-08	2.9e-04	2.7e-04	1.3e-04	1.44
4	5.0e-05	3.1e-05	0.62	5.2e-06	4.5e-04	8.7e-05	3.3e-05	1.62
5	1.5e-05	1.0e-05	0.71	2.7e-06	6.0e-04	2.8e-05	8.2e-06	1.87
6	4.7e-06	3.9e-06	0.82	1.2e-06	9.8e-04	9.9e-06	1.9e-06	2.24
7	1.7e-06	1.6e-06	0.94	5.8e-07	1.3e-03	3.9e-06	3.9e-07	3.17

TABLE 10. Method (c) in case $\mu = 0.1$.

[14] S. Repin. A posteriori error estimation for nonlinear variational problems by duality theory. *Zapiski Nauchn. Semin. POMI*, 243(1997), 201–214.

[15] S. Repin. A posteriori estimates for the Stokes problem. *J. Math. Sci. (New York)*, 109 (2002), no. 5, 1950–1964.

[16] S. Repin. Estimates of deviations from exact solutions of elliptic variational inequalities. *Zapiski Nauchn. Semin. V.A. Steklov Mathematical Institute in St.-Petersburg (POMI)*, 271 (2000), 188–203.

[17] Repin S., Sauter S., Smolianski A., *A posteriori error estimates for the Dirichlét problem with account of the error in the approximation of boundary conditions*, Computing **70** (2003), no. 3, 205–233.

[18] Repin S., Sauter S., Smolianski A., *A posteriori error estimation for the Poisson equation with mixed Dirichlét/Neumann boundary conditions*, J. Comput. Appl. Math. **164–165** (2004), 601–612.

[19] Ciarlet, P. G., *The finite element method for elliptic problems*, North-Holland, Amsterdam, 1978.

[20] M. Ainsworth and J. T. Oden, A posteriori error estimation in finite element analysis, Wiley and Sons, New York, 2000.

[21] R. Verfürth, *A review of a posteriori error estimation and adaptive mesh-refinement techniques* Wiley and Sons, Teubner, New-York, 1996.
[22] O. C. Zienkiewicz and J. Z. Zhu, *A simple error estimator and adaptive procedure for practical engineering analysis,* Internat. J. Numer. Meth. Engrg. 24(1987) pp. 337-357.
[23] I. Babuška and R. Rodriguez, *The problem of the selection of an a posteriori error indicator based on smoothing techniques,* Internat. J. Numer. Meth. Engrg. 36(1993), pp. 539-567.
[24] C. Carstensen and S. Bartels. Each averaging technique yields reliable a posteriori error control in FEM on unstructured grids. I: Low order conforming, nonconforming, and mixed FEM, Math. Comp., 71(2002), pp. 239, 945-969.
[25] I. Babuška and W. C. Rheinboldt, *A-posteriori error estimates for the finite element method.* Internat. J. Numer. Meth. Engrg. 12(1978), pp. 1597-1615.
[26] ———, *Error estimates for adaptive finite element computations.* SIAM J.Numer. Anal. 15(1978), pp. 736-754.
[27] I. Babuška and T. Strouboulis, *The finite element method and its reliability*, Oxford University Press, New York, 2001.
[28] P. Neittaanmäki and S. Repin, *Reliable methods for computer simulation, Error control and a posteriori estimates*, Elsevier, New York, 2004.

V.A. Steklov Institute of Mathematics in St.-Petersburg, 191011, Fontanka 27, St.-Petersburg, Russia
E-mail address: repin@pdmi.ras.ru

SFB F013 'Numerical and Symbolic Scientific Computing', Johannes Kepler University Linz, Altenbergerstrasse 69, 4040 Linz, Austria
E-mail address: jan.valdman@sfb013.uni-linz.ac.at